NOTE ON THE KURATOWSKI THEOREM FOR ABSTRACT MEASURES OF NONCOMPACTNESS

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Abstract. We show in this note whether exactly holds the full version of Kuratowski's Intersection Theorem. Although the technique of proof is standard, this summarizes observations scattered on a few papers of other authors.

1. Notation and definitions

Let us begin with standard concepts from metric topology. By X we mean a complete space furnished with a metric d. The *diameter* of a set $A \subset X$ is $\delta(A) = \sup_{a_1,a_2 \in A} d(a_1,a_2)$, the *open ball* in $x_0 \in X$ with radius $r > 0$ is $B(x_0,r) = \{x \in X : d(x,x_0) < r\}$, the ε-*aureola* (or ε-*neighbourhood*) around A is $\mathcal{O}_\varepsilon A = \bigcup_{a\in A} B(a,\varepsilon)$, and the *Hausdorff distance* between $A, B \subset X$ is $h(A,B) = \inf\{r > 0 : B \subset \mathcal{O}_r A \wedge A \subset \mathcal{O}_r B\}$. Denote by $\mathcal{F}(X)$ the family of nonempty closed sets in X, also called the hyperspace of closed sets. One should recognize that $\mathcal{F}(X)$ with h is an infinite-valued metric space (or generalized metric space in the Luxemburg-Jung sense). However we shall not go in this terminology further, because we need only set-convergence $A_n \underset{n\to\infty}{\longrightarrow} A$ w.r.t. the Hausdorff distance h.

Having a complete space X one can define functionals related to a lack of compactness of its subsets. These functionals are called *measures of noncompactness* (*m.n.c.* for short). Below we collect known general constructions:

— Kuratowski m.n.c. $\alpha(A) = \inf\{r > 0 : \exists_{D_1,D_2,\ldots,D_k \subset X} \text{ s.t. } \delta(D_i) < r, \bigcup_i D_i \supset A\}$,
— Istrătescu m.n.c. $\beta(A) = \inf\{r > 0 : \nexists_{\text{infinite } S\subset A} \text{ s.t. } \forall_{s_1,s_2\in S,\, s_1\neq s_2}\ d(s_1,s_2) \geqslant r\}$,
— Hausdorff m.n.c. $\chi(A) = \inf\{r > 0 : \exists_{x_1,x_2,\ldots,x_k \in X} \text{ s.t. } \bigcup_i B(x_i,r) \supset A\}$,
— Hausdorff m.n.c. relative to $C \subset X$
$\chi_C(A) = \inf\{r > 0 : \exists_{c_1,c_2,\ldots,c_k \in C} \text{ s.t. } \bigcup_i B(c_i,r) \supset A\}$.
— inner Hausdorff m.n.c.
$\chi_i(A) = \chi_A(A) = \inf\{r > 0 : \exists_{a_1,a_2,\ldots,a_k \in A} \text{ s.t. } \bigcup_i B(a_i,r) \supset A\}$.

We also add to this collection the diameter δ (cf. [BG]). All except two of the quoted measures are equivalent: $\chi \leqslant \chi_i \leqslant \beta \leqslant \alpha \leqslant 2\chi$. Furthermore $\alpha \leqslant \delta$, $\chi \leqslant \chi_C$. An *abstract m.n.c.* is simply any functional $\mu : \mathcal{F}(X) \to [0,\infty]$. We call a family $\{A_n\}_{n=1}^{\infty}$ μ-*descending*

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under the conditions:

1° $A_n \neq \emptyset \; \forall n$,
2° A_n is closed $\forall n$,
3° $A_{n+1} \subset A_n \; \forall n$ (decreasing),
4° $\mu(A_n) \longrightarrow 0$.

We shall say that a functional μ *satisfies* $(*)$, if every sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in A_n$ for any μ-descending family $\{A_n\}_{n=1}^{\infty}$ admits a convergent subsequence. We shall say that a functional μ has the *Kuratowski property*, if the intersection $A = \bigcap_n A_n$ of any μ-descending family $\{A_n\}_{n=1}^{\infty}$ fulfills the following:

(nonemptiness) $A \neq \emptyset$;
(compactness) A is compact;
(convergence) $A_n \to A$ w.r.t. h.

Although μ is defined on $\mathcal{F}(X)$ but one can trivially extend it to all subsets of X by putting $\mu(A) \doteq \mu(\overline{A})$ for nonempty $A \subset X$, and $\mu(\emptyset) = 0$. Note that this is not the only possible extension being neutral to property $(*)$, e.g. $\mu(\emptyset) = \infty$ is acceptable.

Concerning Kuratowski's property observe that this implies upper regularity i.e. any nonempty closed set A with $\mu(A) = 0$ must be compact. To see this just take constant sequence $A_n \doteq A$ in the definition. Nevertheless there exist nonregular Kuratowski functionals, namely the diameter (see Example 3).

The functional $\mu : \mathcal{S} \to [0, \infty]$ given on some larger family $\mathcal{F}(X) \subset \mathcal{S} \subset 2^X$ is said to have property $(*)$ (resp. the Kuratowski property), if its restriction to $\mathcal{F}(X)$ has the same property.

2. Theorem

The original result by K. Kuratowski (see [K] chapt.III par.30 pp.318–320) is a joint generalization of the Cantor Intersection Theorem and the characterization of compactness due to Riesz. Thanks to the observation of G. Darbo (e.g. [DG] chapt.III par.4.9.C pp.69-70, [H] chapt.2.6 pp.23–25) the Kuratowski m.n.c. has found applications in the fixed point theory as joint generalization of the Banach and Schauder Fixed Point Principles (because of obvious reasons this generalization makes sense only in the realm of Banach spaces). Actual literature is very large and constitutes a kind of branch in fixed point theory. What matters measures of noncompactness play an important rôle in asymptotic stability of dynamical systems ([H], [BRz], [L]).

The theorem below explains for which measures the Kuratowski theorem holds.

Theorem 1. *The functional μ has the Kuratowski property if and only if it satisfies $(*)$.*

Proof. For the proof let us denote by $\{A_n\}_{n=1}^{\infty}$ some fixed μ-descending family and by A its intersection, $A \doteq \bigcap_n A_n$.

'If' part. Assume $(*)$. Nonemptiness: pick up anyhow $x_n \in A_n$ and find convergent subsequence $x_{n_j} \to \overline{x}$. Since $x_{n_j} \in A_{n_j} \subset A_p$ for j s.t. $n_j \geqslant p$ we get $\overline{x} \in A_p$. Being index p arbitrary $\overline{x} \in A$, so $A \neq \emptyset$. Compactness: fix $\{x_n\}_{n=1}^{\infty} \subset A$. We can think that $x_n \in A_n$, due to $A \subset A_n$, thus we can derive a convergent subsequence. Convergence: suppose on the contrary that A_n does not converge to A w.r.t. h. Hence $A_{n_j} \not\subset \mathcal{O}_\varepsilon A$ for infinitely many n_j and some $\varepsilon > 0$. Putting $B_j \doteq A_{n_j} \cap [X \setminus \mathcal{O}_\varepsilon A]$ defines decreasing family of nonempty closed sets. Similarly as for nonemptiness one sees that $\bigcap_j B_j \neq \emptyset$. (Indeed: $x_j \in B_j \subset A_{n_j}$, $\mu(A_{n_j}) \to 0$ etc.). So we arrived at the contradiction with

$$\bigcap_j B_j = \bigcap_j A_{n_j} \cap [X \setminus \mathcal{O}_\varepsilon A] = \bigcap_n A_n \cap [X \setminus \mathcal{O}_\varepsilon A] = \emptyset.$$

'Only if' part. Assume now Kuratowski's property. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence with $x_n \in A_n$. By convergence $h(A_n, A) \to 0$ one can find in A a sequence $\{a_n\}_{n=1}^{\infty}$ asymptotically close to $\{x_n\}_{n=1}^{\infty}$ i.e. $d(x_n, a_n) \to 0$. By compactness of A there exists a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$. Hence $\{x_{n_k}\}_{k=1}^{\infty}$ is convergent. □

Observe that our proof encompasses the following question: is the family $\{B_j\}_j$ μ-descending itself? A slightly ,,more general" version of the Kuratowski Theorem reads as follows.

Theorem 2. *Let μ satisfy $(*)$, $\{A_n\}_{n=1}^{\infty}$ be a decreasing family of (possibly empty) closed sets with $\mu(A_n) \to 0$ and $A = \bigcap_{n=1}^{\infty} A_n$ be its intersection. Then $A_n \to A$ w.r.t. h and A is compact. If additionally all sets A_n are nonempty, then A is nonempty too.*

We point out that the convergence $A_n \to \emptyset$ w.r.t. h is possible only if $A_n = \emptyset$ starting from some n. (It means that $\emptyset$ is an isolated point in $\mathcal{F}(X) \cup \{\emptyset\}$ furnished with h.) When $\mu(\emptyset) \neq 0$ the Theorems 1 and 2 are exactly the same.

One would like to deal with the convergence even in case the intersection has nonzero measure (e.g. it is noncompact or unbounded). The following statement provides a partial solution.

Theorem 3. *Let μ satisfy $(*)$, $\{A_n\}_{n=1}^{\infty}$ be a decreasing family of closed sets, and $A = \bigcap_{n=1}^{\infty} A_n$ be its intersection. If $\mu(\overline{A_n \setminus A}) \to 0$, then $A_n \to A$ w.r.t. h. In particular A is nonempty, if all sets A_n are so.*

Proof. We can apply Theorem 2 to the sequence $\{\overline{A_n \setminus A}\}_{n=1}^{\infty}$ obtaining its h-convergence to $\bigcap_{n=1}^{\infty} \overline{A_n \setminus A}$. Observe now

$$A_n = \overline{A_n \setminus A} \cup A,$$

$$A = \bigcap_{n=1}^{\infty} \overline{A_n \setminus A} \cup A,$$

and recall that the set-theoretic union ,,$\cup$" is continuous w.r.t. h. Hence the desired convergence of sets follows. □

3. Examples

In this section we are concerned with typical assumptions on abstract m.n.c. $\mu : 2^X \to [0,\infty]$. We show that the condition $(*)$ is fulfilled very often.

Example 1. Let μ satisfy

(regularity) $\mu(A) = 0 \Leftrightarrow A$ is relatively compact,
(nonsingularity) $\mu(A \cup \{x\}) = \mu(A)$,
(monotonicity) $A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$.

Then there holds Kuratowski's theorem (as was observed in [COZ] Lemma 1.3., comp. [ADL]). Due to equivalence from Theorem 1 it is enough to check property $(*)$. To do this fix μ-descending family $\{A_n\}_{n=1}^{\infty}$ and choose $\{x_n\}_{n=1}^{\infty}$ with $x_n \in A_n$. Thus we have

$$\mu(\{x_n : n \geqslant 1\}) = \mu(\{x_n : n \geqslant p\}) \leqslant \mu(A_p) \to 0,$$

so $\{x_n\}_{n=1}^{\infty}$ admits a convergent subsequence. Remember that α (Kuratowski m.n.c.), β (Istrătescu m.n.c.) and χ (Hausdorff m.n.c.) are all regular, nonsingular and monotone. They are even ultraadditive i.e. $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$. $\Diamond$

Example 2. Let μ satisfy

(i) $\mu(\{x_n\}_{n=1}^{\infty}) = 0 \Rightarrow \{x_n : n \geqslant 1\}$ is relatively compact,
(ii) there exists constant $M_1 > 0$ s.t. $\mu(A \cup \{x_1, x_2, \ldots, x_p\}) \leqslant M_1 \cdot \mu(A)$,
(iii) there exists constant $M_2 > 0$ s.t. $A \subset B \Rightarrow \mu(A) \leqslant M_2 \cdot \mu(B)$.

Of course any measure from Example 1 obeys (i)-(iii). Similarly as before every measure with (i)-(iii) has property $(*)$. The measure considered in [B], namely χ_i (inner Hausdorff m.n.c.), is just regular but nonsingularity is weakened to condition (ii) with $M_1 = 2$, and monotonicity is weakened to condition (iii) with $M_2 = 2$. $\Diamond$

Example 3. Let μ satisfy

(a) $\mu(\{x_n : n \geqslant p\}) \underset{p\to\infty}{\longrightarrow} 0 \Rightarrow \{x_n : n \geqslant 1\}$ is relatively compact,
(b) there exists constant $M > 0$ s.t. $A \subset B \Rightarrow \mu(A) \leqslant M \cdot \mu(B)$.

Any measure from Example 2 obeys (a)-(b) ((a) follows from (i)-(ii) and (iii) is exactly (b)). Again each measure with (a)-(b) has property $(*)$. The functionals δ (diameter) and χ_C (relative Hausdorff m.n.c.) are monotone (hence (b)). They fulfill also (a) due to inequalities $\chi \leqslant \chi_C$, $\chi \leqslant \delta$. Moreover, a nonempty set A has $\delta(A) = 0$ iff A is singleton. This is stronger than upper regularity but, at the same time, this shows also nonregularity of δ. $\Diamond$

Remark that for any μ-descending family $\{A_n\}_{n=1}^{\infty}$, μ with the Kuratowski property, one has $\chi(A_n) \to 0$ $(|\chi(A_n) - \chi(A)| \leqslant h(A_n, A) \to 0$, where $A = \bigcap_n A_n$, $\chi(A) = 0$; cf. Theorem 2 in [B]). So every μ-descending family is automatically χ-descending (equivalently α-descending etc.). Obviously the converse is not true (just take $\mu = \delta$). Thus we can think about χ (or any equivalent m.n.c.) as the coarsest possible m.n.c. when formulating the Kuratowski Theorem.

At the end observe that it is enough to check the Kuratowski Theorem just for one measure among α, β, χ, χ_i because they are all equivalent. Then the case of δ and χ_C can be deduced.

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