124. Uritsky VM, Klimas AJ, Vassiliadis D, Chua D, Parks G (2002) Scale-free statistics of spatiotemporal auroral emissions as depicted by POLAR UVI images: Dynamic magnetosphere is an avalanching system. J Geophys Res 107:1426; doi:10.1029/2001JA000281
125. Uritsky VM, Klimas AJ, Vassiliadis D (2002) Multiscale dynamics and robust critical scaling in a continuum current sheet model. Phys Rev E 65:046113
126. Uritsky VM, Paczuski M, Davila JM, Jones SI (2007) Coexistence of self-organized criticality and intermittent turbulence in the solar corona. Phys Rev Lett 99:025001
127. Vahnstein S, Sreenivasan K, Pierrehumbert R, Kashyap V, Juneja A (1994) Scaling exponents for turbulence and other random processes and their relationships with multifractal structure. Phys Rev E 50:1823–1835
128. Vespignani A, Zapperi S (1998) How self-organized criticality works: A unified mean-field picture. Phys Rev E 57:6345–6362
129. Vörös Z (2000) On multifractality of high-latitude geomagnetic fluctuations. Ann Geophys 18:1273–1283
130. Vörös Z, Baumjohann W, Nakamura R, Runov A, Zhang TL, Volwerk M, Eichelberger HU, Balogh A, Horbury TS, Glassmeier KH, Klecker B, Rème H (2003) Multi-scale magnetic field intermittence in the plasma sheet. Ann Geophys 21:1955–1964
131. Vörös Z, Baumjohann W, Nakamura R, Runov A, Volwerk M, Zhang TL, Balogh A (2004) Wavelet analysis of magnetic turbulence in the Earth's plasma sheet. Phys Plasmas 11:1333–1338
132. Vörös Z, Baumjohann W, Nakamura R, Runov A, Volwerk M, Schwarzl H, Balogh A, Rème H (2005) Dissipation scales in the Earth's plasma sheet estimated from cluster measurements. Nonlinear Process Geophys 12:725–732
133. Watkins NW, Chapman SC, Dendy RO, Rowlands G (1999) Robustness of collective behaviour in a strongly driven avalanche model: magnetospheric implications. Geophys Res Lett 26:2617–2620
134. Weygand JM, Kivelson MG, Khurana KK, Schwarzl HK, Thompson SM, McPherron RL, Balogh A, Kistler LM, Goldstein ML, Borovsky J, Roberts DA (2005) Plasma sheet turbulence observed by Cluster, vol II. J Geophys Res 110:A01205; doi:10.1029/2004JA010581
135. Wilson KG, Kogut J (1994) The renormalization group and the ε expansion. Phys Reports 12C:75–200
136. Wu CC, Chang T (2000) 2D MHD simulation of the emergence and merging of coherent structures. Geophys Res Lett 27:863–866
137. Wu CC, Chang T (2001) Further study of the dynamics of two-dimensional MHD coherent structures – a large scale simulation. J Atmos Solar Terr Phys 63:1447–1453
138. Wu CC, Chang T (2005) Intermittent turbulence in 2D MHD simulation. In: Lui ATY, Kamide Y, Consolini G (eds) Multiscale coupling of sun-earth processes. Elsevier, Netherlands, p 321
139. Yordanova E, Grezesiak M, Wernik AW, Popielawska B, Stasiewicz K (2005) Multifractal structure of turbulence in the magnetospheric cusp. Ann Geophys 22:2431–2440

Books and Reviews

Bruno R, Carbone V (2005) The solar wind as a turbulence laboratory. Living Rev Solar Phys 2:4; http://www.livingreviews.org/lrsp-2005-4. Accessed 15 June 2007
Chang T, Chapman S, Klimas A (eds) (2001) Forced and/or self-organized criticality (FSOC) in space plasmas. J Atmos Solar Terr 63:1359–1453
Forster D (1975) Hydrodynamic fluctuations, broken symmetry, and correlation functions. Benjamin, Reading
Frisch U (1995) Turbulence. Cambridge University Press, Cambridge
Lavenda BH (1995) Theromodynamics of Extremes. Albion Publishers, Chichester
Leubner MP, Baumjohann W, Chian ACL (eds) (2006) Advances in space environment research, vol 2. Space Sci Rev 122:1–337
Lui ATY, Kamide Y, Consolini G (eds) (2005) Multiscale coupling of sun-earth processes. Elsevier, the Netherlands
Ma SK (1976) Modern theory of critical phenomena. Benjamin, Reading
Sharma AS, Kaw PK (eds) (2005) Nonequilibrium phenomena in plasmas. Astrophysics and Space Science Library, vol 321. Springer, Dordrecht, pp 1–340
Sornette D (2000) Critical phenomena in natural sciences; Chaos, fractals, selforganization and disorder: concepts and tools. Springer, Berlin

Spectral Theory of Dynamical Systems

MARIUSZ LEMAŃCZYK
Faculty of Mathematics and Computer Science,
Nicolaus Copernicus University, Toruń, Poland

Article Outline

Glossary

Spectral decomposition of a unitary representation If $\mathcal{U} = (U_a)_{a\in\mathbb{A}}$ is a continuous unitary representation of a locally compact second countable (l.c.s.c.)

Abelian group $\mathbb{A}$ in a separable Hilbert space H then a decomposition $H = \bigoplus_{i=1}^{\infty} \mathbb{A}(x_i)$ is called *spectral* if $\sigma_{x_1} \gg \sigma_{x_2} \gg \ldots$ (such a sequence of measures is also called *spectral*); here $\mathbb{A}(x) := \overline{span}\{U_a x : a \in \mathbb{A}\}$ is called the *cyclic space* generated by $x \in H$ and σ_x stands for the spectral measure of x.

Maximal spectral type and the multiplicity function of $\mathcal{U}$ The *maximal spectral type* $\sigma_{\mathcal{U}}$ of $\mathcal{U}$ is the type of σ_{x_1} in any spectral decomposition of H; the *multiplicity function* $M_{\mathcal{U}} : \widehat{\mathbb{A}} \to \{1, 2, \ldots\} \cup \{+\infty\}$ is defined $\sigma_{\mathcal{U}}$-a.e. and $M_{\mathcal{U}}(\chi) = \sum_{i=1}^{\infty} 1_{Y_i}(\chi)$, where $Y_1 = \widehat{\mathbb{A}}$ and $Y_i = \text{supp } d\sigma_{x_i}/d\sigma_{x_1}$ for $i \geq 2$.

A representation $\mathcal{U}$ is said to have *simple spectrum* if H is reduced to a single cyclic space. The multiplicity is *uniform* if there is only one essential value of $M_{\mathcal{U}}$. The essential supremum of $M_{\mathcal{U}}$ is called the *maximal spectral multiplicity*. $\mathcal{U}$ is said to have *discrete spectrum* if H has an orthonormal base consisting of eigenvectors of $\mathcal{U}$; $\mathcal{U}$ has *singular* (*Haar*, *absolutely continuous*) spectrum if the maximal spectral type of $\mathcal{U}$ is singular with respect to (equivalent to, absolutely continuous with respect to) a Haar measure of $\widehat{\mathbb{A}}$.

Koopman representation of a dynamical system $\mathcal{T}$ Let $\mathbb{A}$ be a l.c.s.c. (and not compact) Abelian group and $\mathcal{T} : a \mapsto T_a$ a representation of $\mathbb{A}$ in the group $Aut(X, \mathcal{B}, \mu)$ of (measure-preserving) automorphisms of a standard probability Borel space $(X, \mathcal{B}, \mu)$. The *Koopman representation* $\mathcal{U} = \mathcal{U}_{\mathcal{T}}$ of $\mathcal{T}$ in $L^2(X, \mathcal{B}, \mu)$ is defined as the unitary representation $a \mapsto U_{T_a} \in U(L^2(X, \mathcal{B}, \mu))$, where $U_{T_a}(f) = f \circ T_a$.

Ergodicity, weak mixing, mild mixing, mixing and rigidity of $\mathcal{T}$ A measure-preserving $\mathbb{A}$-action $\mathcal{T} = (T_a)_{a \in \mathbb{A}}$ is called *ergodic* if $\chi_0 \equiv 1 \in \widehat{\mathbb{A}}$ is a simple eigenvalue of $\mathcal{U}_{\mathcal{T}}$. It is *weakly mixing* if $\mathcal{U}_{\mathcal{T}}$ has a continuous spectrum on the subspace $L_0^2(X, \mathcal{B}, \mu)$ of zero mean functions. $\mathcal{T}$ is said to be *rigid* if there is a sequence (a_n) going to infinity in $\mathbb{A}$ such that the sequence $(U_{T_{a_n}})$ goes to the identity in the strong (or weak) operator topology; $\mathcal{T}$ is said to be *mildly mixing* if it has no non-trivial rigid factors. We say that $\mathcal{T}$ is *mixing* if the operator equal to zero is the only limit point of $\{U_{T_a}|_{L_0^2(X,\mathcal{B},\mu)} : a \in \mathbb{A}\}$ in the weak operator topology.

Spectral disjointness Two $\mathbb{A}$-actions $\mathcal{S}$ and $\mathcal{T}$ are called *spectrally disjoint* if the maximal spectral types of their Koopman representations $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\mathcal{S}}$ on the corresponding L_0^2-spaces are mutually singular.

SCS property We say that a Borel measure σ on $\widehat{\mathbb{A}}$ satisfies the *strong convolution singularity* property (SCS property) if, for each $n \geq 1$, in the disintegration (given by the map $(\chi_1, \ldots, \chi_n) \mapsto \chi_1 \cdot \ldots \cdot \chi_n$) $\sigma^{\otimes n} = \int_{\widehat{\mathbb{A}}} \nu_\chi \, d\sigma^{(n)}(\chi)$ the conditional measures ν_χ are atomic with exactly $n!$ atoms ($\sigma^{(n)}$ stands for the nth convolution $\sigma * \ldots * \sigma$). An $\mathbb{A}$-action $\mathcal{T}$ satisfies the SCS property if the maximal spectral type of $\mathcal{U}_{\mathcal{T}}$ on L_0^2 is a type of an SCS measure.

Kolmogorov group property An $\mathbb{A}$-action $\mathcal{T}$ satisfies the *Kolmogorov group property* if $\sigma_{\mathcal{U}_{\mathcal{T}}} * \sigma_{\mathcal{U}_{\mathcal{T}}} \ll \sigma_{\mathcal{U}_{\mathcal{T}}}$.

Weighted operator Let T be an ergodic automorphism of $(X, \mathcal{B}, \mu)$ and $\xi : X \to \mathbb{T}$ be a measurable function. The (unitary) operator $V = V_{\xi,T}$ acting on $L^2(X, \mathcal{B}, \mu)$ by the formula $V_{\xi,T}(f)(x) = \xi(x) f(Tx)$ is called a *weighted operator*.

Induced automorphism Assume that T is an automorphism of a standard probability Borel space $(X, \mathcal{B}, \mu)$. Let $A \in \mathcal{B}$, $\mu(A) > 0$. The *induced automorphism* T_A is defined on the conditional space $(A, \mathcal{B}_A, \mu_A)$, where $\mathcal{B}_A$ is the trace of $\mathcal{B}$ on A, $\mu_A(B) = \mu(B)/\mu(A)$ for $B \in \mathcal{B}_A$ and $T_A(x) = T^{k_A(x)} x$, where $k_A(x)$ is the smallest $k \geq 1$ for which $T^k x \in A$.

AT property of an automorphism An automorphism T of a standard probability Borel space $(X, \mathcal{B}, \mu)$ is called *approximatively transitive* (AT for short) if for every $\varepsilon > 0$ and every finite set $f_1, \ldots, f_n$ of non-negative L^1-functions on $(X, \mathcal{B}, \mu)$ we can find $f \in L^1(X, \mathcal{B}, \mu)$ also non-negative such that $\|f_i - \sum_j \alpha_{ij} f \circ T^{n_j}\|_{L_1} < \varepsilon$ for all $i = 1, \ldots, n$ (for some $\alpha_{ij} \geq 0$, $n_j \in \mathbb{N}$).

Cocycles and group extensions If T is an ergodic automorphism, G is a l.c.s.c. Abelian group and $\varphi : X \to G$ is measurable then the pair (T, φ) generates a *cocycle* $\varphi^{(\cdot)}(\cdot) : \mathbb{Z} \times X \to G$, where

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \ldots + \varphi(T^{n-1}x) & \text{for} \quad n > 0\,, \\ 0 & \text{for} \quad n = 0\,, \\ -(\varphi(T^n x) + \ldots + \varphi(T^{-1}x)) & \text{for} \quad n < 0\,. \end{cases}$$

(That is $(\varphi^{(n)})$ is a standard 1-cocycle in the algebraic sense for the $\mathbb{Z}$-action $n(f) = f \circ T^n$ on the group of measurable functions on X with values in G; hence the function $\varphi : X \to G$ itself is often called a *cocycle*.) Assume additionally that G is compact. Using the cocycle φ we define a *group extension* T_φ on $(X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes \lambda_G)$ (λ_G stands for Haar measure of G), where $T_\varphi(x, g) = (Tx, \varphi(x) + g)$.

Special flow Given an ergodic automorphism T on a standard probability Borel space $(X, \mathcal{B}, \mu)$ and a positive integrable function $f : X \to \mathbb{R}^+$ we put

$$X^f = \{(x, t) \in X \times \mathbb{R} : 0 \leq t < f(x)\}\,,$$
$$\mathcal{B}^f = \mathcal{B} \otimes \mathcal{B}(\mathbb{R})|_{X^f}\,,$$

and define μ^f as normalized $\mu \otimes \lambda_{\mathbb{R}}|_{X^f}$. By a *special flow* we mean the $\mathbb{R}$-action $T^f = (T_t^f)_{t\in\mathbb{R}}$ under which a point $(x,s) \in X^f$ moves vertically with the unit speed, and once it reaches the graph of f, it is identified with $(Tx, 0)$.

Markov operator A linear operator $J\colon L^2(X,\mathcal{B},\mu) \to L^2(Y,C,\nu)$ is called *Markov* if it sends non-negative functions to non-negative functions and $J1 = J^*1 = 1$.

Unitary actions on Fock spaces If H is a separable Hilbert space then by $H^{\odot n}$ we denote the subspace of n-tensors of $H^{\otimes n}$ symmetric under all permutations of coordinates, $n \geq 1$; then the Hilbert space $F(H) := \bigoplus_{n=0}^{\infty} H^{\odot n}$ is called a *symmetric Fock space*. If $V \in U(H)$ then $F(V) := \bigoplus_{n=0}^{\infty} V^{\odot n} \in U(F(H))$ where $V^{\odot n} = V^{\otimes n}|H^{\odot n}$.

Definition of the Subject

Spectral theory of dynamical systems is a study of special unitary representations, called Koopman representations (see the glossary). Invariants of such representations are called spectral invariants of measure-preserving systems. Together with the entropy, they constitute the most important invariants used in the study of measure-theoretic intrinsic properties and classification problems of dynamical systems as well as in applications. Spectral theory was originated by von Neumann, Halmos and Koopman in the 1930s. In this article we will focus on recent progresses in the spectral theory of finite measure-preserving dynamical systems.

Introduction

Throughout $\mathbb{A}$ denotes a non-compact l.c.s.c. Abelian group ($\mathbb{A}$ will be most often $\mathbb{Z}$ or $\mathbb{R}$). The assumption of second countability implies that $\mathbb{A}$ is metrizable, σ-compact and the space $C_0(\mathbb{A})$ is separable. Moreover the dual group $\widehat{\mathbb{A}}$ is also l.c.s.c. Abelian.

General Unitary Representations

We are interested in *unitary*, that is with values in the unitary group $U(H)$ of a Hilbert space H, (weakly) continuous representations $V\colon \mathbb{A} \ni a \mapsto V_a \in U(H)$ of such groups (the scalar valued maps $a \mapsto \langle V_a x, y\rangle$ are continuous for each $x, y \in H$).

Let $H = L^2(\widehat{\mathbb{A}}, \mathcal{B}(\widehat{\mathbb{A}}), \mu)$, where $\mathcal{B}(\widehat{\mathbb{A}})$ stands for the σ-algebra of Borel sets of $\widehat{\mathbb{A}}$ and $\mu \in M^+(\widehat{\mathbb{A}})$ (whenever X is a l.c.s.c. space, by $M(X)$ we denote the set of complex Borel measures on X, while $M^+(X)$ stands for the subset of positive (finite) measures). Given $a \in \mathbb{A}$, for $f \in L^2(\widehat{\mathbb{A}}, \mathcal{B}(\widehat{\mathbb{A}}), \mu)$ put

$$V_a^{\mu}(f)(\chi) = i(a)(\chi)\cdot f(\chi) = \chi(a)\cdot f(\chi) \quad (\chi \in \widehat{\mathbb{A}}),$$

where $i\colon \mathbb{A} \to \widehat{\widehat{\mathbb{A}}}$ is the canonical Pontriagin isomorphism of $\mathbb{A}$ with its second dual. Then $V^{\mu} = (V_a^{\mu})_{a\in\mathbb{A}}$ is a unitary representation of $\mathbb{A}$. Since $C_0(\widehat{\mathbb{A}})$ is dense in $L^2(\widehat{\mathbb{A}}, \mu)$, the latter space is separable. Therefore also direct sums $\bigoplus_{i=1}^{\infty} V^{\mu_i}$ of such type representations will be unitary representations of $\mathbb{A}$ in separable Hilbert spaces.

Lemma 1 (Wiener Lemma) *If $F \subset L^2(\widehat{\mathbb{A}}, \mu)$ is a closed V_a^{μ}-invariant subspace for all $a \in \mathbb{A}$ then $F = 1_Y L^2(\widehat{\mathbb{A}}, \mathcal{B}(\widehat{\mathbb{A}}), \mu)$ for some Borel subset $Y \subset \widehat{\mathbb{A}}$.*

Notice however that since $\mathbb{A}$ is not compact (equivalently, $\widehat{\mathbb{A}}$ is not discrete), we can find μ continuous and therefore V^{μ} has no irreducible (non-zero) subrepresentation. From now on only unitary representations of $\mathbb{A}$ in separable Hilbert spaces will be considered and we will show how to classify them.

A function $r\colon \mathbb{A} \to \mathbb{C}$ is called *positive definite* if

$$\sum_{n,m=0}^{N} r(a_n - a_m) z_n \overline{z_m} \geq 0 \tag{1}$$

for each $N > 0$, $(a_n) \subset \mathbb{A}$ and $(z_n) \subset \mathbb{C}$. The central result about positive definite functions is the following theorem (see e. g. [173]).

Theorem 1 (Bochner–Herglotz) *Let $r\colon \mathbb{A} \to \mathbb{C}$ be continuous. Then r is positive definite if and only if there exists (a unique) $\sigma \in M^+(\widehat{\mathbb{A}})$ such that*

$$r(a) = \int_{\widehat{\mathbb{A}}} \chi(a)\,\mathrm{d}\sigma(\chi) \quad \textit{for each} \quad a \in \mathbb{A}.$$

If now $\mathcal{U} = (U_a)_{a\in\mathbb{A}}$ is a representation of $\mathbb{A}$ in H then for each $x \in H$ the function $r(a) := \langle U_a x, x\rangle$ is continuous and satisfies (1), so it is positive definite. By the Bochner–Herglotz Theorem there exists a unique measure $\sigma_{\mathcal{U},x} = \sigma_x \in M^+(\widehat{\mathbb{A}})$ (called the *Spectral measure* of x) such that

$$\widehat{\sigma}_x(a) := \int_{\widehat{\mathbb{A}}} i(a)(\chi)\,\mathrm{d}\sigma_x(\chi) = \langle U_a x, x\rangle$$

for each $a \in \mathbb{A}$. Since the partial map $U_a x \mapsto i(a) \in L^2(\widehat{\mathbb{A}}, \sigma_x)$ is isometric and equivariant, there exists a unique extension of it to a unitary operator

$W\colon \mathbb{A}(x) \to L^2(\widehat{\mathbb{A}}, \sigma_x)$ giving rise to an isomorphism of $\mathcal{U}|_{\mathbb{A}(x)}$ and V^{σ_x}. Then the existence of a spectral decomposition is proved by making use of separability and a choice of maximal cyclic spaces at every step of an induction procedure. Moreover, a spectral decomposition is unique in the following sense.

Theorem 2 (Spectral Theorem) *If* $H = \bigoplus_{i=1}^{\infty} \mathbb{A}(x_i) = \bigoplus_{i=1}^{\infty} \mathbb{A}(x_i')$ *are two spectral decompositions of* H *then* $\sigma_{x_i} \equiv \sigma_{x_i'}$ *for each* $i \geq 1$.

It follows that the representation $\mathcal{U}$ is entirely determined by types (the sets of equivalent measures to a given one) of a decreasing sequence of measures or, equivalently, $\mathcal{U}$ is determined by its maximal spectral type $\sigma_{\mathcal{U}}$ and its multiplicity function $M_{\mathcal{U}}$.

Notice that eigenvalues of $\mathcal{U}$ correspond to Dirac measures: $\chi \in \widehat{\mathbb{A}}$ *is an eigenvalue* (i. e. *for some* $\|x\| = 1$, $U_a(x) = \chi(a)x$ *for each* $a \in \mathbb{A}$) *if and only if* $\sigma_{\mathcal{U},x} = \delta_\chi$. Therefore $\mathcal{U}$ has a discrete spectrum if and only if the maximal spectral type of $\mathcal{U}$ is a discrete measure.

We refer the reader to [64,105,127,147,156] for presentations of spectral theory needed in the theory of dynamical systems – such presentations are usually given for $\mathbb{A} = \mathbb{Z}$ but once we have the Bochner–Herglotz Theorem and the Wiener Lemma, their extensions to the general case are straightforward.

Koopman Representations

We will consider *measure-preserving* representations of $\mathbb{A}$. It means that we fix a probability standard Borel space $(X, \mathcal{B}, \mu)$ and by $Aut(X, \mathcal{B}, \mu)$ we denote the group of automorphisms of this space, that is $T \in Aut(X, \mathcal{B}, \mu)$ if $T\colon X \to X$ is a bimeasurable (a.e.) bijection satisfying $\mu(A) = \mu(TA) = \mu(T^{-1}A)$ for each $A \in \mathcal{B}$. Consider then a representation of $\mathbb{A}$ in $Aut(X, \mathcal{B}, \mu)$ that is a group homomorphism $a \mapsto T_a \in Aut(X, \mathcal{B}, \mu)$; we write $\mathcal{T} = (T_a)_{a \in \mathbb{A}}$. Moreover, we require that the associated Koopman representation $\mathcal{U}_{\mathcal{T}}$ is continuous. Unless explicitly stated, $\mathbb{A}$-actions are assumed to be *free*, that is we assume that for μ-a.e. $x \in X$ the map $a \mapsto T_a x$ is $1-1$. In fact, since constant functions are obviously invariant for U_{T_a}, that is the trivial character 1 is always an eigenvalue of $\mathcal{U}_{\mathcal{T}}$, the Koopman representation is considered only on the subspace $L_0^2(X, \mathcal{B}, \mu)$ of zero mean functions. We will restrict our attention only to *ergodic* dynamical systems (see the glossary). It is easy to see that $\mathcal{T}$ is ergodic if and only if whenever $A \in \mathcal{B}$ and $A = T_a(A)$ (μ-a.e.) for all $a \in A$ then $\mu(A)$ equals 0 or 1. In case of ergodic Koopman representations, all eigenvalues are simple. In particular, (ergodic) Koopman representations with discrete spectra have simple spectra. The reader is referred to monographs mentioned above as well as to [26,158,177,196,204] for basic facts on the theory of dynamical systems.

The passage $\mathcal{T} \mapsto \mathcal{U}_{\mathcal{T}}$ can be seen as functorial (contravariant). In particular a measure-theoretic isomorphism of $\mathbb{A}$-systems $\mathcal{T}$ and $\mathcal{T}'$ implies spectral isomorphism of the corresponding Koopman representations; hence spectral properties are measure-theoretic invariants. Since unitary representations are completely classified, one of the main questions in the spectral theory of dynamical systems is to decide which pairs $([\sigma], M)$ can be realized by Koopman representations. The most spectacular, still unsolved, is the Banach problem concerning $([\lambda_{\mathbb{T}}], M \equiv 1)$. Another important problem is to give complete spectral classification in some classes of dynamical systems (classically, it was done in the theory of Kolmogorov and Gaussian dynamical systems). We will also see how spectral properties of dynamical systems can determine their statistical (ergodic) properties; a culmination given by the fact that a spectral isomorphism may imply measure-theoretic similitude (discrete spectrum case, Gaussian–Kronecker case). We conjecture that a dynamical system $\mathcal{T}$ either is spectrally determined or there are uncountably many pairwise non-isomorphic systems spectrally isomorphic to $\mathcal{T}$.

We could also consider Koopman representations in L^p for $1 \leq p \neq 2$. However whenever $W\colon L^p(X, \mathcal{B}, \mu) \to L^p(Y, \mathcal{C}, \nu)$ is a surjective isometry and $W \circ U_{T_a} = U_{S_a} \circ W$ for each $a \in \mathbb{A}$ then by the Lamperti Theorem (e. g. [172]) the isometry W has to come from a nonsingular pointwise map $R\colon Y \to X$ and, by ergodicity, R "preserves" the measure ν and hence establishes a measure-theoretic isomorphism [94] (see also [127]). Thus spectral classification of such Koopman representations goes back to the measure-theoretic classification of dynamical systems, so it looks hardly interesting. This does not mean that there are no interesting dynamical questions for $p \neq 2$. Let us mention still open Thouvenot's question (formulated in the 1980s) for $\mathbb{Z}$-actions: *For every ergodic* T *acting on* $(X, \mathcal{B}, \preceq)$, *does there exist* $f \in L^1(X, \mathcal{B}, \mu)$ *such that the closed linear span of* $f \circ T^n$, $n \in \mathbb{Z}$ *equals* $L^1(X, \mathcal{B}, \mu)$?

Iwanik [79,80] proved that if T is a system with positive entropy then its L^p-multiplicity is ∞ for all $p > 1$. Moreover, Iwanik and de Sam Lazaro [85] proved that for Gaussian systems (they will be considered in Sect. "Spectral Theory of Dynamical Systems of Probabilistic Origin") the L^p-multiplicities are the same for all $p > 1$ (see also [137]).

Markov Operators, Joinings and Koopman Representations, Disjointness and Spectral Disjointness, Entropy

We would like to emphasize that spectral theory is closely related to the theory of joinings (see ▶ Joinings in Ergodic Theory for needed definitions). The elements ρ of the set $J(\mathcal{S}, \mathcal{T})$ of joinings of two $\mathbb{A}$-actions $\mathcal{S}$ and $\mathcal{T}$ are in a 1-1 correspondence with Markov operators $J = J_\rho$ between the L^2-spaces equivariant with the corresponding Koopman representations (see the glossary and ▶ Joinings in Ergodic Theory). The set of ergodic self-joinings of an ergodic $\mathbb{A}$-action $\mathcal{T}$ is denoted by $J_2^e(\mathcal{T})$.

Each Koopman representation $\mathcal{U}_\mathcal{T}$ consists of Markov operators (indeed, U_{T_a} is clearly a Markov operator). In fact, if $U \in U(L^2(X, \mathcal{B}, \mu))$ is Markov then it is of the form U_R, where $R \in Aut(X, \mathcal{B}, \mu)$ [133]. This allows us to see Koopman representations as unitary Markov representations, but since a spectral isomorphism does not "preserve" the set of Markov operators, spectrally isomorphic systems can have drastically different sets of self-joinings.

We will touch here only some aspects of interactions (clearly, far from completeness) between the spectral theory and the theory of joinings.

In order to see however an example of such interactions let us recall that the simplicity of eigenvalues for ergodic systems yields a short "joining" proof of the classical isomorphism theorem of Halmos-von Neumann in the discrete spectrum case: *Assume that $\mathcal{S} = (S_\dashv)_{\dashv\in\mathbb{A}}$ and $\mathcal{T} = (T_a)_{a\in\mathbb{A}}$ are ergodic $\mathbb{A}$-actions on $(X, \mathcal{B}, \mu)$ and (Y, C, ν) respectively. Assume that both Koopman representations have purely discrete spectrum and that their group of eigenvalues are the same. Then $\mathcal{S}$ and $\mathcal{T}$ are measure-theoretically isomorphic*. Indeed, each ergodic joining of $\mathcal{T}$ and $\mathcal{S}$ is the graph of an isomorphism of these two systems (see [127]; see also Goodson's proof in [66]). Another example of such interactions appear in the study Rokhlin's multiple mixing problem and its relation with the *pairwise independence property* (PID) for joinings of higher order. We will not deal with this subject here, referring the reader to ▶ Joinings in Ergodic Theory (see however Sect. "Lifting Mixing Properties").

Following [60], two $\mathbb{A}$-actions $\mathcal{S}$ and $\mathcal{T}$ are called *disjoint* provided the product measure is the only element in $J(\mathcal{S}, \mathcal{T})$. It was already noticed in [72] that spectrally disjoint systems are disjoint in the Furstenberg sense; indeed, $Im(J_\rho|_{L_0^2}) = \{0\}$ since $\sigma_{\mathcal{T}, J_\rho f} \ll \sigma_{\mathcal{S}, f}$.

Notice that whenever we take $\rho \in J_2^e(\mathcal{T})$ we obtain a new ergodic $\mathbb{A}$-action $(T_a \times T_a)_{a\in\mathbb{A}}$ defined on the probability space $(X \times X, \rho)$. One can now ask how close spectrally to $\mathcal{T}$ is this new action? It turns out that except of the obvious fact that the marginal σ-algebras are factors, $(\mathcal{T} \times \mathcal{T}, \rho)$ can have other factors spectrally disjoint with $\mathcal{T}$: the most striking phenomenon here is a result of Smorodinsky and Thouvenot [198] (see also [29]) saying that each zero entropy system is a factor of an ergodic self-joining system of a fixed Bernoulli system (Bernoulli systems themselves have countable Haar spectrum). The situation changes if $\rho = \mu \otimes \mu$. In this case for $f, g \in L^2(X, \mathcal{B}, \mu)$ the spectral measure of $f \otimes g$ is equal to $\sigma_{\mathcal{T}, f} * \sigma_{\mathcal{T}, g}$. A consequence of this observation is that an ergodic action $\mathcal{T} = (T_a)_{a\in\mathbb{A}}$ is weakly mixing (see the glossary) if and only if the product measure $\mu \otimes \mu$ is an ergodic self-joining of $\mathcal{T}$.

The entropy which is a basic measure-theoretic invariant does not appear when we deal with spectral properties. We will not give here any formal definition of entropy for amenable group actions referring the reader to [153]. Assume that $\mathbb{A}$ is countable and discrete. We always assume that $\mathbb{A}$ is Abelian, hence it is amenable. For each dynamical system $\mathcal{T} = (T_a)_{a\in\mathbb{A}}$ acting on $(X, \mathcal{B}, \mu)$, we can find a largest invariant sub-σ field $\mathcal{P} \subset \mathcal{B}$, called the *Pinsker σ-algebra*, such that the entropy of the corresponding quotient system is zero. Generalizing the classical Rokhlin-Sinai Theorem (see also [97] for $\mathbb{Z}^d$-actions), Thouvenot (unpublished) and independently Dooley and Golodets [31] proved this theorem for groups even more general than those considered here: *The spectrum of $\mathcal{U}_\mathcal{T}$ on $L^2(X, \mathcal{B}, \mu) \ominus L^2(\mathcal{P})$ is Haar with uniform infinite multiplicity*. This general result is quite intricate and based on methods introduced to entropy theory by Rudolph and Weiss [179] with a very surprising use of Dye's Theorem on orbital equivalence of all ergodic systems. For $\mathbb{A}$ which is not countable the same result was recently proved in [17] in case of unimodular amenable groups which are not increasing union of compact subgroups. It follows that spectral theory of dynamical systems essentially reduces to the zero entropy case.

Maximal Spectral Type of a Koopman Representation, Alexeyev's Theorem

Only few general properties of maximal spectral types of Koopman representations are known. The fact that a Koopman representation preserves the space of real functions implies that its maximal spectral type is the type of a symmetric (invariant under the map $\chi \mapsto \overline{\chi}$) measure.

Recall that the *Gelfand spectrum* $\sigma(\mathcal{U})$ of a representation $\mathcal{U} = (U_a)_{a\in\mathbb{A}}$ is defined as the of *approximative eigenvalues* of $\mathcal{U}$, that is $\sigma(\mathcal{U}) \ni \chi \in \widehat{\mathbb{A}}$ if for a sequence (x_n) bounded and bounded away from zero, $\|U_a x_n - \chi(a)x_n\| \to 0$ for each $a \in \mathbb{A}$. The spectrum is a closed subset in the topology of pointwise convergence, hence in the compact-open topology of $\widehat{\mathbb{A}}$. In case of $\mathbb{A} = \mathbb{Z}$, the above set $\sigma(U)$ is equal to $\{z \in \mathbb{C} : U - z \cdot Id \text{ is not invertible}\}$.

Assume now that $\mathbb{A}$ is countable and discrete (and Abelian). Then there exists a Fölner sequence $(B_n)_{n\geq 1}$ whose elements tile $\mathbb{A}$ [153]. Take a free and ergodic action $\mathcal{T} = (T_a)_{a\in\mathbb{A}}$ on $(X, \mathcal{B}, \mu)$. By [153] for each $\varepsilon > 0$ we can find a set $Y_n \in \mathcal{B}$ such that the sets $T_b Y_n$ are pairwise disjoint for $b \in B_n$ and $\mu(\bigcup_{b\in B_n} T_b Y_n) > 1 - \varepsilon$. For each $\chi \in \widehat{\mathbb{A}}$, by considering functions of the form $f_n = \sum_{b\in B_n} \chi(b) 1_{T_b Y_n}$ we obtain that $\chi \in \sigma(\mathcal{U}_{\mathcal{T}})$. It follows that the topological support of the maximal spectral type of the Koopman representation of a free and ergodic action is full [105,127,147]. The theory of Gaussian systems shows in particular that there are symmetric measures on the circle whose topological support is the whole circle but which cannot be maximal spectral types of Koopman representations.

An open well-known question remains whether an absolutely continuous measure ρ is the maximal spectral type of a Koopman representation if and only if ρ is equivalent to a Haar measure of $\widehat{\mathbb{A}}$ (this is unknown for $\mathbb{A} = \mathbb{Z}$).

Another interesting question was recently raised by A. Katok (private communication): *Is it true that the topological supports of all measures in a spectral sequence of a Koopman representation are full*? If the answer to this question is positive then for example the essential supremum of $M_{\mathcal{U}_{\mathcal{T}}}$ is the same on all balls of $\widehat{\mathbb{A}}$.

Notice that the fact that all spectral measures in a spectral sequence are symmetric means that $\mathcal{U}_{\mathcal{T}}$ is isomorphic to $\mathcal{U}_{\mathcal{T}^{-1}}$. A. del Junco [89] showed that generically for $\mathbb{Z}$-actions, T and its inverse are not measure-theoretically isomorphic (in fact he proved disjointness).

Let $\mathcal{T}$ be an $\mathbb{A}$-action on $(X, \mathcal{B}, \mu)$. One can ask wether a "good" function can realize the maximal spectral type of $\mathcal{U}_{\mathcal{T}}$. In particular can we find a function $f \in L^{\infty}(X, \mathcal{B}, \mu)$ that realizes the maximal spectral type? The answer is given in the following general theorem (see [139]).

Theorem 3 (Alexeyev's Theorem) *Assume that $\mathcal{U} = (U_a)_{a\in\mathbb{A}}$ is a unitary representation of $\mathbb{A}$ in a separable Hilbert space H. Assume that $F \subset H$ is a dense linear subspace. Assume moreover that with some F-norm $|\!|\!|\cdot|\!|\!|$ – stronger than the norm $\|\cdot\|$ given by the scalar product – F becomes a Fréchet space. Then, for each spectral measure σ ($\ll \sigma_{\mathcal{U}}$) there exists $y \in F$ such that $\sigma_y \gg \sigma$. In particular, there exists $y \in F$ that realizes the maximal spectral type.*

By taking $H = L^2(X, \mathcal{B}, \mu)$ and $F = L^{\infty}(X, \mathcal{B}, \mu)$ we obtain the positive answer to the original question. Alexeyev [14] proved the above theorem for $\mathbb{A} = \mathbb{Z}$ using analytic functions. Refining Alexeyev's original proof, Frączek [52] showed the existence of a sufficiently regular function realizing the maximal spectral type depending only on the "regularity" of the underlying probability space, e. g. when X is a compact metric space (compact manifold) then one can find a continuous (smooth) function realizing the maximal spectral type.

By the theory of systems of probabilistic origin (see Sect. "Spectral Theory of Dynamical Systems of Probabilistic Origin"), in case of simplicity of the spectrum, one can easily point out spectral measures whose types are not realized by (essentially) bounded functions. However, it is still an open question whether for each Koopman representation $\mathcal{U}_{\mathcal{T}}$ there exists a sequence $(f_i)_{i\geq 1} \subset L^{\infty}(X, \mathcal{B}, \mu)$ such that the sequence $(\sigma_{f_i})_{i\geq 1}$ is a spectral sequence for $\mathcal{U}_{\mathcal{T}}$. For $\mathbb{A} = \mathbb{Z}$ it is unknown whether the maximal spectral type of a Koopman representation can be realized by a characteristic function.

Spectral Theory of Weighted Operators

We now pass to the problem of possible essential values for the multiplicity function of a Koopman representation. However, one of known techniques is a use of cocycles, so before we tackle the multiplicity problem, we will go through recent results concerning spectral theory of compact group extensions automorphisms which in turn entail a study of weighted operators (see the glossary).

Assume that T is an ergodic automorphism of a standard Borel probability space $(X, \mathcal{B}, \mu)$. Let $\xi: X \to \mathbb{T}$ be a measurable function and let $V = V_{\xi,T}$ be the corresponding weighted operator. To see a connection of weighted operators with Koopman representations of compact group extensions consider a compact (metric) Abelian group G and a cocycle $\varphi: X \to G$. Then U_{T_φ} (see the glossary) acts on $L^2(X \times G, \mu \otimes \lambda_G)$. But

$$L^2(X \times G, \mu \otimes \lambda_G) = \bigoplus_{\chi\in\widehat{G}} L_\chi, \quad \text{where } L_\chi = L^2(X, \mu) \otimes \chi,$$

where L_χ is a U_{T_φ}-invariant (clearly, closed) subspace. Moreover, the map $f \otimes \chi \mapsto f$ settles a unitary isomorphism of $U_{T_\varphi}|_{L_\chi}$ with the operator $V_{\chi\circ\varphi,T}$. Therefore, spectral analysis of such Koopman representations reduces to the spectral analysis of weighted operators $V_{\chi\circ\varphi,T}$ for all $\chi \in \widehat{G}$.

Maximal Spectral Type of Weighted Operators over Rotations

The spectral analysis of weighted operators $V_{\xi,T}$ is especially well developed in case of rotations, i. e. when we additionally assume that T is an ergodic rotation on a compact monothetic group X: $Tx = x + x_0$, where x_0 is a topologically cyclic element of X (and μ will stand for Haar measure λ_x of X). In this case Helson's analysis [74] applies (see also [68,82,127,160]) leading to the following conclusions:

- The maximal spectral type $\sigma_{V_{\xi,T}}$ is either discrete or continuous.
- When $\sigma_{V_{\xi,T}}$ is continuous it is either singular or Lebesgue.
- The spectral multiplicity of $V_{\xi,T}$ is uniform.

We now pass to a description of some results in case when $Tx = x + \alpha$ is an irrational rotation on the additive circle $X = [0, 1)$. It was already noticed in the original paper by Anzai [16] that when $\xi\colon X \to \mathbb{T}$ is an affine cocycle ($\xi(x) = \exp(nx + c)$, $0 \neq n \in \mathbb{Z}$) then $V_{\xi,T}$ has a Lebesgue spectrum. It was then considered by several authors (originated by [123], see also [24,26]) to which extent this property is stable when we perturb our cocycle. Since the topological degree of affine cocyles is different from zero, when perturbing them we consider smooth perturbations by cocycles of degree zero.

Theorem 4 ([82]) *Assume that $Tx = x + \alpha$ is an irrational rotation. If $\xi\colon [0, 1) \to \mathbb{T}$ is of non-zero degree, absolutely continuous, with the derivative of bounded variation then $V_{\xi,T}$ has a Lebesgue spectrum.*

In the same paper, it is noticed that if we drop the assumption on the derivative then the maximal spectral type of $V_{\xi,T}$ is a Rajchman measure (i. e. its Fourier transform vanishes at infinity). It is still an open question, whether one can find ξ absolutely continuous with non-zero degree and such that $V_{\xi,T}$ has singular spectrum. "Below" absolute continuity, topological properties of the cocycle seem to stop playing any role – in [82] a continuous, degree 1 cocycle ξ of bounded variation is constructed such that $\xi(x) = \eta(x)/\eta(Tx)$ for a measurable $\eta\colon [0, 1) \to \mathbb{T}$ (that is ξ is a *coboundary*) and therefore $V_{\xi,T}$ has purely discrete spectrum (it is isomorphic to U_T). For other results about Lebesgue spectrum for Anzai skew products see also [24,53,81] (in [53] $\mathbb{Z}^d$-actions of rotations and so called winding numbers instead of topological degree are considered).

When the cocycle is still smooth but its degree is zero the situation drastically changes. Given an absolutely continuous function $f\colon [0, 1) \to \mathbb{R}$ M. Herman [76], using the Denjoy–Koksma inequality (see e. g. [122]), showed that $f_0^{(q_n)} \to 0$ uniformly (here $f_0 = f - \int_0^1 f \, d\lambda_{[0,1)}$ and (q_n) stands for the sequence of denominators of α). It follows that $T_{e^{2\pi i f}}$ is rigid and hence has a singular spectrum. B. Fayad [37] shows that this result is no longer true if one dimensional rotation is replaced by a multi-dimensional rotation (his counterexample is in the analytic class). See also [130] for the singularity of spectrum for functions f whose Fourier transform satisfies $o(1/|n|)$ condition or to [84], where it is shown that sufficiently small variation implies singularity of the spectrum.

A natural class of weighted operators arises when we consider Koopman operators of rotations on nil-manifolds. We only look at the particular example of such a rotation on a quotient of the Heisenberg group $(\mathbb{R}^3, *)$ (a general spectral theory of nil-actions was mainly developed by W. Parry [157]) – these actions have countable Lebesgue spectrum in the orthocomplement of the subspace of eigenfunctions) that is take the nil-manifold $\mathbb{R}^3/_*\mathbb{Z}^3$ on which we define the nil-rotation $S((x, y, z) * \mathbb{Z}^3) = (\alpha, \beta, 0)*(x, y, z)*\mathbb{Z}^3 = \left(x + \alpha, y + \beta, z + \alpha y\right)* \mathbb{Z}^3$, where α, β and 1 are rationally independent. It can be shown that S is isomorphic to the skew product defined on $[0, 1)^2 \times \mathbb{T}$ by

$$T_\varphi\colon (x, y, z) \mapsto \left(x + \alpha, y + \beta, z \cdot e^{2\pi i \varphi(x,y)}\right) ,$$

where $\varphi(x, y) = \alpha y - \psi(x + \alpha, y + \beta) + \psi(x, y)$ with $\psi(x, y) = x[y]$. Since nil-cocycles can be considered as a certain analog of affine cocycles for one-dimensional rotations, it would be nice to explain to what kind of perturbations the Lebesgue spectrum property is stable.

Yet another interesting problem which is related to the spectral theory of extensions given by so called *Rokhlin cocycles* (see Sect. "Rokhlin Cocycles") arises, when given $f\colon [0, 1) \to \mathbb{R}$, we want to describe spectrally the one-parameter set of weighted operators $W_c := V_{e^{2\pi i c f},T}$; here T is a fixed irrational rotation by α. As proved by quite sophisticated arguments in [84], if we take $f(x) = x$ then for all non-integer $c \in \mathbb{R}$ the spectrum of W_c is continuous and singular (see also [68] and [145] where some special α's are considered). It has been open for some time if at all one can find $f\colon [0, 1) \to \mathbb{R}$ such that for each $c \neq 0$, the operator W_c has a Lebesgue spectrum. The positive answer is given in [205]: for example if $f(x) = x^{-(2+\varepsilon)}$ ($\varepsilon > 0$) and α has bounded partial quotients then W_c has a Lebesgue spectrum for all $c \neq 0$. All functions with such a property considered in [205] are non-integrable. It would be interesting to find an integrable f with the above property.

We refer to [66] and the references therein for further results especially for transformations of the form $(x, y) \mapsto (x+\alpha, 1_{[0,\beta)}(x)+y)$ on $[0, 1)\times\mathbb{Z}/2\mathbb{Z}$. Recall however that earlier Katok and Stepin [104] used this kind of transformations to give a first counterexample to the Kolmogorov group property (see the glossary) for the spectrum.

The Multiplicity Problem for Weighted Operators over Rotations

In case of perturbations of affine cocycles, this problem was already raised by Kushnirenko [123]. Some significant results were obtained by M. Guenais. Before we state her results let us recall a useful criterion to find an upper bound for the multiplicity: *If there exist* $c > 0$ *and a sequence* $(F_n)_{n\geq 1}$ *of cyclic subspaces of* H *such that for each* $y \in H$, $\|y\| = 1$ *we have* $\liminf_{n\to\infty} \|proj_{F_n} y\|^2 \geq c$, *then* $esssup(M_U) \leq 1/c$ which follows from a well-known lemma of Chacon [23,26,111,127]. Using this and a technique which is close to the idea of local rank one (see [44,111]) M. Guenais [69] proved a series of results on multiplicity which we now list.

Theorem 5 *Assume that* $Tx = x + \alpha$ *and let* $\xi\colon [0,1) \to \mathbb{T}$ *be a cocycle.*

(i) *If* $\xi(x) = e^{2\pi icx}$ *then* $M_{V_{\xi,T}}$ *is bounded by* $|c| + 1$.
(ii) *If* ξ *is absolutely continuous and* ξ *is of topological degree zero, then* $V_{\xi,T}$ *has a simple spectrum.*
(iii) *if* ξ *is of bounded variation, then* $M_{V_{\xi,T}} \leq \max(2, 2\pi Var(\xi)/3)$.

Remarks on the Banach Problem

We already mentioned in Introduction the Banach problem in ergodic theory, which is simply the question whether there exists a Koopman representation for $\mathbb{A} = \mathbb{Z}$ with simple Lebesgue spectrum. In fact no example of a Koopman representation with Lebesgue spectrum of finite multiplicity is known. Helson and Parry [75] asked for the validity of a still weaker version: *Can one construct* T *such that* U_T *has a Lebesgue component in its spectrum whose multiplicity is finite*? Quite surprisingly in [75] they give a general construction yielding for each ergodic T a cocycle $\varphi\colon X \to \mathbb{Z}/2\mathbb{Z}$ such that the unitary operator U_{T_φ} has a Lebesgue spectrum in the orthocomplement of functions depending only on the X-coordinate. Then Mathew and Nadkarni [144] gave examples of cocycles over so called dyadic adding machine for which the multiplicity of the Lebesgue component was equal to 2. In [126] this was generalized to so called *Toeplitz* $\mathbb{Z}/2\mathbb{Z}$*-extensions* of adding machines: for each even number k we can find a two-point extension of an adding machine so that the multiplicity of the Lebesgue component is k. Moreover, it was shown that Mathew and Nadkarni's constructions from [144] in fact are close to systems arising from number theory (like the famous Rudin–Shapiro sequence, e. g. [160]), relating the result about multiplicity of the Lebesgue component to results by Kamae [96] and Queffelec [160]. Independently of [126], Ageev [8] showed that one can construct 2-point extensions of the Chacon transformation realizing (by taking powers of the extension) each even number as the multiplicity of the Lebesgue component. Contrary to all previous examples, Ageev's constructions are weakly mixing.

Still an open question remains whether for $\mathbb{A} = \mathbb{Z}$ one can find a Koopman representation with the Lebesgue component of multiplicity 1 (or even odd).

In [70], M. Guenais studies the problem of Lebesgue spectrum in the classical case of Morse sequences (see [107] as well as [124], where the problem of spectral classification in this class is studied). All dynamical sytems arising from Morse sequences have simple spectra [124]. It follows that if a Lebesgue component appears in a Morse dynamical system, it has multiplicity one. Guenais [70] using a Riesz product technique relates the Lebesgue spectrum problem with the still open problem of whether a construction of "flat" trigonometric polynomials with coefficients ± 1 is possible. However, it is proved in [70] that such a construction can be carried out on some compact Abelian groups and it leads, for an Abelian countable torsion group $\mathbb{A}$, to a construction of an ergodic action of $\mathbb{A}$ with simple spectrum and a Haar component in its spectrum.

Lifting Mixing Properties

We now give one more example of interactions between spectral theory and joinings (see Introduction) that gives rise to a quick proof of the fact that r-fold mixing property of T ($r \geq 2$) lifts to a weakly mixing compact group extension T_φ (the original proof of this fact is due to D. Rudolph [175]). Indeed, to prove r-fold mixing for a mixing(=2-mixing) transformation S (acting on (Y, C, ν)) one has to prove that each weak limit of off-diagonal self-joinings (given by powers of S, see ► Joinings in Ergodic Theory) of order r is simply the product measure $\nu^{\otimes r}$. We need also a Furstenberg's lemma [62] about relative unique ergodicity (RUE) of compact group extensions T_φ: *If* $\mu \otimes \lambda_G$ *is an ergodic measure for* T_φ *then it is the only (ergodic) invariant measure for* T_φ *whose projection on the first coordinate is* μ. Now the result about lifting r-fold mixing to compact group extensions follows directly from the fact that whenever T_φ is weakly mixing, $(\mu \otimes \lambda_G)^{\otimes r}$ is an ergodic measure (this approach was shown to me by A. del Junco). In particular if T is mixing and T_φ is weakly mixing then for each $\chi \in \widehat{G} \setminus \{1\}$, the maximal spectral type of $V_{\chi\circ\varphi,T}$ is Rajchman.

See Sect. "Rokhlin Cocycles" for a generalization of the lifting result to Rokhlin cocycle extensions.

The Multiplicity Function

In this chapter only $\mathbb{A} = \mathbb{Z}$ is considered (for other groups, even for $\mathbb{R}$, much less is known; see however the case of so called *product* $\mathbb{Z}^d$*-actions* [50]). Contrary to the case of maximal spectral type, it is rather commonly believed that there are no restrictions for the set of essential values of Koopman representations.

Cocycle Approach

We will only concentrate on some results of the last twenty years. In 1983, E.A. Robinson [164] proved that for each $n \geq 1$ there exists an ergodic transformation whose maximal spectral multiplicity is n. Another important result was proved in [165] (see also [98]), where it is shown that given a finite set $M \subset \mathbb{N}$ containing 1 and closed under the least common multiple one can find (even a weakly mixing) T so that the set of essential values of the multiplicity function equals M. This result was then extended in [67] to infinite sets and finally in [125] (see also [11]) to all subsets $M \subset \mathbb{N}$ containing 1. In fact, as we have already noticed in the previous section the spectral theory for compact Abelian group extensions is reduced to a study of weighted operators and then to comparing maximal spectral types for such operators. This leads to sets of the form

$$M(G, v, H) = \Big\{\sharp(\{\chi \circ v^i : i \in \mathbb{Z}\} \cap anih(H)) : \chi \in anih(H)\Big\}$$

($H \subset G$ is a closed subgroup and v is a continuous group automorphism of G). Then an algebraic lemma has been proved in [125] saying that each set M containing 1 is of the form $M(G, v, H)$ and the techniques to construct "good" cocycles and a passage to "natural factors" yielded the following: *For each $M \subset \{1, 2, \ldots\} \subset \{\infty\}$ containing 1 there exists an ergodic automorphim such that the set of essential values for its Koopman representation equals M.* See also [166] for the case of non-Abelian group extensions.

A similar in spirit approach (that means, a passage to a family of factors) is present in a recent paper of Ageev [13] in which he first applies Katok's analysis (see [98,102]) for spectral multiplicities of the Koopman representation associated with Cartesian products $T^{\times k}$ for a generic transformation T. In a natural way this approach leads to study multiplicities of tensor products of unitary operators. Roughly, the multiplicity is computed as the number of atoms (counted modulo obvious symmetries) for conditional measures (see [98]) of a product measure over its convolution. Ageev [13] proved that for a typical automorphism T the set of the values of the multiplicity function for $U_{T^{\times k}}$ equals $\{k, k(k-1), \ldots, k!\}$ and then he just passes to "natural" factors for the Cartesian products by taking sets invariant under a fixed subgroup of permutations of coordinates. In particular, he obtains all sets of the form $\{2, 3, \ldots, n\}$ on L^2_0. He also shows that such sets of multiplicities are realizable in the category of mixing transformations.

Rokhlin's Uniform Multiplicity Problem

The Rokhlin multiplicity problem (see the recent book by Anosov [15]) was, given $n \geq 2$, to construct an ergodic transformation with uniform multiplicity n on L^2_0. A solution for $n = 2$ was independently given by Ageev [9] and Ryzhikov [188] (see also [15] and [66]) and in fact it consists in showing that for some T (actually, any T with simple spectrum for which $1/2(Id + U_T)$ is in the weak operator closure of the powers of U_T will do) the multiplicity of $T \times T$ is uniformly equal to 2 (see also Sect. "Future Directions").

In [12], Ageev proposed a new approach which consists in considering actions of "slightly non-Abelian" groups; and showing that for a "typical" action of such a group a fixed "direction" automorphism has a uniform multiplicity. Shortly after publication of [12], Danilenko [27], following Ageev's approach, considerably simplified the original proof. We will present Danilenko's arguments.

Fix $n \geq 1$. Denote $\bar{e}_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^n$, $i = 1, \ldots, n$. We define an automorphism L of $\mathbb{Z}^n$ setting $L(\bar{e}_i) = \bar{e}_{i+1}$, $i = 1, \ldots, n-1$ and $L(\bar{e}_n) = \bar{e}_1$. Using L we define a semi-direct product $G := \mathbb{Z}^n \rtimes \mathbb{Z}$ defining multiplication as $(u, k) \cdot (w, l) = (u + L^k w, k + l)$. Put $e_0 = (0, 1)$, $e_i = (\bar{e}_i, 0)$, $i = 1, \ldots, n$ (and $Le_i = (L\bar{e}_i, 0)$). Moreover, denote $e_{n+1} = e_0^n = (0, n)$. Notice that $e_0 \cdot e_i \cdot e_0^{-1} = Le_i$ for $i = 1, \ldots, n$ ($L(e_{n+1}) = e_{n+1}$).

Theorem 6 (Ageev, Danilenko) *For every unitary representation $\mathcal{U}$ of G in a separable Hilbert space H, for which $U_{e_1 - L^r e_1}$ has no non-trivial fixed points for $1 \leq r < n$, the essential values of the multiplicity function for $U_{e_{n+1}}$ are contained in the set of multiples of n. If, in addition, U_{e_0} has a simple spectrum, then $U_{e_{n+1}}$ has uniform multiplicity n.*

It is then a certain work to show that the assumption of the second part of the theorem is satisfied for a typical action of the group G. Using a special (C,F)-construction with all the cut-and-stack parameters explicit Danilenko [27] was also able to show that each set of the form $k \cdot M$, where $k \geq 1$ and M is an arbitrary subset of natural numbers containing 1, is realizable as the set of essential values of a Koopman representation.

Some other constructions based on the solution of the Rokhlin problem for $n = 2$ and the method of [125] are presented in [103] leading to sets different than those pointed above; these sets contain 2 as their minimum.

Rokhlin Cocycles

We consider now a certain class of extensions which should be viewed as a generalization of the concept of compact group extensions. We will focus on $\mathbb{Z}$-actions only.

Assume that T is an ergodic automorphism of $(X, \mathcal{B}, \mu)$. Let G be a l.c.s.c. Abelian group. Assume that this group acts on $(Y, \mathcal{C}, \nu)$, that is we have a G-action $S = (S_g)_{g\in G}$ on $(Y, \mathcal{C}, \nu)$. Let $\varphi\colon X \to G$ be a cocycle. We then define an automorphism $T_{\varphi,S}$ of the space $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ by

$$T_{\varphi,S}(x, y) = (Tx, S_{\varphi(x)}(y)).$$

Such an extension is called a *Rokhlin cocycle extension* (the map $x \mapsto S_{\varphi(x)}$ is called a *Rokhlin cocycle*). Such an operation generalizes the case of compact group extensions; indeed, when G is compact the action of G on itself by rotations preserves Haar measure. (It is quite surprising, that when only we admit G non-Abelian, then, as shown in [28], **each** ergodic extension of T has a form of a Rokhlin cocycle extension.) Ergodic and spectral properties of such extensions are examined in several papers: [63,65,129,131,132,133,167,176]. Since in these papers rather joining aspects are studied (among other things in [129] Furstenberg's RUE lemma is generalized to this new context), we will mention here only few results, mainly spectral, following [129] and [133]. We will constantly assume that G is non-compact. As $\varphi\colon X \to G$ is then a cocycle with values in a non-compact group, the theory of such cocycles is much more complicated (see e. g. [193]), and in fact the theory of Rokhlin cocycle extensions leads to interesting interactions between classical ergodic theory, the theory of cocycles and the theory of non-singular actions arising from cocycles taking values in non-compact groups – especially, the Mackey action associated to φ plays a crucial role here (see the problem of invariant measures for $T_{\varphi,S}$ in [132] and [28]); see also monographs [1,98,101,193]. Especially, two Borel subgroups of $\widehat{G}$ are important here:

$$\Sigma_\varphi = \{\chi \in \widehat{G}\colon \chi \circ \varphi = c \cdot \xi/\xi \circ T \text{ for a measurable } \xi\colon X \to \mathbb{T} \text{ and } c \in \mathbb{T}\}\,.$$

and its subgroup Λ_φ given by $c = 1$. Λ_φ turns out to be the group of L^∞-eigenvalues of the Mackey action (of G) associated to the cocycle φ. This action is the quotient action of the natural action of G (by translations on the second coordinate) on the space of ergodic components of the skew product T_φ – the Mackey action is (in general) not measure-preserving, it is however non-singular. We refer the reader to [2,78,147] for other properties of those subgroups.

Theorem 7 ([**132**,**133**])

(i) $\sigma_{T_{\varphi,S}}|_{L^2(X\times Y,\mu\otimes\nu)\ominus L^2(X,\mu)} = \int_{\widehat{G}} \sigma_{V_{\chi\circ\varphi,T}}\, d\sigma_S$.

(ii) *$T_{\varphi,S}$ is ergodic if and only if T is ergodic and $\sigma_S(\Lambda_\varphi) = 0$.*

(iii) *$T_{\varphi,S}$ is weakly mixing if and only if T is weakly mixing and S has no eigenvalues in Σ_φ.*

(iv) *if T is mixing, S is mildly mixing, φ is recurrent and not cohomologous to a cocycle with values in a compact subgroup of G then $T_{\varphi,S}$ remains mixing.*

(v) *If T is r-fold mixing, φ is recurrent and $T_{\varphi,S}$ is mildly mixing then $T_{\varphi,S}$ is also r-fold mixing.*

(vi) *If T and R are disjoint, the cocycle φ is ergodic and S is mildly mixing then $T_{\varphi,S}$ remains disjoint with R.*

Let us recall [61,195] that an $\mathbb{A}$-action $S = (S_a)_{a\in\mathbb{A}}$ is mildly mixing (see the glossary) if and only if the $\mathbb{A}$-action $(S_a \times \tau_a)_{a\in\mathbb{A}}$ remains ergodic for every properly ergodic non-singular $\mathbb{A}$-action $\tau = (\tau_a)_{a\in\mathbb{A}}$.

Coming back to Smorodinsky–Thouvenot's result about factors of ergodic self-joinings of a Bernoulli automorphism we would like to emphasize here that the disjointness result (*vi*) above was used in [132] to give an example of an automorphism which is disjoint from all weakly mixing transformations but which has an ergodic self-joining whose associated automorphism has a non-trivial weakly mixing factor. In a sense this is opposed to Smorodinsky–Thouvenot's result as here from self-joinings we produced a "more complicated" system (namely the weakly mixing factor) than the original system.

It would be interesting to develop the theory of spectral multiplicity for Rokhlin cocycle extensions as it was done in the case of compact group extensions. However a difficulty is that in the compact group extension case we deal with a countable direct sum of representations of the form

$V_{\chi\circ\varphi,T}$ while in the non-compact case we have to consider an integral of such representations.

Rank-1 and Related Systems

Although properties like mixing, weak (and mild) mixing as well as ergodicity, are clearly spectral properties, they have "good" measure-theoretic formulations (expressed by a certain behavior on sets). Simple spectrum property is another example of a spectral property, and it was a popular question in the 1980s whether simple spectrum property of a Koopman representation can be expressed in a more "measure-theoretic" way. We now recall rank-1 concept which can be seen as a notion close to Katok's and Stepin's theory of cyclic approximation [104] (see also [26]).

Assume that T is an automorphism of a standard probability Borel space $(X, \mathcal{B}, \mu)$. T is said to have *rank one* property if there exists an increasing sequence of Rokhlin towers tending to the partition into points (a *Rokhlin tower* is a family $\{F, TF, \dots, T^{n-1}F\}$ of pairwise disjoint sets, while "tending to the partition into points" means that we can approximate every set in $\mathcal{B}$ by unions of levels of towers in the sequence). Baxter [20] showed that the maximal spectral type of such a T is realized by a characteristic function. Since the cyclic space generated by the characteristic function of the base contains characteristic functions of all levels of the tower, by the definition of rank one, the increasing sequence of cyclic spaces tends to the whole L^2-space, therefore rank one property implies simplicity of the spectrum for the Koopman representation. It was a question for some time whether rank-1 is just a characterization of simplicity of the spectrum, disproved by del Junco [88]. We refer the reader to [46] as a good source for basic properties of rank-1 transformations.

Similarly to the rank one property, one can define *finite rank* automorphisms (simply by requiring that an approximation is given by a sequence of a fixed number of towers) – see e. g. [152], or even, a more general property, namely the *local rank one* property can be defined, just by requiring that the approximating sequence of single towers fills up a fixed fraction of the space (see [44,111]). Local rank one (so the more finite rank) property implies finite multiplicity [111]. Mentzen [146] showed that for each $n \geq 1$ one can construct an automorphism with simple spectrum and having rank n; in [138] there is an example of a simple spectrum automorphism which is not of local rank one. Ferenczi [45] introduced the notion of funny rank one (approximating towers are Rokhlin towers with "holes"). Funny rank one also implies simplicity of the spectrum. An example is given in [45] which is even not loosely Bernoulli (see Sect. "Inducing and Spectral Theory", we recall that local rank one property implies loose Bernoullicity [44]).

The notion of AT (see the glossary) has been introduced by Connes and Woods [25]. They proved that AT property implies zero entropy. They also proved that funny rank one automorphisms are AT. In [32] it is proved that the system induced by the classical Morse-Thue system is AT (it is an open question by S. Ferenczi whether this system has funny rank one property). A question by Dooley and Quas is whether AT implies funny rank one property. AT property implies "simplicity of the spectrum in L^1" which we already considered in Introduction (a "generic" proof of this fact is due to J.-P. Thouvenot).

A persistent question was formulated in the 1980s whether rank one itself is a spectral property. In [49] the authors maintained that this is not the case, based on an unpublished preprint of the first named author of [49] in which there was a construction of a Gaussian–Kronecker automorphism (see Sect. "Spectral Theory of Dynamical Systems of Probabilistic Origin") having rank-1 property. This latter construction turned out to be false. In fact de la Rue [181] proved that no Gaussian automorphism can be of local rank one. Therefore the question whether: *Rank one is a spectral property* remains one of the interesting open questions in that theory. Downarowicz and Kwiatkowski [33] proved that rank-1 is a spectral property in the class of systems generated by generalized Morse sequences.

One of the most beautiful theorems about rank-1 automorphisms is the following result of J. King [110] (for a different proof see [186]).

Theorem 8 (WCT) *If T is of rank one then for each element S of the centralizer $C(T)$ of T there exists a sequence (n_k) such that $U_T^{n_k} \to U_S$ strongly.*

A conjecture of J. King is that in fact for rank-1 automorphisms each indecomposable Markov operator $J = J_\rho$ ($\rho \in J_2^e(T)$) is a weak limit of powers of U_T (see [112], also [186]). To which extent the WCT remains true for actions of other groups is not clear. In [214] the WCT is proved in case of rank one flows, however the main argument seems to be based on the fact that a rank one flow has a non-zero time automorphism T_{t_0} which is of rank one, which is not true. After the proof of the WCT by Ryzhikov in [186] there is a remark that the rank one flow version of the theorem can be proved by a word for word repetition of the arguments. He also proves that if the flow $(T_t)_{t\in\mathbb{R}}$ is mixing, then T_1 does not have finite rank. On the other hand, for $\mathbb{A} = \mathbb{Z}^2$, Downarowicz and Kwiatkowski [34] gave recently a counterexample to the WCT.

Even though it looks as if rank one construction is not complicated, mixing in this class is possible; historically the first mixing constructions were given by D. Ornstein [151] in 1970, using probability type arguments for a choice of spacers. Once mixing was shown, the question arose whether absolutely continuous spectrum is also possible, as this would give automatically the positive answer to the Banach problem. However Bourgain [21], relating spectral measures of rank one automorphisms with some classical constructions of Riesz product measures, proved that a certain subclass of Ornstein's class consists of automorphisms with singular spectrum (see also [5] and [6]). Since in Ornstein's class spacers are chosen in a certain "non-constructive" way, quite a lot of attention was devoted to the rank one automorphism defined by cutting a tower at the nth step into $r_n = n$ subcolumns of equal "width" and placing i spacers over the ith subcolumn. The mixing property conjectured by M. Smorodinsky, was proved by Adams [7] (in fact Adams proved a general result on mixing of a class of staircase transformations). Spectral properties of rank-1 transformations are also studied in [114], where the authors proved that whenever $\sum_{n=1}^{\infty} r_n^{-2} = +\infty$ (r_n stands for the number of subcolumns at the nth step of the construction of a rank-1 automorphism) then the spectrum is automatically singular. H. Abdalaoui [5] gives a criterion for singularity of the spectrum of a rank one transformation; his proof uses a central limit theorem. It seems that still the question whether rank one implies singularity of the spectrum remains the most important question of this theory.

We have already seen in Sect. "Spectral Theory of Weighted Operators" that for a special class of rank one systems, namely those with discrete spectra ([87]), we have a nice theory for weighted operators. It would be extremely interesting to find a rank one automorphism with continuous spectrum for which a substitute of Helson's analysis exists.

B. Fayad [39] constructs a rank one differentiable flow, as a special flow over a two-dimensional rotation. In [40] he gives new constructions of smooth flows with singular spectra which are mixing (with a new criterion for a Rajchman measure to be singular). In [35] a certain smooth change of time for an irrational flows on the 3-torus is given, so that the corresponding flow is partially mixing and has the local rank one property.

Spectral Theory of Dynamical Systems of Probabilistic Origin

Let us just recall that when $(Y_n)_{n=-\infty}^{\infty}$ is a stationary process then its distribution μ on $\mathbb{R}^{\mathbb{Z}}$ is invariant under the shift S on $\mathbb{R}^{\mathbb{Z}}$: $S((x_n)_{n\in\mathbb{Z}}) = (y_n)_{n\in\mathbb{Z}}$, where $y_n = x_{n+1}$, $n \in \mathbb{Z}$. In this way we obtain an automorphism S defined on $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \mu)$. For each automorphism T we can find $f\colon X \to \mathbb{R}$ measurable such that the smallest σ-algebra making the stationary process $(f \circ T^n)_{n\in\mathbb{Z}}$ measurable is equal to $\mathcal{B}$, therefore, for the purpose of this article, by a system of probabilistic origin we will mean (S, μ) obtained from a stationary infinitely divisible process (see e. g. [142,192]). In particular, the theory of Gaussian dynamical systems is indeed a classical part of ergodic theory (e. g. [149,150,211,212]). If $(X_n)_{n\in\mathbb{Z}}$ is a stationary real centered Gaussian process and σ denotes the *spectral measure of the process*, i. e. $\widehat{\sigma}(n) = E(X_n \cdot X_0)$, $n \in \mathbb{Z}$, then by $S = S_\sigma$ we denote the corresponding Gaussian system on the shift space (recall also that for each symmetric measure σ on $\mathbb{T}$ there is exactly one stationary real centered Gaussian process whose spectral measure is σ). Notice that if σ has an atom, then in the cyclic space generated by X_0 there exists an eigenfunction Y for S_σ – if now S_σ were ergodic, $|Y|$ would be a constant function which is not possible by the nature of elements in $\mathbb{Z}(X_0)$. In what follows we assume that σ is continuous.

It follows that U_{S_σ} restricted to $\mathbb{Z}(X_0)$ is spectrally the same as $V = V^\sigma$ acting on $L^2(\mathbb{T}, \sigma)$, and we obtain that $(U_{S_\sigma}, L^2(\mathbb{R}^{\mathbb{Z}}, \mu_\sigma))$ can be represented as the symmetric Fock space built over $H = L^2(\mathbb{T}, \sigma)$ and $U_{S_\sigma} = F(V)$ – see the glossary ($H^{\odot n}$ is called the *n-th chaos*). In other words the spectral theory of Gaussian dynamical systems is reduced to the spectral theory of special tensor products unitary operators. Classical results (see [26]) which can be obtained from this point of view are for example the following:

(A) ergodicity implies weak mixing,
(B) the multiplicity function is either 1 or is unbounded,
(C) the maximal spectral type of U_{S_σ} is equal to $\exp(\sigma)$, hence Gaussian systems enjoy the Kolmogorov group property.

However we can also look at a Gaussian system in a different way, simply by noticing that the variables $e^{2\pi i f}$ (f is a real variable), where $f \in \mathbb{Z}(X_0)$ generate $L^2(\mathbb{R}^{\mathbb{Z}}, \mu_\sigma)$. Now calculating the spectral measure of $e^{2\pi i f}$ is not difficult and we obtain easily (C). Moreover, integrals of type $\int e^{2\pi i f_0} e^{2\pi i f_1 \circ T^n} e^{2\pi i f_2 \circ T^{n+m}}\, d\mu_\sigma$ can also be calculated, whence in particular we easily obtain Leonov's theorem on the multiple mixing property of Gaussian systems [141].

One of the most beautiful parts of the theory of Gaussian systems concerns ergodic properties of S_σ when σ is concentrated on a thin Borel set. Recall that a closed subset $K \subset \mathbb{T}$ is said to be a *Kronecker set* if each $f \in C(K)$ is a uniform limit of characters (restricted to K). Each Kronecker set has no rational relations. Gaussian–Kronecker automorphisms are, by definition, those Gaussian systems for which the measure σ (always assumed to be continuous) is concentrated on $K \cup \overline{K}$, K a Kronecker set. The following theorem has been proved in [51] (see also [26]).

Theorem 9 (Foiaş–Stratila Theorem) *If T is an ergodic automorphism and f is a real-valued element of L^2_0 such that the spectral measure σ_f is concentrated on $K \cup \overline{K}$, where K is a Kronecker set, then the process $(f \circ T^n)_{n\in\mathbb{Z}}$ is Gaussian.*

This theorem is indeed striking as it gives examples of weakly mixing automorphisms which are spectrally determined (like rotations). A relative version of the Foiaş–Stratila Theorem has been proved in [129].

The Foiaş–Stratila Theorem implies that whenever a spectral measure σ is Kronecker, it has no realization of the form σ_f with f bounded. We will see however in Sect. "Future Directions" that for some automorphisms T (having the SCS property) the maximal spectral type σ_T has the property that S_{σ_T} has a simple spectrum.

Gaussian–Kronecker automorphisms are examples of automorphisms with simple spectra. In fact, whenever σ is concentrated on a set without rational relations, then S_σ has a simple spectrum (see [26]). Examples of mixing automorphisms with simple spectra are known [149], however it is still unknown (Thouvenot's question) whether the Foiaş–Stratila property may hold in the mixing class. F. Parreau [154] using independent Helson sets gave an example of mildly mixing Gaussian system with the Foiaş–Stratila property.

In [165] there is a remark that the set of finite essential values of the multiplicity function of U_{S_σ} forms a (multiplicative) subsemigroup of $\mathbb{N}$. However, it seems that there is no "written" proof of this fact.

Joining theory of a class of Gaussian system, called GAG, is developed in [136]. A Gaussian automorphism S_σ with the Gaussian space $H \subset L^2_0(\mathbb{R}^{\mathbb{Z}}, \mu_\sigma)$ is called a GAG if for each ergodic self-joining $\rho \in J^e_2(S_\sigma)$ and arbitrary $f, g \in H$ the variable

$$(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}, \rho) \ni (x, y) \mapsto f(x) + g(y)$$

is Gaussian. For GAG systems one can describe the centralizer and factors, they turn out to be objects close to the probability structure of the system. One of the crucial observations in [136] was that all Gaussian systems with simple spectrum are GAG.

It is conjectured (J.P. Thouvenot) that in the class of zero entropy Gaussian systems the PID property holds true.

For the spectral theory of classical factors of a Gaussian system see [137]; also spectrally they share basic spectral properties of Gaussian systems. Recall that historically one of the classical factors namely the σ-algebra of sets invariant for the map

$$(\ldots, x_{-1}, x_0, x_1, \ldots) \mapsto (\ldots, -x_{-1}, -x_0, -x_1, \ldots)$$

was the first example with zero entropy and countable Lebesgue spectrum (indeed, we need a singular measure σ such that $\sigma * \sigma$ is equivalent to Lebesgue measure [150]). For factors obtained as functions of a stationary process see [83].

T. de la Rue [181] proved that Gaussian systems are never of local rank-1, however his argument does not apply to classical factors. We conjecture that Gaussian systems are disjoint from rank-1 automorphisms (or even from local rank-1 systems).

We now turn the attention to Poissonian systems (see [26] for more details). Assume that $(X, \mathcal{B}, \mu)$ is a standard Borel space, where μ is infinite, σ-finite. The new configuration space $\widetilde{X}$ is taken as the set of all countable subsets $\{x_i : i \geq 1\}$ of X. Once a set $A \in \mathcal{B}$, of finite measure is given one can define a map $N_A : \widetilde{X} \to \mathbb{N}(\cup\{\infty\})$ just counting the number of elements belonging to A. The measure-theoretic structure $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$ is given so that the maps N_A become random variables with Poisson distribution of parameter $\mu(A)$ and such that whenever $A_1, \ldots, A_k \subset X$ are of finite measure and are pairwise disjoint then the variables $N_{A_1}, \ldots, N_{A_k}$ are independent.

Assume now that T is an automorphism of $(X, \mathcal{B}, \mu)$. It induces a natural automorphism on the space $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$ defined by $\widetilde{T}(\{x_i : i \geq 1\} = \{Tx_i : i \geq 1\}$. The automorphism $\widetilde{T}$ is called the *Poisson suspension* of T (see [26]). Such a suspension is ergodic if and only if no set of positive and finite μ-measure is T-invariant. Moreover ergodicity of $\widetilde{T}$ implies weak mixing. In fact the spectral structure of $U_{\widetilde{T}}$ is very similar to the Gaussian one: namely the first chaos equals $L^2(X, \mathcal{B}, \mu)$ (we emphasize that this is about the whole L^2 and not only L^2_0) on which $U_{\widetilde{T}}$ acts as U_T and the $L^2(\widetilde{X}, \widetilde{\mu})$ together with the action of $U_{\widetilde{T}}$ has the structure of the symmetric Fock space $F(L^2(X, \mathcal{B}, \mu))$ (see the glossary).

We refer to [22,86,168,169] for ergodic properties of systems given by symmetric α-stable stationary processes, or more generally infinitely divisible processes. Again, they share spectral properties similar to the Gaussian case: ergodicity implies weak mixing, while mixing implies mixing of all orders.

In [171], E. Roy clarifies the dynamical "status" of such systems. He uses Poisson suspension automorphisms and the Maruyama representation of an infinitely divisible process mixed with basic properties of automorphisms preserving infinite measure (see [1]) to prove that as a dynamical system, a stationary infinitely divisible process (without the Gaussian part), is a factor of the Poisson suspension over the Lévy measure of this process. In [170] a theory of ID-joinings is developed (which should be viewed as an analog of the GAG theory in the Gaussian class). Parreau and Roy [155] give an example of a Poisson suspension with a minimal possible set of ergodic self-joinings.

Many natural problems still remain open here, for example (assuming always zero entropy of the dynamical system under consideration): Are Poisson suspensions disjoint from Gaussian systems? What is the spectral structure for dynamical systems generated by symmetric α-stable process? Are such systems disjoint whenever $\alpha_1 \neq \alpha_2$? Are Poissonian systems disjoint from local rank one automorphisms (cf. [181])? In [91] it is proved that Gaussian systems are disjoint from so called simple systems (see ► Joinings in Ergodic Theory and [93,208]); we will come back to an extension of this result in Sect. "Future Directions". It seems that flows of probabilistic origin satisfy the Kolmogorov group property for the spectrum. One can therefore ask how different are systems satisfying the Kolmogorov group property from systems for which the convolutions of the maximal spectral type are pairwise disjoint (see also Sect. "Future Directions" and the SCS property).

We also mention here another problem which will be taken up in Sect. "Special Flows and Flows on Surfaces, Interval Exchange Transformations" – *Is it true that flows of probabilistic origin are disjoint from smooth flows on surfaces?* Recently A. Katok and A. Windsor announced that it is possible to construct a Kronecker measure so that the corresponding Gaussian system ($\mathbb{Z}$-action (!)) has a smooth representation on the torus.

Yet one more (joining) property seems to be characteristic in the class of systems of probabilistic origin, namely they satisfy so called ELF property (see [30] and ► Joinings in Ergodic Theory). Vershik asked whether the ELF property is spectral – however the example of a cocycle from [205] together with Theorem 7 (i) yields a certain Rokhlin extension of a rotation which is ELF and has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions (see [206]); on the other hand any affine extension of that rotation is spectrally the same, while it cannot have the ELF property.

Prikhodko and Thouvenot (private communication) have constructed weakly mixing and non-mixing rank one automorphisms which enjoy the ELF property.

Inducing and Spectral Theory

Assume that T is an ergodic automorphism of a standard probability Borel space $(X, \mathcal{B}, \mu)$. Can "all" dynamics be obtained by inducing (see the glossary) from one fixed automorphism was a natural question from the very beginning of ergodic theory. Because of Abramov's formula for entropy $h(T_A) = h(T)/\mu(A)$ it is clear that positive entropy transformations cannot be obtained from inducing on a zero entropy automorphism. However here we are interested in spectral questions and thus we ask how many spectral types we obtain when we induce. It is proved in [59] that the family of $A \in \mathcal{B}$ for which T_A is mixing is dense for the (pseudo) metric $d(A_1, A_2) = \mu(A_1 \triangle A_2)$. De la Rue [182] proves the following result: *For each ergodic transformation T of a standard probability space $(X, \mathcal{B}, \mu)$ the set of $A \in \mathcal{B}$ for which the maximal spectral type of U_{T_A} is Lebesgue is dense in $\mathcal{B}$.* The multiplicity function is not determined in that paper. Recall (without giving a formal definition, see [152]) that a zero entropy automorphism is *loosely Bernoulli* (LB for short) if and only if it can be induced from an irrational rotation (see also [43,99]). The LB theory shows that not all dynamical systems can be obtained by inducing from an ergodic rotation. However an open question remained whether LB systems exhaust spectrally all Koopman representations. In a deep paper [180], de la Rue studies LB property in the class of Gaussian–Kronecker automorphisms, in particular he constructs S which is not LB. Suppose now that T is LB and for some $A \in \mathcal{B}$, U_{T_A} is isomorphic to U_S. Then by the Foiaş–Stratila Theorem, T_A is isomorphic to S, and hence T_A is not LB. However an induced automorphism from an LB automorphism is LB, a contradiction.

Special Flows and Flows on Surfaces, Interval Exchange Transformations

We now turn our attention to flows. The cases of the geodesic flow, horocycle flows on homogenous spaces of $SL(2, \mathbb{R})$ and nilflows are classical (we refer the reader to [105] with a nice description of the first two cases, while for nilflows we refer to [157]: these classes of flows on homogenous spaces have countable Lebesgue spectrum, in the third case – in the orthocomplement of the eigenspace). On the other hand the classical cyclic approximation theory of Katok and Stepin [104] (see [26]) leads to examples of smooth flows on the torus with simple continuous singular spectra.

Given an ergodic automorphism T on $(X, \mathcal{B}, \mu)$ and a positive integrable function $f: X \to \mathbb{R}^+$ consider the corresponding special flow T^f (see the glossary). Obviously, such a flow is ergodic. Special flows were introduced to ergodic theory by von Neumann in his fundamental work [148] in 1932. Also in that work he explains how to compute eigenvalues for special flows, namely: $r \in \mathbb{R}$ *is an eigenvalue of* T^f *if and only if the following functional equation*

$$e^{2\pi i r f(x)} = \frac{\xi(x)}{\xi(Tx)}$$

has a measurable solution $\xi: X \to \mathbb{T}$. We recall also that the classical Ambrose-Kakutani theorem asserts that practically each ergodic flow has a special representation ([26], see also Rudolph's theorem on special representation therein).

A classical situation when we obtain "natural" special representations is while considering smooth flows on surfaces (we refer the reader to Hasselblatt's and Katok's monograph [73]). They have transversals on which the Poincaré map is piecewise isometric, and this entails a study of interval exchange transformations (IET), see [26,108,163]. Formally, to define IET of m intervals we need a permutation π of $\{1, \dots, m\}$ and a probability vector $\lambda = (\lambda_1, \dots, \lambda_m)$ (with positive entries). Then we define $T = T_{\lambda,\pi}$ of $[0, 1)$ by putting

$$T_{\lambda,\pi}(x) = x + \beta_i^\pi - \beta_i \quad \text{for} \quad x \in [\beta_i, \beta_{i+1}) ,$$

where $\beta_i = \sum_{j<i} \lambda_j$, $\beta_i^\pi = \sum_{\pi j < \pi i} \beta_j$. Obviously, each IET preserves Lebesgue measure. One of possible approaches to study ergodic properties of IET is an "a.e" approach "seen" in the definition of $T_{\lambda,\pi}$. It is based on the fundamental fact that the induced transformation on a subinterval of $[0, 1)$ is also IET (see [26]). This leads to a very delicate and deep mathematics based on Rauzy induction, which is a way of inducing on special intervals, considering only irreducible permutations whose set is partitioned into orbits of some maps (any such an orbit is called a *Rauzy class*). If now $\mathcal{R}$ is a Rauzy class of permutations and λ lies in the standard simplex Δ_{m-1} then the Rauzy induction together with a natural renormalization leads to a map $\mathcal{P}: \mathcal{R} \times \Delta_{m-1} \to \mathcal{R} \times \Delta_{m-1}$. For a better understanding of the dynamics of the Rauzy map Veech [209] introduced the space of *zippered rectangles*. A zippered rectangle associated to the Rauzy class $\mathcal{R}$ is a quadruple (λ, h, a, π), where $\lambda \in \mathbb{R}_+^m$, $h \in \mathbb{R}_+^m$, $a \in \mathbb{R}_+^m$, $\pi \in \mathcal{R}$ and the vectors h and a satisfy some equations and inequalities. Every zippered rectangle (λ, h, a, π) determines a Riemann structure on a compact connected surface. Denote by $\Omega(\mathcal{R})$ the space of all zippered rectangles, corresponding to a given Rauzy class $\mathcal{R}$ and satisfying the condition $\langle \lambda, h \rangle = 1$. In [209], Veech defined a flow $(P^t)_{t \in \mathbb{R}}$ on the space $\Omega(\mathcal{R})$ putting

$$P^t(\lambda, h, a, \pi) = (e^t \lambda, e^{-t} h, e^{-t} a, \pi)$$

and extended the Rauzy map. On so called *Veech moduli space* of zippered rectangles, the flow (P^t) becomes the *Teichmüller flow* and it preserves a natural Lebesgue measure class; by passing to a transversal and projecting the measure on the space of IETs $\mathcal{R} \times \Delta_{m-1}$ Veech has proved the following fundamental theorem ([209], see also [143]) which is a generalization of the fact that Gauss measure $1/(\ln 2) 1/(1 + x) dx$ is invariant for the Gauss map which sends $t \in (0, 1)$ into the fractional part of its inverse.

Theorem 10 (Veech, Masur, 1982) *Assume that* $\mathcal{R}$ *is a Rauzy class. There exists a* σ*-finite measure* $\mu_{\mathcal{R}}$ *on* $\mathcal{R} \times \Delta_{m-1}$ *which is* $\mathcal{P}$*-invariant, equivalent to "Lebesgue" measure, conservative and ergodic.*

In [209] it is proved that a.e. (in the above sense) IET is then of rank one (and hence is ergodic and has a simple spectrum). He also formulated as an open problem whether we can achieve the weak mixing property a.e. This has been recently answered in positive by A. Avila and G. Forni [19] (for π which is not a rotation).

Katok [100] proved that IET have no mixing factors (in fact his proof shows more: the IET class is disjoint with the class of mixing transformations). By their nature, IET transformations are of finite rank (see [26]) so they are of finite multiplicity. They need not be of simple spectrum (see remarks in [105] pp. 88–90). It remains an open question whether an IET can have a non-singular spectrum. Joining properties in the class of exchange of 3 and more intervals are studied in [47,48]. An important question of Veech [208] whether a.e. IET is simple is still open.

When we consider a smooth flow on a surface preserving a smooth measure, whose only singularity (we assume that we have only finitely many singularities) are simple (non-degenerated) saddles then such a flow has a special representation over an interval exchange automorphism under a smooth function which has finitely many logarithmic singularities (see [73]). In the article by Arnold [18] the quasi-periodic Hamiltonian case is considered: $H: \mathbb{R}^2 \to \mathbb{R}$ satisfies $H(x+m, y+n) = H(x, y) + n\alpha_1 + m\alpha_2$, $\alpha_1/\alpha_2 \notin \mathbb{Q}$, and we then consider the following system of differential equations on $\mathbb{T}^2$

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = \frac{-\partial H}{\partial x} . \tag{2}$$

As Arnold shows the dynamical system arising from the system (2) has one ergodic component which has a special representation over the irrational rotation by $\alpha := \alpha_1/\alpha_2$ under a smooth function with finitely many logarithmic singularities (all other ergodic components are periodic orbits and separatrices). By changing a speed as it is done in [57] so that critical points of the vector field in (2) become singular points, Arnold's special representation is transformed to a special flow over the same irrational rotation however under a piecewise smooth function. If the sum of jumps is not zero then in fact we come back to von Neumann's class of special flows considered in [148]. Similar classes of special flows (when the roof function is of bounded variation) are obtained from ergodic components of flows associated to billiards in convex polygones with rational angles [106]. Kochergin [115] showed that special flows over irrational rotations and under bounded variation functions are never mixing. This has been recently strengthened in [54] to the following: *If T is an irrational rotation and f is of bounded variation then the special flow T^f is spectrally disjoint from all mixing flows*. In particular all such flows have singular spectra. Moreover, in [54] it is proved that whenever the Fourier transform of the roof function f is of order $O(1/n)$ then T^f is disjoint from all mixing flows (see also [55]). In fact in the papers [54,55,56,57] the authors discuss the problem of disjointness of those special flows with all ELF-flows conjecturing that no flow of probabilistic origin has a smooth realization on a surface. In [140] the analytic case is considered leading to a "generic" result on disjointness with the ELF class generalizing the classical Shklover's result on the weak mixing property [197].

Kochergin [117] proved the absence of mixing for flows where the roof function has finitely many singularities, whenever the sum of "left logarithmic speeds" and the sum of "right logarithmic speeds" are equal – this is called a *symmetric logarithmic case*, however some Diophantine restriction is put on α.

In [128], where also the absence of mixing is considered for the symmetric logarithmic case, it was conjectured (and proved for arbitrary rotation) that a necessary condition for mixing of a special flow T^f (with arbitrary T and f) is the condition that the sequence of distributions $((f_0^{(n)})_*)_n$ tends to δ_∞ in the space of probability measures on $\overline{\mathbb{R}}$. K. Schmidt [194] proved it using the theory of cocycles and extending a result from [3] on tightness of cocycles.

A. Katok [100] proved the absence of mixing for special flows over IET when the roof function is of bounded variation (see also [187]). Katok's theorem was strengthened in [56] to the disjointness theorem with the class of mixing flows.

On the other hand there is a lot of (difficult) results pointing out classes of special flows over irrational rotations which are mixing, especially (but not only) in the class of non-symmetric logarithmic singularities: [36,38] (B. Fayad was able to give a speed of convergence to zero for Fourier coefficients), [109,119,120]. Recently mixing property has been proved in a non-symmetric case in [203] when the base transformation is a special class of IETs.

The eigenvalue problem (mainly how many frequencies can have the group of eigenvalues) for special flows over irrational rotations is studied in [41,42,71].

A. Avila and G. Forni [19] proved that a.e. translation flow on a surface (of genus at least two) is weakly mixing (which is a drastic difference with the linear flow case of the torus, where the spectrum is always discrete).

The problem of whether mixing flows indicated in this chapter are mixing of all orders is open (it is also unknown whether they have singular spectra). One of several possible approaches (proposed by B. Fayad and J.-P. Thouvenot) toward positive solution of this problem would be to show that such flows enjoy so called Ratner's property (R-property). This property may be viewed as a particular way of divergence of orbits of close points; it was shown to hold for horocycle flows by M. Ratner [162]. We refer the reader to [162] and the survey article [201] for the formal definitions and basic consequences of R-property. In particular, R-property implies "rigidity" of joinings and it also implies the PID property; hence mixing and R-property imply mixing of all orders. In [57,58] a version of R-property is shown for the class of von Neumann special flows (however α is assumed to have bounded partial quotients). This allowed one to prove there that such flows are even mildly mixing (mixing is excluded by a Kochergin's result). We conjecture that an R-property may also hold for special flows over multidimensional rotations with roof functions given by nil-cocycles which we mentioned in Sect. "Spectral Theory of Weighted Operators".

If indeed the R-property is ubiquitous in the class of smooth flows on surfaces it may also be useful to show that smooth flows on surfaces are disjoint with flows of probabilistic origin – see [91,92,135,190,202].

B. Fayad [40] gives a criterion that implies singularity of the maximal spectral type for a dynamical system on a Riemannian manifold. As an application he gives a class of smooth mixing flows (with singular spectra) on $\mathbb{T}^3$ obtained from linear flows by a time change (again this is a drastic difference with dimension two, where a smooth time change of a linear flow leads to non-mixing flows [26]).

The spectral multiplicity problem for special flows (with sufficiently regular roof functions) over irrational rotations seems to be completely untouched (except for the case of a sufficiently smooth f – the spectrum of T^f is then simple [26]). It would be nice to have examples of such flows with finite bigger than one multiplicity. In particular, is it true that the von Neumann class of special flows have finite multiplicity? This was partially solved by A. Katok (private communication) on certain subspaces in L^2, but not on the whole L^2-space.

Problem. *Given $Tx = x + \alpha$ (with α irrational) can we find $f\colon [0,1) \to \mathbb{R}^+$ sufficiently regular (e. g. with finitely many "controllable" singularities) such that T^f has a Lebesgue spectrum?*

Of course the above is related to the question whether at all one can find a smooth flow on a surface with a Lebesgue spectrum (for $\mathbb{Z}$-actions we can even see positive entropy diffeomorphisms on the torus).

We mention at the end that if we drop here (and in other problems) the assumption of regularity of f then the answers will be always positive because of the LB theory; in particular there is a section of any horocycle flow (it has the LB property [161]) such that in the corresponding special representation T^f the map T is an irrational rotation. Using a Kochergin's result [118] on cohomology (see also [98,176]) the L^1-function f is cohomologous to a positive function g which is even continuous, thus T^f is isomorphic to T^g.

Future Directions

We have already seen several cases where spectral properties interact with measure-theoretic properties of a system. Let us mention a few more cases which require further research and deeper understanding.

We recall that the weak mixing property can be understood as a property complementary to discrete spectrum (more precisely to the distality [62]), or similarly mild mixing property is complementary to rigidity. This can be phrased quite precisely by saying that T is not weakly (mildly) mixing if and only if it has a non-trivial factor with discrete spectrum (it has a non-trivial rigid factor). It has been a question for quite a long time if in a sense mixing can be "built" on the same principle. In other words we seek a certain "highly" non-mixing factor. It was quite surprising when in 2005 F. Parrreau (private communication) gave the positive answer to this problem.

Theorem 11 (Parreau) *Assume that T is an ergodic automorphism of a standard probability space $(X, \mathcal{B}, \mu)$. Assume moreover that T is not mixing. Then there exists a non-trivial factor (see below) of T which is disjoint with all mixing automorphisms.*

In fact, Parreau proved that each factor of T given by $\mathcal{B}_\infty(\rho)$ (this σ-algebra is described in [136]), where $U_T^{n_k} \to J_\rho$, is disjoint from all mixing transformations. This proof leads to some other results of the same type, for example: *Assume that T is an ergodic automorphism of a standard probability space. Assume that there exists a non-trivial automorphism S with a singular spectrum which is not disjoint with T. Then T has a non-trivial factor which is disjoint with any automorphism with a Lebesgue spectrum.*

The problem of spectral multiplicity of Cartesian products for "typical" transformation studied by Katok [98] and then its solution in [13] which we already considered in Sect. "The Multiplicity Function" lead to a study of those T for which

$$(CS) \quad \sigma^{(m)} \perp \sigma^{(n)} \text{ whenever } m \neq n\,,$$

where $\sigma = \sigma_T$ just stands for the reduced maximal spectral type of U_T (which is constantly assumed to be a continuous measure), see also Stepin's article [199].

The usefulness of the above property (CS) in ergodic theory was already shown in [90], where a spectral counterexample machinery was presented using the following observation: *If $\mathcal{A}$ is a $T^{\times\infty}$-invariant sub-σ-algebra such that the maximal spectral type on $L^2(\mathcal{A})$ is absolutely continuous with respect to σ_T then $\mathcal{A}$ is contained in one of the coordinate sub-σ-algebras $\mathcal{B}$*. Based on that in [90] it is shown how to construct two weakly isomorphic action which are not isomorphic or how to construct two non-disjoint automorphisms which have no common non-trivial factors (such constructions were previously known for so called minimal self-joining automorphisms [174]). See also [200] for extensions of those results to $\mathbb{Z}^d$-actions.

Prikhodko and Ryzhikov [159] proved that the classical Chacon transformation enjoys the (CS) property. The SCS property defined in the glossary is stronger than the (CS) condition above; the SCS property implies that the corresponding Gaussian system S_{σ_T} has a simple spectrum. Ageev [10] shows that Chacon's transformation satisfies the SCS property; moreover in [13] he shows that the SCS property is satisfied generically and he gives a construction of a rank one mixing SCS-system (see also [191]). In [134] it is proved that some special flows considered in Sect. "Special Flows and Flows on Surfaces, Interval Exchange Transformations" (including the von Neumann class, however with α having unbounded partial quotients) have the SCS property. Since the corresponding Gaussian systems have simple spectra, it would be interesting to decide whether σ_T (for an SCS-automorphism) can be concentrated on a set without rational relations. It is quite plausible that the SCS property is commonly seen for smooth flows on surfaces.

Katok and Thouvenot (private communication) considered systems called *infinitely divisible*. These are systems T on $(X, \mathcal{B}, \mu)$ which have a family of factors $\mathcal{B}_\omega$ indexed by $\omega \in \bigcup_{n=0}^{\infty}\{0,1\}^n$ ($\mathcal{B}_\varepsilon = \mathcal{B}$) such that $\mathcal{B}_{\omega 0} \perp \mathcal{B}_{\omega 1}$, $\mathcal{B}_{\omega 0} \vee \mathcal{B}_{\omega 1} = \mathcal{B}_\omega$ and for each $\eta \in \{0,1\}^{\mathbb{N}}$, $\bigcap_{n\in\mathbb{N}} \mathcal{B}_{\eta[0,n]} = \{\emptyset, X\}$. They showed (unpublished) that there are discrete spectrum transformations which are ID, and that there are rank one transformations with continuous spectra which are also ID (clearly Gaussian systems are ID). It was until recently that a relationship between ID automorphisms and systems coming from stationary ID processes was unclear. In [135] it is proved that dynamical systems coming from stationary ID processes are factors of ID automorphisms; moreover, ID automorphisms are disjoint with all systems having the SCS property. It would be nice to decide whether Koopman representations associated to ID automorphisms satisfy the Kolmogorov group property.

Acknowledgments

Research supported by the EU Program Transfer of Knowledge "Operator Theory Methods for Differential Equations" TODEQ and the Polish Ministry of Science and High Education.

Bibliography

1. Aaronson J (1997) An introduction to infinite ergodic theory, mathematical surveys and monographs. Am Math Soc 50:284
2. Aaronson J, Nadkarni MG (1987) L^∞ eigenvalues and L^2 spectra of nonsingular transformations. Proc London Math Soc 55(3):538–570
3. Aaronson J, Weiss B (2000) Remarks on the tightness of cocycles. Coll Math 84/85:363–376
4. Abdalaoui H (2000) On the spectrum of the powers of Ornstein transformations Ergodic theory and harmonic analysis (Mumbai, 1999). Sankhya Ser. A 62:291–306
5. Abdalaoui H (2007) A new class of rank 1 transformations with singular spectrum. Ergodic Theor Dynam Syst 27:1541–1555
6. Abdalaoui H, Parreau F, Prikhodko A (2006) A new class of Ornstein transformations with singular spectrum. Ann Inst H Poincaré Probab Statist 42:671–681
7. Adams T (1998) Smorodinsky's conjecture on rank one systems. Proc Am Math Soc 126:739–744
8. Ageev ON (1988) Dynamical systems with a Lebesgue component of even multiplicity in the spectrum. Mat Sb (NS) 136(178):307–319, (Russian) 430 (1989) Translation in Math. USSR-Sb. 64:305–317
9. Ageev ON (1999) On ergodic transformations with homogeneous spectrum. J Dynam Control Syst 5:149–152
10. Ageev ON (2000) On the spectrum of cartesian powers of classical automorphisms. Mat Zametki 68:643–647, (2000) Translation Math. Notes 68:547–551, (Russian)
11. Ageev ON (2001) On the multiplicity function of generic group extensions with continuous spectrum. Ergodic Theory Dynam Systems 21:321–338
12. Ageev ON (2005) The homogeneous spectrum problem in ergodic theory. Invent Math 160:417–446
13. Ageev ON (2007) Mixing with staircase multiplicity function. preprint
14. Alexeyev VM (1958) Existence of bounded function of the maximal spectral type. Vestnik Mosc Univ 5:13-15 and (1982) Ergodic Theory Dynam Syst 2:259–261
15. Anosov DV (2003) On Spectral multiplicities in ergodic theory. Institut im. V.A. Steklova, Moscow
16. Anzai H (1951) Ergodic skew product transformations on the torus. Osaka J Math 3:83–99
17. Arni N (2005) Spectral and mixing properties of actions of amenable groups. Electron Res Announc AMS 11:57–63
18. Arnold VI (1991) Topological and ergodic properties of closed 1-forms with incommensurable periods. Funktsional Anal Prilozhen 25:1–12, (Russian)
19. Avila A, Forni G (2007) Weak mixing for interval exchange transformations and translation flows. Ann Math 165:637–664
20. Baxter JR (1971) A class of ergodic transformations having simple spectrum. Proc Am Math Soc 27:275–279
21. Bourgain J (1993) On the spectral type of Ornstein's class of transformations. Isr J Math 84:53–63
22. Cambanis S, Podgórski K, Weron A (1995) Chaotic behaviour of infinitely divisible processes. Studia Math 115:109–127
23. Chacon RV (1970) Approximation and spectral multiplicity. In: Dold A, Eckmann B (eds) Contributions to ergodic theory and probability. Springer, Berlin, pp 18–27
24. Choe GH Products of operators with singular continuous spectra. In: Operator theory: operator algebras and applications, Part 2. (Durham, 1988), pp 65–68, Proc Sympos Pure Math, 51, Part 2, Amer Math Soc, Providence, 1990
25. Connes A, Woods E (1985) Approximately transitive flows and ITPFI factors. Ergod Theory Dynam Syst 5:203–236
26. Cornfeld IP, Fomin SV, Sinai YG (1982) Ergodic theory. Springer, New York
27. Danilenko AL (2006) Explicit solution of Rokhlin's problem on homogeneous spectrum and applications. Ergod Theory Dyn Syst 26:1467–1490
28. Danilenko AL, Lemańczyk M (2005) A class of multipliers for $\mathcal{W}^\perp$. Isr J Math 148:137–168
29. Danilenko AI, Park KK (2002) Generators and Bernouillian factors for amenable actions and cocycles on their orbits. Ergod Theory Dyn Syst 22:1715–1745
30. Derriennic Y, Frączek K, Lemańczyk M, Parreau F (2007) Ergodic automorphisms whose weak closure of off-diagonal measures consists of ergodic self-joinings. Coll Math 110:81–115
31. Dooley A, Golodets VY (2002) The spectrum of completely positive entropy actions of countable amenable groups. J Funct Anal 196:1–18
32. Dooley A, Quas A (2005) Approximate transitivity for zero-entropy systems. Ergod Theory Dyn Syst 25:443–453

33. Downarowicz T, Kwiatkowski J (2000) Spectral isomorphism of Morse flows. Fundam Math 163:193–213
34. Downarowicz T, Kwiatkowski J (2002) Weak Cosure Theorem fails for $\mathbb{Z}^d$-actions. Stud Math 153:115–125
35. Fayad B (2001) Partially mixing and locally rank 1 smooth transformations and flows on the torus $\mathbb{T}^d, d \geq 3$. J London Math Soc 64(2):637–654
36. Fayad B (2001) Polynomial decay of correlations for a class of smooth flows on the two torus. Bull Soc Math France 129:487–503
37. Fayad B (2002) Skew products over translations on $T^d, d \geq 2$. Proc Amer Math Soc 130:103–109
38. Fayad B (2002) Analytic mixing reparametrizations of irrational flows. Ergod Theory Dyn Syst 22:437–468
39. Fayad B (2005) Rank one and mixing differentiable flows. Invent Math 160:305–340
40. Fayad B (2006) Smooth mixing flows with purely singular spectra. Duke Math J 132:371–391
41. Fayad B, Katok AB, Windsor A (2001) Mixed spectrum reparameterizations of linear flows on $\mathbb{T}^2$, dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary. Mosc Math J 1:521–537
42. Fayad B, Windsor A (2007) A dichotomy between discrete and continuous spectrum for a class of special flows over rotations. J Mod Dyn 1:107–122
43. Feldman J (1976) New K-automorphisms and a problem of Kakutani. Isr J Math 24:16–38
44. Ferenczi S (1984) Systèmes localement de rang un. Ann Inst H Poincaré Probab Statist 20:35–51
45. Ferenczi S (1985) Systèmes de rang un gauche. Ann Inst H Poincaré Probab Statist 21:177–186
46. Ferenczi S (1997) Systems of finite rank. Colloq Math 73:35–65
47. Ferenczi S, Holton C, Zamboni LQ (2005) Joinings of three-interval exchange transformations. Ergod Theory Dyn Syst 25:483–502
48. Ferenczi S, Holton C, Zamboni LQ (2004) Structure of three-interval exchange transformations III: ergodic and spectral properties. J Anal Math 93:103–138
49. Ferenczi S, Lemańczyk M (1991) Rank is not a spectral invariant. Stud Math 98:227–230
50. Filipowicz I (1997) Product $\mathbb{Z}^d$-actions on a Lebesgue space and their applications. Stud Math 122:289–298
51. Foiaş C, Stratila S (1968) Ensembles de Kronecker dans la théorie ergodique. CR Acad Sci Paris, Ser A-B 267:A166–A168
52. Frączek K (1997) On a function that realizes the maximal spectral type. Stud Math 124:1–7
53. Frączek K (2000) Circle extensions of Z^d-rotations on the d-dimensional torus. J London Math Soc 61(2):139–162
54. Frączek K, Lemańczyk M (2004) A class of special flows over irrational rotations which is disjoint from mixing flows. Ergod Theory Dyn Syst 24:1083–1095
55. Frączek K, Lemańczyk M (2003) On symmetric logarithm and some old examples in smooth ergodic theory. Fund Math 180:241–255
56. Frączek K, Lemańczyk M (2005) On disjointness properties of some smooth flows. Fund Math 185:117–142
57. Frączek K, Lemańczyk M (2006) On mild mixing of special flows over irrational rotations under piecewise smooth functions. Ergod Theory Dyn Syst 26:719–738
58. Frączek K, Lemańczyk M, Lesigne E (2007) Mild mixing property for special flows under piecewise constant functions. Disc Contin Dyn Syst 19:691–710
59. Friedman NA, Ornstein DS (1973) Ergodic transformations induce mixing transformations. Adv Math 10:147–163
60. Furstenberg H (1967) Disjointness in ergodic theory, minimal sets, and a problem of Diophantine approximation. Math Syst Theory 1:1–49
61. Furstenberg H, Weiss B (1978) The finite multipliers of infinite ergodic transformations. Lect Notes Math 668:127–132
62. Furstenberg H (1981) Recurrence in ergodic theory and combinatorial number theory. Princeton University Press, Princeton
63. Glasner E (1994) On the class of multipliers for $\mathcal{W}^{\perp}$. Ergod Theory Dyn Syst 14:129–140
64. Glasner E (2003) Ergodic theory via joinings. Math Surv Monogr, AMS, Providence, RI 101:384
65. Glasner E, Weiss B (1989) Processes disjoint from weak mixing. Trans Amer Math Soc 316:689–703
66. Goodson GR (1999) A survey of recent results in the spectral theory of ergodic dynamical systems. J Dyn Control Syst 5:173–226
67. Goodson GR, Kwiatkowski J, Lemańczyk M, Liardet P (1992) On the multiplicity function of ergodic group extensions of rotations. Stud Math 102:157–174
68. Gromov AL (1991) Spectral classification of some types of unitary weighted shift operators. Algebra Anal 3:62–87, (Russian) (1992) Translation in St. Petersburg Math J 3:997–1021
69. Guenais M (1998) Une majoration de la multiplicité spectrale d'opérateurs associés à des cocycles réguliers. Isr J Math 105:263–284
70. Guenais M (1999) Morse cocycles and simple Lebesgue spectrum. Ergod Theory Dyn Syst 19:437–446
71. Guenais M, Parreau F (2005) Valeurs propres de transformations liées aux rotations irrationnelles et aux fonctions en escalier. (in press)
72. Hahn F, Parry W (1968) Some characteristic properties of dynamical systems with quasi-discrete spectra. Math Syst Theory 2:179–190
73. Hasselblatt B, Katok AB (2002) Principal structures. In: Handbook of dynamical systems, vol 1A. North-Holland, Amsterdam, pp 1–203
74. Helson H (1986) Cocycles on the circle. J Oper Theory 16:189–199
75. Helson H, Parry W (1978) Cocycles and spectra. Arkiv Math 16:195–206
76. Herman M (1979) Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. Publ Math IHES 49:5–234
77. Host B (1991) Mixing of all orders and pairwise independent joinings of systems with singular spectrum. Isr J Math 76:289–298
78. Host B, Méla F, Parreau F (1991) Non-singular transformations and spectral anaysis of measures. Bull Soc Math France 119:33–90
79. Iwanik A (1991) The problem of L^p-simple spectrum for ergodic group automorphisms. Bull Soc Math France 119:91–96
80. Iwanik A (1992) Positive entropy implies infinite L^p-multiplicity for $p > 1$. Ergod Theory Rel Top (Güstrow 1990) III:124–127. Lecture Notes in Math, vol 1514. Springer, Berlin

81. Iwanik A (1997) Anzai skew products with Lebesgue component of infinite multiplicity. Bull London Math Soc 29:195–199
82. Iwanik A, Lemańczyk M, Rudolph D (1993) Absolutely continuous cocycles over irrational rotations. Isr J Math 83:73–95
83. Iwanik A, Lemańczyk M, de Sam Lazaro J, de la Rue T (1997) Quelques remarques sur les facteurs des systèmes dynamiques gaussiens. Stud Math 125:247–254
84. Iwanik A, Lemańczyk M, Mauduit C (1999) Piecewise absolutely continuous cocycles over irrational rotations. J London Math Soc 59(2):171–187
85. Iwanik A, de Sam Lazaro J (1991) Sur la multiplicité L^p d'un automorphisme gaussien. CR Acad Sci Paris Ser I 312:875–876
86. Janicki A, Weron A (1994) Simulation and chaotic behavior of α-stable stochastic processes. Monographs and Textbooks in Pure and Applied Mathematics, vol 178. Marcel Dekker, New York, pp 355
87. del Junco A (1976) Transformations with discrete spectrum are stacking transformations. Canad J Math 28:836–839
88. del Junco A (1977) A transformation with simple spectrum which is not of rank one. Canad J Math 29:655–663
89. del Junco A (1981) Disjointness of measure-preserving transformations, minimal self-joinings and category. Prog Math 10:81–89
90. del Junco A, Lemańczyk M (1992) Generic spectral properties of measure-preserving maps and applications. Proc Amer Math Soc 115:725–736
91. del Junco A, Lemańczyk M (1999) Simple systems are disjoint with Gaussian systems. Stud Math 133:249–256
92. del Junco A, Lemańczyk M (2005) Joinings of distally simple systems. (in press)
93. del Junco A, Rudolph D (1987) On ergodic actions whose self-joinings are graphs. Ergod Theory Dyn Syst 7:531–557
94. Kachurovskii AG (1990) A property of operators generated by ergodic automorphisms. Optimizatsiya 47(64):122–125, (Russian)
95. Kalikow S (1984) Two fold mixing implies three fold mixing for rank one tranformations. Ergod Theory Dyn Syst 4:237–259
96. Kamae T Spectral properties of automata generating sequences. (in press)
97. Kamiński B (1981) The theory of invariant partitions for $\mathbb{Z}^d$-actions. Bull Acad Polon Sci Ser Sci Math 29:349–362
98. Katok AB Constructions in ergodic theory. (in press)
99. Katok AB (1977) Monotone equivalence in ergodic theory. Izv Akad Nauk SSSR Ser Mat 41:104–157, (Russian)
100. Katok AB (1980) Interval exchange transformations and some special flows are not mixing. Isr J Math 35:301–310
101. Katok AB (2001) Cocycles, cohomology and combinatorial constructions in ergodic theory. In: Robinson Jr. EA (ed) Proc Sympos Pure Math 69, Smooth ergodic theory and its applications, Seattle, 1999. Amer Math Soc, Providence, pp 107–173
102. Katok AB (2003) Combinatorial constructions in ergodic theory and dynamics. In: University Lecture Series 30. Amer Math Soc, Providence
103. Katok AB, Lemańczyk M (2007) Some new cases of realization of spectral multiplicity function for ergodic transformations. (in press)
104. Katok AB, Stepin AM (1967) Approximations in ergodic theory. Uspekhi Mat Nauk 22(137):81–106, (Russian)
105. Katok AB, Thouvenot J-P (2006) Spectral Properties and Combinatorial Constructions in Ergodic Theory. In: Handbook of dynamical systems, vol 1B. Elsevier, Amsterdam, pp 649–743
106. Katok AB, Zemlyakov AN (1975) Topological transitivity of billiards in polygons. Mat Zametki 18:291–300, (Russian)
107. Keane M (1968) Generalized Morse sequences. Z Wahr Verw Geb 10:335–353
108. Keane M (1975) Interval exchange transformations. Math Z 141:25–31
109. Khanin KM, Sinai YG (1992) Mixing of some classes of special flows over rotations of the circle. Funct Anal Appl 26:155–169
110. King JL (1986) The commutant is the weak closure of the powers, for rank-1 transformations. Ergod Theory Dyn Syst 6:363–384
111. King JL (1988) Joining-rank and the structure of finite rank rank mixing transformations. J Anal Math 51:182–227
112. King JL (2001) Flat stacks, joining-closure and genericity. (in press)
113. Klemes I (1996) The spectral type of the staircase transformation. Tohoku Math J 48:247–248
114. Klemes I, Reinhold K (1997) Rank one transformations with singular spectral type. Isr J Math 98:1–14
115. Kocergin AV (1972) On the absence of mixing in special flows over the rotation of a circle and in flows on a two-dimensional torus. Dokl Akad Nauk SSSR 205:949–952, (Russian)
116. Kochergin AV (1975) Mixing in special flows over rearrangment of segments and in smooth flows on surfaces. Math USSR Acad Sci 25:441–469
117. Kochergin AV (1976) Non-degenerate saddles and absence of mixing. Mat Zametky 19:453–468, (Russian)
118. Kochergin AV (1976) On the homology of function over dynamical systems. Dokl Akad Nauk SSSR 231:795–798
119. Kochergin AV (2002) A mixing special flow over a rotation of the circle with an almost Lipschitz function. Sb Math 193:359–385
120. Kochergin AV (2004) Nondegenerate fixed points and mixing in flows on a two-dimensional torus II. Math Sb 195:15–46, (Russian)
121. Koopman BO (1931) Hamiltonian systems and transformations in Hilbert space. Proc Nat Acad Sci USA 17:315–318
122. Kuipers L, Niederreiter H (1974) Uniform distribution of sequences. Wiley, London, pp 407
123. Kushnirenko AG (1974) Spectral properties of some dynamical systems with polynomial divergence of orbits. Vestnik Moskovskogo Univ 1–3:101–108
124. Kwiatkowski J (1981) Spectral isomorphism of Morse dynamical systems Bull Acad Polon Sci Ser Sci Math 29:105–114
125. Kwiatkowski Jr. J, Lemańczyk M (1995) On the multiplicity function of ergodic group extensions II. Stud Math 116:207–215
126. Lemańczyk M (1988) Toeplitz Z_2–extensions. Ann Inst H Poincaré 24:1–43
127. Lemańczyk M (1996) Introduction to ergodic theory from the point of view of the spectral theory. In: Lecture Notes of the tenth KAIST mathematics workshop, Taejon, 1996, pp 1–153
128. Lemańczyk M (2000) Sur l'absence de mélange pour des flots spéciaux au dessus d'une rotation irrationnelle. Coll Math 84/85:29–41
129. Lemańczyk M, Lesigne E (2001) Ergodicity of Rokhlin cocycles. J Anal Math 85:43–86

130. Lemańczyk M, Mauduit C (1994) Ergodicity of a class of cocycles over irrational rotations. J London Math Soc 49:124–132
131. Lemańczyk M, Mentzen MK, Nakada H (2003) Semisimple extensions of irrational rotations. Stud Math 156:31–57
132. Lemańczyk M, Parreau F (2003) Rokhlin extensions and lifting disjointness. Ergod Theory Dyn Syst 23:1525–1550
133. Lemańczyk M, Parreau F (2004) Lifting mixing properties by Rokhlin cocycles. (in press)
134. Lemańczyk M, Parreau F (2007) Special flows over irrational rotation with simple convolution property. (in press)
135. Lemańczyk M, Parreau F, Roy E (2007) Systems with simple convolutions, distal simplicity and disjointness with infinitely divisible systems. (in press)
136. Lemańczyk M, Parreau F, Thouvenot J-P (2000) Gaussian automorphisms whose ergodic self–joinings are Gaussian. Fundam Math 164:253–293
137. Lemańczyk M, de Sam Lazaro J (1997) Spectral analysis of certain compact factors for Gaussian dynamical systems. Isr J Math 98:307–328
138. Lemańczyk M, Sikorski A (1987) A class of not local rank one automorphisms arising from continuous substitutions. Prob Theory Rel Fields 76:421–428
139. Lemańczyk M, Wasieczko M (2006) A new proof of Alexeyev's theorem. (in press)
140. Lemańczyk M, Wysokińska M (2007) On analytic flows on the torus which are disjoint from systems of probabilistic origin. Fundam Math 195:97–124
141. Leonov VP (1960) The use of the characteristic functional and semi–invariants in the ergodic theory of stationary processes. Dokl Akad Nauk SSSR 133:523–526, (1960) Sov Math 1:878–881
142. Maruyama G (1970) Infinitely divisible processes. Theory Prob Appl 15(1):1–22
143. Masur H (1982) Interval exchange transformations and measured foliations. Ann Math 115:169–200
144. Mathew J, Nadkarni MG (1984) Measure-preserving transformation whose spectrum has Lebesgue component of multiplicity two. Bull London Math Soc 16:402–406
145. Medina H (1994) Spectral types of unitary operators arising from irrational rotations on the circle group. Michigan Math J 41:39–49
146. Mentzen MK (1988) Some examples of automorphisms with rank r and simple spectrum. Bull Pol Acad Sci 7–8:417–424
147. Nadkarni MG (1998) Spectral theory of dynamical systems. Hindustan Book Agency, New Delhi
148. von Neumann J (1932) Zur Operatorenmethode in der Klassichen Mechanik. Ann Math 33:587–642
149. Newton D (1966) On Gaussian processes with simple spectrum. Z Wahrscheinlichkeitstheorie Verw Geb 5:207–209
150. Newton D, Parry W (1966) On a factor automorphism of a normal dynamical system. Ann Math Statist 37:1528–1533
151. Ornstein D (1970) On the root problem in ergodic theory. In: Proc. 6th Berkeley Symp Math Stats Prob, University California Press, Berkeley, 1970, pp 348–356
152. Ornstein D, Rudolph D, Weiss B (1982) Equivalence of measure preserving transformations. Mem Amer Math Soc 37(262):116
153. Ornstein D, Weiss B (1987) Entropy and isomorphism theorems for actions of amenable groups. J Anal Math 48:1–141
154. Parreau F (2000) On the Foiaş and Stratila theorem. In: Proceedings of the conference on Ergodic Theory, Toruń, 2000
155. Parreau F, Roy E (2007) Poissson suspensions with a minimal set od self-joinings. (in press)
156. Parry W (1981) Topics in ergodic theory. Cambridge Tracts in Mathematics, 75. Cambridge University Press, Cambridge-New York
157. Parry W (1970) Spectral analysis of G-extensions of dynamical systems. Topology 9:217–224
158. Petersen K (1983) Ergodic theory. Cambridge University Press, Cambridge
159. Prikhodko AA, Ryzhikov VV (2000) Disjointness of the convolutions for Chacon's automorphism. Dedicated to the memory of Anzelm Iwanik. Colloq Math 84/85:67–74
160. Queffelec M (1988) Substitution dynamical systems – spectral analysis. In: Lecture Notes in Math 1294. Springer, Berlin, pp 240
161. Ratner M (1978) Horocycle flows are loosely Bernoulli. Isr J Math. 31:122–132
162. Ratner M (1983) Horocycle flows, joinings and rigidity of products. Ann Math 118:277–313
163. Rauzy G (1979) Echanges d'intervalles et transformations induites. Acta Arith 34:315–328
164. Robinson EA Jr (1983) Ergodic measure preserving transformations with arbitrary finite spectral multiplicities. Invent Math 72:299–314
165. Robinson EA Jr (1986) Transformations with highly nonhomogeneous spectrum of finite multiplicity. Isr J Math 56:75–88
166. Robinson EA Jr (1988) Nonabelian extensions have nonsimple spectrum. Compos Math 65:155–170
167. Robinson EA Jr (1992) A general condition for lifting theorems. Trans Amer Math Soc 330:725–755
168. Rosiński J, Zak T (1996) Simple condition for mixing of infinitely divisible processes. Stoch Process Appl 61:277–288
169. Rosiński J, Zak T (1997) The equivalence of ergodicity and weak mixing for infinitely divisible processes. J Theor Probab 10:73–86
170. Roy E (2005) Mesures de Poisson, infinie divisibilité et propriétés ergodiques. In: Thèse de doctorat de l'Université Paris 6
171. Roy E (2007) Ergodic properties of Poissonian ID processes. Ann Probab 35:551–576
172. Royden HL (1968) Real analysis. McMillan, New York
173. Rudin W (1962) Fourier analysis on groups. In: Interscience Tracts in Pure and Applied Mathematics, No. 12 Interscience Publishers. Wiley, New York, London, pp 285
174. Rudolph DJ (1979) An example of a measure-preserving map with minimal self-joinings and applications. J Anal Math 35:97–122
175. Rudolph D (1985) k-fold mixing lifts to weakly mixing isometric extensions. Ergod Theor Dyn Syst 5:445–447
176. Rudolph D (1986) $\mathbb{Z}^n$ and $\mathbb{R}^n$ cocycle extension and complementary algebras. Ergod Theor Dyn Syst 6:583–599
177. Rudolph D (1990) Fundamentals of measurable dynamics. Oxford Sci Publ, pp 169
178. Rudolph D (2004) Pointwise and L^1 mixing relative to a sub-sigma algebra. Illinois J Math 48:505–517
179. Rudolph D, Weiss B (2000) Entropy and mixing for amenable group actions. Ann Math 151(2):1119–1150
180. de la Rue T (1996) Systèmes dynamiques gaussiens d'entropie nulle, lâchement et non lâchement Bernoulli. Ergod Theor Dyn Syst 16:379–404

181. de la Rue T (1998) Rang des systèmes dynamiques gaussiens. Isr J Math 104:261–283
182. de la Rue T (1998) L'ergodicité induit un type spectral maximal équivalent à la mesure de Lebesgue. Ann Inst H Poincaré Probab Statist 34:249–263
183. de la Rue T (1998) L'induction ne donne pas toutes les mesures spectrales. Ergod Theor Dyn Syst 18:1447–1466
184. de la Rue T (2004) An extension which is relatively twofold mixing but not threefold mixing. Colloq Math 101:271–277
185. Ryzhikov VV (1991) Joinings of dynamical systems. Approximations and mixing. Uspekhi Mat Nauk 46 5(281):177–178 (in Russian); Math Surv 46(5)199–200
186. Ryzhikov VV (1992) Mixing, rank and minimal self-joining of actions with invariant measure. Math Sb 183:133–160, (Russian)
187. Ryzhikov VV (1994) The absence of mixing in special flows over rearrangements of segments. Math Zametki 55:146–149, (Russian)
188. Ryzhikov VV (1999) Transformations having homogeneous spectra. J Dyn Control Syst 5:145–148
189. Ryzhikov VV (2000) The Rokhlin problem on multiple mixing in the class of actions of positive local rank. Funkt. Anal Prilozhen 34:90–93, (Russian)
190. Ryzhikov VV, Thouvenot J-P (2006) Disjointness, divisibility, and quasi-simplicity of measure-preserving actions. Funkt Anal Prilozhen 40:85–89, (Russian)
191. Ryzhikov VV (2007) Weak limits of powers, simple spectrum symmetric products and mixing rank one constructions. Math Sb 198:137–159
192. Sato K-I (1999) Lévy Processes and infinitely divisible distributions. In: Cambridge Studies in Advanced Mathematics, vol 68. Cambridge University Press, Cambridge, pp 486
193. Schmidt K (1977) Cocycles of Ergodic Transformation Groups. In: Lecture Notes in Math. 1. Mac Millan, India
194. Schmidt K (2002) Dispersing cocycles and mixing flows under functions. Fund Math 173:191–199
195. Schmidt K, Walters P (1982) Mildly mixing actions of locally compact groups. Proc London Math Soc 45(3)506–518
196. Sinai YG (1994) Topics in ergodic theory. Princeton University Press, Princeton
197. Shklover M (1967) Classical dynamical systems on the torus with continuous spectrum. Izv Vys Ucebn Zaved Matematika 10:(65)113–124, (Russian)
198. Smorodinsky M, Thouvenot J-P (1979) Bernoulli factors that span a transformation. Isr J Math 32:39–43
199. Stepin AM (1986) Spectral properties of generic dynamical systems. Izv Akad Nauk SSSR Ser Mat 50:801–834, (Russian)
200. Tikhonov SV (2002) On a relation between the metric and spectral properties of $\mathbb{Z}^d$-actions. Fundam Prikl Mat 8:1179–1192
201. Thouvenot J-P (1995) Some properties and applications of joinings in ergodic theory. London Math Soc Lect Note Ser 205:207–235
202. Thouvenot J-P (2000) Les systèmes simples sont disjoints de ceux qui sont infiniment divisibles et plongeables dans un flot. Coll Math 84/85:481–483
203. Ulcigrai C (2007) Mixing of asymmetric logarithmic suspension flows over interval exchange transformations. Ergod Theor Dyn Syst 27:991–1035
204. Walters P (1982) An Introduction to Ergodic Theory. In: Graduate Texts in Mathematics, vol 79. Springer, Berlin, pp 250
205. Wysokińska M (2004) A class of real cocycles over an irrational rotation for which Rokhlin cocycle extensions have Lebesgue component in the spectrum. Topol Meth Nonlinear Anal 24:387–407
206. Wysokińska M (2007) Ergodic properties of skew products and analytic special flows over rotations. In: Ph D thesis. Toruń, 2007.
207. Veech WA (1978) Interval exchange transformations. J Anal Math 33:222–272
208. Veech WA (1982) A criterion for a process to be prime. Monatshefte Math 94:335–341
209. Veech WA (1982) Gauss measures for transformations on the space of interval exchange maps. Ann Math 115(2):201–242
210. Veech WA (1984) The metric theory of interval exchange transformations, I Generic spectral properties. Amer J Math 106:1331–1359
211. Vershik AM (1962) On (1962) the theory of normal dynamic systems. Math Sov Dokl 144:625–628
212. Vershik AM (1962) Spectral and metric isomorphism of some normal dynamical systems. Math Sov Dokl 144:693–696
213. Yassawi R (2003) Multiple mixing and local rank group actions. Ergod Theor Dyn Syst 23:1275–1304
214. Zeitz P (1993) The centralizer of a rank one flow. Isr J Math 84:129–145

Spin Dependent Exchange and Correlation in Two-Dimensional Electron Layers

M. W. CHANDRE DHARMA-WARDANA
Institute of Microstructural Sciences,
National Research Council of Canada, Ottawa, Canada

Article Outline

Glossary

Atomic units, a. u. The electron-charge $|e|$, and the mass m_e are taken as unity. The unit of time is fixed by setting the Plank constant $\hbar$ to unity. The Bohr radius $a_0 = \hbar^2/(m_e e^2)$ is one a.u. of length in the