Hausdorff metric in formulae

- d – metric in X, $d(a,B)=\inf_{b\in B} d(a,b)$ – distance,
- $e_d(A,B)=\sup_{a\in A} d(a,B)$ – excess, $h_d(A,B)=\max\{\, e(A,B), e(B,A)\,\}$ – Hausdorff metric,
- $B(a,r)=\{x\in X:\ d(x,a)<r\}$ – open ball, $\mathcal{O}_r(A)=\{x\in X:\ d(x,A)<r\}$ – ε-neighbourhood

Equivalent description

1. $B(a,\varepsilon)=\mathcal{O}_\varepsilon\{a\}$ $\qquad \mathcal{O}_\varepsilon A=\bigcup_{a\in A} B(a,\varepsilon)$ $\qquad x\in\mathcal{O}_\varepsilon A \Leftrightarrow B(x,\varepsilon)\cap A\neq\emptyset$
2. $e(A,B)=\inf\{\, r>0: A\subset\mathcal{O}_r B\,\}=\sup_{x\in X}(d(x,B)-d(x,A))=\inf_{f:A\to B}\ \sup_{a\in A} d(a,f(a))$
3. $h(A,B)=\inf\{\, r>0\ :\ A\subset\mathcal{O}_r B,\ B\subset\mathcal{O}_r A\,\}=\sup_{x\in X}|d(x,A)-d(x,B)|$

Properties of e and h

1. $e(A,B)=0\Leftrightarrow A\subset\overline{B}$ $\qquad h(A,B)=0\Leftrightarrow \overline{A}=\overline{B}$
2. $h(A,B)=h(B,A)$ $\qquad e(A,B)\neq e(B,A)$ (in general)
3. $e(A,B)\leqslant e(A,C)+e(C,B)$ $\qquad h(A,B)\leqslant h(A,C)+h(C,B)$
4. $e(A,B)=e(A,\overline{B})=e(\overline{A},B)=e(\overline{A},\overline{B})$ $\qquad h(A,B)=h(A,\overline{B})=h(\overline{A},B)=h(\overline{A},\overline{B})$
5. $A\subset B\ \Rightarrow\ e(A,C)\leqslant e(B,C)$
6. $e(A\cup C,B\cup C)\leqslant e(A,B)$ $\qquad h(A\cup C,B\cup C)\leqslant h(A,B)$
7. $e(\bigcup_{i\in I}A_i,\bigcup_{i\in I}B_i)\leqslant \sup_{i\in I} e(A_i,B_i)$ $\qquad h(\bigcup_{i\in I}A_i,\bigcup_{i\in I}B_i)\leqslant \sup_{i\in I} h(A_i,B_i)$
8. $e(\bigcup_{i\in I}A_i,B)=\sup_{i\in I} e(A_i,B)$ $\qquad e(A,\bigcup_{i\in I}B_i)\leqslant \inf_{i\in I} e(A,B_i)$
9. $d(x,\bigcup_{i\in I}B_i)=\inf_{i\in I} d(x,B_i)$ $\qquad \inf_{a\in A\ b\in B}\ d(a,b)\leqslant\min\{e(A,B),e(B,A)\}\leqslant h(A,B)$
10. $e(\lambda A,\lambda B)=|\lambda|\cdot e(A,B)$ $\qquad e(\lambda A,\mu A)\leqslant|\lambda-\mu|\cdot\|A\|$ $\quad$ (where $\|A\|=\sup_{a\in A}\|a\|$)
11. $e(A+C,B+C)\leqslant e(A,B)$ $\qquad h(A+C,B+C)\leqslant h(A,B)$
12. $e[\,A,\lambda B+(1-\lambda)C\,]\leqslant\lambda\cdot e(A,B)+(1-\lambda)\cdot e(A,C)\quad\forall_{\lambda\in(0,1)}$
13. $h[\,A,\lambda A+(1-\lambda)C\,]\leqslant h(A,C)\quad\forall_{\lambda\in(0,1)}\ \forall_{A,C}$ – compact $\quad$ [in normed spaces]
14. $e(\operatorname{conv}A,\operatorname{conv}B)=e(A,\operatorname{conv}B)\leqslant e(A,B)$ $\qquad h(\operatorname{conv}A,\operatorname{conv}B)\leqslant h(A,B)$
15. $e_{d\wedge1}(A,B)=e_d(A,B)\wedge1$ $\qquad h_{d\wedge1}(A,B)=h_d(A,B)\wedge1$ $\quad$ (where $d\wedge1=\min\{1,d\}$)
16. d – ultrametric $\Rightarrow$ e_d, h_d – satisfy ultrametric triangle inequality

Properties of $\mathcal{O}$

1. $\overline{A}=\bigcap_{\varepsilon>0}\mathcal{O}_\varepsilon(A)$ $\qquad \mathcal{O}_\varepsilon\overline{A}=\mathcal{O}_\varepsilon A$ $\qquad \overline{\mathcal{O}_{\varepsilon_1}A}\subset\mathcal{O}_{\varepsilon_2}A\quad\forall_{\varepsilon_1<\varepsilon_2}$
2. $A\subset B\ \Rightarrow\ \mathcal{O}_\varepsilon A\subset\mathcal{O}_\varepsilon B$ $\qquad \mathcal{O}_\varepsilon A\cap B\subset\mathcal{O}_\varepsilon(\,A\cap\mathcal{O}_\varepsilon B\,)$ $\qquad \mathcal{O}_\varepsilon A\cap B\neq\emptyset\Leftrightarrow A\cap\mathcal{O}_\varepsilon B\neq\emptyset$
3. $\mathcal{O}_\varepsilon\left(\bigcup_{i\in I}A_i\right)=\bigcup_{i\in I}\mathcal{O}_\varepsilon A_i$ $\qquad \mathcal{O}_\varepsilon\,(\mathcal{O}_\varepsilon A)^{\mathrm{c}}\subset A^{\mathrm{c}}$
4. $\mathcal{O}_{\varepsilon_1}[\,\mathcal{O}_{\varepsilon_2}A\,]\subset\mathcal{O}_{\varepsilon_1+\varepsilon_2}(A)$ $\qquad \mathcal{O}_{\varepsilon_1}(\mathcal{O}_{\varepsilon_2}A)\neq\mathcal{O}_{\varepsilon_2}(\mathcal{O}_{\varepsilon_1}A)$ (in general)
5. $\mathcal{O}_{\varepsilon_1}[\,\mathcal{O}_{\varepsilon_2}A\,]=\mathcal{O}_{\varepsilon_1+\varepsilon_2}(A)$ $\quad$ [in normed spaces]

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6. $\mathcal{O}_{\varepsilon_1}(A) + \mathcal{O}_{\varepsilon_2}(B) \subset \mathcal{O}_{\varepsilon_1+\varepsilon_2}(A+B)$ $\qquad$ $\mathcal{O}_\varepsilon(A+B) = A + \mathcal{O}_\varepsilon B$ $\qquad$ $\mathcal{O}_\varepsilon(-A) = -\mathcal{O}_\varepsilon A$

7. $\lambda \mathcal{O}_\varepsilon(A) \subset \mathcal{O}_{|\lambda|\cdot\varepsilon}(\lambda A) \ \forall_{\lambda\neq 0}$ $\qquad$ $\lambda \cdot \mathcal{O}_\varepsilon(A) \subset \mathcal{O}_\varepsilon(\lambda A) \ \forall_{|\lambda|\leqslant 1}$

8. $\mathrm{conv}(\mathcal{O}_\varepsilon A) \subset \mathcal{O}_\varepsilon(\mathrm{conv}\, A)$ $\qquad$ $\mathrm{conv}(\mathcal{O}_\varepsilon \,\mathrm{conv}\, A) = \mathcal{O}_\varepsilon \,\mathrm{conv}\, A$

Other properties

1. $e(A,B) < \varepsilon \ \Rightarrow\ A \subset \mathcal{O}_\varepsilon B$ $\qquad$ $A \subset \overline{\mathcal{O}_\varepsilon B} \ \Rightarrow\ e(A,B) \leqslant \varepsilon$

2. $h(\mathcal{O}_{\varepsilon_1} A_1, \mathcal{O}_{\varepsilon_2} A_2) \leqslant h(A_1, A_2) + \max(\varepsilon_1, \varepsilon_2)$ $\qquad$ $h(A, \mathcal{O}_\varepsilon A) \leqslant \varepsilon$

3. $h(\mathcal{O}_{\varepsilon_1} A_1, \mathcal{O}_{\varepsilon_2} A_2) = h(A_1, A_2) + |\varepsilon_1 - \varepsilon_2|$ [in normed spaces]

4. C – convex $\Rightarrow$ $\mathcal{O}_\varepsilon C$ – convex [in normed spaces]

5. C – (path-)connected $\Rightarrow$ $\mathcal{O}_\varepsilon C$ – (path-)connected [in spaces with (path-)connected open balls]

6. $\varphi\colon X \multimap Y$, $h\,[\,\varphi(x_1), \varphi(x_2)\,] \leqslant L \cdot d(x_1, x_2)$ (i.e. φ – multivalued Lipschitz) $\Rightarrow$
   $e[\,\varphi(A), \varphi(B)\,] \leqslant L \cdot e(A,B)$ $\qquad$ $\varphi\,(\,\mathcal{O}_\varepsilon A\,) \subset \mathcal{O}_{L\cdot\varepsilon}\,\varphi(A)$

7. $K = \bigcap_{n=1}^\infty K_n$ — decreasing sequence of compacta $\Rightarrow$ $h(K_n, K) \underset{n\to\infty}{\longrightarrow} 0$

8. $A_n \underset{n\to\infty}{\overset{h}{\longrightarrow}} A$, A_n – connected, A – compact $\Rightarrow$ A – connected

9. $\mathrm{Li}\, A_n = \bigcap_{\varepsilon>0} \bigcup_{m=1}^\infty \bigcap_{n=m}^\infty \mathcal{O}_\varepsilon A_n \ \subset\ \mathrm{Ls}\, A_n = \bigcap_{m=1}^\infty \overline{\bigcup_{n=m}^\infty A_n}$
   $A_n \underset{n\to\infty}{\overset{h}{\longrightarrow}} A \ \Rightarrow\ \mathrm{Li}\, A_n = A = \mathrm{Ls}\, A_n$

Hausdorff measure of noncompactness

- $\beta(A) = \inf\{\, r > 0 \,:\, \exists_{x_1, x_2, \ldots, x_k \in X} \bigcup_{i=1}^k B(x_i, r) \supset A \,\}$ — Hausdorff measure of noncompactness

1. $\beta(A) = 0 \Leftrightarrow A$ – precompact (= totally bounded)

2. $A_1 \subset A_2 \ \Rightarrow\ \beta(A_1) \leqslant \beta(A_2)$ $\qquad$ $\beta(A_1 \cup A_2) = \max\{\beta(A_1), \beta(A_2)\}$

3. (seminorm) $\quad$ $\beta(\,\lambda A\,) = |\lambda| \cdot \beta(A)$ $\qquad$ $\beta(A_1 + A_2) \leqslant \beta(A_1) + \beta(A_2)$

4. (Lipschitz continuity) $\quad$ $\beta(\,\mathcal{O}_\delta A\,) \leqslant \beta(A) + \delta$ $\qquad$ $|\beta(A_1) - \beta(A_2)| \leqslant h(A_1, A_2)$

5. $\beta(\,B(0,1)\,) = 1$ $\qquad$ $\beta(\,\mathcal{O}_\delta A\,) = \beta(A) + \delta$ [in infinite dimensional normed spaces]

6. $\beta(A_1 \cap A_2) \leqslant \min\{\beta(A_1), \beta(A_2)\}$ $\qquad$ $\beta(A) \leqslant \min\{\,\mathrm{diam}\, A,\, 2 \cdot \|A\|\,\}$

7. $\beta(A + C) = \beta(A)$ $\qquad$ $\beta(A \cup C) = \beta(A)$ for precompact C

8. $\beta(\overline{A}) = \beta(A)$ $\qquad$ (Darbo) $\quad$ $\beta(\mathrm{conv}\, A) = \beta(A)$

Other properties

1. (Kuratowski's Intersection Theorem) X – complete, $\{B_n\}_{n=1}^\infty$ – decreasing sequence of nonempty closed sets with $\beta(B_n) \underset{n\to\infty}{\longrightarrow} 0$ $\Rightarrow$ $B_n \underset{n\to\infty}{\overset{h}{\longrightarrow}} \bigcap_{n=1}^\infty B_n$ – nonempty compact

2. $\varphi\colon X \multimap Y$, $h\,[\,\varphi(x_1), \varphi(x_2)\,] \leqslant L \cdot d(x_1, x_2)$ $\forall_{x_1, x_2 \in X}$, $\forall_{x\in X}$ $\varphi(x)$ – compact
   (i.e. φ – multivalued Lipschitz with compact values) $\Rightarrow$ $\beta\,[\varphi(A)] \leqslant L \cdot \beta(A)$ $\forall_{A\subset X}$

3. $\beta(A) = \inf_{K\in\mathcal{K}(X)} h(A, K)$ $\quad$ (where $\mathcal{K}(X)$ – family of compacta in X)

4. $\{A_t\}_{t\in T}$ – precompact w.r.t. h $\Rightarrow$ $\beta\left(\bigcup_{t\in T} A_t\right) = \sup_{t\in T} \beta(A_t)$

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