An algorithmic solution of a Birkhoff type problem

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ABSTRACT. We give an algorithmic solution in a simple combinatorial data of Birkhoff's type problem studied in [14], for the category $\operatorname{rep}_{ft}(I, K[t]/(t^m))$ of filtered I-chains of modules over the K-algebra $K[t]/(t^m)$, where $m \geq 2$, I is a finite poset with a unique maximal element, and K is an algebraically closed field.

1. Introduction

Throughout we denote by K an algebraically closed field, and by $\operatorname{mod}(R)$ the category of finitely generated unitary right R-modules, where R is a ring with an identity element. We assume that

$$I \equiv (I, \preceq)$$

is a finite **poset** (that is, partially ordered set) with a unique maximal element $*$, called a peak of I. We fix an integer $m \geq 1$ and consider the K-algebra

$$F_m = K[t]/(t^m).$$

Obviously F_m is a uniserial algebra of K-dimension m, and $F_m = K$, for $m = 1$. Following Gabriel [7], we study the additive category $\operatorname{rep}_{ft}(I, F_m)$ of filtered F_m-representations of I (or filtered I-chains of F_m-modules) whose objects are systems $U = (U_j)_{j\in I}$ of finitely generated F_m-modules $U_j \subseteq U_*$ such that $U_s \subseteq U_j \subseteq U_*$, if $s \preceq j$ in I, see also [11] and [12]. In case the poset I is the chain $1 \to *$, the category $\operatorname{rep}_{ft}(I, F_m)$ is just the category $\mathcal{C}(2, F_m)$ of 2-chains $C = (C_1 \subseteq C_*)$ of F_m-modules studied in [13]. Following Birkhoff [4], the problem of determining the indecomposable objects and the representation type of the category $\mathcal{C}(2, F_m)$ is called the **Birkhoff problem**, see [14]. One of the aims of this paper is to get an algorithmic solution of a more general problem, called **Birkhoff type problem** [14], that is, the problem of determining the indecomposable objects and the representation type (finite, tame, or wild) of the category $\operatorname{rep}_{ft}(I, F_m)$ of I-chains, for an arbitrary poset I with a unique maximal element. We do it in Sections 2 and 3 by proving Theorems 2.4 and 2.5 that, in view of the results of [14], reduce the problem to a combinatorial one. Moreover, the proof given in Section 3 provides with algorithms and computer accessible procedures that construct the list of pairs (I, m) satisfying the conditions required in Theorems 2.4 and 2.5.

In case $m = 1$, the algebra F_m is the field K and $\operatorname{rep}_{ft}(I, F_m) = \operatorname{rep}_{ft}(I, K)$ is the category of $I\setminus\{*\}$-spaces in the sense of Gabriel [7], and the solution of the problem is given in [8], see also [11, Chapter 15]. For $m \geq 2$, the problem is studied by Plohotnik in [9] and by Simson in [12] and [13], where a characterization of finite type is presented. A classification of the pairs (I, m) such that the category $\operatorname{rep}_{ft}(I, F_m)$ is of finite representation type is given in [13, Theorem 3.4]. Here we present similar criteria for $\operatorname{rep}_{ft}(I, F_m)$ to be wild representation type or tame representation type.

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The main results of the present paper (Theorems 2.4 and 2.5) provide with a combinatorial characterization of the Birkhoff type problems that are of wilde type and of tame type, for arbitrary I and $m \geq 2$, because it is shown in [14] that, given $m \geq 2$,

• the category $\mathrm{rep}_{ft}(I, F_m)$ is of tame represaentation type (see [11, Chapter 14] for details) and admits a classification of indecomposable objects if and only if the statement (i) of Theorem 2.5 holds, and

• the category $\mathrm{rep}_{ft}(I, F_m)$ is of wild represaentation type (see [11, Chapter 14] for details) and does not admit a classification of indecomposable objects if and only if the statement (i) of Theorem 2.4 holds.

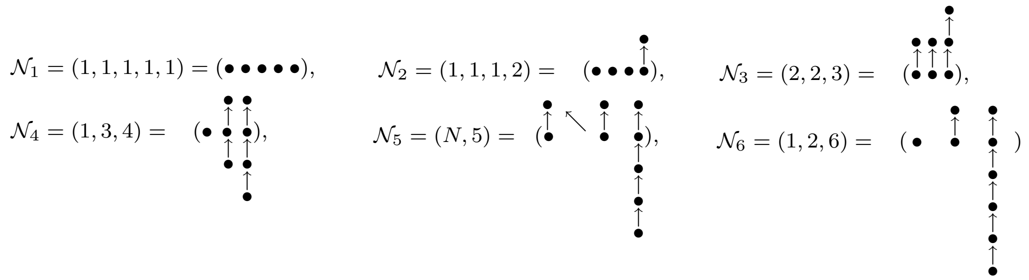
It follows from [14, 2.3 and 5.3] that, for arbitrary I and $m \geq 1$, the categories $\mathrm{rep}_{ft}(I, F_m)$ provide an important class of bimodule matrix problems in the sense of Drozd [5] and [6].

Except of the motivation presented above, one of our main motivalions for the study is the fact that the category $\mathrm{rep}_{ft}(I, F_m)$ is playing an important role in the representation theory of finite dimensional algebras (see [11]), in the study of lattices over orders (see [11, Chapter 13], [12], [15]) and in the investigation of categories of abelian groups (see [1], [10]). Some application of the results presented here are given in the recent papers [2] and [3].

2. A formulation of main results

In the formulation of the main results of the paper we use the following six hypercritical posets of Nazarova [8] (see also [11])

(2.0)



Following [13], given a pair (I, m), where $I \equiv (I, \preceq)$ is a finite poset with a unique maximal element $*$ and $m \geq 1$ is an integer, we define the infinite poset $\widehat{I_m}$ with a $\mathbb{Z}$-action to be the infinite poset

$$\widehat{I_m} = \bigcup_{s \in \mathbb{Z}} I \times \{s\} \tag{2.1}$$

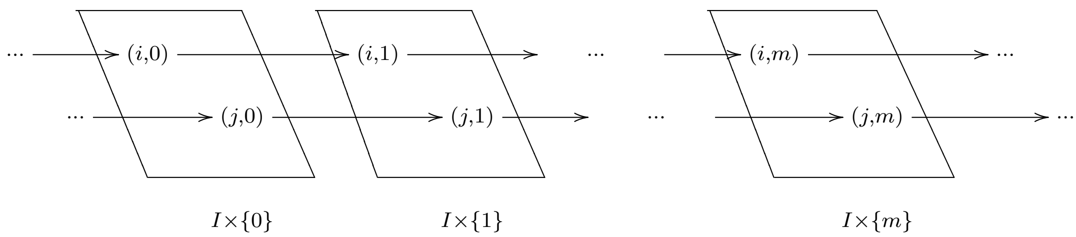
with the partial order relation $\preceq$ defined by the formulae:

(i) $(u, s) \preceq (v, s) \quad \Leftrightarrow \quad u \preceq v \in I$,

(ii) $(i, t) \prec (i, s)$ for all $s < t$ in $\mathbb{Z}$ and $i \in I$,

(iii) $(j, t) \prec (i, t + m)$, for all $j \not\preceq i$ in I and $t \in \mathbb{Z}$.

This means that the poset $I \times \{s\}$ is isomorphic to I and $\widehat{I_m}$ is a disjoint union of countably many copies of the poset $I \cong I \times \{s\}$, $s \in \mathbb{Z}$, with the relations (ii) and (iii). We view $\widehat{I_m}$ as follows (compare with the poset of Zavadskij-Kirichenko in [15]):



where we draw the skew arrow from (j, r) to $(i, r+m)$, if $j \not\preceq i$ in I. The infinite cyclic group $\mathbb{Z}$ acts on $\widehat{I_m}$ by shift in a natural way.

An important role is also played in this paper by the following two finite subposets of the infinite poset $\widehat{I_m}$

$$\widehat{I}_{[0,m]} = \bigcup_{s=0}^{m} I \times \{s\} \quad \supseteq \quad \widehat{I}_{[1,m]} = \bigcup_{s=1}^{m} I \times \{s\}. \tag{2.2}$$

The following important result was established in [13] and [14].

Proposition 2.3. *Assume that I is a finite poset with a unique maximal element $*$, $m \geq 2$ is an integer and $\widehat{I_m}$ is the infinite poset ,(2.1) associated to I and m.*

The infinite poset $\widehat{I_m}$ contains any of the six hypercritical posets $(1,1,1,1,1)$, $(1,1,1,2)$, $(2,2,3)$, $(1,3,4)$, $(N,5)$ and $(1,2,6)$ of Nazarova if and only if the finite subposet $\widehat{I}_{[0,m]}$ (2.2) of $\widehat{I_m}$ contains, as a subposet, any of the six hypercritical posets of Nazarova.

The proposition can be also proved by applying the technique we use in the proof of our main results of this paper, that is, the following two theorems and a corollary proved in Section 3.

Theorem 2.4. *Assume that $m \geq 2$ is an integer, I is a finite poset with a unique maximal element $*$, and $\widehat{I}_{[1,m]} \subseteq \widehat{I}_{[0,m]}$ are the finite posets (2.2) associated to I and m. Then the following three conditions are equivalent.*

(i) *The finite subposet $\widehat{I}_{[0,m]}$ (2.2) of $\widehat{I_m}$ contains, as a subposet, any of the six hypercritical posets of Nazarova (2.0).*

(ii) *Either $m \geq 3$ and the finite subposet $\widehat{I}_{[1,m]}$ (2.2) of $\widehat{I_m}$ contains, as a subposet, any of the six hypercritical posets of Nazarova, or else $1 \leq m \leq 2$ and the finite subposet $\widehat{I}_{[0,m]}$ (2.2) of $\widehat{I_m}$ contains, as a subposet, any of the six hypercritical posets of Nazarova (2.0).*

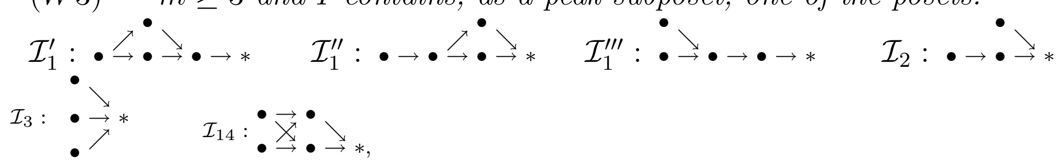
(iii) *The poset I is a chain and the pair (I, m) satisfies any of the following four conditions:*

$(W0_1)$ $m \geq 7$ *and* $|I| \geq 3$, $(W0_3)$ $m \geq 4$ *and* $|I| \geq 5$, *or*
$(W0_2)$ $m \geq 5$ *and* $|I| \geq 4$, $(W0_4)$ $m \geq 3$ *and* $|I| \geq 7$,

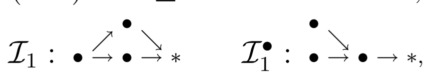
or else the poset I is not a chain and any of the following conditions is satisfied:

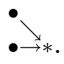
$(W2)$ $m \geq 2$ *and I contains, as a peak subposet, one of the 36?? minimal hypercritical posets of Table 2.6 below*

$(W3)$ $m \geq 3$ *and I contains, as a peak subposet, one of the posets:*



$(W4)$ $m \geq 4$ *and I contains, as a peak subposet, one of the posets:*



$(W5)$ $m \geq 5$ and I contains, as a peak subposet, the poset $\mathcal{F}_0$: 

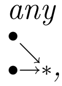
Theorem 2.5. *Assume that $m \geq 2$ is an integer, I is a finite poset with a unique maximal element $*$, and $\widehat{I}_{[0,m]}$ is the finite poset (2.2) associated to I and m. Then the following two conditions are equivalent.*

(i) *The finite subposet $\widehat{I}_{[0,m]}$ of $\widehat{I_m}$ does not contain any of the six hypercritical posets of Nazarova (2.0).*

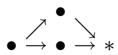
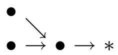
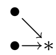
(ii) *The poset I is a chain and $(|I|, m)$ is one of the pairs*

- $(1, m)$, $(2, m)$, *with* $m \geq 2$,
- $(3, 3)$, $(3, 4)$, $(3, 5)$, $(3, 6)$,
- $(4, 3)$, $(4, 4)$,
- $(5, 3)$, $(6, 3)$,
- $(|I|, 2)$, *with* $|I| \geq 3$*; or else*

the poset I *is not a chain and any of the following conditions is satisfied:*

$(t4)$ $m = 4$ *and* I *is the poset* $\mathcal{F}_0$: ,

$(t3)$ $m = 3$ *and* I *is any of the posets*

$\mathcal{I}_1$:  $\mathcal{I}_1^{\bullet}$:  $\mathcal{F}_0$: 

$(t2)$ $m = 2$ *and* I *is a peak subposet of one of the* 18 *tame posets listed in Table* 2.7 *below, or* I *is a peak subposet of the poset*

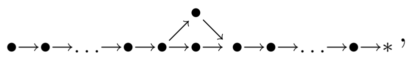
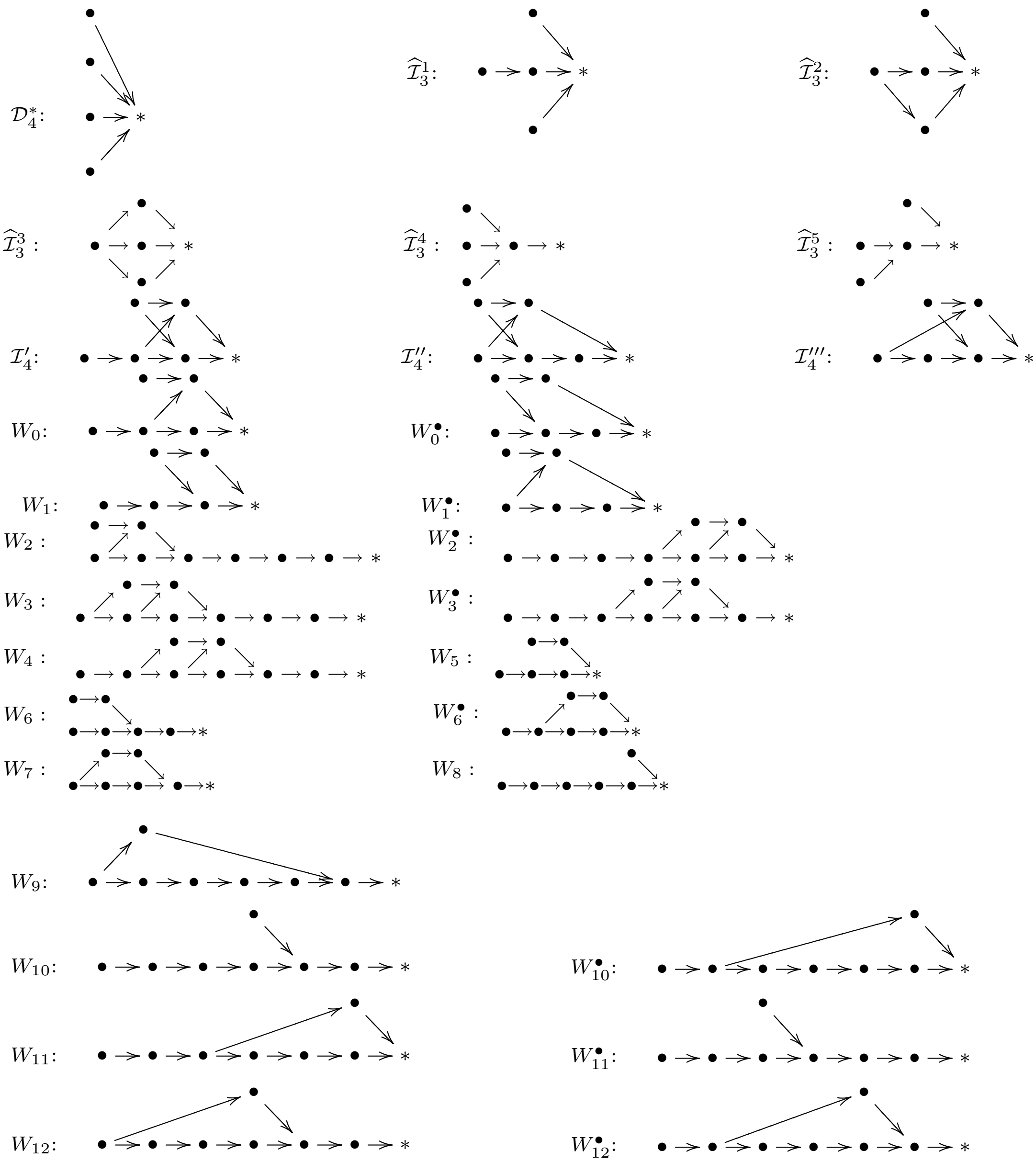
$\mathcal{F}_{0,s}$: , $s + 1$ points, $s \geq 3$,

Table 2.6. Minimal hypercritical wild posets, for $m = 2$



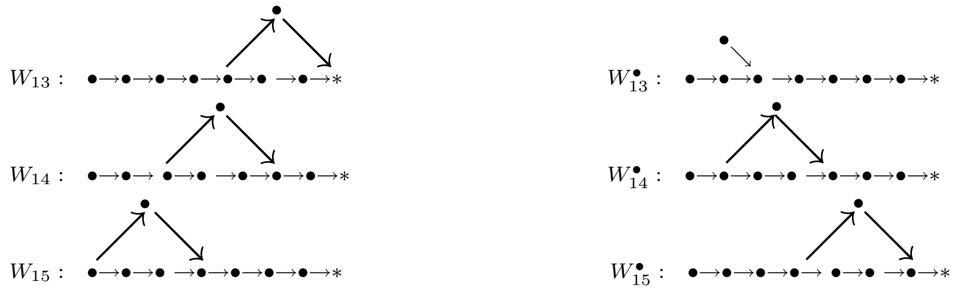
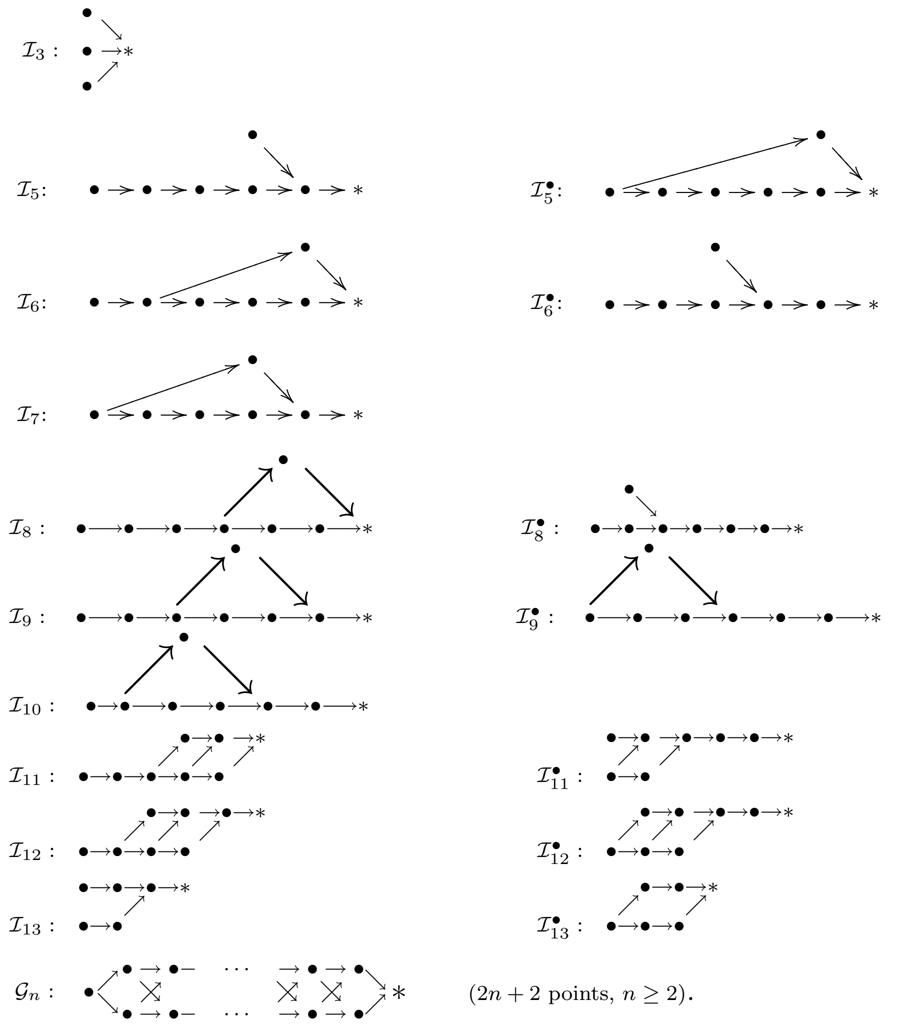


Table 2.7. Maximal tame posets, for $m = 2$



The posets $\mathcal{I}_3$-$\mathcal{I}_{13}^{\bullet}$ of Table 2.7 (different from the garland $\mathcal{G}_n$) are called **maximal tame posets**. In Tables 2.6 and 2.7 we follow the notation introduced in [14].

3. Proof of the main results

Consider the set

$$\mathcal{X} = \{(I, m);\ \ I \text{ is a poset and } \ m \geq 2\} \tag{3.1}$$

consisting of all pairs (I, m), where $m \geq 2$ an integer and $I \equiv (I, \preceq)$ is a finite poset with a unique maximal element $*$, called the peak of I. We say that I is a **peak subposet** of $I' \equiv (I', \preceq')$ if $I \subseteq I'$, the inclusion preserves the partial order relations, and $* = *'$.

We equip the set $\mathcal{X}$ with a partial order relation $\preceq$ defined by the formula

$$(I,m) \preceq (I',m') \quad \Leftrightarrow \quad I \text{ is a peak subposet of } I' \text{ and } m \leq m',$$

for any pair of elements (I,m) and (I',m') of $\mathcal{X}$. We view the poset $\mathcal{X}$ as a disjoint union

$$\mathcal{X} = \mathcal{X}_{\ell in} \cup \mathcal{X}_2 \cup \mathcal{X}_3 \cup \mathcal{X}_4 \cup \ldots \cup \mathcal{X}_m \cup \ldots$$

of subposets, where

- $\mathcal{X}_{\ell in}$ consists of the pair $(I,s) \in \mathcal{X}$ such that I is linearly ordered and $s \geq 2$,
- given $m \geq 2$, $\mathcal{X}_m$ is the set of pair $(I,m) \in \mathcal{X}$ such that I is not linearly ordered.

We denote by $\mathcal{W} \supseteq \check{\mathcal{W}}$ the subposets of $\mathcal{X}$ consisting of all pairs $(I,m) \in \mathcal{X}$ such that the finite subposet $\widehat{I}_{[0,m]}$ (2.2) of $\widehat{I_m}$ (resp. the finite subposet $\widehat{I}_{[1,m]}$ of $\widehat{I}_{[0,m]}$) contains, as a subposet, some of the six hypercritical posets of Nazarova (2.0).

It follows that the subset $\min \mathcal{W}$ of all minimal elements in the posets $\mathcal{W}$ is a disjoint union

$$\min \mathcal{W} = \mathcal{W}^{\vee}_{\ell in} \cup \mathcal{W}^{\vee}_2 \cup \mathcal{W}^{\vee}_3 \cup \mathcal{W}^{\vee}_4 \cup \ldots \cup \mathcal{W}^{\vee}_m \cup \ldots, \tag{3.2}$$

where $\mathcal{W}^{\vee}_{\ell in} = \mathcal{X}_{\ell in} \cap \min \mathcal{W}$ and $\mathcal{W}^{\vee}_m = \mathcal{X}_m \cap \min \mathcal{W}$, for each $m \geq 2$.

Proof of Theorem 2.4. Consider the subposet $\mathcal{W}^{\blacklozenge}$ of $\mathcal{X}$ consisting of all pairs (I,m) satisfying any of the conditions $(W0_1)$-$(W0_4)$, $(W2)$, $(W3)$, $(W4)$, and $(W5)$.

First we prove the equivalence (i)⇔(iii). Note that (i)⇔(iii) holds if and only if

$$\mathcal{W}^{\blacklozenge} = \mathcal{W}. \tag{3.3}$$

To prove the equality (3.3), we consider the subset $\mathcal{W}^{\bullet} = \mathcal{W}^{\bullet}_{\ell in} \cup \mathcal{W}^{\bullet}_2 \cup \mathcal{W}^{\bullet}_3 \cup \mathcal{W}^{\bullet}_4 \cup \mathcal{W}^{\bullet}_5$ of $\mathcal{W}^{\blacklozenge}$, where

1° $\mathcal{W}^{\bullet}_5 = \{(\mathcal{F}_0, 5),$

2° $\mathcal{W}^{\bullet}_4 = \{(\mathcal{I}_1, 4), (\mathcal{I}^{\bullet}_1, 4),$

3° $\mathcal{W}^{\bullet}_3 = \{(\mathcal{I}'_1, 3), (\mathcal{I}''_1, 3), (\mathcal{I}'''_1, 3), (\mathcal{I}_2, 3), (\mathcal{I}_3, 3), (\mathcal{I}_{14}, 3),$

4° $\mathcal{W}^{\bullet}_2 = \{(I,2), \text{where } I \text{ is any of the poset of Table 2.6}\}$, and

5° $\mathcal{W}^{\bullet}_{\ell in} = \{(C_3, 7), (C_4, 5), (C_5, 4), (C_7, 3)\}$,

and $C_s : \circ\!\to\!\circ\!\to\!\ldots \to\!\circ\!\to\!\circ\!\to\!\ast$ is a chain with $s \geq 2$ vertices.

<u>Step A</u> We show that $\mathcal{W}^{\bullet} \subseteq \mathcal{W}$ and the inclusions 1°-5° hold, by proving that

(A1) $\mathcal{W}^{\bullet}_{\ell in} \cup \mathcal{W}^{\bullet}_3 \cup \mathcal{W}^{\bullet}_4 \cup \mathcal{W}^{\bullet}_5 \subseteq \check{\mathcal{W}} \subseteq \mathcal{W}$,

(A2) $\mathcal{W}^{\bullet}_2 \subseteq \mathcal{W}$, and

First we prove (A1). To show that $\mathcal{W}^{\bullet}_5 \subseteq \mathcal{W}$, we note that if $I = \mathcal{F}_0$ is the poset $\mathcal{F}_0 : \begin{smallmatrix} \circ & & \\ & \searrow & \\ \circ & \to & \ast \end{smallmatrix}$ then the finite poset $\widehat{I}_{[1,5]}$ has the form

$$\begin{array}{ccccccccc}
\circ & \to & \circ & \to & \bullet & \to & \bullet & \to & \bullet \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bullet & \to & \bullet & \to & \ast & \to & \ast & \to & \ast \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\circ & \to & \circ & \to & \bullet & \to & \bullet & \to & \bullet
\end{array}$$

and contains the subposet of the hypercritical type $(2,3,3)$ marked by the bullet points. It follows that $(\mathcal{F}_0, 5) \in \mathcal{W}$, that is, $\mathcal{W}^{\bullet}_5 = \{(\mathcal{F}_0, 5)\} \subseteq \mathcal{W}$. The inclusion $\mathcal{W}^{\bullet}_3 \cup \mathcal{W}^{\bullet}_4 \subseteq \mathcal{W}$ follows in a similar way.

For the proof of the inclusion $\mathcal{W}^{\bullet}_{\ell in} \subseteq \mathcal{W}$, we only show that the pair $(C_5, 4) \in \mathcal{W}^{\bullet}_{\ell in}$ belongs to $\mathcal{W}$, because the proof for the remaining pairs of $\mathcal{W}^{\bullet}_{\ell in}$ is analogous. Suppose that $m = 4$ and I is the chain $C_5 : \circ\!\to\!\circ\!\to\!\circ\!\to\!\circ\!\to\!\ast$. Then the finite subposet $\widehat{I}_{[1,4]}$ of the infinite poset $\widehat{I}_4$ associated to the pair $(I,4) = (C_5, 4) \in \mathcal{W}^{\bullet}_{\ell in}$ has the form

$$\widehat{I}_{[1,4]} : \qquad \begin{array}{ccccccc} \circ & \to & \circ & \to & \circ & \to & \bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \circ & \to & \circ & \to & \bullet & \to & \circ \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \circ & \to & \bullet & \to & \circ & \to & \circ \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bullet & \to & \circ & \to & \circ & \to & \circ \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bullet & \to & * & \to & * & \to & * \end{array}$$

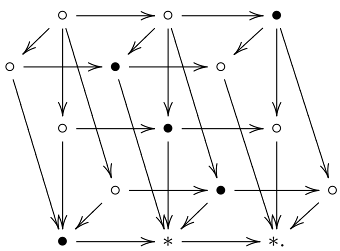
and contains a subposet of the hypercritical type $(2,1,1,1)$ marked by the bullet points. It follows that the pair $(C_5, 4) \in \mathcal{W}^{\bullet}_{\ell in}$ belongs to $\mathcal{W} \cap \mathcal{X}_{\ell in}$. This completes the proof of (A1).

For the proof of the inclusion $\mathcal{W}^{\bullet}_2 \subseteq \mathcal{W}$ in (A2), we only show that the pair $(\widehat{\mathcal{I}}^3_3, 2) \in \mathcal{W}^{\bullet}_2$ belongs to $(\mathcal{W} \cap \mathcal{X}_2) \setminus \check{\mathcal{W}}$, because the proof for the remaining pairs of $\mathcal{W}^{\bullet}_2$ is analogous. Suppose that $m = 2$ and I is the poset

$$I = \widehat{\mathcal{I}}^3_3 : \qquad \begin{array}{ccccc} & \nearrow & \circ & \searrow & \\ \circ & \to & \circ & \to & * . \\ & \searrow & \circ & \nearrow & \end{array}$$

Then the poset $\widehat{I}_{[0,2]}$ associated with the pair $(I, 2) = (\widehat{\mathcal{I}}^3_3, 2) \in \mathcal{W}^{\bullet}_2$ has the form

$\widehat{I}_{[0,2]}$:



Since the subposet $\mathcal{N}$ of $\widehat{I}_{[0,2]}$ marked by the five bullet points is a hypercritical poset of Nazarova of type $(1,1,1,2)$ then the pair $(I,2) = (\widehat{\mathcal{I}}^3_3, 2)$ belongs to $\mathcal{W}^{\bullet}_2$. Since the subposet $\widehat{I}_{[1,2]}$ of $\widehat{I}_{[0,2]}$ does not contain any of the hypercritical posets of Nazarova then the pair $(I,2) = (\widehat{\mathcal{I}}^3_3, 2)$ does not belong to $\check{\mathcal{W}}$.

For the sake of completeness of the proof of the inclusion(A2), we apply the following simple algorithm.

Algorithm W.1. Input: The set $\mathcal{W}^{\bullet}_2$.

```
begin
  twr ← true;
  for (each pair (I, m) from W•_2) do
    if (the poset Î_[0,m] does not contain any Nazarowa posets)
    then
      twr ← false;
      break;
  if (twr = true) then
    print(W•_2 is contained in W∨_2);
 else
    print(W•_2 is not contained in W∨_2);
end
```

The algorithm verifies that the inclusion $\mathcal{W}^{\bullet}_2 \subseteq \mathcal{W}^{\vee}_2$ holds by showing that, given a pair $(I,2) \in \mathcal{W}^{\bullet}_2$, the finite poset $\widehat{I}_{[0,2]}$ (2.2) contains any of the hypercritical posets of Nazarova (2.0). The algorithm uses:

• the function $\mathrm{Imm}(I,m)$ (4.4) described in Section 4 that constructs the finite poset $\widehat{I}_{[0,m]}$ (2.2), for a positive integer $m \geq 2$ and a finite poset I with a unique maximal element, and

• the function pswild(p) (4.5) of the package CREP that checks whether or not, for a poset I, the finite poset $p = \widehat{I}_{[0,m]}$ contains any of the hypercritical posets of Nazarova, see Section 4 for more details.

This finishes the proof of (A2).

Step B Recall that $\mathcal{W}^{\blacklozenge}$ is the subposet of $\mathcal{X}$ consisting of all pairs (I,m) satisfying any of the conditions $(W0_1)$-$(W0_4)$, $(W2)$, $(W3)$, $(W4)$, and $(W5)$. We show that

(B1) the union $\mathcal{W}^{\bullet} = \mathcal{W}^{\bullet}_{\ell in} \cup \mathcal{W}^{\bullet}_2 \cup \mathcal{W}^{\bullet}_3 \cup \mathcal{W}^{\bullet}_4 \cup \mathcal{W}^{\bullet}_5$ is the set of all minimal elements (I,m) in the poset $\mathcal{W}^{\blacklozenge}$ and

(B2) $\mathcal{W}^{\blacklozenge} \subseteq \mathcal{W}$.

We prove (B1), by showing that

- $\mathcal{X}_{\ell in} \cap \min \mathcal{W}^{\blacklozenge} = \mathcal{W}^{\bullet}_{\ell in}$,
- $\mathcal{X}_m \cap \min \mathcal{W}^{\blacklozenge} = \mathcal{W}^{\bullet}_m$, for $m = 2,3,4,5$, and
- the set $\mathcal{X}_m \cap \min \mathcal{W}^{\blacklozenge}$ is empty, for all $m \geq 6$.

Assume that $(I',m') \preceq (I,m)$, the pairs $(I',m'),(I,m)$ belong to $\mathcal{W}^{\blacklozenge}$, and $(I,m) \in \min \mathcal{W}^{\blacklozenge}$.

If I is linearly ordered then (I,m) satisfies any of the conditions $(W0_1)$-$(W0_4)$. Hence, obviously, I' is linearly ordered and (I',m') also satisfies any of the conditions $(W0_1)$-$(W0_4)$. Hence easily follows that $(I',m') = (I,m)$ and therefore, given a linearly ordered poset I, $(I,m) \in \min \mathcal{W}^{\blacklozenge}$ if and only if $(I,m) \in \mathcal{W}^{\bullet}_{\ell in}$. Consequently, we get $\mathcal{X}_{\ell in} \cap \min \mathcal{W}^{\blacklozenge} = \mathcal{W}^{\bullet}_{\ell in}$.

Assume now that I is not linearly ordered. If $m = 2$ then $m' = 2$, by our assumption, and I' is a subposet of I. It follows that I' is not linearly ordered, because otherwise $(I',2) \in \min \mathcal{W}^{\blacklozenge} \cap \mathcal{X}_{\ell in} = \mathcal{W}^{\bullet}_{\ell in}$ and we get a contradiction. Then, by the definition of $\mathcal{W}^{\blacklozenge}$, I' satisfies (W2), that is, I' contains, as a peak subposet, one of the hypercritical wild posets listed in Table 2.6. Hence easily follows that $I' = I$ is one of the hypercritical wild posets listed in Table 2.6 and, consequently, we get $\mathcal{X}_2 \cap \min \mathcal{W}^{\blacklozenge} = \mathcal{W}^{\bullet}_2$. The equality $\mathcal{X}_m \cap \min \mathcal{W}^{\blacklozenge} = \mathcal{W}^{\bullet}_m$, for $m = 3,4,5$, follows in a similar way, by consulting Table 2.6 and the posets listed in (W3), (W4), and (W5) of Theorem 2.4.

Assume that $m \geq 6$. To prove that the set $\mathcal{X}_m \cap \min \mathcal{W}^{\blacklozenge}$ is empty, assume that there exists $(I,m) \in \mathcal{X}_m \cap \min \mathcal{W}^{\blacklozenge}$. By the definition of $\mathcal{W}^{\blacklozenge}$, I contains, as a peak subposet, one of the posets listed in Table 2.6 or one of the posets listed in (W3), (W4), and (W5) of Theorem 2.4. It follows that I contains, as a peak subposet, the poset $\mathcal{F}_0$ and, consequently, $(\mathcal{F}_0,5) \prec (I,m)$. This contradicts the relation $(I,m) \in \min \mathcal{W}^{\blacklozenge}$, because $(\mathcal{F}_0,5) \in \mathcal{W}^{\bullet}_5 \subseteq \min \mathcal{W}^{\blacklozenge}$, and finishes the proof of (B1).

To prove (B2), we note that if $(I'm') \preceq (I,m)$ and $(I',m') \in \mathcal{W}$ then $(I,m' \in \mathcal{W}$. Since $\mathcal{W}^{\bullet} = \min \mathcal{W}^{\blacklozenge} \subseteq \mathcal{W}$, by (B1) and Step A, then $\mathcal{W}^{\blacklozenge} \subseteq \mathcal{W}$ and (B2) follows. This finishes the proof of Step B.

Step C It follows from Step B that, to prove the equivalence (i)$\Leftrightarrow$(iii), it is sufficient to show that

$$\min \mathcal{W} = \mathcal{W}^{\bullet}_{\ell in} \cup \mathcal{W}^{\bullet}_2 \cup \mathcal{W}^{\bullet}_3 \cup \mathcal{W}^{\bullet}_4 \cup \mathcal{W}^{\bullet}_5,$$

or equivalently, that the following equalities hold:

(a) the set $\mathcal{W}^{\vee}_m$ is empty, for all $m \geq 6$,

(b) $\mathcal{W}^{\vee}_5 = \mathcal{W}^{\bullet}_5$,

(c) $\mathcal{W}^{\vee}_4 = \mathcal{W}^{\bullet}_4$,

(d) $\mathcal{W}^{\vee}_3 = \mathcal{W}^{\bullet}_3$,

(e) $\mathcal{W}^{\vee}_2 = \mathcal{W}^{\bullet}_2$, and

(f) $\mathcal{W}^{\vee}_{\ell in} = \mathcal{W}^{\bullet}_{\ell in}$.

To prove (a)-(f), we need to show that the inclusion $\mathcal{W}^{\vee} \subseteq \mathcal{W}^{\bullet}$ holds, because the inverse inclusion was established above.

First we prove (f) by showing that the inclusion $\mathcal{W}^{\vee}_{\ell in} \subseteq \mathcal{W}^{\bullet}_{\ell in}$ holds. Assume that $(I,m) \in \mathcal{W}$ is a minimal element of $\mathcal{W}$, $m \geq 2$, and $I = C_s$ is a chain, with $s \geq 1$. Note that $(I',m') \prec (I,m)$, where (I',m') is any of the pairs $(C_3,6)$, $(C_4,4)$, and $(C_6,3)$, because the finite poset $\widehat{I'}_{[0,m']}$ contains one of the Nazarova's hypercitical poset (2.0), if $(I', m') \in \{(C_3, 6), (C_4, 4), (C_6, 3)\}$. Since, by (A1), we have $\mathcal{W}^{\bullet}_{\ell in} = \{(C_3, 7), (C_4, 5), (C_5, 4), (C_7, 3)\} \subseteq \mathcal{W}$ and (I, m) is a minimal element of $\mathcal{W}$ then (I, m) is one of the pairs listed in $\mathcal{W}^{\bullet}_{\ell in}$, and (f) follows.

To prove (a)-(e), assume that $(I, m) \in \mathcal{W}$ is a minimal element of $\mathcal{W}$, $m \geq 2$, and I is not a chain, that is, I contains, as a peak subposet, the poset

$$\mathcal{F}_0 : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[dr] & \\ \bullet \arrow[r] & * \end{tikzcd}.$$

Hence $(\mathcal{F}_0, m) \preceq (I, m)$. Since we have shown earlier that $(\mathcal{F}_0, 5) \in \mathcal{W}$ then the set $\mathcal{W}^{\vee}_m$ is empty, for all $m \geq 6$, and $\mathcal{W}^{\vee}_5 = \{(\mathcal{F}_0, 5)\}$. Hence (a) and (b) follow.

(c) Assume that $m = 4$ and $(I, 4)$ is a minimal element of $\mathcal{W}$. Since I contains, as a peak subposet, the poset $\mathcal{F}_0$ and a direct checking shows that $(\mathcal{F}_0, 4) \notin \mathcal{W}$ then the peak poset embedding $\mathcal{F}_0 \hookrightarrow I$ is proper and we need to describe all such posets I with a unique maximal element and with four vertices. It is clear that they are just the following four posets

$$\mathcal{I}_1 : \begin{tikzcd}[row sep=tiny, column sep=small] & \bullet \arrow[dr] & \\ a \arrow[ur] \arrow[r] & \bullet \arrow[r] & * \end{tikzcd}, \quad \mathcal{I}_1^{\bullet} : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[dr] & & \\ \bullet \arrow[r] & a \arrow[r] & * \end{tikzcd}, \quad \mathcal{I}_2 : \begin{tikzcd}[row sep=tiny, column sep=small] & \bullet \arrow[dr] & \\ \bullet \arrow[r] & a \arrow[r] & * \end{tikzcd}, \quad \mathcal{I}_3 : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[dr] & \\ \bullet \arrow[r] & * \\ a \arrow[ur] & \end{tikzcd}$$

presented in (W3) and (W4) of Theorem 2.4, and I contains as a peak subposet one of them. Since $(\mathcal{I}_2, 3), (\mathcal{I}_3, 3) \in \mathcal{W}^{\bullet}_3$ and $(I, 4)$ is chosen to be minimal then $(I, 4)$ is one of the pairs $(\mathcal{I}_1, 4)$, $(\mathcal{I}_1^{\bullet}, 4)$. This shows the inclusion $\mathcal{W}^{\vee}_4 = \{(\mathcal{I}_1, 4), (\mathcal{I}_1^{\bullet}, 4)\} \subseteq \mathcal{W}^{\bullet}_4$ and finishes the proof of (c).

(d) Assume that $m = 3$ and $(I, 3)$ is a minimal element of $\mathcal{W}$. First we show by a direct calculation that none of the pairs $(\mathcal{I}_1, 3)$, $(\mathcal{I}_1^{\bullet}, 3)$, $(\mathcal{I}_2, 2)$, $(\mathcal{I}_3, 2)$ belongs to $\mathcal{W}$. Since $(\mathcal{I}_2, 3), (\mathcal{I}_3, 3) \in \mathcal{W}^{\bullet}_3$ then they are minimal. It follows that I contains, as a proper peak subposet, any the posets $\mathcal{I}_1$, $\mathcal{I}_1^{\bullet}$ and we need to describe all such posets I with a unique maximal element and with five vertices. It is clear that such enlargements I of $\mathcal{I}_1$ are just the following five posets

$$\mathcal{I}_1' : \begin{tikzcd}[row sep=tiny, column sep=small] & \bullet \arrow[dr] & & \\ \bullet \arrow[ur] \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & * \end{tikzcd} \quad \mathcal{I}_1'' : \begin{tikzcd}[row sep=tiny, column sep=small] & & \bullet \arrow[dr] & \\ \bullet \arrow[r] & \bullet \arrow[ur] \arrow[r] & \bullet \arrow[r] & * \end{tikzcd}, \quad \mathcal{I}_{14} : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[r] \arrow[dr] & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[r] & \bullet \arrow[r] & * \end{tikzcd} \quad \mathcal{I}_{15} : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[r] \arrow[dr] & \bullet \arrow[dr] & \\ \bullet \arrow[r] & \bullet \arrow[r] & * \end{tikzcd}$$

$$\widehat{\mathcal{I}}_3^3 : \begin{tikzcd}[row sep=tiny, column sep=small] & \bullet \arrow[dr] & \\ \bullet \arrow[r] \arrow[dr] & \bullet \arrow[r] & * \\ & \bullet \arrow[ur] & \end{tikzcd},$$

and such enlargements I of $\mathcal{I}_1^{\bullet}$ are just the following five posets

$$\mathcal{I}_1''' : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[dr] & & & \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & * \end{tikzcd} \quad \mathcal{I}_1'''' : \begin{tikzcd}[row sep=tiny, column sep=small] & \bullet \arrow[dr] & & \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & * \end{tikzcd} \quad \mathcal{I}_1' : \begin{tikzcd}[row sep=tiny, column sep=small] & \bullet \arrow[dr] & & \\ \bullet \arrow[ur] \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & * \end{tikzcd}$$

$$\widehat{\mathcal{I}}_3^4 : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[dr] & & \\ \bullet \arrow[r] & \bullet \arrow[r] & * \\ \bullet \arrow[ur] & & \end{tikzcd} \qquad \widehat{\mathcal{I}}_3^5 : \begin{tikzcd}[row sep=tiny, column sep=small] & \bullet \arrow[dr] & \\ \bullet \arrow[r] & \bullet \arrow[r] & * \\ \bullet \arrow[ur] & & \end{tikzcd}.$$

It is easy to check that

- the pairs $(\mathcal{I}_1', 2)$, $(\mathcal{I}_1''', 2)$, $(\mathcal{I}_1'''', 2)$, and $(\mathcal{I}_{14}, 2)$ do not belong to $\mathcal{W}$, whereas
- each of the pairs $(\mathcal{I}_{15}, 2)$, $(\widehat{\mathcal{I}}_3^4, 2)$, $(\widehat{\mathcal{I}}_3^5, 2)$, $(\widehat{\mathcal{I}}_3^3, 2)$, $(\mathcal{I}_1'''', 2)$ belongs to $\mathcal{W}$.

Hence we easily conclude that the minimal pair $(I, 3)$ is one of the pairs listed in $\mathcal{W}^{\bullet}$. This finishes the proof of (d).

(e) Assume that $m = 2$. Because of a high combinatorial complexity of the problem, we list the minimal elements $(I, 2)$ of $\mathcal{W}$ by applying the Algorithm W.2 defined as follows.

Input: the set $\mathcal{W}^{\bullet}_3 = \mathcal{W}^{\vee}_3 = \{(\mathcal{I}_1', 3), (\mathcal{I}_1'', 3), (\mathcal{I}_1''', 3), (\mathcal{I}_2, 3), (\mathcal{I}_3, 3), (\mathcal{I}_{14}, 3)\}$.

W.2.1. Given a finite poset I, with a unique maximal element, we denote by $\mathcal{E}_I$ the set of all poset J, with a unique maximal element, such that $|J| = 1 + |I|$ and I is a peak subposet of J. For each I such that $(I, 3) \in \mathcal{W}^{\bullet}_3$, we construct the set $\mathcal{E}_I$.

W.2.2. For each I of W.2.2, we find a disjoint union decomposition $\mathcal{E}_I = \mathcal{I}_I^1 \cup \mathcal{I}_I^2$, where $\mathcal{E}_I^1$ consists of the posets $J \in \mathcal{E}_I$ such that $(J, 2) \in \mathcal{W}$, and we set $\mathcal{E}_I^2 = \mathcal{E}_I \setminus \mathcal{E}_I^1$[2]. We do it by applying the procedure $\mathrm{Imm}(J, 2)$ and ext(p), see Section 4.

W.2.3. For each I of W.2.1, we list the pairs $(J, 2) \in \mathcal{X}$, with $J \in \mathcal{I}_I^1$, that are minimal in $\mathcal{W}$, by applying the procedure $\mathrm{mini}(L)$, see Section 4.

W.2.4. For each I of W.2.1 and for each $(L, 2) \in \mathcal{X}$ such that $L \in \mathcal{E}_I^2$, we find the finite set $\mathcal{E}_L$ and apply the steps W.2.2 and W.2.3, with I and L interchanged.

W.2.5. As an output we get the list $\mathcal{W}_2^\vee$ of the minimal pairs $(I, 2)$ in $\mathcal{W}$. It turns out that the set $\mathcal{W}_2^\vee$ consists of the following pairs
$(\widehat{\mathcal{I}}_3^2, 2), (\widehat{\mathcal{I}}_3^4, 2), (\widehat{\mathcal{I}}_3^5, 2), (\mathcal{W}_1^\bullet, 2), (\widehat{\mathcal{I}}_3^1, 2), (\mathcal{W}_5, 2), (\mathcal{I}_4'', 2), (\mathcal{W}_6^\bullet, 2), (\mathcal{W}_7, 2), (\mathcal{W}_6, 2), (\mathcal{I}_4''', 2), (\mathcal{W}_0^\bullet, 2), (\mathcal{W}_{13}, 2), (\mathcal{W}_{15}^\bullet, 2), (\mathcal{W}_{14}, 2), (\mathcal{W}_{14}^\bullet, 2), (\mathcal{W}_{15}, 2), (\mathcal{W}_1, 2), (\mathcal{W}_2^\bullet, 2), (\mathcal{W}_3^\bullet, 2), (\mathcal{W}_4, 2), (\mathcal{W}_3, 2), (\mathcal{W}_0, 2), (\mathcal{W}_2, 2), (CW_{11}, 2), (\mathcal{W}_{12}^\bullet, 2), (\mathcal{W}_{12}, 2), (\mathcal{W}_{10}^\bullet, 2), (\mathcal{W}_9, 2), (\mathcal{W}_8, 2), (\mathcal{W}_{10}, 2), (\mathcal{W}_{11}^\bullet, 2), (\mathcal{W}_{13}^\bullet, 2), (\widehat{\mathcal{I}}_3^3, 2), (\mathcal{D}_4^*, 2), (\widehat{\mathcal{I}}_4', 2)$.

Obviously, this is just the list $\mathcal{W}^\bullet$ given in 4°, and hence we get the equality (e).
The above description leads to the following recursion procedure used in the algorithm.

Procedure(I)
construct the set $\mathcal{E}_I$
for(each poset J from $\mathcal{E}_I$) **do**
if($\hat{J}_{[1,2]}$ contains one of the posets $\mathcal{N}_1, \ldots, \mathcal{N}_6$) **then**
search the pairs $(J, 2)$ that are minimal in $\mathcal{W}$,
else
Procedure(J);
end

The following easy modification of the above description leads to the following more general algorithm that determines the set $\mathcal{W}_2^\vee$, and the sets $\mathcal{W}_3^\vee$, $\mathcal{W}_4^\vee$, $\mathcal{W}_5^\vee$ we have already described above.

Algorithm W.2. **Input:** The poset $\mathcal{F}_0$.
Pass:

- Global variables: $\mathcal{W}_2^\vee$, $\mathcal{W}_3^\vee$, $\mathcal{W}_4^\vee$, $\mathcal{W}_5^\vee$, $\mathcal{K}^{\mathcal{W}}$, G, m, MAX ;
- Local variables: n, I;
- Meaning of particular variables:
  - $\mathcal{W}_2^\vee$, $\mathcal{W}_3^\vee$, $\mathcal{W}_4^\vee$, $\mathcal{W}_5^\vee$ - lists of pairs $(J, j) \in \mathcal{W}_j^\vee$, for $j = 2 \ldots 5$, that are minimal in $\mathcal{W}$.
  - $\mathcal{K}^{\mathcal{W}}$ - list of pairs (J, j) that do not belong to any of the sets $\mathcal{W}_2^\vee$, $\mathcal{W}_3^\vee$, $\mathcal{W}_4^\vee$, $\mathcal{W}_5^\vee$.
  - $m = 2..5$.
  - MAX - constant that determines the number of maximal elements of posets.
  - G - the garland $\mathcal{G}_r$, with $r \geq 2$.

begin
enroll the pair $(\mathcal{F}_0, 5)$ in $\mathcal{W}_5^\vee$;
for $m := 4$ **to** 2 **do**
for $k := 1$ **to** length$(\mathcal{W}_{m+1}^\vee)$ **do**
WProcedure$(\mathcal{W}_{m+1}^\vee[k])$;
end

Output: the set $\mathcal{K}^{\mathcal{W}}$ and the set $\mathcal{W}_2^\vee \cup \mathcal{W}_3^\vee \cup \mathcal{W}_4^\vee \cup \mathcal{W}_5^\vee$ of all minimal elements in $\mathcal{W}$.

The algorithm uses the following procedure that modifies the previous one.

Procedure W.2a.

[2]tego nie ma w procedurach???

```
WProcedure($J$)
  if($|J| \leq MAX$) then
    construct the set $\mathcal{E}_J$;
    for (each poset $I$ from $\mathcal{E}_J$) do
    begin
      $Min := false$;
      if($\widehat{I}_{[0,m]}$ contains one of the posets $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$, $\mathcal{N}_5$, $\mathcal{N}_6$)
       then
        for $n := m-1$ to $1$ do
        begin
          if (any of $\mathcal{N}_1, \ldots, \mathcal{N}_6$ is not a subposet of $\widehat{I}_{[0,n]}$)
           then
            $Min := true$;
            break;
        end
        if ($Min = true$) then
          if ($\mathcal{W}^{\vee}_{n+1}$ does not contain any pair $(I', n+1)$,
             with $I'$ a peak subposet of $I$) then
            if ($I$ is not a peak subposet of any $J$,
              with $(J, n+1)$ from the list $\mathcal{W}^{\vee}_{n+1}$) then
              enroll $(I, n+1)$ in $\mathcal{W}^{\vee}_{n+1}$;
            else
              enroll $(I, n+1)$ in $\mathcal{W}^{\vee}_{n+1}$ instead of the pair $(I', n+1)$,
              with $I'$ containing $I$ as a peak subposet.
      else
        WProcedure($I$);
    end
  else
    if ($J$ is not a subposet of $G$) then
      enroll $(J, n)$ in $\mathcal{K}^{\mathcal{W}}$;
end
```

(i)⇔(ii) The implication (i)⇐(ii) is obvious. To prove the inverse implication (i)⇒(ii) it is sufficient to show that (iii) implies (i), because of the equivalence (i)⇔(iii) proved earlier. But this follows from the fact that $\mathcal{W}^{\bullet} = \min \mathcal{W}^{\blacklozenge}$ is the set of all minimal elements (I, m) in the poset $\mathcal{X}$ satisfying any of the conditions $(W0_1)$-$(W0_4)$, $(W2)$, $(W3)$, $(W4)$, and $(W5)$, proved in Step B, and the inclusion $\mathcal{W}^{\bullet}_{\ell in} \cup \mathcal{W}^{\bullet}_3 \cup \mathcal{W}^{\bullet}_4 \cup \mathcal{W}^{\bullet}_5 \subseteq \check{\mathcal{W}}$, proved above. This finishes the proof of Theorem 2.4. □

<u>Proof of Theorem 2.5.</u> Consider the following subposet $\mathcal{T} = \mathcal{X} \setminus \mathcal{W}$ of the poset $\mathcal{X}$, and note that $\mathcal{T}$ consists of all pairs (I, m) in $\mathcal{X}$ such that the finite subposet $\widehat{I}_{[0,m]}$ (2.2) of $\widehat{I_m}$ does not contain, as a subposet, any of the six hypercritical posets of Nazarova (2.0).

Note that the subset $\max \mathcal{T}$ of all maximal elements in the posets $\mathcal{T}$ is a disjoint union

$$\max \mathcal{T} = \mathcal{T}^{\vee}_{\ell in} \cup \mathcal{T}^{\vee}_2 \cup \mathcal{T}^{\vee}_3 \cup \mathcal{T}^{\vee}_4 \cup \ldots \cup \mathcal{T}^{\vee}_m \cup \ldots, \tag{3.4}$$

where $\mathcal{T}^{\vee}_{\ell in} = \mathcal{X}_{\ell in} \cap \max \mathcal{T}$ and $\mathcal{T}^{\vee}_m = \mathcal{X}_m \cap \max \mathcal{T}$, for each $m \geq 2$.

Consider the subposet $\mathcal{T}^{\blacklozenge}$ of $\mathcal{X}$ consisting of all pairs $(I, m) \in \mathcal{X}$ satisfying any of the conditions listed in the statement (ii) of Theorem 2.5

First we prove the equivalence (i)⇔(ii). Note that (i)⇔(ii) holds if and only if

$$\mathcal{T}^{\blacklozenge} = \mathcal{T}. \tag{3.5}$$

To prove the equality (3.5), we consider the subposet $\mathcal{T}^{\bullet} = \mathcal{T}^{\bullet}_{\ell in} \cup \mathcal{T}^{\bullet}_{2} \cup \mathcal{T}^{\bullet}_{3} \cup \mathcal{T}^{\bullet}_{4}$ of $\mathcal{T}^{\blacklozenge}$, where

- $\mathcal{T}^{\bullet}_{\ell in} = \{(C_3, 6), (C_4, 4), (C_6, 3)\}$ and C_s means a chain with $s \geq 3$ vertices,
- $\mathcal{T}^{\bullet}_{2} = \{(I, 2);\ I$ is one of the posets of Table 2.7 different from the garland $\mathcal{G}_n\}$,
- $\mathcal{T}^{\bullet}_{3} = \{(\mathcal{I}_1, 3),\ (\mathcal{I}^{\bullet}_1, 3)\}$,
- $\mathcal{T}^{\bullet}_{4} = \{(\mathcal{F}_0, 4)\}$,

and $\mathcal{F}_0$, $\mathcal{I}_1$, and $\mathcal{I}^{\bullet}_1$ are as in Theorem 2.5.

Now we prove the following four statements:

(A) $\mathcal{T}^{\bullet} = \max \mathcal{T}^{\blacklozenge}$ is the set of all maximal elements in $\mathcal{T}^{\blacklozenge}$.

(B) $\mathcal{T}^{\blacklozenge} \ \subseteq\ \mathcal{T}$,

(C) $\mathcal{T}^{\bullet} = \max \mathcal{T}$ is the subset all maximal elements in the poset $\mathcal{T}$, and

(D) Given a pair $(I, m) \in \mathcal{T}$, there is no $(I', m') \in \max \mathcal{T}$ such that $(I, m) \prec (I', m')$ if and only if $m = 2$ and I is a peak subposet of a garland $\mathcal{G}_n$ of Table 2.7, with $n \geq 2$, or there is an $m \geq 3$ such that $(I, m) = (C_1, m)$ or $(I, m) = (C_2, m)$.

The equality (A) easily follows by a case by case inspection of the posets in the finite set $\mathcal{T}^{\bullet}$. The details are left to the reader.

To prove (B), first we note (by looking at the set $\mathcal{T}^{\blacklozenge}$) that a pair $(I, m) \in \mathcal{T}^{\blacklozenge}$ has no $(I', m') \in \max \mathcal{T}^{\blacklozenge}$ if and only if $m = 2$ and I is a peak subposet of a garland $\mathcal{G}_n$ of Table 2.7, with $n \geq 2$, or there is an $m \geq 3$ such that $(I, m) = (C_1, m)$ or $(I, m) = (C_2, m)$. It is easy to check, by applying the definition of the poset $\widehat{I}_{[0,m]}$, that $(\mathcal{G}_n, 2) \in \mathcal{T}$, for $n \geq 2$, and $(C_1, m), (C_2, m) \in \mathcal{T}$, for each $m \geq 2$.

Next we note that if $(I, m) \prec (I', m')$ and $(I', m') \in \mathcal{T}$ then $(I, m) \in \mathcal{T}$. Hence, in view of (A) and the remark above, the inclusion $\mathcal{T}^{\blacklozenge} \subseteq \mathcal{T}$ holds if $\mathcal{T}^{\bullet} = \max \mathcal{T}^{\blacklozenge}$ is a subset of $\mathcal{T}$. To prove it, assume that (I, m) lies in $\mathcal{T}^{\bullet}$, and recall from (3.3) and Step B of the proof of Theorem 2.4 that $\mathcal{W} = \mathcal{W}^{\blacklozenge}$ and $\min \mathcal{W} = \mathcal{W}^{\bullet}$. It follows that (I, m) does not belongs to $\mathcal{W}$ (that is, (I, m) belongs to $\mathcal{T} = \mathcal{X} \setminus \mathcal{W}$ and, consequently, the inclusion $\mathcal{T}^{\blacklozenge} \subseteq \mathcal{T}$ holds), because a case by case inspection of the elements $(I, m) \in \mathcal{T}^{\bullet}$ and $(I'', m'') \in \mathcal{W}^{\bullet}$ shows that, given $(I, m) \in \mathcal{T}^{\bullet}$, there is no $(I'', m'') \in \min \mathcal{W} = \mathcal{W}^{\bullet}$ such that $(I'', m'') \preceq (I, m)$, or equivalently, $(I, m) \notin \mathcal{W}$, that is, $(I, m) \in \mathcal{T}$.

If $(I, m) \in \mathcal{T}^{\bullet}_{\ell in} = \{(C_3, 6), (C_4, 4), (C_6, 3)\}$, or $(I, m) = (C_s, 2)$ and $s \geq 2$, then one easily shows that the poset $\widehat{I}_{[0,m]}$ does not contain any of hypercitical posets of Nazarova (2.0). Hence, to finish the proof, it remains to show that $\mathcal{T}^{\bullet}_{2} \cup \mathcal{T}^{\bullet}_{3} \cup \mathcal{T}^{\bullet}_{4} \subset \mathcal{T}$. We do it by applying the following algorithm.

Algorithm T.1. Input: The sets $\mathcal{T}^{\bullet}_{2}$, $\mathcal{T}^{\bullet}_{3}$, $\mathcal{T}^{\bullet}_{4}$.

begin
  $twr \leftarrow true$;
  for (each pair (I, m) belongs to $\mathcal{T}^{\bullet}_{2} \cup \mathcal{T}^{\bullet}_{3} \cup \mathcal{T}^{\bullet}_{4}$) **do**
    if (the poset $\widehat{I}_{[0,m]}$ contains one of Nazarova's posets)
    then
      $twr \leftarrow false$;
      break;
  if (twr = true) **then**
    print($\mathcal{T}^{\bullet}_{2} \cup \mathcal{T}^{\bullet}_{3} \cup \mathcal{T}^{\bullet}_{4}$ is contained in $\mathcal{T}$);
  else
    print($\mathcal{T}^{\bullet}_{2} \cup \mathcal{T}^{\bullet}_{3} \cup \mathcal{T}^{\bullet}_{4}$ is not contained in $\mathcal{T}$);
end

This finishes the proof of (B).

Finally, we prove the statements (C) and (D). We show that

- the set $\mathcal{T}_m = \mathcal{X}_m \cap \mathcal{T}$ is empty, for all $m \geq 5$, and
- $\mathcal{T}^{\vee}_{\ell in} = \mathcal{T}^{\bullet}_{\ell in}$, $\mathcal{T}^{\vee}_2 = \mathcal{T}^{\bullet}_2$, $\mathcal{T}^{\vee}_3 = \mathcal{T}^{\bullet}_3$, and $\mathcal{T}^{\vee}_4 = \mathcal{T}^{\bullet}_4$.

Assume, to the contrary, that $m \geq 5$ and $\mathcal{T}_m$ is not empty. Then there is an $(I, m) \in \mathcal{T}_m$ and I is not a chain, that is, I contains, as a peak subposet, the poset

$$\mathcal{F}_0 : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[dr] & \\ \bullet \arrow[r] & * \end{tikzcd}.$$

Hence $(\mathcal{F}_0, m) \preceq (I, m)$. Since $(\mathcal{F}_0, 5) \in \mathcal{W}^{\bullet} = \min \mathcal{W}$ then $(I, m) \in \mathcal{W} \cap \mathcal{T} = \emptyset$ and we get a contradiction. It follows that $\mathcal{T}_m$ is empty, for all $m \geq 5$.

To prove the equality $\mathcal{T}^{\vee}_{\ell in} = \mathcal{T}^{\bullet}_{\ell in}$, we note that the inclusion $\mathcal{T}^{\vee}_{\ell in} \supseteq \mathcal{T}^{\bullet}_{\ell in}$ follows, because given a relation $(I, m) \prec (I', m')$, with $(I, m) \in \mathcal{T}^{\bullet}_{\ell in}$, the pair (I', m') belongs to $\mathcal{W}^{\blacklozenge} = \mathcal{W}$, by (3.3). To prove the inverse inclusion $\mathcal{T}^{\vee}_{\ell in} \subseteq \mathcal{T}^{\bullet}_{\ell in}$, assume that $(I, m) \in \mathcal{T}_{\ell in} = \mathcal{T} \cap \mathcal{X}_{\ell in}$, that is, $I = C_s$ is a chain with $s \geq 1$ vertices.

We recall from the proof of (B) that if $s = 1$ or $s = 2$, then $(I, m) = (C_s, m) \in \mathcal{T}$, for each $m \geq 2$.

Assume that $s \geq 3$ and $(I, m) = (C_s, m) \in \mathcal{T}$. It follows that $m \leq 6$, because otherwise $(C_3, 7) \preceq (I, m)$ and, hence, $(I, m) \in \mathcal{W}^{\blacklozenge} \cap \mathcal{T} = \mathcal{W} \cap \mathcal{T} = \emptyset$; a contradiction. This shows that if $(I, m) = (C_s, m) \in \mathcal{T}$ and $s \geq 3$ then $m \in \{3, 4, 5, 6\}$. Hence easily follows that $(I, m) = (C_s, m) \in \mathcal{T}^{\bullet}_{\ell in}$, because $\mathcal{W} \cap \mathcal{T} = \emptyset$ and, by 5° in the proof of Theorem 2.4, we have $\mathcal{W}^{\vee}_{\ell in} = \mathcal{W}^{\bullet}_{\ell in} = \{(C_3, 7), (C_4, 5), (C_5, 4), (C_7, 3)\} \subseteq \mathcal{W}$. Then the inclusion inclusion $\mathcal{T}^{\vee}_{\ell in} \subseteq \mathcal{T}^{\bullet}_{\ell in}$ follows.

Assume that $(I, m) \in \mathcal{T}$, $m \geq 2$, and I is not linearly ordered, that is, the poset

$$\mathcal{F}_0 : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[dr] & \\ \bullet \arrow[r] & * \end{tikzcd}$$

is a peak subposet of I. Since the set $\mathcal{T}_m$ is empty, for all $m \geq 5$, then $m \leq 4$, that is, $m \in \{2, 3, 4\}$.

Since I contains $\mathcal{F}_0$ then either $I = \mathcal{F}_0$, or $|I| \geq 4$, there is a non-maximal element $a \in I \setminus \mathcal{F}_0$ and one can show that I contains as a peak subposet any of the following four posets

$$\mathcal{I}_1 : \begin{tikzcd}[row sep=tiny, column sep=small] & \bullet \arrow[dr] & \\ a \arrow[ur] \arrow[r] & \bullet \arrow[r] & * \end{tikzcd}, \quad \mathcal{I}^{\bullet}_1 : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[dr] & & \\ \bullet \arrow[r] & a \arrow[r] & * \end{tikzcd}, \qquad \mathcal{I}_2 : \begin{tikzcd}[row sep=tiny, column sep=small] & \bullet \arrow[dr] & \\ \bullet \arrow[r] & a \arrow[r] & * \end{tikzcd}, \qquad \mathcal{I}_3 : \begin{tikzcd}[row sep=tiny, column sep=small] \bullet \arrow[dr] & \\ \bullet \arrow[r] & * \\ a \arrow[ur] & \end{tikzcd}$$

presented in (W3) and (W4) of Theorem 2.4.

Assume that $m = 4$. It follows that $I = \mathcal{F}_0$, because otherwise $(I', 4) \preceq (I, 4)$, for some $I' \in \{\mathcal{I}_1, \mathcal{I}^{\bullet}_1, \mathcal{I}_2, \mathcal{I}_3\}$, and we get the contradiction $(I', 4) \in \mathcal{W} \cap \mathcal{T} = \emptyset$. This shows that $(I, 4) = (\mathcal{F}_0, 4)$ and $\mathcal{T}^{\vee}_4 = \mathcal{T}^{\bullet}_4$.

Assume that $m = 3$ and $(I, 3) \in \max \mathcal{T}$. By applying the poset extension type arguments as for $m = 4$, we show that $(I, 3) = (\mathcal{I}_1, 3)$ or $(I, 3) = (\mathcal{I}^{\bullet}_1, 3)$, that is, $\mathcal{T}^{\vee}_3 = \mathcal{T}^{\bullet}_3$.

Finally, assume that $m = 2$, I is not a chain, and $(I, 2) \in \mathcal{T}$. By applying the poset extension type arguments as for $m = 4$ and the Algorithm T.2 presented below, we show that

(D1) $(I, 2) \in \max \mathcal{T}$ if and only if I is not a peak subposet of a garland $\mathcal{G}_n$, with $n \geq 2$, and I is one of the maximal tame posets of Table 2.7.

(D2) I is a peak subposet of a garland $\mathcal{G}_n$ of Table 2.7, with $n \geq 2$, and there is an infinite chain

$$\mathcal{F}_0 \hookrightarrow I \hookrightarrow I_2 \hookrightarrow I_3 \hookrightarrow \ldots\ldots \hookrightarrow I_r \hookrightarrow I_{r+1} \hookrightarrow \ldots\ldots$$

of proper peak embeddings, where I_r is a peak subposet of a garland $\mathcal{G}_n$, with $n \geq 2$, for each $r \geq 2$.

Algorithm T.2. Input: The poset $\mathcal{F}_0$ and the garland $\mathcal{G}_r$.

Pass:
- Global variables: $\mathcal{T}_2^\vee$, $\mathcal{T}_3^\vee$, $\mathcal{T}_4^\vee$, $\mathcal{K}^\mathcal{T}$, G, m, MAX.
- Local variables: n, I.
- Meaning of particular variables:
  - $\mathcal{T}_2^\vee, \mathcal{T}_3^\vee, \mathcal{T}_4^\vee$ - lists of pairs $(J, j) \in \mathcal{T}_j^\vee$ for $j = 2 \dots 4$, that are maximal in $\mathcal{T}$.
  - $\mathcal{K}^\mathcal{T}$ - list of pairs (J, j) that do not belong to any of the sets $\mathcal{T}_2^\vee, \mathcal{T}_3^\vee, \mathcal{T}_4^\vee$.
  - $m = 2..4$,
  - MAX - constant that determines the number of maximal elements of posets.
  - G - the garland $\mathcal{G}_r$, with $r \geq 2$.

```
begin
  enroll the pair (F_0, 4) in T_4^∨;
  m ← 3;
  G ← the garland;
  TProcedure(T_4^∨[1]);
end
```

As an output we get the set $\mathcal{K}^\mathcal{T}$ defined above and the set $\mathcal{T}_2^\vee \cup \mathcal{T}_3^\vee \cup \mathcal{T}_4^\vee$ of all maximal elements in $\mathcal{T}$.

The algorithm uses the following two procedures, where the first one is applied by the second.

Procedure T.2a.

```
on_list(I, T_n^∨)
  if( T_n^∨ does not contain any (J, n), with J a peak subposet of I) then
    if (I does not contain a peak subposet of any posets J, with (J, n) ∈ T_n^∨)
    then
      enroll (I, n) in T_n^∨;
  else
    enroll (I, n) in T_n^∨ and replace it with (I', n) ∈ T_n^∨
    if I' is a proper peak subposet of I;
end
```

Procedure T.2b.

```
TProcedure(J)
  if(|J| ≤ MAX) then
    construct the set E_J;
    for (each I from E_J) do
    begin
        for n ← m to 2 do
          if (Î_[0,n] does not contain any of the posets N_1,...,N_6)
          then
            if (I is not a subposet of G) then
              on_list(I, T_n^∨);
              TProcedure(I);
              break;
            else
              if (m = 3) then
                on_list(I, T_n^∨);
              TProcedure(I);
              break;
    end;
  else
    if(J is not a subposet of G) then
       enroll (J, n) in K^T;
end
```

The algorithm relay on the following observations. If $(I, 2) \in \mathcal{T}$, I is not a chain and is not a peak subposet of a garland $\mathcal{G}_n$, with $n \geq 2$, then

(i) I contains $\mathcal{F}_0$ as a peak subposet, and

(ii) if I' is any of the 36 hypercitical posets of Table 2.6, then I' is not a peak subposet of I,

because otherwise $(I', 2) \preceq (I, 2)$ and, by Theorem 2.4, we get the contradiction $(I, 2) \in \mathcal{W} \cap \mathcal{T} = \emptyset$.

The algorithm produces the list of all maximal posets I (with respect to the peak embedding) that are not peak subposets of garlands $\mathcal{G}_n$, with $n \geq 2$, satisfying the condition (i), and satisfying (ii) (equivalently, the poset $\widehat{I}_{[0,2]}$ (2.2) does not contain any hypercritical poset of Nazarova (2.0)). As an output we get just the set $\mathcal{K}^{\mathcal{T}}$ defined above and and the list of the first 17 posets of Table 2.7.

Since, by a direct checking, we show that the garland $\mathcal{G}_n$, with $n \geq 2$, does not contain as a peak subposet any of the 36 posets of Table 2.6 then Theorem 2.4 yields $(\mathcal{G}_n, 2) \notin \mathcal{W}$, that is, $(\mathcal{G}_n, 2) \in \mathcal{T}$ and consequently $(I, 2) \in \mathcal{T}$, for any peak subposet I of a garland. Hence, by Theorem 2.4, the the posets I containing $\mathcal{F}_0$ as a peak subposet that are not peak subposets of any of the first 17 posets of Table 2.7, but are peak subposets of a garland $\mathcal{G}_n$, with $n \geq 2$, are just the remaining posets of $\mathcal{T}$. This shows that $\mathcal{T}_2^{\vee} = \mathcal{T}_2^{\bullet}$ and finishes the proof of the statements (C) and (D).

In view of (B), to finish the proof of the equality (3.5), it remains to show that the inclusion $\mathcal{T}^{\blacklozenge} \supseteq \mathcal{T}$ holds. Let (I, m) be an element of $\mathcal{T}$. If there is an $(I', m') \in \max \mathcal{T} = \mathcal{T}^{\bullet}$ such that $(I, m) \preceq (I', m')$ then $(I, m) \in \mathcal{T}^{\blacklozenge}$, because $(I', m') \in \mathcal{T}^{\bullet} = \max \mathcal{T}^{\blacklozenge}$ (see (A)) and obviously the poset $\mathcal{T}^{\blacklozenge}$ is closed under the predecessors in $\mathcal{T}$. If there is no $(I', m') \in \max \mathcal{T} = \mathcal{T}^{\bullet}$ such that $(I, m) \preceq (I', m')$ then (D) yields $(I, m) \in \mathcal{T}^{\blacklozenge}$, because each of the pairs (C_1, m) and (C_2, m), with $m \geq 2$, belongs to $\mathcal{T}^{\blacklozenge}$ and any pair $(I, 2)$, with I a peak subposet of a garland $\mathcal{G}_n$, belong to $\mathcal{T}^{\blacklozenge}$. This finishes the proof of the equality (3.5) and of Theorem 2.5. $\square$

4. Appendix

We collect in this section some explanations concerning the computational programs we use in the proof of Theorems 2.4 and 2.5 in Section 3.

4.1. In most of the programs we use several functions from the package CREP, which is a package of programs allowing us to work with particular problems that appear in representation theory of finite dimensional algebras over a field. In particular, CREP contains several data bases containing some classifications that appear in the theory. It can be retrived via ftp from the server **ftp.uni-bielefeld.de** under the directory **pub/math/f-d-alg**.

4.2. Throughout this section, by a poset $I \equiv (I, \preceq)$ we mean a finite partially ordered set with a unique maximal element. Following the CREP data format for posets, we represent any poset I by a pair $[n, l]$, where n is the number of elements of I (that is identified with the set $\{1, 2, \ldots, n\}$) and $\ell = [\ell_1, \ldots, \ell_r]$ is a set describing the Hasse quiver of I (see [11, p. 281]) by providing, for each $j \in \{1, \ldots, r\}$, a list $\ell_j = [\ell_{j,1}, \ldots, \ell_{j,m_j}]$ of the upper neighbours $\ell_{j,1}, \ldots, \ell_{j,m_j}$ of $\ell_{j,1}$ in the Hasse quiver of I. For instance, any of the following two different descriptions

$[6, [\,[1, 2, 3], [2, 4], [3, 4, 5], [4, 6], [5, 6]\,]]$ and $[6, [\,[1, 2], [1, 3], [2, 4], [3, 4], [3, 5], [4, 6], [5, 6]\,]]$

define the poset I whose Hasse quiver has the form $\begin{tikzcd}[sep=small] & 2 \arrow[r] & 4 \arrow[r] & 6 \\ 1 \arrow[ur] \arrow[r] & 3 \arrow[ur] \arrow[r] & 5 \arrow[ur] & \end{tikzcd}$.

4.3. We say that I is a peak subposet of I' if there is a poset embedding $I \hookrightarrow I'$ that carries the unique maximal element of I to the unique maximal element of I'. We denote by $\mathcal{E}_I$ the set of all one-element peak extensions $I \hookrightarrow I'$ of I, see W.2.1.

4.4. Our programs use the following functions:

- $\operatorname{ext}(I)$ that returns the set $\mathcal{E}_I$ of all one-element peak extensions $I \hookrightarrow I'$ of the poset I.
- $\operatorname{Imm}(I, m)$ presented below, that constructs the finite poset $\widehat{I}_{[0,m]}$ defined in (2.2), for each pair (I, m), with $m \geq 2$.

- subposet(I, I') that tests whether the poset I is a peak subposet of I'.

4.5. We also freely use the following functions available in the CREP package:

- pswild(p) that tests, whether a given poset p contains one of the hypercitical posets of Nazarova (2.0). **Hint:** In order to work (in our situation) with the function pswild$(-)$ from the CREP package in Maple V release 5.1, one has to delete the lines 556 and 558 in **crep.src** file.
- cartanm(p) that computes the Cartan matrix of the poset p.
- mini(p) that returns all minimal of the poset p.
- minEl(p) that returns all maximal of the poset p.

4.6. All programs we use in this paper are written in Maple V release 5.1, because most of them use some functions from the CREP package.

In order to use CREP with Maple as a surface, we have to start with a Maple session from the CREP home directory (otherwise Maple is not able to execute CREP commands properly). Then to start CREP with Maple, we need to change first the current directory to the CREP home directory and then we start Maple from there.

```
> Imm:=proc(p,m)
>    local n,f,l,elem_min,elem_max,r,rr,pmm;
>    n:=p[1];
>    elem_max:=maxEl(p);
>    elem_min:=minEl(p);
>    f:=x->x+n*m;
>    elem_min:=map(f,op({elem_min}));
>    l:=[];
>    for r in elem_max do
>       for rr in elem_min do
>          l:=[l[],[rr,r]];
>       od;
>    od;
>    pmm:=Imm(p,m+1);
>    RETURN([pmm[1],[op(pmm[2]),l[]]]);
> end:
```

Complete source codes of all implementations used in this paper and an instruction on "how to start the programs in Maple with the CREP package" can be found in

`www.mat.uni.torun.pl/~simson`

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