Disjunctive sequences, discontinuous systems and fast basins

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I – finite alphabet

Def. $i_1 i_2 i_3 \cdots \in I^\infty$ – **disjunctive**, if

$$\forall\ L\ \ \forall\ \tau_1\tau_2 ... \tau_L \in I^L\ \ \exists\ n \quad \text{s.t.} \quad \tau_1 ... \tau_L = i_{n+1} ... i_{n+L}.$$

Algorithmic randomness

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Theorem. Endow I^∞ with the Baire metric. Then

$$\{(i_n)_{n=1}^\infty \in I^\infty : (i_n)_{n=1}^\infty \ \text{ is } \textbf{not} \text{ disjunctive}\}$$

is σ-**porous**.

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Def. $P \subset Y$ is **porous** when

$$\exists\ 0 < \lambda < 1,\ r_0 > 0\ \ \forall\ p \in P,\ 0 < r < r_0\ \ \exists\ y \in Y \quad B(y, \lambda r) \subset B(p, r) \setminus P.$$

σ-**porous** $\equiv$ countable union of porous sets.

Theorem.
$(Z_n)_{n=1}^{\infty}$ – stochastic process with states in I
$\forall n \geq 1\ \forall i_1, \ldots, i_n \in I$

$$\Pr(Z_n = i_n \,|\, Z_{n-1} = i_{n-1}, \ldots, Z_1 = i_1) \geq \alpha_n.$$

If either of the conditions holds:

1. $\alpha_n \equiv \alpha > 0$ (Barnsley & Vince 2011),
2. $\alpha_n \approx 1/(\ln n)^b$, for some $b > 0$,
3. $\alpha_n \gg 1/n^c$, uniformly for all $c > 0$,

then $(Z_n)_{n=1}^{\infty}$ a.s. generates a disjunctive sequence.

Randomness

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Prehistory (Onicescu & Mihoc, Mandelbrot, Hutchinson, Barnsley, Elton): homogeneous Bernoulli schemes and Markov chains

Randomness

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$\forall n \geq 1 \; \forall i_1, \ldots, i_n \in I$

$$\Pr(Z_n = i_n \mid Z_{n-1} = i_{n-1}, \ldots, Z_1 = i_1) \geq \alpha_n.$$

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Counterexamples:
(i) pairwise i.i.d. processes,
(ii) some strongly ergodic homogeneous Markov chains.

How to get disjunctiveness (deterministically)?

Champernowne

Write all 1-letter words, 2-letter words, 3-letter words, etc.

Not optimal!

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Write all 1-letter words, 2-letter words, 3-letter words, etc.

DeBruijn

Given m there exists a shortest sequence w so that each word of the length $\leq m$ appears in w.

IFS

X – metric space, $(X; f_i : i \in I)$ IFS of $f_i : X \to X$, $F : 2^X \to 2^X$ – Hutchinson operator,

$$F(S) := \overline{\bigcup_{i\in I} f_i(S)} \quad \forall\, S \subset X,$$

$C = F(C)$ – invariant (in closure) set.

Def. A – **strict attractor**, when A – compact,

$$(*) \qquad F^k(S) \to A \text{ for all nonempty compact } S \subset B(A);$$

$B(A)$ – basin of attraction = maximal open nbd of A fulfilling $(*)$.

Chaos game

$(X; f_i : i \in I)$, $F(C) = C$ – compact

$$\begin{cases} \text{Start} \quad x_0 \quad \text{"near"} \quad C; \\ \text{Iterate} \quad x_n := f_{i_n}(x_{n-1}). \end{cases}$$

Observation

Often, $(x_n)_{n=0}^{\infty}$ recovers C:

$$\omega((x_n)) := \bigcap_{k=0}^{\infty} \overline{\{x_n : n \geq k\}} = C.$$

Chaos game

$(X; f_i : i \in I)$, $F(C) = C$ – compact, $(i_n)_{n=1}^{\infty}$ – disjunctive

$$\begin{cases} \text{Start} \quad x_0 \quad \text{“near”} \quad C; \\ \text{Iterate} \quad x_n := f_{i_n}(x_{n-1}). \end{cases}$$

Theorems. Assume either of the set of conditions below:

1. C – strict attractor **strongly-fibred** by f_i's: for every $a \in C$ and open nbd $U_a \ni a$ there exists $j_1 j_2 \ldots j_k \in I^k$ s.t.

$$f_{j_1} \circ \cdots \circ f_{j_k}(C) \subset U_a,$$

2. C – strict attractor, f_i – nonexpansive for all $i \in I$,

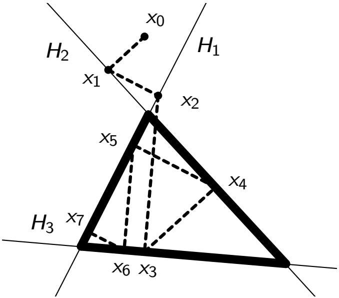
$$d(\, f_i(x), f_i(\widetilde{x})\,) \leq d(x, \widetilde{x}) \quad \forall x, \widetilde{x},$$

3. C – minimal invariant set s.t. $d(x_n, C) \to 0$, f_i – nonexpansive, $f_{j_1} \circ \cdots \circ f_{j_k}$ – Lipschitz contraction for some $j_1 \ldots j_k \in I^k$, $k \geq 1$.

Then $C = \omega((x_n))$.

Illustration: Kaczmarz algorithm in the infeasible case

Randomly applied projections onto three lines



History: G.S. Goodman; Hoggar & McFarlane; Angelos & Grossman & Kaufman & Lenker & Rakesh; Barnsley & L.

Illustration: Discontinuous IFS

$$\beta(x) := \begin{cases} \frac{1-b}{p} \cdot x + b, & x < p, \\ \frac{a}{1-p} \cdot x - \frac{p}{1-p}, & x \geq p \end{cases}$$

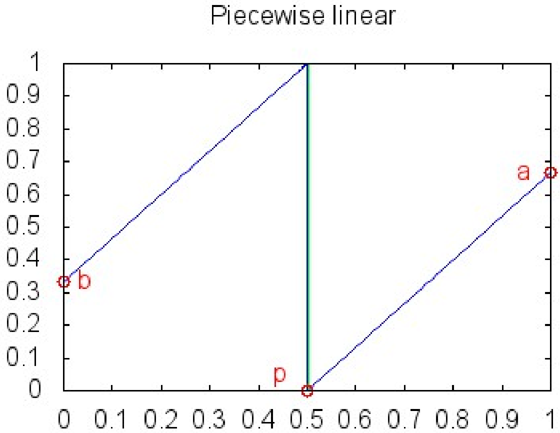
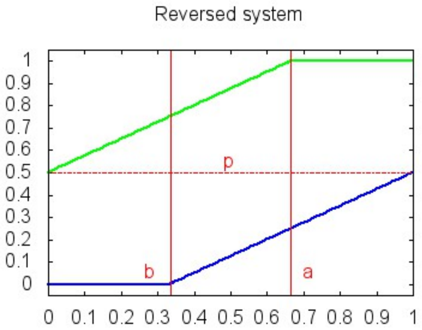


Illustration: Discontinuous IFS

Reversed system $\mathcal{F} = ([0,1]; f_0, f_1)$ (invert monotone branches of β and extend them; contractive with the attractor $[0,1]$)

$$f_0(x) := \max\left(0, p \cdot (x-b)/(1-b)\right),$$
$$f_1(x) := \min\left(1, (1-p) \cdot x/a + p\right).$$

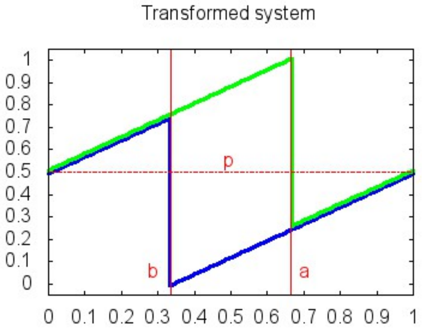


Barnsley:
What happens, if we play the place dependent chaos game for $\mathcal{F}$ so that to dismiss flat branches?

Illustration: Discontinuous IFS

Place dependent realization of $\mathcal{F}$: $\mathcal{G} = ([0,1]; g_0, g_1)$

$$g_i(x) := \begin{cases} f_i(x), & x \in [0,1] \setminus J_i, \\ f_{1-i}(x), & x \in J_i, \end{cases}$$
$$J_0 = [0,b), J_1 = (a,1], i = 0,1.$$



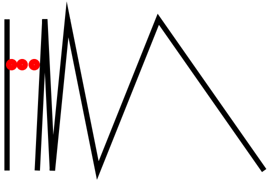
Depending on p, a, b the chaos game for $\mathcal{G}$ yields various results. For example:
$p = \frac{1}{2}, a = \frac{2}{3}, b = \frac{1}{3} \Rightarrow$

1. $g_0 \circ g_1,\ g_1 \circ g_0$ – local contractions
2. $C = \{2/7, 5/7\}$, $U = B(C, 1/21)$.

Less practical story...

Non-point-fibred peculiarities

1. $(X; f_i : i \in I)$ — IFS of contractions "tiling" $X = \bigcup_{i=1}^{N} f_i(X)$, $\mathrm{card} X \geq 2 \quad \Rightarrow \quad (X \times X; id \times f_i, f_i \times id \; : \; i \in I)$ is **strongly-fibred**, **noncontractive** under remetrization and admits $X \times X$ as a strict attractor.
2. (Sivak's system) Connected strict attractor which is NOT arcwise connected, NOT locally connected. [Hata's theorem violated]



Homoclinic attractors in discontinuous IFSs

- $\mathcal{F} = (X; f_i : i \in I)$ – IFS,
- A – strict attractor, $A \neq B(A)$ – basin, i.e.,

$$(*) \qquad F^k(S) \to A \text{ for all nonempty compact } S \subset B(A),$$

- $b \in B(A) \setminus A$,
- $\widetilde{f_i}|A \equiv b$, $\widetilde{f_i} = f_i$ outside A,
- $\widetilde{\mathcal{F}_b} = (X; \widetilde{f_i} : i \in I)$ – discontinuous modification of $\mathcal{F}$.

Question

Whether/when A persists a strict attractor after the modification of $\mathcal{F}$?

We would have then a homoclinic attractor $\widetilde{F}(A) \not\subset A$ (udergoing an expulsion of its content)

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Question

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Answer

It depends upon the fast basin $\widehat{B}(A)$ of the original system $\mathcal{F}$.

Fast basin

$\mathcal{F} = (X; f_i : i \in I)$ – IFS, A – strict attractor,
Def. **Fast basin** of A:

$$\widehat{B}(A) := \{x \in X \ : \ \exists_k \ F^k(x) \cap A \neq \emptyset\}.$$

Remark: The fast basin is a projection of the fractal manifold onto X.

Criteria for a homoclinic attractor

- $\mathcal{F} = (X; f_i : i \in I)$, A – strict attractor,
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Necessary condition

If A is a strict attractor of $\widetilde{F_b}$, then $b \notin \widehat{B}(A)$.

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Sufficient condition

If $b \notin \widehat{B}(A)$ and
(**nonresonance**) there exists an open nbd $A \subset U(A) \subset B(A)$ s.t.

$$\kappa(S) := \sup\{k : \ \ F^k(S) \cap (\widehat{B}(A) \setminus A) \neq \emptyset \ \} < \infty$$

for all nonempty compact $S \subset U(A)$,
then A is a strict attractor of $\widetilde{F_b}$.

THANK YOU!

- M.F. Barnsley, K. Leśniak, *The chaos game on a general iterated function system from a topological point of view*. International Journal of Bifurcation and Chaos
- K. Leśniak, *On discrete stochastic processes with disjunctive outcomes*. Bulletin of the Australian Mathematical Society
- K. Leśniak, *Random iteration for nonexpansive iterated function systems: derandomised algorithm*. International Journal of Applied Nonlinear Science
- megaloxantha, *Random Kaczmarz algorithm*, http://www.geogebratube.org/student/m93219
- arXiv:1307.5920, in progress (FGS Tabarz 2014, PODE Białystok 2013)