A NOTE ON INVARIANT SUBSPACES

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The purpose of this note is to make some comments on the invariant subspace problem and give some (possibly new) formulations of the problem.

Let $\mathcal{H}$ be a complex separable Hilbert space and $H^2_{\mathcal{H}} = H^2$ be the usual l Hilbert space of $\mathcal{H}$ valued square integrable analytic functions on the circle group S^1. Let χ denote the character on S^1 given by $\chi(z) = z$ and let S denote the isometry on H^2 given by

$$Sf = \chi f, \ \ f \in H^2.$$

A subspace (always supposed to be closed) $\mathcal{M} \subset H^2$ is said to be S invariant if $S\mathcal{M} \subset \mathcal{M}$. An S-invariant subspace is of the form $\mathbf{J}H^2$ where the linear operator $\mathbf{J}$ is a multiplication operator defined by

$$(\mathbf{J}f) = J(z)f(z), \ \ f \in H^2,$$

with J a measurable function on S^1 whose values are partial isometries on $\mathcal{H}$. Since $\mathbf{J}$ maps H^2 into itself, the Fourier expansion of J is necessarily one sided ([2]).

We shall assume in the rest of this article that the partial isometries $J(z), z \in S^1$, are all isometries in which case $\mathbf{J}$ is an isometry of H^2. Such J's are called operator inner functions. We also assume that the isometry $\mathbf{J}$ is *pure* in the sense that $\cap_{n=1}^{\infty} \mathbf{J}^n H^2 = \{0\}$. Pure isometries are also called *completely non-unitary* ([5]). The isometry S is pure in this sense.

It is known that every contraction T on $\mathcal{H}$ is similar to the adjoint S^* acting on the $\mathcal{K} = (\mathbf{J}H^2)^\perp$ for a suitable pure isometry $\mathbf{J}$ on H^2. If S^* acting on $\mathcal{K}$ admits a proper invariant subspace $\mathcal{L}$ then the S invariant subspace $\mathbf{J}H^2$ admits a proper S invariant superspace, namely the space $\mathcal{L}^\perp$ ([3] p 103).

The invariant subspace problem asks if every bounded linear operator T on a complex separable Hilbert space of dimension at least two admits a proper T invariant subspace. Since invariant subspaces of T are same as those for any of the contractions cT, with $0 < c < \frac{1}{||T||}$, we see that the invariant subspace problem is equivalent to showing that for any pure isometry $\mathbf{J}$ on H^2 commuting with S the S invariant subspace $\mathbf{J}H^2$ admits a proper S invariant superspace.

We now view things slightly differently. For a pure isometry $\mathbf{J}$ as above we can speak of $\mathbf{J}$-invariant subspaces. In particular SH^2 is one such since J commutes with S. Let

$$J(z) = J_0 + J_1\chi + J_2\chi^2 \cdots$$

be the Fourier expansion of J.

Proposition: SH^2 admits a proper $\mathbf{J}$ invariant superspace if and only if J_0, which acts on $\mathcal{H}$, admits a proper J_0 invariant subspace. Moreover, $J_0^* = \mathbf{J}^*$ restricted to $\mathcal{H}$

Proof: If J_0 admits a proper J_0 invariant subspace, say $\mathcal{L}$, then the direct sum of SH^2 and $\mathcal{L}$ is a proper $\mathbf{J}$ invariant superspace for SH^2. On the other hand if $\mathcal{M}$ is a proper $\mathbf{J}$ invariant superspace of SH^2, then the orthogonal complement of SH^2 in $\mathcal{M}$ is necessarily a proper subspace of $\mathcal{H}$ and invariant under J_0.

For the second part note that for $x, y \in \mathcal{H}$

$$(J_0^* x, y)_{\mathcal{H}} = (x, J_0 y)_{\mathcal{H}} = \int_{S^1} (x, J(z)y)_{\mathcal{H}} dz$$

$$= \int_{S^1} (J^*(z)x, y)_{\mathcal{H}} dz = (\mathbf{J}^* x, y)_{H^2}.$$

This proves the proposition.

We now show that every contraction appears, up to similarity, as J_0 of some isometry valued analytic function J. To this end let T^* (rather than T) be any contraction and view it as S^* acting on the orthogonal complement of an S invariant subspace which is of the type $\mathbf{K}H^2$ for some pure isometry $\mathbf{K}$ commuting with S. Let $W = (\mathbf{K}H^2)^\perp$ which is the wandering subspace for $\mathbf{K}$. Then $H^2 = \sum_{k=0}^{\infty} \mathbf{K}^n W$, and we can realize $\mathbf{K}$ as the shift S_W acting on H^2_W. If ϕ is a unitary map from H^2 to H^2_W such that

$$\phi \circ \mathbf{K} \circ \phi^{-1} = S_W$$

then, since S and $\mathbb{K}$ commute,

$$\phi \circ S \circ \phi^{-1} = \mathbf{J},$$

for some isometry (on W) valued analytic function J. As seen above J_0^* on W is same as $\mathbf{J}^*$ acting W. But $\mathbf{J}^*$ acting on W is unitarily equivalent to S^* acting $(\mathbb{K}H^2)^\perp$, which in turn is similar to T^*. Thus J_0 is similar to T, and since T to begin with was arbitrary, the assertion is proved.

Assume now that dimension of $\mathcal{H}$ is greater than one. It follows that the statement " for any pure isometry $\mathbf{J}$ commuting with S, SH^2 admits a proper $\mathbf{J}$ invariant superspace" is equivalent to the positive answer to the invariant subspace problem.

Let $\mathcal{H}$ and $(\mathbf{J}H^2)^\perp$ have dimension greater than one. Consider the two statements:

1. SH^2 admits a proper $\mathbf{J}$ invariant superspace $\leftrightarrow$ J_0 admits a proper invariant subspace.

2. $\mathbf{J}H^2$ admits a proper S invariant superspace $\leftrightarrow$ J_0 admits a proper invariant subspace.

The two statements look similar and could be true, for any given $\mathbf{J}$, by both sides of the implication being simultaneously true or false. One of the point of this note is to observe that while the first statement is a proved statement, the second statement is equivalent to the positive answer to the invariant subspace problem. Indeed if the invariant subspace problem has a positive solution then each side of the second statement is true, hence the statement is true.

To prove the converse assume that the second statement is true and let $J(z) = J_0 + J_1\chi + \cdots$ any isometry valued analytic function such that the resulting $\mathbf{J}$ is pure. We can assume that J_0 is not the zero operator. Let $J_0 = UA$ be the polar decomposition ([6] p 154) of J_0 with U a partial isometry and A self-adjoint and non-negative. If A has a non-trivial null space then it is also the null space of J_0 and so non-trivial invariant subspace for J_0. By the leftward implication of the second statement $\mathbf{J}H^2$ admits a proper S invariant superspace. Now U^*U is projection on the closure of the range of A. If A does not admit a null space then$U$ is either an isometry which is not unitary or it is a unitary operator. In the first case U^* admits a no-trivial null space which is then also the null space of J_0^*. Clearly the ortho-compliment of this null space is a non-trivial invariant subspace for J_0 and again by the leftward implication of the second statement $\mathbf{J}H^2$ admits a proper S invariant superspace. Finally if U is unitary then U^*J has constant term A in its Fourier expansion which, being Hermitian, admits a proper invariant subspace. Again by our leftward implication in the second statement, the subspace $U^*\mathbf{J}H^2$ admits an S invariant proper superspace, say $\mathcal{L}$,

$$U^*\mathbf{J}H^2 \subsetneq \mathcal{L} \subsetneq H^2$$

so multiplying by U

$$\mathbf{J}H^2 \subsetneq U\mathcal{L} \subsetneq H^2$$

Then $U\mathcal{L}$ is a proper S invariant superspace for $\mathbf{J}H^2$. Thus truth of the second statement above implies the truth of the invariant subspace problem.

Since any pair of commuting pure isometries can be realized as an S and a $\mathbf{J}$ on a suitable H^2, we can state the invariant subspace problem in terms of commuting isometries as follows: If R and T are commuting pure isometries on a complex separable Hilbert space $\mathcal{H}$ such that dimensions of $(R\mathcal{H})^\perp$ and $(T\mathcal{H})^\perp$ are greater than one then $R\mathcal{H}$ admits a proper T invariant superspace *and* $T\mathcal{H}$ admits a proper R invariant superspace.

Thus positive solution to the invariant subspace problem implies a certain symmetry or mutuality between any pair of commuting pure isometries in the sense that each admits a proper invariant superspace with respect to the other, while the negative solution implies the existence of a non-symmetric pair.

We now discuss some examples of commuting pairs of isometries. The first non-trivial example is when one of $(R\mathcal{H})^\perp$, $(T\mathcal{H})^\perp$, say $(R\mathcal{H})^\perp$, has finite dimension. Under this assumption the fact that $T\mathcal{H}$ admits a proper R invariant superspace is a consequence of the fact that every finite dimensional linear operator has a proper invariant subspace. However the fact that $R\mathcal{H}$ has a proper T invariant superspace is a consequence of the more advanced theorem stating that a non-constant finite dimensional inner function admits a non-trivial factoring into inner functions ([1] Theorem 16 p 107).

Let $\mathcal{H} = L_0^2(S^1)$, the square integrable function (with respect to the Lebesgue measure) with zero means. Consider the commuting measure preserving maps $\rho(z) = z^2$ and $\tau(z) = z^3$. The isometries $R : f \to f \circ \rho$ and $T : f \to f \circ \tau$ commute, and are pure since the maps ρ and τ are exact endomorphisms. The spaces $(R\mathcal{H})^\perp$, $(T\mathcal{H})^\perp$ are infinite dimensional. Finally it is easy to show that $R\mathcal{H}$ and $T\mathcal{H}$ admit proper invariant superspaces with respect to T and S respectively.

One can generalize the above example by taking ρ and τ to be any two commuting exact measure preserving endomorphisms on S^1. The resulting S and T are commuting pure isometries on $L_0^2(S^1)$. The spaces $(R\mathcal{H})^\perp$, $(T\mathcal{H})^\perp$ are infinite dimensional. In this case we do not know if the conclusions of the last paragraph are valid.

We now discuss a dynamical formulation of the problem which is the second main point of this note. Let T be an exact endomorphism on a standard probability space $(X, \mathcal{B}, m)$, which means that T preserves m and that $\cap_{n=0}^{\infty} T^{-n}\mathcal{B} = \{\emptyset, X\}$, the equality being modulo m-null sets. Clearly, T is necessarily non-invertible. Let $\mathbb{H} = L_0^2(X, \mathcal{B}, m)$, the space of square integrable functions on X with mean zero. The map $Sf = f \circ T, f \in L_0^2(X, m)$ is an isometry since T is measure preserving and it is pure since T is exact. $(S^n\mathbb{H} = L_0^2(X, T^{-n}\mathcal{B}, m))$.

The requirement that $\mathbb{W} = \mathbb{H} - S\mathbb{H}$ be infinite dimensional comes gratis, being a fact well known in ergodic theory. ([1], Chapter 13, Lemma 1)

Let $\mathbb{M}$ be an S- invariant closed subspace of $\mathbb{H}$. Let $\mathcal{M}$ be the σ- algebra generated by a countable collection of Borel measurable functions dense in $\mathbb{M}$. We call $\mathcal{M}$ the σ-algebra generated by $\mathbb{M}$. Any two such $\sigma-$ algebras are equal modulo m-null sets.. Clearly, $\mathbb{M} \subset L_0^2(X, \mathcal{M}, m)$. Moreover $T^{-1}\mathcal{M} \subset \mathcal{M}$ since $T^{-1}\mathcal{M}$ is the σ-generated by functions in $S\mathbb{M} \subset \mathbb{M}$. Consequently $L_0^2(X, \mathcal{M}, m)$

is an S-invariant closed subspace of $\mathbb{H}$. If $\mathcal{M}$ does not separate points (i.e. $\mathcal{M} \neq \mathcal{B}$) and the inclusion $\mathbb{M} \subset L_0^2(X, \mathcal{M}, m)$ is strict then $\mathbb{M}$ admits a proper closed S-invariant superspace, viz, $L_0^2(X, \mathcal{M}, m)$.

A countably generated sub-σ-algebra of $\mathcal{B}$ is called a factor if $T^{-1}\mathcal{F} \subset \mathcal{F}$. Since T is an exact endomorphism the inclusion is strict except when $\mathcal{F}$ is equal to $\{\emptyset, X\}$ (modulo m-null) sets. A factor is called proper

if it is not equal to $\mathcal{B}$ and also not equal to $(\emptyset, X)$, modulo m-null sets. A proper factor $\mathcal{F}$ is called maximal if any factor containing $\mathcal{F}$ coincides (modulo m-null sets) with $\mathcal{F}$ or $\mathcal{B}$.

Clearly, if a proper factor $\mathcal{F}$ is not maximal then the closed S-invariant subspace $L_0^2(X, \mathcal{F}, m)$ admits a proper closed S-invariant superspace. If $\mathcal{F}$ is maximal and $L_0^2(X, \mathcal{F}, m)$ admits a proper closed S-invariant superspace, say $\mathbb{M}$, then functions in $\mathbb{M}$ must separate points of X, i.e., the σ-algebra generated by $\mathbb{M}$ equals $\mathcal{B}$ modulo m-null sets. Call a proper factor $\mathcal{F}$ strongly maximal if there is no proper $S - \textit{invariant}$ closed subspace of $\mathbb{H}$ strictly containing $L_0^2(X, \mathcal{F}, m)$.

From the above discussion it is clear that the statement ' there are no strongly maximal factors and every proper closed S-invariant subspace of $L_0^2(X, \mathcal{B}, m)$ separating points,and with orthocompliment of dimension at least two, admits a proper closed S-invariant superspace' is equivalent to the statement that 'every bounded linear operator on a Hilbert space of dimension at least two admits a proper invariant closed subspace'.

We note that maximal factors are necessarily of full entropy. For let T be an exact endomorphism of entropy α on $(X, \mathcal{B}, m)$ and $\mathcal{F}$ is a factor such that T acting on $(X, \mathcal{F}, m)$ has entropy $\beta < \alpha$, then $\mathcal{F}$ is not maximal. To see this, let $A_n, n = 0, 1, 2, 3, \cdots$ be a countable generator for $\mathcal{F}$, which means that $A_n, n = 0, 1, 2, 3, \cdots$ is a partition of X such that the collection$\{T^{-}kA_i : 0 \leq k, i < \infty\}$ generates $\mathcal{F}$. By a theorem of Rokhlin countable generators for factors exist. Now β, being strictly less than α, is finite. Since entropy moves continuously with small perturbation of the generating partition, we can partition A_1 as $C \cup D$, with $C, D \in \mathcal{B}$, but not in $\mathcal{F}$, in such a way that the entropy of the factor $\mathcal{G}$

generated by the partition $C, D, A_2, A_3, \cdots$ is less than α. Since $A_1 = C \cup D$, and $C, D \notin \mathcal{F}$, we see that $\mathcal{G}$ is strictly bigger than $\mathcal{F}$, and since the entropy of T acting on $(X, \mathcal{G}, m)$ is smaller than α, $\mathcal{G} \neq \mathcal{B}$. So $\mathcal{F}$ is not a maximal factor.

So, from the point of view of the invariant subspace problem, only the factors of full entropy are of interest.

For an ergodic measure preserving *automorphism* T on $(X, \mathcal{B}, m)$ a factor is defined to be a countably generated sub-σ algebra $\mathcal{F}$ such that $T^{-1}\mathcal{F} = \mathcal{F}$, and such factors are relatively well understood. ([4]). Maximal factors for automorphism exist, indeed any factor $\mathcal{F}$ whose atoms are two point sets is a maximal factor. However, for endomorphisms a factor whose atoms are two point sets may or may not be maximal. We give two examples, one of each kind.

Let T be defined on S^1 by $Tz = z^2$, and equip S with its Borel σ-algebra $\mathcal{B}$, and Haar measure. It is known that T is an exact endomorphism. Let $\mathcal{F}$ be the collection of Borel subsets B of

S^1 such that if $z \in B$ then $-z \in B$, i.e., pairs of antipodal points form the atoms of $\mathcal{F}$. Note that $T^{-1}\mathcal{B} = \mathcal{F}$, so it is a factor. It is not a maximal factor. To see this let $\mathcal{G}$ consist of Borel subsets of

$A = \{e^{ix} : 0 \leq x < \frac{\pi}{2}\} \cup \{e^{ix} : \pi \leq x < \frac{3}{2}\pi\}$ together with the Borel subsets B of $S^1 - A$ which are in $\mathcal{F}$, i.e., B contains $-z$ whenever it contains z. It can be verified that $\mathcal{G}$ is a proper factor obviously strictly bigger than $\mathcal{F}$.

The σ-algebra $\mathcal{F}$ is also a factor for the endomorphism T_1 defined by $T_1 z = z^3, z \in S^1$. However, unlike T, T_1 preserves atoms of $\mathcal{F}$. Indeed, $T_1(\{-z, z\}) = \{-z^3, z^3\}$. Further, $\mathcal{F}$ is a maximal factor for T_1. For let $\mathcal{G}$ be a factor strictly bigger than $\mathcal{F}$. Let $A = \{x : \{x\} \in \mathcal{G}\}$. We show that $A = S^1$ up to an h-null set. Now $h(A) > 0$ since

$\mathcal{G}$ is strictly bigger than $\mathcal{F}$. Since atoms of $\mathcal{F}$ are two point sets, $\mathcal{G}$ and $\mathcal{F}$ agree on $S^1 - A$. For for any $x \in A$ the points in $T_1^{-1}(\{x\})$ form the vertices of an equilateral triangle in S^1, so they can not contain a pair of antipodal points.

Since $\mathcal{G}$ is a factor for T_1, we see that $T_1^{-1}\{x\} \subset A$ for all $x \in A$. Since T_1 is ergodic and $h(A) > 0$, we have $h(S^1 - A) = 0$, proving that $\mathcal{F}$ is maximal for T_1.

We note, however, that $\mathcal{F}$ is not strongly maximal for T_1. The space $\mathbb{M} = L_0^2(S^1, \mathcal{F}, h)$ consists of square integrable functions on S^1 whose odd Fourier coefficients together constant term of the Fourier expansion vanish. It is invariant under the map $Sf = f \circ T_1$.

The closed subspace of $L_0^2(S^1, h)$ spanned by $\mathbb{M}$ together with the functions $f_n(z) = z^{3^n}, z \in S^1, n \in \mathbb{Z}$ is a proper superspace of $\mathbb{M}$ invariant under S. It is interesting to note that $L_0^2(X, \mathcal{F}, m)$ is invariant under both S and S^*. Further S^* acting on this space is not compact, since $S^*\{z^{2\cdot 3^n} : n \geq 1\} = \{z^{2\cdot 3^{n-1}}, n \geq 1\}$.

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