Some topology for iterated function systems

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IFS

X — metric (or topological) space, $f_i : X \to X$ — continuous functions
$\mathcal{F} = (X; f_i \ : \ i \in I)$ — iterated function system, I — finite.

The Hutchinson operator

$F : 2^X \to 2^X$,

$$F(S) := \overline{\bigcup_{i \in I} f_i(S)}$$

for $S \subset X$.

Invariant set

$$A = F(A) = \bigcup_{i \in I} \overline{f_i(A)}$$

$(\emptyset \neq A \subset X)$.

Attractors

X – metric space, $X \supset A$ – nonempty closed,
$\mathcal{F} = (X; f_i \,:\, i \in I)$ – IFS

$$\textbf{(Attr)} \qquad F^n(S) = \underbrace{F \circ .. \circ F}_{n}(S) \xrightarrow{\text{Hausdorff metric}} A$$

- **(Attr)** for all nonempty closed bounded sets $S \subset X$
  $\rightsquigarrow A$ — Hutchinson **attractor**;
- there exists open $U \supset A$
  (Attr) for all nonempty compact sets $S \subset U$
  $\rightsquigarrow A$ — Barnsley–Vince **strict attractor**;
- there exists open $U \supset A$
  (Attr) for all singletons $S = \{x\}$, $x \in U$
  $\rightsquigarrow A$ — **pointwise strict attractor**.

Properties of attractors

A – (Hutchinson | BV strict | pointwise strict) attractor

- (*Invariance*) $A = F(A)$;
- (*Compactness*) A – compact (and separable);
- (*Uniqueness*) $F(A) = A$ – unique among

  Hutchinson all nonempty closed and bounded invariant sets $A \subset X$;

  BV all nonempty compact invariant sets $A \subset U$;

  pointwise NOT unique!
- (*Chaos Game Representation*) $A = \bigcap_{m=1}^{\infty} \overline{\{x_n : n \geq m\}}$ with probability 1, where

$$\begin{cases} x_0 \in U, \\ x_n = f_{i(n)}(x_{n-1}), \end{cases}$$

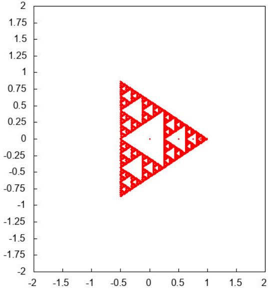
$\exists_{\alpha>0} \quad \Pr(Z_n = i(n) \mid Z_{n-1} = i(n-1), \ldots, Z_1 = i(1)) \geq \alpha$,
Z_n – chain with complete connections with states in I.

Hutchinson vs BV: remetrization

$\mathcal{F} = (\mathbb{C}; f_1, f_2, f_3)$,

$f_i(z) = \frac{1}{2} \cdot (z + a_i),\ i = 1, 2, 3,$

$A = \mathrm{SG}(a_1, a_2, a_3)$ – Sierpiński gasket.



After changing the metric in $\mathbb{C}$,

$$|\cdot| \quad \rightsquigarrow \quad d_1(z, w) = \min\{1, |z - w|\}\ \forall\, z, w \in \mathbb{C},$$

- $\mathcal{F}$ **does not have** a Hutchinson attractor.

  Reason: $F(A) = A,\ F(\mathbb{C}) = \mathbb{C},\quad A, \mathbb{C}$ – closed bounded in $(\mathbb{C}, d_1)$.

- However, $\mathcal{F}$ **does have** a strict attractor regardless of remetrizations.

Hutchinson vs BV: infinite dim

$\ell^2 \supset X = \left\{(\lambda_k)_{k=1}^{\infty} \in \mathbb{R}^{\infty} \ : \ \sum_{k=1}^{\infty} |\lambda_k|^2 \leq 1\right\}$ – closed unit ball in Hilbert space,

$$f : X \to X, \qquad f\,(\lambda_k)_{k=1}^{\infty} = \left(\left(1 - \tfrac{1}{k+1}\right) \cdot \lambda_k \right)_{k=1}^{\infty},$$

$\|f(x) - f(y)\| < \|x - y\|, \qquad \forall\, x \neq y \in X$ – Edelstein contraction,

$\mathcal{F} = (X; f)$.

- $A = \{0\}$ is a strict attractor of $\mathcal{F}$ with basin $U = X$.
- $\mathcal{F}$ has **no** Hutchinson attractor.
  Indeed: $F^n(D(0,r)) \not\to A, \ \forall 0 < r \leq 1$.

BV vs pointwise

(Fitzsimmons & Kunze 2020)

$X = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ – circle, $\quad f : X \to X$,

$f(\ \exp(2\pi\sqrt{-1} \cdot t)\ ) = \exp(2\pi\sqrt{-1} \cdot t^2), \quad t \in [0, 1)$,

$\mathcal{F} = (X; f)$.

- $A = \{1\}$ is a pointwise strict attractor of $\mathcal{F}$ with basin $U = X$.
  Indeed: $t^{2^n} \to 0 \ \Rightarrow \ f^n(z) \to 1 = exp(2\pi\sqrt{-1} \cdot 0)$.
- $A = \{1\}$ is **not unique** compact $\mathcal{F}$-invariant set within U;
  $X = f(X) = F(X)$ is another one.
- $\mathcal{F}$ has **no** strict attractor.

Attractors in topological spaces

X – normal **topological** space, $X \supset A$ – nonempty closed,
$\mathcal{F} = (X; f_i \ : \ i \in I)$ – IFS

$$\textbf{(Attr)} \qquad F^n(S) = \underbrace{F \circ .. \circ F}_{n}(S) \ \overset{\text{Vietoris}}{\longrightarrow} \ A$$

- there exists open $U \supset A$
  (Attr) for all nonempty compact sets $S \subset U$
  $\rightsquigarrow A$ — Barnsley–Vince **strict attractor**;
- there exists open $U \supset A$
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  $\rightsquigarrow A$ — **pointwise strict attractor**.

Note: Vietoris topology will be recalled 4 slides later...

Properties of attractors in topological spaces

X – normal topological space;
A – (BV strict | pointwise strict) attractor

- (*Invariance*) $A = F(A)$;
- (*Compactness+*) A – compact and separable;
- (*Uniqueness*) $F(A) = A$ – unique among

  BV all nonempty compact invariant sets $A \subset U$;

  pointwise NOT unique!
- (*Chaos Game Representation*) $A = \bigcap_{m=1}^{\infty} \overline{\{x_n : n \geq m\}}$ with probability 1, where

$$\begin{cases} x_0 \in U, \\ x_n = f_{i(n)}(x_{n-1}), \end{cases}$$

$\exists_{\alpha>0} \quad \Pr(Z_n = i(n) \,|\, Z_{n-1} = i(n-1), \dots, Z_1 = i(1)) \geq \alpha$,
Z_n – chain with complete connections with states in I.

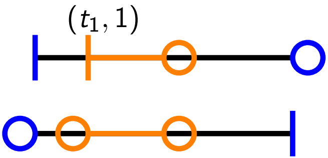
Ex: Nonmetrizable strict attractor

- $X = (0,1] \times \{0\} \cup [0,1) \times \{1\} \subset \mathbb{R}^2$ – Alexandrov's double arrow.
- Subbase for the topology: $t_0 \in (0,1], t_1 \in [0,1), r > 0,$

$$((t_0 - r, t_0] \times \{0\} \cup (t_0 - r, t_0) \times \{1\}) \cap X,$$
$$((t_1, t_1 + r) \times \{0\} \cup [t_1, t_1 + r) \times \{1\}) \cap X.$$

- $\mathcal{F} = (X; f_i : i = 1,2,3), \quad f_1(t,j) = \left(\frac{t}{2}, j\right),\ f_2(t,j) = \left(\frac{t+1}{2}, j\right),$
  $f_3(t,j) = (1-t, 1-j) \quad \forall\, (t,j) \in X$

1. $A = X$ – strict attractor of $\mathcal{F}$;
2. X – compact separable first countable non-metrizable space.



$\mathcal{F}$ behaves like a mosaic IFS $([0,1]; f_1, f_2)$, $f_1(x) = x/2$, $f_1(x) = (x+1)/2$.

More nonmetrizable attractors?

Problem:
Is the cube of continuum weight $X = [0,1]^{\mathfrak{c}}$ a strict attractor of a finite IFS?

Note: X is a separable compact space.

Vietoris topology

X — normal topological space
$\mathcal{C}\ell(X)$ — nonempty closed subsets of X
$\mathcal{K}(X)$ — nonempty compact subsets

Vietoris topology in $\mathcal{C}\ell(X)$ is generated by subbasic sets U^-, U^+,

$$U^- := \{C : C \cap U \neq \emptyset\},$$
$$U^+ := \{C : C \subset U\},$$

where $U \subset X$ is open. $(\mathcal{C}\ell(X), \mathit{Vietoris}) \in T_2$ thanks to $X \in T_4$.

Theorem. If X — metric space, then Vietoris = Hausdorff metric topology on $\mathcal{K}(X)$.

Example:

$$\mathcal{K}(X) \ni \{1, 2, ..., n\} \xrightarrow[n\to\infty]{\mathit{Vietoris}} \mathbb{N} \notin \mathcal{K}(X) \qquad \text{but} \quad \overset{\mathit{Hausdorff}}{\underset{n\to\infty}{\not\longrightarrow}}$$

Kuratowski–Painleve limits

$X \supset S_n$ – closed sets, $n \geq 1$.

Lower Kuratowski lim

$\mathrm{Li}\, S_n = \{y \in X : \exists x_n \in S_n, \ \ x_n \to y\}$

Upper Kuratowski lim

$\mathrm{Ls}\, S_n = \{z \in X : \exists n_k \nearrow \infty, \ \exists x_{n_k} \in S_{n_k}, \ \ x_{n_k} \to z\}$

If $\mathrm{Li}\, S_n = \mathrm{Ls}\, S_n =: S$, then $\mathrm{Lt}\, S_n := S$ – **Kuratowski limit**.

Property: $\overline{\bigcup_{n=1}^{\infty} S_n} = \bigcup_{n=1}^{\infty} S_n \cup \mathrm{Ls}\, S_n, \qquad \mathrm{Ls}\, S_n = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} S_n}$.

Example:

$$\mathcal{K}(X) \ni \{1, n\} \xrightarrow[n\to\infty]{Kuratowski} \{1\} \in \mathcal{K}(X) \qquad \text{but} \quad \underset{n\to\infty}{\overset{Vietoris}{\not\longrightarrow}}$$

Set-convergence: toward synthesis

X – **compact** metric space, $\quad S_n \in \mathcal{C}\ell(X) = \mathcal{K}(X)$

1. If S_n descends, i.e., $S_n \supset S_{n+1}, \forall n \geq 1$, then $S_n \xrightarrow[n\to\infty]{H,V,K} \bigcap_{n=1}^{\infty} S_n$.

2. If S_n ascends, i.e., $S_n \subset S_{n+1}, \forall n \geq 1$, then $S_n \xrightarrow[n\to\infty]{H,V,K} \overline{\bigcup_{n=1}^{\infty} S_n}$.
3. Hausdorff = Vietoris = Kuratowski convergence.

X – complete, $X \supset S_n$ – compact, S – closed.
Consider

(H) $S_n \to S$ w.r.t. the Hausdorff distance,

(V) $S_n \to S$ in the Vietoris topology,

(K) $\mathrm{Lt}\, S_n = S$ (Kuratowski limit).

Then

(i) If S is a compact set, then $(H) \Leftrightarrow (V) \Rightarrow (K)$.
If additionally $\overline{(\bigcup_{n=1}^{\infty} S_n)}$ is compact, then $(H) \Leftrightarrow (V) \Leftrightarrow (K)$.

(ii) For (H) we have that the sets $\overline{(\bigcup_{n=1}^{\infty} S_n)}$ and S are compact.

(iii) For (V) we have: S – compact $\Leftrightarrow$ $\overline{(\bigcup_{n=1}^{\infty} S_n)}$ – compact.

(iv) For (K) we have: $\overline{(\bigcup_{n=1}^{\infty} S_n)}$ – compact $\Rightarrow$ S – compact.

Lasota–Myjak semiattractor

A nonempty closed set $A_\flat \subset X$ is a **semiattractor** of the IFS $\mathcal{F} = (X; f_i \,:\, i \in I)$ provided

$$(LM\,At) \qquad \bigcap_{x \in X} \operatorname{Li} F^n(\{x\}) = A_\flat.$$

Properties:

- (*Invariance*) $A_\flat = F(A_\flat)$;
- (*Minimality* – instead of Uniqueness)

$$\forall C \subset X, \text{ nonempty closed}, \ (F(C) = C \Rightarrow C \supset A_\flat);$$

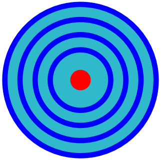
- (*Semiattractivity*) $\forall \emptyset \neq S \subset X, \ \operatorname{Li} F^n(S) \supset A_\flat = \bigcap_{\emptyset \neq S \subset X} \operatorname{Li} F^n(S)$;
- (*Self-regeneration*, or inner attractivity) $\forall \emptyset \neq S \subset A_\flat, \ \operatorname{Lt} F^n(S) = A_\flat$.

Ex: Semiattractor and several coexisting invariant sets

$\mathcal{F} = (X = \mathbb{R}^2; f_1 = R_\alpha, f_2(v) = v/2)$, $\quad \frac{\alpha}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$,

$A_\flat = \{0\}$ – semiattractor of $\mathcal{F}$,

$F(A) = A$ for every disk $A = D(0; r)$, $r \geq 0$.



Ex: Symmetric gasket as semiattractor

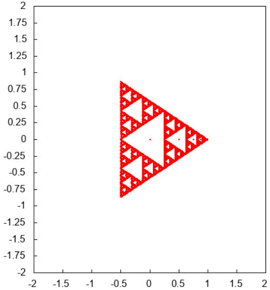
$(\mathbb{C}; f_1, f_2, f_3)$,

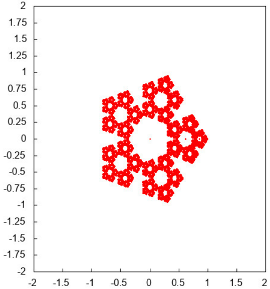
$f_i(z) = \frac{1}{2} \cdot (z + a_i)$, $i = 1, 2, 3$,

$A = \mathrm{SG}(a_1, a_2, a_3)$ – Sierpiński gasket, a_1, a_2, a_3 — 3rd roots of unity

$(\mathbb{C}; f_1, R_{120^\circ})$ yields $A_\flat = \mathrm{SG}(a_1, a_2, a_3)$

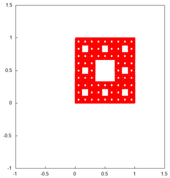
Conclusion: Rotations may be handy for introducing symmetry, e.g., pentagasket.



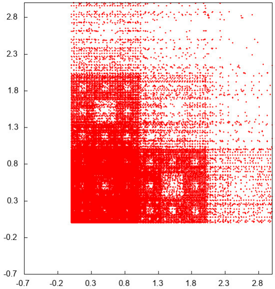


Ex: Unbounded semiattractors

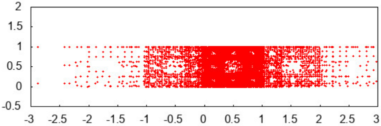
$(\mathbb{R}^2; f_1, \ldots, f_8)$ – standard encoding of A = Sierpiński carpet



$(\mathbb{R}^2; f_1, .., f_8,\ T_{[0,1]},\ T_{[1,0]})$ yields...
$A_\flat$ = quarter-infinite Sierpiński carpet



$(\mathbb{R}^2; f_1, .., f_8,\ T_{[-1,0]},\ T_{[1,0]})$ yields...
$A_\flat$ = infinite Sierpiński scarf



(Lasota & Myjak 1996): Infinite Sierpiński chessboard

Theorem (Lasota–Myjak Criterion, Reverse Self-Regeneration):

Let $\mathcal{F} \subset \mathcal{G}$ be two IFSs.

If $\mathcal{F}$ admits a semiattractor, denoted $A_\flat$,
then $\mathcal{G}$ admits a semiattractor $A_{\flat\flat}$, which is given by the formula

$$A_{\flat\flat} = \operatorname{Lt} G^n(A_\flat) = \overline{\bigcup_{n=0}^{\infty} G^n(A_\flat)} \supset A_\flat.$$

$A_\flat$ – **nucleus** of $A_{\flat\flat}$; G – the Hutchinson operator induced by $\mathcal{G}$.

Relation between various kinds of attractors

$\mathcal{F} = (X; f_i : i \in I)$ (Hu At) $\to$ (BV At) $\to$ (pt At) $\to$ (LM At)

(Hu At) vs (BV At)

1. A – Hutchinson attr. $\Rightarrow A_* := A$ – strict attr., $B(A_*) = X$.
2. A_* – strict attr., $\emptyset \neq F(C) \subset C$ – compact, $C \subset B(A_*)$ $\Rightarrow A := A_* \subset C$ – Hutchinson attr. of $\mathcal{F}|\,C$.

(BV At) vs (pt At)

3. A_* – strict attr. $\Rightarrow A_*$ – pt attr., $B_1(A_*) = B(A_*)$.
4. A_* – pt attr., $\mathcal{F}$ – nonexpansive ($d[f(x), f(y)] \leq d(x, y)$ $\forall\, x, y \in X\ \forall\, f \in \mathcal{F}) \Rightarrow A_*$ – strict attr., $B(A_*) = B_1(A_*)$.

(LM At) vs rest

5. A_* – pt attr. $\Rightarrow A_\flat := A_*$ – semiattr. of $\mathcal{F}|\,B_1(A_*)$.
6. $A_\flat$ – compact semiattr. $\Rightarrow A_* := A_\flat$ – strict attr. of $\mathcal{F}|\,A_\flat$.

The type of a compact attractor is important only outside but not inside an attractor.

Markov theory for IFSs

X – complete separable metric space,

$(X;(f_i,p_i): i\in I)$, $f_i : X\to X$, $\sum_{i\in I} p_i = 1$, $p_i\geq 0$ – probabilistic IFS.

Markov operator $M:\mathcal{M}(X)\to\mathcal{M}(X)$ – Borel probab. measures,

$M(\mu)(B) := \sum_{i\in I} p_i\cdot \mu[f_i^{-1}(B)]$, $\forall\, B\in \mathrm{Borel}(X)$, $\mu\in\mathcal{M}(X)$.

Theorem (Barnsley & Elton 1988, ..., Lasota, Myjak & Szarek 1996–2004, ...):

$\sum_{i\in I} p_i\cdot \mathrm{Lip}(f_i) < 1$ (average contractivity) $\Rightarrow \exists\, \mu_\flat\in\mathcal{M}(X)$:

1. (*Invariance*) $M(\mu_\flat) = \mu_\flat$ – unique;
2. (*Asympt. stabil.*) $M^n(\mu) \overset{w*}{\rightharpoonup} \mu_\flat$, $\forall\,\mu\in\mathcal{M}(X)$;
3. $A_\flat := \mathrm{supp}\,\mu_\flat$ – semiattractor of $\mathcal{F} = (X; f_i : i\in I)$.

THANK YOU