Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 6, 1995, 81–95

DEGREE FORMULAS FOR MAPS WITH NONINTEGRABLE JACOBIAN

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Dedicated to Professor Louis Nirenberg

1. Introduction

This paper arose from a discussion sparked between the authors after the lecture of Louis Nirenberg at the Conference in Naples on June 1, 1995. He presented a joint work with Haïm Brezis [BN] on the degree theory for VMO (vanishing mean oscillation) mappings $f : X \to Y$ between n-dimensional smooth manifolds. Their results include a variety of discontinuous maps. We soon realized that we can contribute to their work by studying some Orlicz–Sobolev classes weaker than $W^{1,n}(X,Y)$. Our approach relies on new estimates for the Jacobians [IS], [GIM] and most recent improvements [I] concerning non-linear commutators. Also L^p-Hodge theory [S], [ISS] plays a crucial role in this paper.

Let us begin with the well known formula for the degree of a C^1-map $f : X \to Y$:

$$\deg(f; X, Y) = \int_X f^\sharp \omega, \tag{1.1}$$

1991 *Mathematics Subject Classification.* 49F05, 49F20, 14H40.

Research of the first, third and fourth authors supported by M.U.R.S.T. 40% (1990).

Research of the second author supported by NSF Grant DMS-9401104, and by GNAFA-CNR.

where ω is a smooth n-form on Y such that $\int_Y \omega = 1$. Here and subsequently $f^\sharp\omega$ denotes the pullback of ω via the map f. The degree is an integer which does not depend on the choice of the n-form ω. It is always convenient to introduce a Riemannian structure on X and Y. Let dx and dy denote the canonical (induced by the metric tensors) oriented volume forms on X and Y, respectively. We then have

$$\deg(f;X,Y)\int_Y dy = \int_X J(x,f)\,dx, \tag{1.2}$$

where $J(x,f)$ stands for the Jacobian of f, that is, $J(x,f)\,dx = f^\sharp(dy)$.

In this paper we shall discuss mappings of Orlicz–Sobolev classes whose Jacobian is not integrable. The aim is to establish an integral formula for the degree which is free of any approximation by smooth mappings. Our main result is contained in Theorem 1 of Section 7. We should point out here that in this result the target manifold Y is assumed to have nontrivial lth cohomology, for some $l = 1,\dots,n-1$. This, unfortunately, excludes the case $Y = S^n$.

2. Spaces of differential forms

Here and subsequently, X is a closed (compact without boundary) oriented C^∞-smooth Riemannian manifold of dimension $n \geq 2$. The lth exterior power of the cotangent bundle will be denoted by $\Lambda^l X$, $l = 0,1,\dots,n$. Each fiber $\Lambda^l_a X$, $a \in X$, is furnished with an inner product induced by the metric tensor on X, which we denote by $\langle \xi,\zeta\rangle$, for l-covectors $\xi,\zeta \in \Lambda^l_a X$. Observe that in this notation we ignore the dependence of the inner product on the point $a \in X$. We will use the symbol $\Gamma(\Lambda^l X)$ to denote sections of $\Lambda^l X$ (l-forms). When we wish to denote a particular subspace of $\Gamma(\Lambda^l X)$, we simply replace Γ by the familiar notation for the space. For example, the infinitely differentiable l-forms on X are denoted by $C^\infty(\Lambda^l X)$, and those which are L^p-integrable by $L^p(\Lambda^l X)$.

The measure on X will be the one induced by the volume form $dx = *1 \in \Gamma(\Lambda^n X)$, where $* : \Lambda^l X \to \Lambda^{n-l} X$ stands for the Hodge star operator. We shall omit notation of the volume form under the integral sign. Accordingly, the norm of an l-form $\omega \in L^p(\Lambda^l X)$ is defined by

$$\|\omega\|_p = \left(\int_X |\omega|^p\right)^{1/p} \qquad \text{if } 1 \leq p < \infty, \tag{2.1}$$

$$\|\omega\|_\infty = \operatorname*{ess\,sup}_X |\omega|. \tag{2.2}$$

If $1 \leq p,q \leq \infty$ is a Hölder conjugate pair, then the scalar product of $\alpha \in L^p(\Lambda^l X)$ and $\beta \in L^q(\Lambda^l X)$ is given by

$$(\alpha,\beta) = \int_X \alpha\wedge *\beta = \int_X \beta\wedge *\alpha = \int_X \langle\alpha,\beta\rangle. \tag{2.3}$$

Of fundamental concern to us will be the exterior derivative $d : C^\infty(\Lambda^l X) \to C^\infty(\Lambda^{l+1} X)$ and its formal adjoint $d^* = (-1)^{1+nl} * d * : C^\infty(\Lambda^{l+1} X) \to C^\infty(\Lambda^l X)$, also known as the Hodge codifferential. The duality between these operators is emphasized by the formula of integration by parts

$$(d\varphi, \psi) = (\varphi, d^*\psi) \tag{2.4}$$

for $\varphi \in C^\infty(\Lambda^l M)$ and $\psi \in C^\infty(\Lambda^{l+1} M)$. Of course, d and d^* can be extended to more general spaces of differential forms.

An $\omega \in L^1(\Lambda^l X)$ is said to have a *generalized exterior derivative* in case there exists an integrable $(l+1)$-form on X, denoted by $d\omega$, such that $(\omega, d^*\eta) = (d\omega, \eta)$ for every test form $\eta \in C^\infty(\Lambda^{l+1} X)$. The notion of generalized exterior coderivative is defined analogously. We refer to $\ker d = \{\omega \in L^1(\Lambda^l X) : d\omega = 0\}$ as the *closed l-forms* and to $\ker d^* = \{\omega \in L^1(\Lambda^l X) : d^*\omega = 0\}$ as the *co-closed l-forms*. A form $\omega \in L^1(\Lambda^l X)$ which is both closed and co-closed will be called a *harmonic field* of degree l. We denote by $\mathcal{H}^l(X)$ the space of all harmonic fields on X, and regard it as well known that such forms are C^∞-smooth.

Each de Rham cohomology class of X is uniquely represented by a harmonic field. Clearly, the Hodge star operator preserves harmonic fields. Precisely, we have $*\mathcal{H}^l(X) = \mathcal{H}^{n-l}(X)$.

For $1 \le p < \infty$, the Sobolev space $W^{1,p}(\Lambda^l X)$ is defined in the usual fashion by using coordinate systems. Then the Meyers–Serrin approximation theorem states that $C^\infty(\Lambda^l X)$ is dense in $W^{1,p}(\Lambda^l X)$.

One special feature of the questions we shall discuss is that partial differentiation will occur only via the operators d or d^*. Therefore, the natural spaces of differential forms for these problems will not require that all partials exist. For the space $W^{d,p}(\Lambda^l X)$, we only require that both a form and its generalized exterior derivative are L^p-integrable:

$$W^{d,p}(\Lambda^l X) = \{\omega \in L^p(\Lambda^l X) : d\omega \in L^p(\Lambda^{l+1} X)\}. \tag{2.5}$$

This is a Banach space equipped with the norm

$$\|\omega\|_{d,p} = \|\omega\|_p + \|d\omega\|_p. \tag{2.6}$$

Similarly, we define $W^{d^*,p}(\Lambda^l X)$ and the norm

$$\|\omega\|_{d^*,p} = \|\omega\|_p + \|d^*\omega\|_p. \tag{2.7}$$

Note that both $W^{d,p}(\Lambda^l X)$ and $W^{d^*,p}(\Lambda^l X)$ are modules over the ring $C^\infty(X)$ and that $C^\infty(\Lambda^l X)$ is dense in these spaces.

A Gaffney-type inequality (see [S] and [ISS]) represents a critical L^p-estimate for the operators d and d^*:

$$\|\omega\|_{1,p} \le C_p(X)(\|\omega\|_p + \|d\omega\|_p + \|d^*\omega\|_p), \qquad 1 < p < \infty. \tag{2.8}$$

Hence $W^{1,p}(\Lambda^l X) = W^{d,p}(\Lambda^l X) \cap W^{d^*,p}(\Lambda^l X)$. The right hand side of this estimate can be used as a norm in the Sobolev class $W^{1,p}(\Lambda^l X)$, $1 < p < \infty$.

We now introduce some subspaces of $\bigcap_{1\le s<p} L^s(\Lambda^l X)$, $1 < p < \infty$. The space $\mathcal{A}^p(\Lambda^l X)$ consists of l-forms ω on X such that

$$(2.11)\qquad [\![\omega]\!]_p = \sup_{0<\varepsilon\le 1-1/p} \left[\varepsilon \int_X |\omega|^{p-\varepsilon p}\right]^{1/(p-\varepsilon p)} < \infty.$$

This expression is a norm and $\mathcal{A}^p(\Lambda^l X)$ is a Banach space. The closure of $C^\infty(\Lambda^l X)$ in this norm is a proper subspace of $\mathcal{A}^p(\Lambda^l X)$ which we denote by $\mathcal{C}^p(\Lambda^l X)$. It is not difficult to see that

$$(2.12)\qquad \mathcal{C}^p(\Lambda^l X) = \left\{\omega \in \bigcap_{0\le s<p} L^s(\Lambda^l X) : \lim_{\varepsilon\downarrow 0} \varepsilon \int_X |\omega|^{p-\varepsilon p} = 0\right\}.$$

Two subspaces of $\mathcal{A}^p(\Lambda^l X)$ are worth discussing here. The Marcinkiewicz space, denoted by weak-$L^p(\Lambda^l X)$, consists of forms ω such that

$$\sup_{t>0} t^p \operatorname{meas}\{x : |\omega(x)| > t\}$$

is finite. We have

$$(2.13)\qquad [\![\omega]\!]_p \le C_p \sup_{t>0} t^p \operatorname{meas}\{|\omega| > t\}.$$

Hence weak-$L^p(\Lambda^l X) \subset \mathcal{A}^p(\Lambda^l X)$. The Marcinkiewicz space is not contained in $\mathcal{C}^p(\Lambda^l X)$.

We say that ω belongs to the Orlicz space $L^p \log^{-1} L(\Lambda^l X)$ if

$$(2.14)\qquad [\omega]_{L^p \log^{-1} L} = \left[\int_X \frac{|\omega|^p}{\log(e + |\omega|/\|\omega\|_1)}\right]^{1/p} < \infty.$$

This expression is not a norm. However, we have

$$(2.15)\qquad [\![\omega]\!]_p \le C_p [\omega]_{L^p \log^{-1} L}.$$

The class $C^\infty(\Lambda^l X)$ is dense in $L^p \log^{-1} L(\Lambda^l X)$. In particular, $L^p \log^{-1} L(\Lambda^l X)$ is a subspace of $\mathcal{C}^p(\Lambda^l X)$. For more details see [IS].

Lemma 2.1. *The space $C^\infty(\Lambda^l X) \cap \ker d$ is dense in $\mathcal{C}^p(\Lambda^l X) \cap \ker d$.*

Proof. Fix $\omega \in \mathcal{C}^p(\Lambda^l X) \cap \ker d$. By L^p-Hodge decomposition we have

$$(2.16)\qquad \omega = d\alpha + \gamma,$$

where $\alpha \in W^{1,s}(\Lambda^{l-1} X)$ for all $1 < s < p$ and γ is a harmonic field, thus $\gamma \in C^\infty(\Lambda^l X) \cap \ker d$ (see [ISS]). We only need to approximate α by smooth forms. To this end, we proceed as follows. Let $\{\chi_j\}$, $j = 1, \dots, m$, be a partition of unity subordinate to the coordinate neighborhoods of X. We then decompose $\alpha = \chi_1\alpha + \ldots + \chi_m\alpha$, where we notice that each term $\chi_j\alpha$ belongs to $W^{1,s}(\Lambda^{l-1} X)$

and is supported in a coordinate neighborhood. Its $W^{1,s}$-norm is controlled by $\|\alpha\|_{1,s}$ with a constant independent of s, as long as $1 < s < p$. Next, with the aid of coordinate functions, we pull back $\chi_j \alpha$ to $\mathbb{R}^n$ and then mollify it by the familiar convolution technique. As a result we obtain a sequence $\{\alpha_k\}_{k=1,2,\ldots}$ of forms $\alpha_k \in C^\infty(\Lambda^{l-1}X)$ such that

$$\lim_{k\to\infty} \|\alpha - \alpha_k\|_{1,s} = 0 \qquad \text{for all } 1 < s < p \tag{2.17}$$

and

$$\|\alpha - \alpha_k\|_{1,s} \le C(X)\|\alpha\|_{1,s} \le C(p,X)\|\omega\|_s \tag{2.18}$$

for all $k = 1, 2, \ldots$ and a constant $C(p, X)$ independent of s, $1 < s < p$.

Following formula (2.16) we now define $\omega_k = d\alpha_k + \gamma \in C^\infty(\Lambda^l X) \cap \ker d$, $k = 1, 2, \ldots$ For each $0 < \delta < 1 - 1/p$ we can write

$$\begin{aligned}\sup_{0<\varepsilon<1-1/p} \varepsilon \int_X |\omega_k - \omega|^{p-\varepsilon p} \le & \sup_{\delta<\varepsilon<1-1/p} \varepsilon \int_X |d\alpha_k - d\alpha|^{p-\varepsilon p} \\ & + \sup_{0<\varepsilon\le\delta} \varepsilon \int_X |d\alpha_k - d\alpha|^{p-\varepsilon p}.\end{aligned} \tag{2.19}$$

The latter term, in view of (2.18), is uniformly controlled by $\sup_{0<\varepsilon\le\delta} \varepsilon \int_X |\omega|^{p-\varepsilon p}$. Since $\omega \in \mathcal{C}^p(\Lambda^l X)$ this term can be made as small as one wishes, provided δ is chosen to be sufficiently close to zero (see (2.12)). The remaining term on the right hand side of (2.19) converges to zero as $k \to \infty$, by (2.17). This proves the lemma. □

3. Weak wedge product

Let $\alpha \in L^a(\Lambda^l X)$ and $\beta \in L^b(\Lambda^{n-l}X)$ be forms on X, $l = 1, \ldots, n-1$. We shall be concerned with the wedge product $\alpha \wedge \beta \in \Gamma(\Lambda^n X)$ and its integral

$$\int_X \alpha \wedge \beta. \tag{3.1}$$

This presents no difficulty if the exponents a, b are Hölder conjugate. In fact, $\alpha \wedge \beta$ becomes an integrable n-form and for each test function $\eta \in C^\infty(X)$ we have a trivial estimate

$$\left| \int_X \eta(\alpha \wedge \beta) \right| \le \|\eta\|_\infty \|\alpha\|_a \|\beta\|_b. \tag{3.2}$$

Thus $\alpha \wedge \beta$ can be viewed as a Schwartz distribution of order 0. In order to obtain something more interesting we assume from now on that both α and β are closed forms. In this case it is evident that the integral (3.1) depends only on the cohomology class of α and β, thus defines a bilinear form on $\mathcal{H}^l(X) \times \mathcal{H}^{n-l}(X)$.

It is of interest to know whether the wedge product $\alpha \wedge \beta$ of closed forms can be defined as a Schwartz distribution under a weaker hypothesis on the exponents a and b. The following improvement of inequality (3.2) enables us to accomplish this:

Proposition 3.1. *Suppose $\alpha \in C^\infty(\Lambda^l X) \cap \ker d$ and $\beta \in C^\infty(\Lambda^{n-l} X) \cap \ker d$. Then for each test function $\eta \in C^\infty(X)$ we have*

$$\left| \int_X \eta(\alpha \wedge \beta) \right| \le C(\eta) \, \|\alpha\|_a \|\beta\|_b, \tag{3.3}$$

where a, b is an arbitrary Sobolev conjugate pair, that is, $1 \le a, b < \infty$ and $1/a + 1/b = 1 + 1/n$. Moreover,

$$C(\eta) \le C(a, b, X)(\|\eta\|_\infty + \|d\eta\|_\infty).$$

This can be found in [I] (see also [RRT], [IL]). Now, for $\alpha^1, \alpha^2 \in C^\infty(\Lambda^l X) \cap \ker d$ and $\beta^1, \beta^2 \in C^\infty(\Lambda^{n-l} X) \cap \ker d$ we can write

$$\alpha^1 \wedge \beta^1 - \alpha^2 \wedge \beta^2 = (\alpha^1 - \alpha^2) \wedge \beta^2 + \alpha^1 \wedge (\beta^1 - \beta^2).$$

Hence

$$\left| \int_X \eta(\alpha^1 \wedge \beta^1 - \alpha^2 \wedge \beta^2) \right| \le C(\eta) \|\alpha^1 - \alpha^2\|_a \|\beta^2\|_b + C(\eta) \|\beta^1 - \beta^2\|_b \|\alpha^1\|_a. \tag{3.4}$$

Next recall that the space $C^\infty(\Lambda^l X) \cap \ker d$ is dense in $L^a(\Lambda^l X) \cap \ker d$. Having inequality (3.4) we can define the wedge product $\alpha \wedge \beta$ of forms $\alpha \in L^a(\Lambda^l X) \cap \ker d$ and $\beta \in L^b(\Lambda^{n-l} X) \cap \ker d$, $1/a + 1/b = 1 + 1/n$, as a Schwartz distribution on a test function $\eta \in C^\infty(X)$ by the rule

$$(\eta, \alpha \wedge \beta) = \lim_{j \to \infty} (\eta, \alpha_j \wedge \beta_j), \tag{3.5}$$

where $\{\alpha_j\}$ and $\{\beta_j\}$ are two sequences of closed forms from $C^\infty(\Lambda^l X)$ and $C^\infty(\Lambda^{n-l} X)$, converging to α and β in $L^a(\Lambda^l X)$ and $L^b(\Lambda^{n-l} X)$, respectively. Clearly, this definition does not depend on the approximation. The product

$$\alpha \wedge \beta \in \mathcal{D}'(\Lambda^n X)$$

will be referred to as *weak wedge product*.

4. Perturbations of closed forms

In this section we are concerned with a nonlinear perturbation of the forms appearing in inequalities (3.3) and (3.4). Let $\alpha \in C^\infty(\Lambda^l X)$ and $\beta \in C^\infty(\Lambda^{n-l} X)$ be closed forms and let ε be a small positive number. We wish to investigate the integrals of the n-form

$$\frac{\alpha \wedge \beta}{|\alpha|^\varepsilon |\beta|^\varepsilon}. \tag{4.1}$$

It is therefore natural to try to estimate the L^p-distance of $|\alpha|^{-\varepsilon}\alpha$ from the space of closed forms, that is, $L^p(\Lambda^l X)\cap\ker d$. We refer to [IS], [IL] and [I] for relevant material and latest developments. Accordingly, there exists a closed form $\alpha_0\in L^p(\Lambda^l X)$ such that

$$\||\alpha|^{-\varepsilon}\alpha-\alpha_0\|_p\le C_p(X)\varepsilon\|\alpha\|_{p-\varepsilon p}^{1-\varepsilon} \tag{4.2}$$

provided $p>1$ and $\varepsilon<1-1/p$. However, the proof of this inequality exceeds the scope of this paper. More detailed information together with new ingredients are available in [I]. We shall now compile Theorem 6.1 of [I] to extend inequality (3.4) as follows:

Proposition 4.1. *Let $1<a,b<\infty$ be Sobolev conjugate and $1<p,q<\infty$ be Hölder conjugate exponents and let $a<\varepsilon<\min\{1/p,1/q,1-1/a,1-1/b\}$. Then for each test function $\eta\in C^\infty(X)$ we have*

$$\begin{aligned}
&\left|\int_X\eta\left(\frac{\alpha^1\wedge\beta^1}{|\alpha^1|^\varepsilon|\beta^1|^\varepsilon}-\frac{\alpha^2\wedge\beta^2}{|\alpha^2|^\varepsilon|\beta^2|^\varepsilon}\right)\right| \\
&\quad\le C(\eta)\|\alpha^1-\alpha^2\|_{a-\varepsilon a}^{1-\varepsilon}\|\beta^2\|_{b-\varepsilon b}^{1-\varepsilon}+C(\eta)\|\beta^1-\beta^2\|_{b-\varepsilon b}^{1-\varepsilon}\|\alpha^1\|_{a-\varepsilon a}^{1-\varepsilon} \\
&\qquad+\varepsilon C_p(X)\|\eta\|_\infty\|\alpha^1-\alpha^2\|_{p-\varepsilon p}^{(1-\varepsilon)/2}(\|\alpha^1\|_{p-\varepsilon p}+\|\alpha^2\|_{p-\varepsilon p})^{(1-\varepsilon)/2}\|\beta^2\|_{q-\varepsilon q}^{1-\varepsilon} \\
&\qquad+\varepsilon C_p(X)\,\|\eta\|_\infty\|\beta^1-\beta^2\|_{q-\varepsilon q}^{(1-\varepsilon)/2}(\|\beta^1\|_{q-\varepsilon q}+\|\beta^2\|_{q-\varepsilon q})^{(1-\varepsilon)/2}\|\alpha^1\|_{p-\varepsilon p}^{1-\varepsilon}.
\end{aligned} \tag{4.3}$$

Here $C(\eta)$ is the same constant as in (3.3) and (3.4).

Critical to our next step is the presence of the factor ε in front of the last two terms. By the definition of the norm $[\![\ ]\!]_p$ (see (2.11)), these two terms are controlled by

$$[\![\alpha^1-\alpha^2]\!]_p^{(1-\varepsilon)/2}([\![\alpha^1]\!]_p+[\![\alpha^2]\!]_p)^{(1-\varepsilon)/2}[\![\beta^2]\!]_q^{1-\varepsilon} \tag{4.4}$$

and

$$[\![\beta^1-\beta^2]\!]_q^{(1-\varepsilon)/2}([\![\beta^1]\!]_q+[\![\beta^2]\!]_q)^{(1-\varepsilon)/2}[\![\alpha^1]\!]_p^{1-\varepsilon}. \tag{4.5}$$

Concerning the first two terms of the right hand side of (4.3) we can choose $1<a<p$ and $1<b<q$ so that these terms will also be controlled by (4.4) and (4.5), respectively, as is easy to check.

Note that we did not really have to use smoothness of the forms α^1, α^2, β^1 and β^2; we could have applied the above arguments to forms of the class $\mathcal{A}^p(\Lambda^l X)$ and $\mathcal{A}^q(\Lambda^{n-l}X)$, respectively. Summarizing, we have

Proposition 4.2. *Let $1 < p, q < \infty$ be Hölder conjugate exponents, $\alpha^1, \alpha^2 \in \mathcal{A}^p(\Lambda^l X) \cap \ker d$, $\beta^1, \beta^2 \in \mathcal{A}^q(\Lambda^{n-l} X) \cap \ker d$ and $0 < \varepsilon < \min\{1/p, 1/q\}$. Then for each test function $\eta \in C^\infty(X)$ we have*

$$\tag{4.6} \begin{aligned} &\bigg| \int_X \eta \bigg(\frac{\alpha^1 \wedge \beta^1}{|\alpha^1|^\varepsilon |\beta^1|^\varepsilon} - \frac{\alpha^2 \wedge \beta^2}{|\alpha^2|^\varepsilon |\beta^2|^\varepsilon} \bigg) \bigg| \\ &\quad \le C_p(X)(\|\eta\|_\infty + \|d\eta\|_\infty) [\![\alpha^1 - \alpha^2]\!]_p^{(1-\varepsilon)/2} ([\![\alpha^1]\!]_p + [\![\alpha^2]\!]_p)^{(1-\varepsilon)/2} [\![\beta^2]\!]_q^{1-\varepsilon} \\ &\qquad + C_p(X)(\|\eta\|_\infty + \|d\eta\|_\infty) [\![\beta^1 - \beta^2]\!]_q^{(1-\varepsilon)/2} ([\![\beta^1]\!]_q + [\![\beta^2]\!]_q)^{(1-\varepsilon)/2} [\![\alpha^1]\!]_p^{1-\varepsilon}. \end{aligned}$$

In particular,

$$\tag{4.7} \bigg| \int_X \eta \frac{\alpha^1 \wedge \beta^1}{|\alpha^1|^\varepsilon |\beta^1|^\varepsilon} \bigg| \le C_p(X)(\|\eta\|_\infty + \|d\eta\|_\infty) [\![\alpha^1]\!]_p^{1-\varepsilon} [\![\beta^1]\!]_q^{1-\varepsilon}.$$

It is natural to try ε go to zero.

Corollary 4.1. *For $\alpha \in \mathcal{A}^p(\Lambda^l X) \cap \ker d$ and $\beta \in \mathcal{A}^q(\Lambda^{n-l} X) \cap \ker d$ and $\eta \in C^\infty(X)$ the integrals*

$$\tag{4.8} \int_X \eta \frac{\alpha \wedge \beta}{|\alpha|^\varepsilon |\beta|^\varepsilon}$$

stay bounded as ε decreases to zero, while the n-form $\alpha \wedge \beta$ need not be integrable.

One may ask whether the integral (4.8) has a limit as $\varepsilon \to 0$. The affirmative answer is given by the following

Proposition 4.3. *Given $\alpha \in \mathcal{C}^p(\Lambda^l X) \cap \ker d$ and $\beta \in \mathcal{C}^q(\Lambda^{n-l} X) \cap \ker d$, $1 < p, q < \infty$, $1/p + 1/q = 1$. The weak wedge product $\alpha \wedge \beta \in \mathcal{D}'(X)$ can be given by the following formula:*

$$\tag{4.9} (\eta, \alpha \wedge \beta) = \lim_{\varepsilon \downarrow 0} \int_X \eta \frac{\alpha \wedge \beta}{|\alpha|^\varepsilon |\beta|^\varepsilon}.$$

The advantage of using this formula lies in the fact that we may compute $(\eta, \alpha \wedge \beta)$ without approximating α and β by smooth closed forms.

Proof. It follows from Lemma 2.1 that there exist $\alpha_j \in C^\infty(\Lambda^l X) \cap \ker d$ and $\beta_j \in C^\infty(\Lambda^{n-l} X) \cap \ker d$ such that $[\![\alpha - \alpha_j]\!]_p \to 0$ and $[\![\beta - \beta_j]\!]_q \to 0$ as j goes to infinity. We then write

$$\begin{aligned} (\eta, \alpha \wedge \beta) - \int_X \eta \frac{\alpha \wedge \beta}{|\alpha|^\varepsilon |\beta|^\varepsilon} &= (\eta, \alpha \wedge \beta - \alpha_j \wedge \beta_j) \\ &\quad + \int_X \eta \bigg(\frac{\alpha_j \wedge \beta_j}{|\alpha_j|^\varepsilon |\beta_j|^\varepsilon} - \frac{\alpha \wedge \beta}{|\alpha|^\varepsilon |\beta|^\varepsilon} \bigg) \\ &\quad + \int_X \eta \bigg(\alpha_j \wedge \beta_j - \frac{\alpha_j \wedge \beta_j}{|\alpha_j|^\varepsilon |\beta_j|^\varepsilon} \bigg). \end{aligned}$$

For j sufficiently large we can make the first two terms as small as we wish regardless of ε. Indeed, the first term is small by the definition of the weak wedge product (see (3.5)). The second term can be uniformly estimated in terms of $[\![\alpha_j - \alpha]\!]_p$ and $[\![\beta_j - \beta]\!]_q$ by using Proposition 4.2, thus is small as well. When j is fixed, the third term goes to zero as $\varepsilon \to 0$. This proves formula (4.9). $\square$

5. Orlicz–Sobolev classes of mappings

We shall consider, together with X, another closed oriented C^∞-smooth Riemannian manifold, say Y, of the same dimension n. For the purpose of our study there will be no loss of generality in assuming that Y is connected. Furthermore, by a theorem of J. Nash [N] we may also assume that Y is isometrically imbedded in some Euclidean space $\mathbb{R}^N$. This is in order to make the definition of a Sobolev map between manifolds a little easier.

A mapping $f : X \to Y$ is said to belong to the *Sobolev class* $W^{1,p}(X,Y)$ if $f \in W^{1,p}(X,\mathbb{R}^N)$ and $f(x) \in Y$ for a.e. $x \in X$. Thus the differential

$$Df(x) : T_xX \to T_yY \subset \mathbb{R}^N, \qquad y = f(x), \tag{5.1}$$

is defined at almost every $x \in X$.

The Sobolev class $W^{1,p}(X,Y)$ does not depend on the imbedding $Y \hookrightarrow \mathbb{R}^N$. It is a complete metric subspace of $W^{1,p}(X,\mathbb{R}^N)$. The metric depends on the imbedding. However, a different choice of the imbedding leads to an equivalent metric.

Observe that it is legitimate to speak of a weakly converging sequence $\{f_j\}$ of mappings $f_j \in W^{1,p}(X,Y)$, $j = 1, 2, \dots$ Indeed, by the Sobolev compactness theorem, $\{f_j\}$ converges to f in $L^p(X,\mathbb{R}^N)$ and therefore $f(x) \in Y$ for a.e. $x \in X$.

A natural question arises as to whether a given map $f \in W^{1,p}(X,Y), 1 \leq p < \infty$, can be approximated by C^∞-smooth mappings. One has to be a little careful here because for $1 \leq p < n$, there is no assumption about continuity of f. Hence, there is no chance of expressing f in local coordinates as a map of subdomains of $\mathbb{R}^N$. It is not difficult, however, to show that $C^\infty(X,Y)$ is dense in $W^{1,p}(X,Y)$ for $p \geq n$ [BN]. Only four years ago was the case $p < n$ settled by F. Bethuel [B]. Unfortunately, a deeper discussion of the approximation problem would lead us too far astray.

We say that a mapping $f : X \to Y$ belongs to $\mathcal{A}^p(X,Y)$, $1 < p < \infty$, if

(i) $f \in W^{1,s}(X,Y)$ for all $1 \leq s < p$,

(ii) $[\![Df]\!]_p = \sup\limits_{0<\varepsilon<1-1/p} \left[\varepsilon \int_X |Df|^{p-\varepsilon p}\right]^{1/(p-\varepsilon p)} < \infty.$

Recall that this class contains the weak-$W^{1,p}(X,Y)$ as well as the Orlicz–Sobolev class $W^p \log^{-1} W(X,Y)$ of mappings $f : X \to Y$ satisfying

$$\int_X \frac{|Df|^p}{\log(e+|Df|)} < \infty. \tag{5.2}$$

Both classes $\mathcal{A}^p(X,Y)$ and $W^p\log^{-1}W(X,Y)$ are complete metric spaces. The completion of $C^\infty(X,Y)$ in $\mathcal{A}^p(X,Y)$ will be denoted by $\mathcal{C}^p(X,Y)$. We do not know, however, if smooth mappings are dense in $W^p\log^{-1}W(X,Y)$ if $1\le p\le n$. The completion of $C^\infty(X,Y)$ in the metric of the space $W^n\log^{-1}W(X,Y)$ will be of special interest to us. We denote it by

$$\mathcal{W}^n\log^{-1}\mathcal{W}(X,Y). \tag{5.3}$$

Let us point out, in connection with [BN], that mappings of this class need not belong to $\mathrm{BMO}(X,Y)$ or $\mathrm{VMO}(X,Y)$. We close this section with two examples showing that the inclusions fail. To this end, we introduce the function

$$h(x)=\begin{cases} 0 & \text{if } |x|\ge 1,\\ -\log|x| & \text{if } 1/2\le |x|\le 1,\\ \log 2 & \text{if } |x|\le 1/2.\end{cases}$$

Clearly, h has support in the unit ball of $\mathbb{R}^n$.

Now let Ω be an open subset of $\mathbb{R}^n$. If we set $r_j=2^{-j^2}$ for $j\in\mathbb{N}$, then $\sum_j r_j<\infty$ and therefore we can find a sequence of points $x_j\in\Omega$ such that the balls $B(x_j;r_j)$ are pairwise disjoint and contained in Ω (at least for j large enough). Next we define

$$h_j(x)=a_j h\bigg(\frac{x-x_j}{r_j}\bigg)$$

with a_j suitable constants that we shall choose later. Finally, we set $f=\sum_j h_j$. Notice that $f(x)=h_j(x)$ if $|x-x_j|<r_j$. Hence, we have

$$\|f\|_{\mathrm{BMO}} \ge \fint_{B_j}|h_j-(h_j)_{B_j}| = a_j\fint_B |h-(h)_B|. \tag{5.4}$$

On the other hand,

$$|\nabla h_j|\le \begin{cases} a_j/|x-x_j| & \text{if } r_j/2\le |x-x_j|\le r_j,\\ 0 & \text{if } |x-x_j|\le r_j/2.\end{cases}$$

Setting $k_j=1+j^{-2}$ we have $r_j/2=r_j^{k_j}$. Hence, for $a_j\ge 1$,

$$\begin{aligned}\int_{|x-x_j|\le r_j}\frac{|\nabla h_j|^n}{\log(e+|\nabla h_j|)} &\le a_j^n\int_{r_j^{k_j}\le|x-x_j|\le r_j}\frac{|x-x_j|^{-n}}{-\log|x-x_j|}\\ &= n\omega_n a_j^n\int_{r_j^{k_j}}^{r_j}\frac{dt}{-t\log t} = n\omega_n a_j^n\log k_j.\end{aligned} \tag{5.5}$$

Now we are in a position to produce our examples by choosing a_j.

EXAMPLE 5.1. Since $\log k_j \sim j^{-2}$, we can find $\{a_j\}$ so that $\sum_j a_j^n \log k_j < \infty$ and $a_j \to \infty$. It follows from (5.5) and (5.4) that $|\nabla f| \in L^n \log^{-1} L$, but $f \notin \mathrm{BMO}(\Omega)$.

EXAMPLE 5.2. If we choose $a_j = 1$, for all j, then we find that $|\nabla f| \in L^n \log^{-1} L$. Moreover, $f \in L^\infty$, but $f \notin \mathrm{VMO}(\Omega)$, as follows easily from (5.5).

6. The Jacobian

Given a C^∞-smooth n-form ω on Y. Let $f : X \to Y$ be a mapping of Sobolev class $W^{1,p}(X, Y)$. The pullback of ω via the map f, denoted by $f^\sharp\omega$, belongs to $L^{p/n}(\Lambda^n X)$. For $p = n$ and $\int_Y \omega = 1$ we recall the degree formula for f:

$$\deg(f; X, Y) = \int_X f^\sharp\omega. \tag{6.1}$$

We want to obtain integral formulas for the degree of mappings of weaker Sobolev classes. To this end we shall need an n-form ω of the form

$$\omega = \alpha \wedge \beta, \tag{6.2}$$

where $\alpha \in C^\infty(\Lambda^l X) \cap \ker d$ and $\beta \in C^\infty(\Lambda^{n-l} X) \cap \ker d, 1 < l \leq n - 1$, such that

$$\int_Y \omega = 1. \tag{6.3}$$

An obstruction to the existence of such a form is the lth cohomology group of Y. Indeed, if $\mathcal{H}^l(Y) = 0$, then every closed form, in particular $\omega = \alpha \wedge \beta$ as above, is exact. This implies that $\int_Y \omega = 0$.

On the other hand, if $\mathcal{H}^l(Y) \neq 0$ then there exists a nonzero harmonic field $\mu \in \mathcal{H}^l(Y) \subset C^\infty(\Lambda^l Y) \cap \ker d$. We may assume that $\int_Y |\mu|^2 = 1$. Then $*\mu \in \mathcal{H}^{n-l}(Y) \subset C^\infty(\Lambda^{n-l} Y) \cap \ker d$ and we put $\omega = \mu \wedge *\mu$ to obtain $\int_Y \omega = \int_Y |\mu|^2 = 1$. From now on we shall work with the n-form ω given by (6.2) and (6.3). We find at once that

$$f^\sharp\omega = (f^\sharp\alpha) \wedge (f^\sharp\beta), \tag{6.4}$$

where $f^\sharp\alpha \in L^{p/l}(\Lambda^l X)$ and $f^\sharp\beta \in L^{p/(n-l)}(\Lambda^{n-l} X)$. Next, we assume that $p \geq \max\{l, n - l\}$. This makes it legitimate to apply the commutation rule

$$d(f^\sharp\alpha) = f^\sharp(d\alpha) = 0 \quad \text{and} \quad d(f^\sharp\beta) = f^\sharp(d\beta) = 0. \tag{6.5}$$

The situation is particularly interesting if

$$p \geq \frac{n^2}{n+1}. \tag{6.6}$$

Indeed, regardless of l, the exponents p/l and $p/(n-l)$ exceed Sobolev conjugate numbers, that is,

$$\frac{l}{p} + \frac{n-l}{p} \leq 1 + \frac{1}{n}. \tag{6.7}$$

We can, therefore, speak of $f^\sharp\omega$ as a weak wedge product of closed forms $f^\sharp\alpha$ and $f^\sharp\beta$ (see Section 3). The degree of f can be defined as

$$\deg(f; X, Y) = (1, f^\sharp\alpha \wedge f^\sharp\beta). \tag{6.8}$$

In order to establish basic properties of the degree given by (6.8) one has to approximate f by smooth mappings in the metric of the Sobolev class $W^{1,p}(X,Y)$. For $n^2/(n+1) \leq p < n$ such approximation is always possible if the homotopy class $\pi_{n-1}(Y)$ is zero (see [B]).

7. The degree formula

Our final goal is to establish an integral formula for the degree of maps $f : X \to Y$ with nonintegrable Jacobian. We shall do it here for maps of class $\mathcal{C}^n(X,Y)$ making no appeal to any approximation of f by smooth maps. For other approaches, see [EM] and [H].

To keep the formula symmetric it is worth while introducing another useful pullback of $f : X \to Y$, defined by the rule

$$f^\flat = (-1)^{ln-l} * f^\sharp * : \Gamma(\Lambda^l Y) \to \Gamma(\Lambda^l X). \tag{7.1}$$

This pullback provides for the general formulation of the Jacobian determinant of $f : X \to Y$, with respect to the canonical volume form $dy \in \Gamma(\Lambda^n Y)$, namely

$$\det Df = f^\flat(1).$$

Theorem 1. *Suppose $\mathcal{H}^l(Y) \neq 0$ for some $1 \leq l \leq n-1$. Let $\mu \in \mathcal{H}^l(Y)$ be a harmonic field with $\int_Y |\mu|^2 = 1$. Then for each map $f \in \mathcal{C}^n(X,Y)$ we have*

$$\deg(f; X, Y) = \lim_{\varepsilon \downarrow 0} \int_X \frac{\langle f^\sharp\mu, f^\flat\mu\rangle}{|f^\sharp\mu|^\varepsilon |f^\flat\mu|^\varepsilon}. \tag{7.2}$$

The degree is an integer and is invariant under homotopy within the class $\mathcal{C}^n(X,Y)$.

By choosing ε close to zero this result allows one to evaluate the degree of f by simply computing the integral on the right hand side of (7.2). One does not need to be very precise for this computation because the limit is known to be an integer. The magnitude of ε is also not difficult to perceive by inequality

(4.6). Besides, this result may have some theoretical interest. The proof follows immediately from what we have done in the previous sections.

Proof of Theorem 1. First note that $*\langle f^\sharp\mu, f^\flat\mu\rangle = f^\sharp\mu \wedge f^\sharp(*\mu)$. Define $\alpha = f^\sharp\mu$ and $\beta = f^\sharp(*\mu)$. Clearly, α and β are closed forms. Since $f \in \mathcal{C}^n(X,Y)$ we see at once that

$$\alpha \in \mathcal{C}^{n/l}(\Lambda^l X) \quad \text{and} \quad \beta \in \mathcal{C}^{n/(n-l)}(\Lambda^{n-l} X)$$

and the exponents n/l and $n/(n-l)$ are Hölder conjugate. Proposition 4.3 now shows that the limit in (7.2) exists and equals the weak wedge product (distribution) $f^\sharp\mu \wedge f^\sharp(*\mu)$ evaluated at the test function $\eta \equiv 1$.

Next, let $\{f_j\}$ be a sequence of smooth mappings converging to f in the metric of $\mathcal{A}^n(X,Y)$. It then follows that the corresponding pullbacks $\alpha_j = f_j^\sharp\mu$ and $\beta_j = f_j^\sharp(*\mu)$ converge to α and β, respectively. That is,

$$\lim_{j\to\infty} [\![\alpha - \alpha_j]\!]_{n/l} = 0 \quad \text{and} \quad \lim_{j\to\infty} [\![\beta - \beta_j]\!]_{n/(n-l)} = 0.$$

By Proposition 4.2 we see that for each $\delta > 0$ there is j such that

$$\left| \int_X \frac{\alpha\wedge\beta}{|\alpha|^\varepsilon|\beta|^\varepsilon} - \int_X \frac{\alpha_j\wedge\beta_j}{|\alpha_j|^\varepsilon|\beta_j|^\varepsilon} \right| \le \delta$$

for all $0 < \varepsilon < \max\{l/n, (n-l)/n\}$. Since

$$\lim_{\varepsilon\downarrow 0} \int_X \frac{\alpha_j\wedge\beta_j}{|\alpha_j|^\varepsilon|\beta_j|^\varepsilon} = \int_X \alpha_j\wedge\beta_j$$

is an integer we conclude that

$$\lim_{\varepsilon\downarrow 0} \int_X \frac{\alpha\wedge\beta}{|\alpha|^\varepsilon|\beta|^\varepsilon}$$

is also an integer.

That the degree does not change under small perturbations of the mappings within the class $\mathcal{C}^n(X,Y)$ also follows from inequality (4.6). $\square$

Note that the proof gives more, namely there is $\delta = \delta(X,Y) > 0$ with the property that if two mappings $f_1, f_2 : X \to Y$ are at a distance smaller than δ, with respect to the metric in $\mathcal{A}^n(X,Y)$, then they have the same degree. In other words, the degree function

$$\deg : \mathcal{C}^n(X,Y) \to \mathbb{Z}$$

is uniformly continuous.

Of course, our theorem, with a slight change in the proof, remains valid if we replace $\mathcal{C}^n(X,Y)$ by the Orlicz–Sobolev class $\mathcal{W}\log^{-1}\mathcal{W}(X,Y)$ (see (5.3)).

Unfortunately, the arguments above fail for mappings of the class weak-$W^{1,n}(X,Y)$, though the integrals

$$\int_X \frac{\langle f^\sharp\mu, f^\flat\mu\rangle}{|f^\sharp\mu|^\varepsilon|f^\flat\mu|^\varepsilon}$$

stay bounded as $\varepsilon\downarrow 0$. It is no longer true that such mappings can be approximated by smooth mappings in the metric of $\mathcal{A}^n(X,Y)$.

One more case merits mentioning here. Suppose $f : X\to Y$ is of class $\mathcal{A}^n(X,Y)$ and has non-negative Jacobian determinant, that is,

$$\det Df = f^\flat(1)\geq 0 \qquad \text{a.e. on } X.$$

Then the Jacobian is actually integrable (see [IS] and compare with Corollary 4.1). In this case we may define

$$\deg(f;X,Y)=\int_X\langle f^\sharp\mu, f^\flat\mu\rangle=\int_X|\mu|^2(f)\det Df.$$

For $f\in\mathcal{C}^n(X,Y)$ this formula agrees with (7.2) and, therefore, represents an integer.

Recall that μ is a harmonic field on Y, $\mu\in\mathcal{H}^l(Y)$, such that $\int_Y|\mu|^2=1$.

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Manuscript received September 29, 1995

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