A SALAD OF COCYCLES

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Abstract. We study the centraliser of locally compact group extensions of ergodic probability preserving transformations. New methods establishing ergodicity of group extensions are introduced, and new examples of squashable and non-coalescent group extensions are constructed. Smooth versions of some of the constructions are also given.

§0 Introduction

Let T be an ergodic probability preserving transformation of the probability space $(X, \mathcal{B}, m)$.

Let $(G, \mathcal{T})$ be a locally compact, second countable, topological group ($\mathcal{T} = \mathcal{T}(G)$ denotes the family of open sets in the topological space G), and let $\varphi : X \to G$ be a measurable function.

The (left) *skew product* or *G-extension* $T_\varphi : X \times G \longrightarrow X \times G$, is defined by

$$T_\varphi(x, g) = (Tx, \varphi(x)g).$$

The skew product preserves the measure $\mu = m \times m_G$ where m_G is left Haar measure on G. There is an ergodic skew product $T_\varphi : X \times G \longrightarrow X \times G$ iff the group G is amenable (see [G-S], references therein, and [Zim]). In this paper, we are mainly concerned with Abelian G. Recall that on any locally compact, Abelian, second countable, topological group G, there is defined a *norm* $\| \cdot \|_G$ (satisfying $\|x\| = \|-x\| \geq 0$ with equality iff $x = 0$, and $\|x+y\| \leq \|x\| + \|y\|$) which generates the topology of G.

The centraliser.

Recall that the *centraliser* of a non-singular transformation $R : X \to X$ is the collection of *commutors* of R, that is, non-singular transformations of X which commute with R. The collection of invertible commutors (the *invertible centraliser*) is denoted by $C(R)$.

We study those commutors Q of T_φ, of form

$$Q(x, y) = (Sx, f(x)w(y)) \qquad (*)$$

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where $w : G \longrightarrow G$ is a surjective, continuous group endomorphism, S is a commutor of T, and $f : X \longrightarrow G$ is measurable. It is shown in proposition 1.1 of [A-L-M-N] that if T is a Kronecker transformation, and T_φ is ergodic, then every commutor of T_φ is of form $(*)$.

Let End (G) denote the collection of surjective, continuous group endomorphisms of G (a semigroup under composition) and let

$$\mathcal{E}_\varphi = \{w \in \text{End}\,(G) : \exists \text{ a commutor } Q \text{ of } T_\varphi \text{ of form } (*) \text{ with } w = w_Q\},$$

a sub-semigroup of End (G).

The study of $\mathcal{E}_\varphi$ yields counterexamples:
if $\mathcal{E}_\varphi$ contains non-invertible endomorphisms, then T_φ is not *coalescent*, i.e. its centraliser contains some non-invertible transformation (see [H-P]); and
if $\mathcal{E}_\varphi$ contains endomorphisms which do not preserve m_G (a possibility only for non-compact G), then T_φ is *squashable*, i.e. i.e. its centraliser contains some non-singular transformation which is not measure preserving (see [Aa1] and below). Counterexamples like these (and others) will be discussed below.

In case T is an odometer, for any Abelian, locally compact, second countable G, the collection

$$\{\varphi : X \to G : T_\varphi \text{ ergodic },\ \mathcal{E}_\varphi = \{\text{Id}\}\}$$

is residual in the collection of measurable functions $\varphi : X \to G$ considered in the topology of convergence in measure (see below). Analogous results hold, when T is a rotation of the circle, for smooth $\varphi : \mathbb{T} \to \mathbb{R}$.

Semigroup homomorphisms.

Let $\mathcal{L}_\varphi$ denote the collection of those commutors S of T, for which $\exists$ a commutor Q of T_φ of form $(*)$ with $S = S_Q$.

When G is Abelian and T_φ is ergodic, there is a surjective semigroup homomorphism $\pi_\varphi : \mathcal{L}_\varphi \to \mathcal{E}_\varphi$ such that if $S \in \mathcal{L}_\varphi$, and Q is a commutor of T_φ of form $(*)$ with $S = S_Q$, then $w_Q = \pi_\varphi(S)$. This result (called the *semigroup embedding lemma*) is proved at the end of this introduction.

It implies that $\mathcal{E}_\varphi$ is Abelian whenever the commutors of T form an Abelian semigroup, for instance when T is a Kronecker transformation.

The restriction of π_φ to $L_\varphi(T) = \{S_Q : Q \in C(T_\varphi) \text{ of form } (*)\}$ is continuous with respect to the relevant Polish topologies by the continuous embedding lemma established in §1 (c.f. [G-L-S] for the case where G is compact).

The question arises as to when a homomorphism π from a sub-semigroup $\mathcal{S}$ of commutors of T into End (G) occurs in this manner. That is, when does there exist a measurable function $\varphi : X \to G$ such that T_φ is ergodic, $\mathcal{S} \subset \mathcal{L}_\varphi$, and $\pi = \pi_\varphi|_{\mathcal{S}}$.

In [L-L-T] it is shown that for T an invertible, ergodic probability preserving transformation with some invertible commutor S so that $\{S^m T^n : m, n \in \mathbb{Z}\}$ acts freely, and $G = \mathbb{T}$, $\exists\ \varphi : X \to \mathbb{T}$ such that $S \in \mathcal{L}_\varphi$, $\mathcal{E}_\varphi \ni [x \mapsto 2x \mod 1]$, and indeed, $\pi_\varphi(S) = [x \mapsto 2x \mod 1]$. This includes the first example of a non-coalescent Anzai skew product (i.e. $\mathbb{T}$-extension of a rotation of $\mathbb{T}$).

The main results. We generalise this to all Abelian, locally compact, second countable G:

Theorem 1. *Suppose that T is an ergodic probability preserving transformation, $d \le \infty$, and $S_1, \dots, S_d \in C(T)$ $(d \le \infty)$ are such that $(T, S_1, \dots, S_d)$ generate a free $\mathbb{Z}^{d+1}$ action of probability preserving transformations of X.*

If $w_1, \dots, w_d \in End(G)$ commute (i.e. $w_i \circ w_j = w_j \circ w_i \ \forall\ 1 \le i, j \le d$), then there is a measurable function $\varphi : X \to G$ such that

$$T_\varphi \text{ is ergodic,}$$

$$S_1, \dots, S_d \in \mathcal{L}_\varphi, \quad w_1, \dots, w_d \in \mathcal{E}_\varphi;$$

and

$$\pi_\varphi(S_i) = w_i \qquad (1 \le i \le d).$$

Any Kronecker transformation of an uncountable compact group satisfies the preconditions of theorem 1.

Theorem 2. *Suppose that T is an ergodic probability preserving transformation, and that $\{S_t : t \in \mathbb{R}\} \subset C(T)$ is such that T and $\{S_t : t \in \mathbb{R}\}$ generate a free $\mathbb{Z} \times \mathbb{R}$ action of probability preserving transformations of X.*

There is a measurable function $\varphi : X \to \mathbb{R}$ such that

$$T_\varphi \text{ is ergodic;}$$

and there is a flow $\{Q_t : t \in \mathbb{R}\} \subset C(T_\varphi)$ of form

$$Q_t(x, y) = (S_t x, e^t y + \psi_t(x)).$$

In particular,

$$S_t \in \mathcal{L}_\varphi, \quad w_t \in \mathcal{E}_\varphi \quad \forall\ t \in \mathbb{R}$$

where $w_t(y) = e^t y$; and

$$\pi_\varphi(S_t) = w_t \qquad \forall\ t \in \mathbb{R}.$$

Remarks.

1) Theorem 1 can be extended (with analogous proof) to enable "realisation" of a semigroup homomorphism defined on a discrete subgroup of the centraliser which is amenable, and which has F∅lner sets which tile (see [O-W]).

2) In view of the reliance here on Rokhlin lemmas (see the proof of lemmas 4.1 and 4.2), we ask if there is an ergodic probability preserving transformation $(X, \mathcal{B}, m, T)$, and an ergodic $\varphi : X \to \mathbb{T}^2$ such that $SL(2, \mathbb{Z}) \subset \mathcal{E}_\varphi$.

Note that $SL(2, \mathbb{Z}) \subset \mathrm{End}(\mathbb{T}^2)$, and that if T is the 4-shift with symmetric product measure, then $SL(2, \mathbb{Z}) \subset C(T)$.

Squashability and laws of large numbers.

Let $T = (X_T, \mathcal{B}_T, m_T)$ be a conservative, ergodic measure preserving transformation of the σ-finite measure space $(X_T, \mathcal{B}_T, m_T)$. Each commutor of T is a measure multiplying transformation. This is because for Q a commutor of T, the measure $m_T \circ Q^{-1}$ is m_T-absolutely continuous, T-invariant, and hence

$$\frac{dm_T \circ Q^{-1}}{dm_T} \circ T = \frac{dm_T \circ Q^{-1}}{dm_T}$$

which is constant by ergodicity.

The *dilation* of a measure multiplying transformation Q is defined by

$$D(Q) = \frac{dm \circ Q}{dm} \in (0, \infty].$$

Recall from [Aa1] that the transformation T is called *squashable* if it has a commutor with non-unit dilation.

Let $\mathcal{C} \subset \mathcal{B}_T$, $\mathcal{C} =$ either $\mathcal{B}_T$ or $\mathcal{F}_T = \{B \in \mathcal{B}_T : m_T(B) < \infty\}$. A *law of large numbers* for T with respect to $\mathcal{C}$ is a function $L : \{0,1\}^{\mathbb{N}} \to [0, \infty]$ such that

$$L(1_A, 1_A \circ T, \dots) = m_T(A) \text{ a.e.}$$

$\forall\ A \in \mathcal{C}$. If L is a law of large numbers for T with respect to $\mathcal{C}$, and Q is a commutor of T such that $Q^{-1}\mathcal{C} \subseteq \mathcal{C}$, then $D(Q) = 1$, as $\forall\ A \in \mathcal{C}$, $Q^{-1}A \in \mathcal{C}$, and for a.e. $x \in X$,

$$m_T(A) = L(1_A(Qx), 1_A(TQx), \dots) = L(1_{Q^{-1}A}(x), 1_{Q^{-1}A}(Tx), \dots) = m_T(Q^{-1}A).$$

Consequently,
if T has a law of large numbers with respect to $\mathcal{B}_T$, then T is non-squashable,
and *if T has a law of large numbers with respect to $\mathcal{F}_T$, then no commutor of T has non-unit, finite dilation.*

It was shown in [Aa2, corollary 2.3, & theorem 3.4] that *if G is a countable group without arbitrarily large finite normal subgroups*
(e.g. $G = \mathbb{Z}^k \times \mathbb{Q}^\ell$ or $G = \mathbb{Z}^\infty = \{(n_1, n_2, \dots) \in \mathbb{Z}^{\mathbb{N}} : n_k \to 0\}$),
then any ergodic G-extension of a Kronecker transformation has a law of large numbers with respect to $\mathcal{F}$.

Example 1.

Let T be a Kronecker transformation, then $\exists\ S \in C(T)$ so that $\{S,\ T\}$ generate a free $\mathbb{Z}^2$ action. Let $G = \mathbb{Z}^\infty$, and let $w = w_1 \in \text{End}\,(G)$ be the shift $w((n_1, n_2, \dots)) = (n_2, n_3, \dots)$. By theorem 1, $\exists\ \varphi : X \to G$ such that T_φ is ergodic, and $w \in \mathcal{E}_\varphi$.

It follows that:
T_φ has a law of large numbers with respect to $\mathcal{F}_{T_\varphi}$,
but also a commutor $Q(x, y) = (Sx, w(y) + g(x))$ which has infinite dilation since (as shown in the proof of proposition 1.1 of [A-L-M-N]) $D(Q) = D(w) = \infty$, whence *T_φ has no law of large numbers with respect to $\mathcal{B}_{T_\varphi}$.*

Complete squashability and Maharam transformations.

Evidently $D : C(T) \to \mathbb{R}_+$ is a multiplicative homomorphism. Set $\Delta_0(T) = D(C(T))$. The group $\Delta_0(T)$ was first considered in [H-I-K] (see also [Aa2]). If T has a law of large numbers with respect to $\mathcal{F}_T$, then $\Delta_0(T) = \{1\}$. In particular, ([Aa2], or [A-L-M-N]) if T is a $\mathbb{Z}$-extension of a Kronecker transformation, then $\Delta_0(T) = \{1\}$. Our results on $\mathbb{R}$-extensions show that this result fails dramatically for other transformations T, an $\mathbb{R}$-extension of T being a $\mathbb{Z}$-extension of a $\mathbb{T}$-extension of T. Moreover, (see proposition 2.5) for Bernoulli T, any ergodic $\mathbb{R}$-extension of T is isomorphic to a $\mathbb{Z}$-extension of T.

It is standard (see §1 where we recall some well known facts about Polish groups of measure multiplying transformations) that $\Delta_0(T)$ is a Borel subgroup of $\mathbb{R}_+$. The class

$$\{\Delta_0(T) : T \text{ a conservative, ergodic measure preserving transformation}\}$$

includes

$\mathbb{R}_+$, all countable subgroups of $\mathbb{R}_+$, and *subgroups of $\mathbb{R}_+$ with any Hausdorff dimension* (see [Aa2]).

Call a conservative, ergodic measure preserving transformation T with the property that $\Delta_0(T) = \mathbb{R}_+$ *completely squashable* Any ergodic *Maharam transformation* (defined below) is completely squashable.
For a non-singular conservative, ergodic transformation R of $(\Omega, \mathcal{A}, p)$, the transformation $T : X = \Omega \times \mathbb{R} \to X$ defined by

$$T(x, y) = (Rx, y - \log \frac{dp \circ R}{dp})$$

preserves the measure $dm_T(x, y) = dp(x)e^y dy$, and is called the *Maharam transformation* of R, as it was shown to be conservative in [Mah]. If $Q_t(x, y) = (x, y + t)$, then $Q_t \in C(T)$ and $D(Q_t) = e^t$.
Conservative, ergodic Maharam transformations were constructed in [Kr], and smooth Maharam transformations of $\mathbb{T} \times \mathbb{R}$ are constructed in [H-S].

We show in §5 that the transformations T_φ constructed in theorem 2 are isomorphic to Maharam transformations (proposition 5.1), and we obtain $\mathbb{Z}$-extensions of Bernoulli transformations which are Maharam transformations (see the remarks after proposition 5.1).

In §-§6 and 7, we present completely squashable $\mathbb{R}$-extensions T_φ which are not isomorphic to any Maharam transformation, for T an odometer, and T a rotation of $\mathbb{T}$. For T a rotation of $\mathbb{T}$, our examples are as smooth as possible given the Diophantine properties of the rotation number of T (including C^∞, ergodic, completely squashable $\mathbb{R}$-extensions for suitable rotation numbers).

It is not hard to construct real analytic, ergodic, completely squashable $\mathbb{R}$-extensions of a suitable irrational rotation using §5 and [Kw-Le-Ru1] and [Kw-Le-Ru2].

Conditions for ergodicity, and non-squashability of skew products.

Let T be an ergodic probability preserving transformation of the probability space $(X, \mathcal{B}, m)$, assume that G is Abelian, and let $\varphi : X \to G$ be measurable.

Recall from [Sch], that the *essential values* of φ are defined by

$$E(\varphi) = \{a \in G : \forall\ A \in \mathcal{B}_+,\ a \in U \in \mathcal{T},\ \exists\ n \geq 1\ \ni\ m(A \cap T^{-n}A \cap [\varphi_n \in U]) > 0\},$$

which is a closed subgroup of G. It is shown in [Sch] that T_φ is ergodic iff $E(\varphi) = G$.

Set

$$\widetilde{D}(\varphi) = \{a \in G : \exists\ q_n\ \ni\ T^{q_n} \overset{\mathcal{M}(X)}{\longrightarrow} \text{Id},\ \&\ \varphi_{q_n} \to a \text{ a.e.}\}$$

where $\xrightarrow{\mathcal{M}(X)}$ denotes convergence in the topology of measure preserving transformations on X (see §1 below), then (see [A-L-M-N]) $E(\varphi) \supset \mathrm{Gp}\,(\widetilde{D}(\varphi))$ (the group generated by $\widetilde{D}(\varphi)$).

If $\mathrm{Gp}\,(\widetilde{D}(\varphi))$ is dense in G, then T_φ is not only ergodic, but also non-squashable.

If T is an odometer, then (see [A-L-M-N]) for any Abelian, locally compact, second countable G, there is a measurable function $\varphi : X \to G$ such that $\mathrm{Gp}\,(\widetilde{D}(\varphi))$ is dense in G. Because of the density of coboundaries, such functions are residual in the collection of measurable functions $\varphi : X \to G$ considered in the topology of convergence in measure. It is also shown in [A-L-M-N] that there are rotations of $\mathbb{T}$ for which there is an ergodic, real analytic $\varphi : \mathbb{T} \to \mathbb{R}$ with $\mathrm{Gp}\widetilde{D}(\varphi)$ dense in $\mathbb{R}$, whence such functions are residual in any space (containing real analytic functions) where the coboundaries are dense.

It is shown in [L-V], that for a rotation T of $\mathbb{T}$, there exist dense G_δ sets in the spaces of absolutely continuous, Lipschitz, k times continuously or infinitely differentiable functions f with zero mean on $\mathbb{T}$ for which the distributions of f_{n_k} converge to a continuous distribution along a rigid sequence $n_k \to \infty$ whenever theses spaces contain non-trivial cocycles (i.e. not T-cohomologous to a constants).

For irrational rotations with bounded partial quotients, nontrivial cocycles exist in the space of absolutely continuous functions (while every zero mean Lipschitz function is a coboundary). For rotations with unbounded partial quotients, nontrivial cocycles exist in the space $\mathcal{C}^p$ of p times continuously differentiable functions if and only if
$\limsup_{n\to\infty} q_{n+1}/q_n^p = \infty$ where $\{q_n : n \in \mathbb{N}\}$ are the principal denominators of the rotation (cf. [Ba-Me]).
For irrational rotations satisfying this condition $\forall\ p \in \mathbb{N}$, there are non-trivial infinitely differentiable cocycles.

Every such cocycle (the distributions of which converge to a continuous one along a rigid sequence) is ergodic (see [L-V]). But also $\mathcal{E}_f \subset \{\pm 1\}$, since, if $f \circ S = cf + g \circ T - g$, where S is another rotation of $\mathbb{T}$;
on the one hand, the distributions $f_{n_k} \circ S$ converge to the same limit as the distributions of f_{n_k},
while on the other hand, $(g - g \circ T)_{n_k} \to 0$ in measure, hence the limit distribution is invariant under multiplication by c which implies $c = \pm 1$ and T_f non-squashable.

New conditions for ergodicity allowing squashability. The conditions for ergodicity of skew products discussed in [A-L-M-N] and [L-V] are unsuitable for our constructions of squashable skew products as they eliminate squashability.

We need new conditions for the ergodicity of a measurable function $\varphi : X \to G$ which are flexible enough to allow squashability.

Such conditions, called *essential value conditions* are introduced in §3.

Cocycles are constructed as infinite sums of coboundaries. Each coboundary "contributes" a particular essential value condition, which the subsequent coboundaries are "too small" to destroy. The essential value conditions remaining for the infinite sum gives its ergodicity.

The simplest version of our essential value conditions is the *rigid* one (see proposition 6.2) used in the constructions of §6 and §7 which could form an introduction to the proofs of theorems 1 and 2 in §4.

To conclude this introduction, we prove the

Semigroup embedding lemma. *Suppose that G is Abelian, and that $\varphi : X \to G$ is such that T_φ is ergodic. There is a surjective semigroup homomorphism*

$$\pi_\varphi : \mathcal{L}_\varphi \to \mathcal{E}_\varphi$$

such that if $Q(x, y) = (Sx, f(x)w(y))$ defines a commutor of T_φ, then $w = \pi_\varphi(S)$.

Proof. We must show that if $S \in \mathcal{L}_\varphi,\ w_1,\ w_2 \in \mathcal{E}(G),\ f_i : X \to G,\ (i = 1, 2)$ are measurable, and $Q_i(x, y) = (Sx, f_i(x)w_i(y))$ are such that $Q_i \circ T_\varphi = T_\varphi \circ Q_i,\ (i = 1, 2)$, then $w_1 = w_2$.

To this end, let $U = w_1 - w_2$, then $T_{U\circ\varphi}$ is an ergodic transformation of $X \times U(G)$ (being a factor of T_φ via $(x, y) \mapsto (x, U(y))$). The condition $Q_i \circ T_\varphi = T_\varphi \circ Q_i$ means that

$$\varphi \circ S = w_i \circ \varphi + f_i \circ T - f_i \quad (i = 1, 2),$$

whence

$$U \circ \varphi = g \circ T - g$$

where $g = f_1 - f_2$. Define $\tilde{g} : X \to G/U(G)$ by $\tilde{g}(x) = g(x) + U(G)$. It follows that $\tilde{g} \circ T = \tilde{g}$, whence by ergodicity of T, $\exists\ \gamma \in G$ such that $\tilde{g} = \gamma + U(G)$ a.e. Therefore $h := g - \gamma : X \to U(G)$ is measurable and satisfies

$$U \circ \varphi = h \circ T - h.$$

The ergodicity of $T_{U\circ\varphi}$ on $X \times U(G)$ now implies $U(G) = \{0\}$, i.e. $U \equiv 0$, or $w_1 = w_2$.

We've shown that $\forall\ S \in \mathcal{L}_\varphi,\ \exists!\ w =: \pi_\varphi(S) \in \mathcal{E}_\varphi$ such that $\exists\ f_S : X \to G$ measurable so that $Q(x, y) = (Sx, f_S(x)\pi_\varphi(S)(y))$ defines a commutor of T_φ. The rest of the lemma follows easily from this. $\square$

§1 Polish groups of measure multiplying transformations

Given a σ-finite measure space $(Y,\mathcal{C},\nu)$ let $\mathcal{M}(Y,\mathcal{C},\nu)$ denote the group of invertible measure multiplying transformations of $(Y,\mathcal{C},\nu)$, i.e. non-singular transformations $Q : Y \to Y$ such that $\nu\circ Q^{-1} = c\nu$ for some constant $c \in \mathbb{R}_+$. This is a Polish group when equipped with the weak topology inherited from that of the invertible, bounded linear operators on $L^2(Y,\mathcal{C},\nu)$. A metric for this topology is defined by

$$\rho(Q,R) = \sum_{n=1}^{\infty}\Big(\|f_n\circ Q - f_n\circ R\|_2 + \|f_n\circ Q^{-1} - f_n\circ R^{-1}\|_2\Big)$$

where $\{f_n : n \in \mathbb{N}\}$ is a C.O.N.S. in $L^2(m_G)$. The dilation function $D : \mathcal{M} \to \mathbb{R}_+$ defined (as above) by $Q \mapsto \frac{d\nu\circ Q^{-1}}{d\nu} := D(Q)$ is a continuous homomorphism.

Let T be a conservative, ergodic measure preserving transformation of the standard σ-finite measure space $(X,\mathcal{B},m)$, then $C(T)$ is a closed subgroup of $\mathcal{M}(X,\mathcal{B},m)$ and hence Polish. The multiplicative homomorphism $D : C(T) \to \mathbb{R}$ is continuous, and so $\operatorname{Ker} D$ is a closed, normal subgroup of $C(T)$. The natural topology on $C(T)/\operatorname{Ker} D$ is Polish, and $D : C(T)/\operatorname{Ker} D \to \mathbb{R}$ is continuous and injective. By Souslin's theorem (see [Kur, §36, IV]),

$$\Delta_0(T) = D(C(T)/\operatorname{Ker} D)$$

is a Borel set in $\mathbb{R}$.

Let G be a locally compact, second countable topological group, then G is σ-compact, and Polish. Let m_G be left Haar measure on G.

The action of G on $(G,\mathcal{B}(G),m_G)$ by left translation is ergodic. To see this, suppose $A \in \mathcal{B}(G)_+$ and $m_G(gA\Delta A) = 0\ \forall\ g \in G$. The measure m' defined by $dm' = 1_A dm_G$ is a left Haar measure on G, and by unicity of such, $m' = m_G$ whence $A = G \bmod m_G$. The ergodicity of the action of G by right translation is obtained in a similar manner, as right Haar measure is equivalent to m_G.

The maps $L,\ R : G \to \mathcal{M}(G)$ given by $L_g f(x) := f(gx),\ R_g f(x) := f(xg)$ are continuous, and their ranges are closed in $\mathcal{M}(G)$. This follows from the ergodicity of the actions of G by translation; indeed, if $R = \lim_{n\to\infty} R_{g_n}$, set $f(x) = x^{-1}R(x)$, which is L_g-invariant $\forall\ g \in G$, hence constant, and $R = R_h$ for some $h \in G$. Let the ranges of these maps in $\mathcal{M}(G)$ be $\tilde{G}_L$ and $\tilde{G}_R$, considered with their inherited (Polish) topologies. By Souslin's theorem ([Kur, §36, IV]), the inverse maps $L^{-1} : \tilde{G}_L \to G$ and $R^{-1} : \tilde{G}_R \to G$ are both measurable, and (being group isomorphisms) are continuous by Banach's theorem ([Ban, p.20]). In particular, the metric d on G, defined by $d(x,y) = \rho(L_x,L_y)$, generates the topology of G.

In case G is Abelian, $\|x\|_G := d(L_x,\mathrm{Id})$ defines a topology generating norm on G.

Let $\operatorname{Aut}(G)$ denote the group of continuous group automorphisms of G. For example, $\tau_g(x) := g^{-1}xg$ is a continuous group automorphisms of G (called an *inner* automorphism). Also, consider $\operatorname{Aff}(G)$, the group of invertible, *affine* transformations of G of form $L_g\circ w \equiv R_g\circ w'$ where $g \in G$ and $w,w' \in \operatorname{Aut}(G)$ (and $w' = \tau_g\circ w$).

Both $\operatorname{Aff}(G)$ and $\operatorname{Aut}(G)$ are closed subgroups of $\mathcal{M}(G)$. To show that $\operatorname{Aff}(G)$ is closed in $\mathcal{M}(G)$, we note first that if $Q \in \mathcal{M}(G)$, then $Q \in \operatorname{Aff}(G)$ iff $\exists\ w \in \operatorname{Aut}(G)$ such that

$$Q\circ L_g = L_{w(g)}\circ Q\ \ \forall\ g \in G.$$

Indeed, supposing this condition, the function $x \mapsto w(x)^{-1}Q(x)$ is L_g-invariant $\forall\, g \in G$, hence constant, and $Q = R_h \circ w \in$ for some $h \in G$.

Also, the topology on $\mathrm{Aff}(G)$ inherited from $\mathcal{M}(G)$ coincides with the compact-open topology. This follows because $\mathrm{Aff}(G)$, equipped with the compact-open topology, is a Polish space continuously embedded (by the identity) onto $\mathrm{Aff}(G)$ equipped with the topology inherited from $\mathcal{M}(G)$; this identity necessarily being a homeomorphism (as shown above by the theorems of Banach and Souslin).

It is not hard to show that the Polish topology on $\mathrm{Aff}(G)$ also coincides with the topology of pointwise convergence.

It follows from the above that $\mathrm{Aut}(G)$ is closed in $\mathrm{Aff}(G)$. We'll assume throughout that $\mathrm{Inn}(G) := \{\tau_g : g \in G\}$ is also closed in $\mathrm{Aff}(G)$. As pointed out to us by Danilenko and Glasner, this is the case e.g. when G is Abelian, compact or a connected Lie group; but not in general. Let T be an ergodic probability preserving transformation of the standard non-atomic probability space $(X, \mathcal{B}, m)$, and let G a locally compact, second countable, topological group with left Haar measure m_G, and consider the σ-finite measure space $(X \times G, \mathcal{B} \times \mathcal{B}(G), \mu)$ where $\mu = m \times m_G$.

The main point of this section is to establish the continuous embedding lemma (see below) which is a topological version of the semigroup embedding lemma. Let $\mathcal{M} = \mathcal{M}(X \times G, \mathcal{B} \times \mathcal{B}(G), \mu)$, and let $\widetilde{\mathcal{M}}$ denote those $Q \in \mathcal{M}$ of form

$$Q(x, y) = (Sx, h(x)w(y)) \tag{$*$}$$

where $w \in \mathrm{Aut}(G)$ is a continuous group automorphism, $S \in \mathcal{M}(X)$ and $h : X \longrightarrow G$ is measurable.

Write

$$S_f(x, y) = (Sx, f(x)y)$$

for $S \in \mathcal{M}(X)$ and $f : X \to G$ measurable. Also, for $w \in \mathrm{Aut}(G)$, write

$$W_w(x, y) = (x, w(y)).$$

If $Q \in \widetilde{\mathcal{M}}$ is as in $(*)$, then

$$Q = S_f \circ W_w.$$

It is clear that the representation $S_f(x, y) = (Sx, f(x)y)$ is unique in the sense that $S_f = S'_{f'}$ implies $S = S'$ and $f = f'$. The unicity of the representation of $Q \in \widetilde{\mathcal{M}}$ by $(*)$ follows from this, and $\sigma_{w(g)} = Q \circ \sigma_g \circ Q^{-1}$ where $\sigma_g(x, y) = (x, yg)$.

Now let $T : X \to X$ be an ergodic probability preserving transformation, and let $\varphi : X \to G$ be measurable.

We study $\widetilde{C}(T_\varphi) := C(T_\varphi) \cap \widetilde{\mathcal{M}}$ - a closed subgroup of $\mathcal{M}$. If $Q \in \widetilde{\mathcal{M}}$, then $Q \in C(T_\varphi)$ iff $S_Q \in C(T)$, and

$$\varphi \circ S_Q = h_Q \circ T \cdot w_Q \circ \varphi \cdot h_Q^{-1}.$$

Set

$$\mathcal{A}_\varphi(T) = \{w_Q : Q \in \widetilde{C}(T_\varphi)\},\ \&\ L_\varphi(T) = \{S_Q : Q \in \widetilde{C}(T_\varphi)\} = \mathcal{L}_\varphi \cap C(T).$$

Continuous embedding lemma. *Suppose that T is an ergodic probability preserving transformation of the standard probability space $(X, \mathcal{B}, m)$, that G is a locally compact, second countable topological group, and that $\varphi : X \to G$ is measurable such that T_φ is ergodic.*

There is a Polish topology on L_φ, stronger than the topology inherited from $\mathcal{M}(X)$, and a continuous homomorphism

$$\pi_\varphi : L_\varphi \to Aut(G)/Inn(G)$$

such that if $Q \in C(T_\varphi)$ is of form $(*)$, then

$$w_Q Inn(G) = \pi_\varphi(S_Q).$$

In case G is Abelian, then π_φ is the restriction to L_φ of the homomorphism in the semigroup embedding lemma, and there is a Polish topology on $\mathcal{A}_\varphi(T)$ stronger than that inherited from $Aut(G)$ such that

$$\pi_\varphi : L_\varphi(T) \to \mathcal{A}_\varphi(T)$$

is continuous.

The proof of the continuous embedding lemma uses four lemmas, two of which concern the structure of $\widetilde{\mathcal{M}}$.

Let $\overline{G} := \{\sigma_g : g \in G\}$, then $\overline{G} = \{\mathrm{Id}\} \times \tilde{G}_R$ is a closed subgroup of $\mathcal{M}$, and the embedding $g \mapsto \sigma_g$ is a homeomorphism ($G \leftrightarrow \overline{G}$). Also, $\overline{G} \subset \widetilde{\mathcal{M}}$ because $\sigma_g(x, y) = (x, g(g^{-1}yg))$. We'll need

Lemma 1.1. *If Z is a separable metric space, and $f : X \times G \to Z$ is measurable and $f \circ \sigma_g = f$ a.e. $\forall\, g \in G$, then $\exists\, g : X \to Z$ such that $f(x, y) = g(x)$ a.e.*

Proof. Choose $h : Z \to [0, 1]$ injective, and (Borel) measurable. For $A \in \mathcal{B}$, the function $f_A : G \to I\!R$ defined by

$$f_A(y) = \int_A h(f(x, y))dm(x)$$

is R_g-invariant $\forall\, g \in G$, and hence, by ergodicity of $\tilde{G}_R$ on G, $\exists\, c(A) \in I\!R$ such that $f_A = c(A)$ a.e. Since $c : \mathcal{B} \to I\!R$ is a m-absolutely continuous signed measure, $\exists\, k : X \to I\!R$ such that $h(f(x, y)) = k(x)$ a.e., and the required function is $g(x) = h^{-1}(k(x))$. □

Suppose that $Q \in \widetilde{\mathcal{M}}$, then, as mentioned above

$$\forall\, g \in G,\ \exists\, g' \in G\ \ni\ Q \circ \sigma_g = \sigma_{g'} \circ Q.$$

We obtain the converse statement as an immediate consequence of lemma 1.1.

Supposing that $Q \in \mathcal{M}$ and that

$$\forall\, g \in G,\ \exists\, g' \in G\ \ni\ Q \circ \sigma_g \circ Q^{-1} = \sigma_{g'},$$

we obtain a continuous group endomorphism $w : G \to G$ such that

$$Q \circ \sigma_g \circ Q^{-1} = \sigma_{w(g)}.$$

Writing

$$Q(x,y) = (S(x,y), F(x,y)),$$

we have that

$$S(x,yg) = S(x,y), \quad F(x,yg) = F(x,y)w(g).$$

The functions $(x,y) \mapsto S(x,y)$, and $(x,y) \mapsto F(x,y)w(y)^{-1}$ are σ_g-invariant $\forall\, g \in G$, and hence, by lemma 1.1, for a.e. $(x,y) \in X \times G$

$$S(x,y) = S(x) \text{ and } F(x,y) = f(x)w(y).$$

The assumption that $Q \in \mathcal{M}$ now gives that $Q \in \widetilde{\mathcal{M}}$.

We now discuss the topology of $\widetilde{\mathcal{M}}$. It can easily be shown that the topology inherited by

$$\{\mathrm{Id}_f : f : X \to G \text{ measurable}\}$$

from $\mathcal{M}$ is the topology of convergence in measure.

Lemma 1.2. *$\widetilde{\mathcal{M}}$ is closed in $\mathcal{M}$, and the projections*

$$Q \mapsto S_Q,\ Q \mapsto f_Q \text{ and } Q \mapsto w_Q$$

are continuous.

Proof. Firstly, suppose that $Q_n \in \widetilde{\mathcal{M}}$, and $Q_n \to Q$ in $\mathcal{M}$. We have that $\sigma_{w_n(g)} = Q_n \circ \sigma_g \circ Q_n^{-1}$ converges, necessarily to $Q \circ \sigma_g \circ Q^{-1} = \sigma_{w(g)}$ since $\overline{G}$ is closed in $\mathcal{M}$, and by the above application of lemma 1.1, $Q \in \widetilde{\mathcal{M}}$. This also proves that

$$w_{Q_n}^{\pm 1} \to w_Q^{\pm 1} \text{ pointwise,}$$

and hence in $\mathrm{Aut}(G)$.

To see that $Q \mapsto S_Q$ is continuous, let $Q, R \in \widetilde{\mathcal{M}}$. Let $A \in \mathcal{B}(X)$, then $\exists\, C \subset A,\ C \in \mathcal{B}(X)$ such that $m(C) > \frac{m(S_Q^{-1}A \Delta S_R^{-1}A)}{2}$, and either (a) $S_Q^{-1}C \cap S_R^{-1}A = \emptyset$, or (b) $S_Q^{-1}A \cap S_R^{-1}C = \emptyset$.

Now let $F \in \mathcal{B}(G),\ m_G(F) < \infty$.

In case (a), $Q^{-1}(C \times F) \cap R^{-1}(A \times F) = \emptyset$, and

$$\mu(Q^{-1}(A \times F)\Delta R^{-1}(A \times F)) \geq \mu(Q^{-1}(C \times F)) = D(Q)m(C)m_G(F),$$

and similarly,in case (b), $Q^{-1}(A \times F) \cap R^{-1}(C \times F) = \emptyset$, and

$$\mu(Q^{-1}(A \times F)\Delta R^{-1}(A \times F)) \geq \mu(R^{-1}(C \times F)) = D(R)m(C)m_G(F).$$

This shows that

$$m(S_Q^{-1}A \Delta S_R^{-1}A) \leq \frac{2\mu(Q^{-1}(A \times F)\Delta R^{-1}(A \times F))}{\min\{D(Q), D(R)\}m_G(F)}.$$

The continuity of $Q \mapsto D(Q) = D(w_Q)$ now shows the continuity of $Q \mapsto S_Q$.

Finally, the continuity $Q \mapsto f_Q$ follows from

$$(\mathrm{Id})_{f_Q} = (S_Q)_0^{-1} \circ Q \circ W_{w_Q}^{-1}.$$

□

Lemma 1.3.

There is a topology on $L_\varphi(T)$ with respect to which it is a Polish group, continuously embedded in $C(T)$, and

$$L_\varphi(T) \cong \widetilde{C}(T_\varphi)/\overline{G}.$$

Proof. Write $p(Q) = S_Q$, then $p : \widetilde{\mathcal{M}} \to \mathcal{M}(X)$ is continuous, and $p(\widetilde{C}(T_\varphi)) = L_\varphi(T)$. Clearly, $p \circ \sigma_g = p$ for all $g \in G$, whence $p : \widetilde{C}(T_\varphi)/\overline{G} \to L_\varphi(T)$ is well-defined, onto, and continuous when $\widetilde{C}(T_\varphi)/\overline{G}$ is equipped with the quotient (Polish) topology. We claim that $p|_{\widetilde{C}(T_\varphi)/\overline{G}}$ is actually injective.

To see this, suppose that $Q \in \widetilde{C}(T_\varphi)$ and $Q(x, y) = (x, h(x)w(y))$, then

$$h \circ T \cdot w \circ \varphi = \varphi \cdot h.$$

Now set $F(x, y) = y^{-1}h(x)w(y)$, then

$$\begin{aligned} F \circ T_\varphi(x, y) &= y^{-1}\varphi(x)^{-1}[h(Tx)w(\varphi(x))]w(y) \\ &= y^{-1}\varphi(x)^{-1}[\varphi(x)h(x)]w(y) \\ &= F(x, y) \end{aligned}$$

and F is constant by ergodicity of T_φ, whence $Q \in \overline{G}$.

The group isomorphism $p : \widetilde{C}(T_\varphi)/\overline{G} \to L_\varphi(T)$ can be used to transport the Polish structure to $L_\varphi(T)$ which is a Polish group, continuously embedded in $C(T)$. □

Remark. By Souslin's theorem (see [Kur, §36, IV]: $L_\varphi(T)$ is a Borel subset of $C(T)$, and

$$\mathcal{B}(L_\varphi(T)) = \mathcal{B}(C(T)) \cap L_\varphi(T).$$

Let $\widetilde{C}_0(T_\varphi) = \{Q \in \widetilde{C}(T_\varphi) : w_Q = \mathrm{Id}\}$, a normal, closed subgroup of $\widetilde{C}(T_\varphi)$.

Lemma 1.4.

There is a topology on $\mathcal{A}_\varphi(T) := \mathcal{E}_\varphi \cap Aut(G)$ with respect to which it is a Polish group, continuously embedded in $Aut(G)$, and, as Polish groups,

$$\mathcal{A}_\varphi(T) \cong \widetilde{C}(T_\varphi)/\widetilde{C}_0(T_\varphi).$$

Proof. Write $q(Q) = w_Q$ for $Q \in \widetilde{C}(T_\varphi)$. By lemma 1.2, $q : \widetilde{C}(T_\varphi) \to \mathrm{Aut}(G)$ is continuous, and $q(\widetilde{C}(T_\varphi)) = \mathcal{A}_\varphi(T)$. Clearly,

$$\mathrm{Ker}\ q = \widetilde{C}_0(T_\varphi),$$

whence $q : \widetilde{C}(T_\varphi)/\widetilde{C}_0(T_\varphi) \to \mathcal{A}_\varphi(T)$ is well-defined and bijective.

As before, the group isomorphism can be used to transport the quotient Polish topology on $\widetilde{C}(T_\varphi)/\widetilde{C}_0(T_\varphi)$ to $\mathcal{A}_\varphi(T)$ which thus becomes a Polish group, continuously embedded in $\mathrm{Aut}(G)$. □

Proof of the continuous embedding lemma. Let

$$\widetilde{C}_I(T_\varphi) = \{Q \in \widetilde{C}(T_\varphi) : w_Q \in \mathrm{Inn}(G)\}.$$

Note that $\widetilde{C}_I(T_\varphi)$ is a closed normal subgroup of $\mathcal{M}$, and is generated by $\widetilde{C}_0(T_\varphi)$, and $\overline{G}$.

It now follows that
$\widetilde{C}(T_\varphi)/\widetilde{C}_I(T_\varphi)$ is a Polish group, continuously embedded in $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ by

$$Q\widetilde{C}_I(T_\varphi) \mapsto w_Q\mathrm{Inn}(G).$$

A natural map $L_\varphi(T) \to \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is now generated by

$$L_\varphi(T) \cong \widetilde{C}(T_\varphi)/\overline{G} \to \widetilde{C}(T_\varphi)/\widetilde{C}_I(T_\varphi) \to \mathrm{Aut}(G)/\mathrm{Inn}(G).$$

It follows from the above that this map is continuous.

In case G is Abelian, $\mathrm{Inn}(G) = \{\mathrm{Id}\}$, and the above becomes a statement of the continuity of:

$$L_\varphi(T) \cong \widetilde{C}(T_\varphi)/\overline{G} \to \widetilde{C}(T_\varphi)/\widetilde{C}_0(T_\varphi) \cong \mathcal{A}_\varphi(T).$$

□

§2 Eigenvalues of skew products

Let T be an ergodic probability preserving transformation of the probability space $(X, \mathcal{B}, m)$, let G be an locally compact, second countable, topological group, and let $\varphi : X \to G$ be a cocycle with T_φ ergodic on $X \times G$.

Recall that if R is non-singular, conservative, ergodic and $f : X \to \mathbb{C}$ is measurable such that

$$f \circ R = \lambda f,$$

where $\lambda \in \mathbb{C}$, then $|f|$ is constant (w.l.o.g. $= 1$), $|\lambda| = 1$.

We consider the situation where the measurable function $\varphi : X \to G$ is *aperiodic* in the sense that all eigenvalues for the skew product T_φ are eigenvalues for T (that is, if $f : X \times G \to \mathbb{T}$ is measurable and $f \circ T_\varphi = \lambda f$ where $\lambda \in \mathbb{T}$, then $\exists\, g : X \to \mathbb{T}$ measurable such that $f(x, y) = g(x)$ a.e.). The main result of this section is

Proposition 2.1.
If $G = \mathbb{R}$ or $\mathbb{T}$, T_φ is ergodic, and $\mathcal{E}_\varphi \neq \{Id\}$, then φ is aperiodic.

Lemma 2.2.
Suppose that T_φ is ergodic and $f : X \times G \to \mathbb{T}$ is measurable such that $f \circ T_\varphi = \lambda_0 f$ where $\lambda_0 \in \mathbb{T}$, then

$$f(x, y) = f_0(x)\gamma(y) \text{ where } \gamma \in \widehat{G} \text{ and } f_0 : X \to \mathbb{T} \text{ is measurable.}$$

Remark. Note that we do not assume that G is Abelian here.

Proof. For $Q \in C(T_\varphi)$, we have that

$$(f \circ Q) \circ T_\varphi = f \circ T_\varphi \circ Q = \lambda_0 f \circ Q,$$

whence, by ergodicity of T_φ, $\exists\ \lambda(Q) \in \mathbb{T}$ such that $f \circ Q = \lambda(Q) f$ (note that $\lambda(T_\varphi) = \lambda_0$. The mapping $\lambda(Q) : C(T_\varphi) \to \mathbb{T}$ is a continuous homomorphism.

Since $\overline{G} \subset C(T_\varphi)$, we obtain $\gamma \in \widehat{G}$ by taking $\gamma(g) := \lambda(\sigma_g)$. Thus

$$f \circ \sigma_g = \gamma(g) f \quad \forall\ g \in G.$$

Set $F(x, y) = \gamma(y)^{-1} f(x, y)$, then $F \circ \sigma_g = F\ \forall\ g \in G$, whence by lemma 1.1, for a.e. fixed $x \in X$, $F(x, \cdot)$ is constant. This proves the lemma. □

Lemma 2.3.
Suppose that T_φ is ergodic and $f = f_0 \otimes \mu$ where $f_0 : X \to \mathbb{T}$ is measurable, $\gamma \in \widehat{G}$, and $f \circ T_\varphi = \lambda_0 f$ for some $\lambda_0 \in \mathbb{T}$, then

$$\gamma \circ w = \gamma \quad \forall\ w \in \mathcal{E}_\varphi.$$

Proof. Let $\lambda : C(T_\varphi) \to \mathbb{T}$ be such that

$$f \circ Q = \lambda(Q) f \quad \forall\ Q \in C(T_\varphi).$$

Suppose that $w \in \mathcal{E}_\varphi$, and $Q \in \widetilde{C}(T_\varphi)$ with $Q(x,y) = (Sx, h(x)w(y))$, then

$$\begin{aligned}\lambda(Q)f_0 \otimes \gamma(x,y) &= \lambda(Q)f(x,y)\\ &= f \circ Q(x,y)\\ &= f_0(Sx)\gamma(h(x))\gamma(w(y))\\ &= [(f_0 \circ S)\cdot(\gamma \circ h)] \otimes \gamma \circ w(x,y),\end{aligned}$$

and since the character $\gamma \in \widehat{G}$ appearing in the eigenfunction $f_0 \otimes \gamma$ is unique,

$$\gamma \circ w = \gamma.$$

□

Proof of Proposition 2.1. This now follows from lemma 2.3, as if $G = \mathbb{T},\ \mathbb{R}$, and $\gamma \in \widehat{G},\ w \in$End$(G)$, then $\gamma \circ w = \gamma$ iff either $\gamma \equiv 1$ or $w =$Id. □

Corollary 2.4 (cf [Rob]**).**
If T_φ is ergodic, then the only eigenvalues of T_φ are the eigenvalues of $T_{\varphi[G,G]}$: $X \times G/[G,G] \to X \times G/[G,G]$.

Proof. By lemma 2.2, an eigenfunction of T_φ must be of form $f_0 \otimes \mu$ where $\mu \in \widehat{G}$. But

$$\widehat{G} = \widehat{G/[G,G]},$$

so that any eigenfunction of T_φ is actually an eigenfunction of $T_{\varphi[G,G]}$. □

Proposition 2.5.
If T is Bernoulli, then any ergodic $\mathbb{R}$-extension of T is (isomorphic to) a $\mathbb{Z}$-extension of T.

Proof. Let $\varphi : X \to \mathbb{R}$ be an measurable such that T_φ is ergodic. For $c > 0$ let $\varphi^{(c)} : X \to \mathbb{T} \cong [0,c)$ be defined by $\varphi^{(c)} = \varphi \bmod c$. For each $c > 0$ there is a measurable function $\psi^{(c)} : X \times \mathbb{T} \to \mathbb{Z}$ such that

$$T_\varphi \cong (T_{\varphi^{(c)}})_{\psi^{(c)}}.$$

It is known that for some $c > 0$, $\varphi^{(c)}$ is not cohomologous to a constant, else (see [M-S] and [H-O-O]) φ would be cohomologous to a constant and not ergodic. For this $c > 0$, $T_{\varphi^{(c)}}$ is weakly mixing, whence by theorem 1 of [Rud] $T_{\varphi^{(c)}}$ is Bernoulli, and since $h(T_{\varphi^{(c)}}) = h(T)$, we have by [Or] that $T_{\varphi^{(c)}} \cong T$. The conclusion is that $(T_{\varphi^{(c)}})_{\psi^{(c)}}$ is a $\mathbb{Z}$-extension of T. □

§3 ESSENTIAL VALUE CONDITIONS

Let T be an invertible, ergodic probability preserving transformation of the standard probability space $(X, \mathcal{B}, m)$, let G be a locally compact, second countable Abelian group, and let $\varphi : X \to G$ be measurable. We develop here a countable condition for ergodicity of T_φ. The EVC's to be defined are best understood in terms of orbit cocycles, and the *groupoid* of T (see [Fe-Mo]).

A *partial probability preserving transformation* of X is a pair (R, A) where $A \in \mathcal{B}$ and $R : A \to RA$ is invertible and $m|_{RA} \circ R^{-1} = m|_A$. The set A is called the *domain* of (R, A). We'll sometimes abuse this notation by writing $R = (R, A)$ and $A = \mathcal{D}(R)$. Similarly, the *image* of (R, A) is the set $\Im(R) = RA$.

The *equivalence relation* generated by T is

$$\mathcal{R} = \{(x, T^n x) : x \in X,\ n \in \mathbb{Z}\}.$$

For $A \in \mathcal{B}(X)$ and $\phi : A \to \mathbb{Z}$, define $T^\phi : A \to X$ by $T^\phi(x) := T^{\phi(x)}x$.

The *groupoid* of T is

$$[T] = \{T^\phi :\ T^\phi \text{ is a partial probability preserving transformation}\}.$$

It's not hard to see that

$$[T] = \{R : R \text{ a partial probability preserving transformation, \& } (x, Rx) \in \mathcal{R} \text{ a.e.}\}.$$

For $R = T^\phi \in [T]$, write $\phi^{(R)} := \phi$. Let

$$[T]_+ = \{R \in [T] :\ \phi^{(R)} \geq 1 \text{ a.e.}\}.$$

Recall from [Halm1]:

E. Hopf's Equivalence lemma. *If T is an ergodic measure preserving transformation of $(X, \mathcal{B}, m)$ and $A,\ B \in \mathcal{B}$ with $m(A) = m(B)$, then*

$$\exists\ R \in [T]_+ \ \text{ such that } \mathcal{D}(R) = A,\ \Im(R) = B.$$

We'll also need a quantitative version of this lemma when $A = B$.

Lemma 3.1.
Suppose that T is an ergodic probability preserving transformation of $(X, \mathcal{B}, m)$, $A \in \mathcal{B}_+$, and $c,\ \varepsilon > 0$, then $\forall\ p, q \in \mathbb{N}$ large enough, $\exists\ R \in [T]_+$ such that

$$\mathcal{D}(R),\ \Im(R) \subset A,\ m(A \setminus \mathcal{D}(R)) < \varepsilon, \quad \text{and } \phi^{(R)} = cpq(1 \pm \varepsilon).$$

The proof of lemma 3.1 will be given at the end of this section.

Let $\mathcal{R}$ be the equivalence relation generated by T. An *orbit cocycle* is a measurable function $\tilde{\varphi} : \mathcal{R} \to G$ such that if $(x, y),\ (y, z) \in \mathcal{R}$, then

$$\tilde{\varphi}(x, z) = \tilde{\varphi}(x, y) + \tilde{\varphi}(y, z).$$

Let $\varphi : X \to G$ be measurable, and let φ_n $(n \in \mathbb{Z})$ denote the cocycle generated by φ under T. The orbit cocycle $\tilde{\varphi} : \mathcal{R} \to G$ corresponding to φ is defined by

$$\tilde{\varphi}(x, T^n x) = \varphi_n(x).$$

For $R \in [T]$, the function $\varphi_R : \mathcal{D}(R) \to G$ is defined by

$$\varphi_R(x) = \tilde{\varphi}(x, Rx).$$

Clearly $\varphi(R \circ S, x) = \varphi(S, x) + \varphi(R, Sx)$ on $\mathcal{D}(R \circ S) = \mathcal{D}(S) \cap S^{-1}\mathcal{D}(R)$.

Definition. Let α be a measurable partition of X, U a subset of G, and $\varepsilon > 0$. We say that the measurable cocycle $\varphi : X \to \Gamma$ satisfies $\mathrm{EVC}_T(U, \varepsilon, \alpha)$ if
for ε-almost every $a \in \alpha$, $\exists\ R = R_a \in [T]_+$ such that

$$\mathcal{D}(R),\ \Im(R) \subset a,\ \varphi_R \in U \text{ on } \mathcal{D}(R_a),\ m(\mathcal{D}(R))) > (1-\varepsilon)m(a).$$

Definition. We say that the partitions $\{\alpha_k : k \geq 1\}$ *approximately generate* $\mathcal{B}$ if

$$\forall\ B \in \mathcal{B}(X),\ \varepsilon > 0\ \exists\ k_0 \geq 1 \ni\ \forall\ k \geq k_0,\ \exists\ A_k \in \mathcal{A}(\alpha_k)\ \ni\ m(B\Delta A_k) < \varepsilon.$$

Here $\mathcal{A}(\alpha)$ denotes the algebra generated by α. It is not hard to see that the partitions $\{\alpha_k : k \geq 1\}$ approximately generate $\mathcal{B}$, if and only if $E(1_B|\mathcal{A}(\alpha_k)) \to 1_B$ in probability $\forall\ B \in \mathcal{B}$, and in this case,

$$\forall\ \varepsilon > 0,\ B \in \mathcal{B},\ \exists\ k_0 \text{ such that } \sum_{a\in\alpha_k,\ 1-m(B|a)\leq\varepsilon} m(a) \geq (1-\varepsilon)m(B)\ \forall\ k \geq k_0.$$

Proposition 3.1. *Suppose that the partitions $\{\alpha_k : k \geq 1\}$ approximately generate $\mathcal{B}$, and let $\varepsilon_k \downarrow 0$, $\gamma \in \Gamma$, and $U_k \subset G$ satisfy $U_n \downarrow \{\gamma\}$, and diam$\,U_n \downarrow 0$.*
If φ satisfies $EVC_T(U_k, \varepsilon_k, \alpha_k)$ $\forall\ k \geq 1$, then

$$\gamma \in E(\varphi).$$

Proof. Suppose that $B \in \mathcal{B}_+$, and that $V \subset G$ is an open neighbourhood of γ. We'll show that

$$\exists\ n \geq 1\ \ni\ m(B \cap T^{-n}B \cap [\varphi_n \in V]) > 0.$$

Evidently, $V \supset U_k$ for all k sufficiently large. It follows from the definitions, that $\forall\ k$ sufficiently large, $\exists\ a \in \alpha_k$ such that

$$m(a \setminus B) < 0.1m(a),$$

and $\exists\ R = R_a \in [T]_+$ such that

$$\mathcal{D}(R),\ \Im(R) \subset a,\quad \varphi_R \in U_k \text{ on } \mathcal{D}(R), \text{ and } m(a \setminus \mathcal{D}(R)) < 0.1m(a).$$

It follows that

$$m(B \setminus \mathcal{D}(R)) < 0.2m(a).$$

Let $R = T^\phi$, where $\phi : \mathcal{D}(R) \to \mathbb{Z}$. We have that

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} m(B \cap [\phi = n] \cap T^{-n}B \cap [\varphi_n \in U_k])\\ &\geq m(B \cap \mathcal{D}(R) \cap R^{-1}(B \cap \Im(R)) \cap [\varphi_R \in U_k])\\ &\geq 0.6m(a),\end{aligned}$$

whence $\exists\ n \in \mathbb{Z}$ such that

$$m(B \cap T^{-n}B \cap [\varphi_n \in V]) \geq m(B \cap [\phi = n] \cap T^{-n}B \cap [\varphi_n \in U_k]) > 0.$$

□

Corollary 3.2. *Suppose that the partitions $\{\alpha_k : k \ge 1\}$ approximately generate $\mathcal{B}$, let $\{U_k : k \ge 1\}$ be a basis of neighbourhoods for the topology of G, and let $\varepsilon_k \downarrow 0$.*
If φ satisfies $EVC_T(U_k, \varepsilon_k, \alpha_k)$ $\forall$ $k \ge 1$, then T_φ is ergodic.

This sufficient condition for ergodicity is actually necessary.

Proposition 3.3. *If T_φ is ergodic, then $\forall$ $A \in \mathcal{B}_+$ $U \neq \emptyset$ open in G, $\exists$ $R \in [T]_+$ such that*

$$\mathcal{D}(R) = \Im(R) = A, \ \& \ \varphi_R \in U \ \text{a.e. on } A,$$

and hence, φ satisfies $EVC_T(U, \varepsilon, \alpha)$ for any measurable partition α of X, U open in G, $\varepsilon > 0$.

Proof. This follows from the ergodictity of T_φ. Let U be open in G. Choose $g \in U$, then $V := U - g$ is a neighbourhood of $0 \in G$. Choose W open in G such that $W + W \subset V$. By ergodicity of T_φ, for every $A, B \in \mathcal{B}_+$, $\exists$ $n \in \mathbb{N}$ such that $\mu((A \times W) \cap T_\varphi^{-n}(B \times (W + g))) > 0$, whence $m(A \cap T^{-n}B \cap [\varphi_n \in U]) > 0$. The proposition follows from this via a standard exhaustion argument. □

We'll need a finite version of EVC more suited to sequential constructions.

Definition. Let α be a measurable partition of X, U open in G, $\varepsilon > 0$, and $N \ge 1$. We say that the measurable cocycle $\varphi : X \to G$ satisfies $\text{EVC}^T(U, \varepsilon, \alpha, N)$ if:
for ε-almost every $a \in \alpha$, $\exists$ $R = R_a \in [T]_+$ with $\phi^{(R)} \le N$ such that

$$\mathcal{D}(R),\ \Im(R) \subset a,\ \varphi_R \in U \text{ on } \mathcal{D}(R), \text{ and } m(a \setminus \mathcal{D}(R)) < \varepsilon m(a).$$

Proposition 3.4. *Let α be a measurable partition of X, U open in G, $\varepsilon > 0$. The measurable cocycle $\varphi : X \to \Gamma$ satisfies $EVC_T(U, \varepsilon, \alpha)$ iff it satisfies $EVC^T(U, \varepsilon, \alpha, N)$ for some $N \ge 1$.*

The next lemma shows that addition of a sufficiently small cocycle does not affect EVC^T conditions too much.

Lemma 3.5. *Let α be a partition, $\varepsilon, \delta > 0$, $N \in \mathbb{N}$, $V \subset G$, and $\phi : X \to G$ be a cocycle satisfying $EVC^T(U, \varepsilon, \alpha, N)$ where $U \subset G$.*
If $\varphi : X \to \Gamma$ is measurable, and

$$m([\varphi \notin V]) < \frac{\delta^2}{N},$$

then $\phi + \varphi$ satisfies $EVC(U + V, \varepsilon + \delta, \alpha, N)$.

Proof. Let $B = [\varphi \circ T^j \in V \ \forall\ 0 \le j \le N - 1]$, then since

$$\varphi_n \in V \text{ on } B \ \forall\ 1 \le n \le N,$$

$$\varphi_R \in V \text{ on } B \cap \mathcal{D}(R) \ \forall\ R \in [T]_+ \text{ with } \phi^{(R)} \le N.$$

Let α_1 consist of those $a \in \alpha$ such that $\exists$ $R = R_a \in [T]_+$ with $\phi^{(R)} \le N$ such that

$$\mathcal{D}(R),\ \Im(R) \subset a,\ \varphi_R \in U \text{ on } \mathcal{D}(R), \text{ and } m(a \setminus \mathcal{D}(R)) < \varepsilon m(a).$$

We have that

$$m(\bigcup_{a \in \alpha_1} a) > 1 - \varepsilon.$$

Let α_2 consist of those $a \in \alpha$ for which

$$m(B \cap a) > (1-\delta)m(a).$$

It follows from Chebyshev's inequality that

$$m(\bigcup_{a\in\alpha_2} a) > 1 - \frac{m(B)}{\delta} > 1-\delta.$$

Therefore

$$m(\bigcup_{a\in\alpha_1\cap\alpha_2} a) > 1-\varepsilon-\delta.$$

If $a \in \alpha_1 \cap \alpha_2$, and $R' = R'_a := (R_a, \mathcal{D}(R_a)\cap B) \in [T]_+$, then:

$$\mathcal{D}(R'),\ \Im(R') \subset a,\ (\phi+\varphi)_{R'} \in U+V \text{ on } \mathcal{D}(R'),\text{ and } m(a\setminus\mathcal{D}(R')) < (\varepsilon+\delta)m(a).$$

□

Our main result in this section is a sufficient condition for a group element to be an essential value of a sum of coboundaries.

Theorem 3.6.
Suppose that $g \in G$, the partitions $\{\alpha_j\}$ approximately generate $\mathcal{B}$; $N_k \in I\!N,\ N_k \uparrow \infty$, and $\varepsilon_k > 0,\ \sum_{k\geq 1}\varepsilon_k < \infty$.
If for $k \in I\!N,\ f_k : X \to G$ is measurable,

$$\sum_{j=1}^{k}(f_j\circ T - f_j) \text{ satisfies } EVC^T(N(g,\varepsilon_k),\varepsilon_k,\alpha_k,N_k),$$

and

$$m([|f_k\circ T - f_k| \geq \frac{\varepsilon_{k-1}}{N_{k-1}}]) \leq \frac{\varepsilon_{k-1}^2}{N_{k-1}},$$

then

$$\sum_{k=1}^{\infty}|f_k\circ T - f_k| < \infty \text{ a.e., and } g \in E\Bigg(\sum_{k=1}^{\infty}(f_k\circ T - f_k)\Bigg).$$

Proof. By the Borel Cantelli lemma, $\sum_{k=1}^{\infty}|f_k\circ T - f_k| < \infty$ a.e.. Write

$$\phi := \sum_{k=1}^{\infty}(f_k\circ T - f_k),\ \tilde{\phi}_k = \sum_{j=1}^{k}(f_j\circ T - f_j),\ \hat{\phi}_k = \sum_{j=k+1}^{\infty}(f_j\circ T - f_j).$$

Since

$$\phi = \tilde{\phi}_k + \hat{\phi}_k\ \forall\ k \geq 1,$$

$\tilde{\phi}_k$ satisfies $\mathrm{EVC}^T(N(g,\varepsilon_k),\varepsilon_k,\alpha_k,N_k)$, and

$$\begin{aligned} m([|\hat{\phi}_k| \ge \frac{1}{N_k}\sum_{j=k+1}^{\infty}\varepsilon_j]) &\le \sum_{j=k+1}^{\infty} m([f_j\circ T - f_j| \ge \frac{\varepsilon_j}{N_k}]) \\ &\le \sum_{j=k+1}^{\infty} m([f_j\circ T - f_j| \ge \frac{\varepsilon_j}{N_{j-1}} \\ &< \sum_{j=k+1}^{\infty}\frac{\varepsilon_{j-1}^2}{N_{j-1}} \\ &\le \frac{1}{N_k}\sum_{j=k}^{\infty}\varepsilon_k^2, \end{aligned}$$

it follows from lemma 3.5 that ϕ satisfies
$\mathrm{EVC}^T(N(g,\sum_{j=k}^{\infty}\varepsilon_j), 2\sqrt{\sum_{j=k}^{\infty}\varepsilon_k^2}, \alpha_k, N_k)$. □

As promised above, we conclude this section with the

Proof of lemma 3.1. Let

$$A_n = \left[\left|\frac{1}{n}\sum_{k=0}^{n-1} 1_A\circ T^k - m(A)\right| < \varepsilon m(A)\right].$$

By Birkhoff's ergodic theorem, $\exists\ p_0 \in \mathbb{N}$ such that $m(A_p^c) < \frac{\varepsilon^4}{2}\ \forall\ p \ge p_0$. Fix $p \ge p_0$. Now fix $q \ge \frac{p}{c\varepsilon} := q_0$. Set

$$B = A_p \cap T^{-[cq]p}A_p.$$

Evidently $m(B) > 1-\varepsilon^2$.
By Birkhoff's ergodic theorem $\exists\ N_0 \in \mathbb{N}$ such that

$$m(C_n^c) < \frac{\varepsilon^2}{2p}\ \ \forall\ n \ge N_0$$

where

$$C_n = \left[\frac{1}{n}\sum_{k=0}^{n-1} 1_B\circ T^{pk} \ge E(1_B|\mathcal{I}_{T^p}) - \varepsilon^2\right].$$

Let $N > \frac{pq}{\varepsilon} \vee pN_0$. By Rokhlin's theorem, $\exists\ F \in \mathcal{B}$ such that

$$\{T^jF : 0 \le j \le N-1\} \text{ are disjoint, and } m\left(X \setminus \bigcup_{j=0}^{N-1} T^jF\right) < \frac{\varepsilon}{p}.$$

Note that since $E(1_B|\mathcal{I}_{T^p})$ is T^p-invariant, we have

$$\begin{aligned} \frac{N}{p}\sum_{k=0}^{p-1}\int_{T^kF} E(1_{B^c}|\mathcal{I}_{T^p})dm &\le \int_X E(1_{B^c}|\mathcal{I}_{T^p})dm \\ &= m(B^c) < \varepsilon^2, \end{aligned}$$

whence $\exists\ 0 \le k \le p-1$ such that

$$\int_{T^kF} E(1_{B^c}|\mathcal{I}_{T^p})dm < \varepsilon^2 m(F).$$

There is no loss of generality in assuming $k = 0$ as this merely involves taking T^kF as the base for a slightly shorter Rokhlin tower, and adding $\bigcup_{j=0}^{k-1} T^jF$ to the "error set".

Set

$$X_0 = \bigcup_{j=0}^{N-pq} T^jF, \quad \text{and } J = X_0 \cap \bigcup_{j\ge 0,\ jp\le N} T^{jp}F,$$

then $m(J) > \frac{1}{2p}$ so

$$m(C_n^c \cap J) \le \varepsilon^2 m(J) \ \ \forall\ n \ge N_0.$$

For $y \in J$, set $\kappa(y) = \#\{0 \le j \le p-1 : T^jy \in A\}$ and write

$$\{T^jy : 0 \le j \le p-1,\ T^jy \in A\} = \{T^{j_i(y)}y : 1 \le i \le \kappa(y)\}$$

in case $\kappa(y) \ge 1$, where $j_i(y) < j_{i+1}(y)$. Note that

$$\kappa = pm(A)(1 \pm \varepsilon) \text{ on } J \cap A_p.$$

To estimate $m(J \cap B)$:

$$\begin{aligned}\sum_{0\le j\le \frac{N}{p}:\ m(C^c_{\frac{N}{p}}|T^{jp}F)\ge\varepsilon} m(T^{jp}F) &\le \sum_{0\le j\le\frac{N}{p}} \frac{m(C^c_{\frac{N}{p}} \cap T^{jp}F)}{\varepsilon} \\ &= \frac{m(C^c_{\frac{N}{p}} \cap J)}{\varepsilon} \\ &\le \frac{m(C^c_{\frac{N}{p}})}{\varepsilon} \le \frac{\varepsilon}{2p} \le \varepsilon m(J),\end{aligned}$$

whence, $\exists\ i \le \frac{\varepsilon N}{p}$ such that

$$m(C_{\frac{N}{p}} \cap T^{ip}F) = m(T^{-ip}C_{\frac{N}{p}} \cap F) \ge (1-\varepsilon)m(F).$$

For $y \in T^{-ip}C_{\frac{N}{p}} \cap F$,

$$\begin{aligned}\#\{0 \le j \le \frac{N}{p} : T^{jp}y \in B\} &\ge \#\{0 \le j \le \frac{N}{p} : T^{(i+j)p}y \in B\} - \varepsilon\frac{N}{p} \\ &\ge \frac{N}{p}(E(1_B|\mathcal{I}_{T^p}) - 2\varepsilon).\end{aligned}$$

Therefore,

$$\begin{aligned} m(J\cap B) &= \sum_{k=0}^{\frac{N}{p}-1} m(T^{jp}F\cap B)\\ &= \int_F \Bigg(\sum_{k=0}^{\frac{N}{p}-1} 1_B\circ T^{jp}\Bigg)dm\\ &\ge \frac{N}{p}\int_{T^{-ip}C_{\frac{N}{p}}\cap F}(E(1_B|\mathcal{I}_{T^p})-2\varepsilon)dm\\ &\ge \frac{N}{p}\int_F (E(1_B|\mathcal{I}_{T^p})-3\varepsilon)dm\\ &\ge (1-4\varepsilon)m(F)\frac{N}{p} = (1-4\varepsilon)m(J). \end{aligned}$$

For $x\in\bigcup_{j=0}^{p-1}T^jJ$, let $j(x)$ be such that $T^{-j(x)}x\in J$, and let $y(x)=T^{-j(x)}x$. Define $\psi: A\cap\bigcup_{j=0}^{p-1}T^jJ\to\{1,2,\dots p\}$ by

$$\psi(x)=\sum_{k=0}^{j(x)}1_A(T^{-k}x)=\sum_{k=0}^{j(x)}1_A(T^ky(x)).$$

Note that

$$x=T^{j_{\psi(x)}(y(x))}y(x).$$

Now define $D\subset A\cap X_0$ by

$$D\cap\bigcup_{j=0}^{p-1}T^jJ_0=\{x\in A\cap J_0:\psi(x)\le\kappa(y(T^{[cq]p}x))\},$$

and define $\phi: D\to I\!N$ by

$$\phi(x)=[cq]p+j_{\psi(x)}(y(T^{[cq]p}x))\quad x\in D\cap\bigcup_{j=0}^{p-1}T^jJ.$$

We claim that if $R\in[T]_+$ is defined by $\mathcal{D}(R)=D$ and $\phi^{(R)}=\phi$, then ϕ is as advertised. To see this, check that $\kappa\ge(1-\varepsilon)m(A)p$ on $J\cap B$, whence

$$m(D)\ge m(J_0\cap B)(1-\varepsilon)m(A)p\ge(1-6\varepsilon)m(J)pm(A)\ge(1-7\varepsilon)m(A).$$

□

§4 PROOF OF THEOREMS 1 AND 2

In this section, we prove theorems 1 and 2. The proofs are sequential using theorem 3.6. The inductive steps are lemmas 4.1 and 4.2. Their proofs use the Rohlin lemmas for Abelian group actions of Katznelson and Weiss [K-W], and Lind [Lin] respectively (see also [O-W] for a general Rohlin lemma for amenable group actions implying these).

Let G be a locally compact, second countable Abelian group with invariant metric d, and let T be an ergodic probability preserving transformation of the standard probability space $(X,\mathcal{B},m)$.

Definition.

A measurable function $f : X \to G$ is called a *T-coboundary* if $f = h - h \circ T$ for some measurable function $h : X \to G$.

Measurable functions $f, g : X \to G$ are said to be *T-cohomologous*, written $f \overset{T}{\sim} g$, if $f - g$ is a T-coboundary.

Let $\varphi : X \to G$ be measurable. Suppose $S \in \mathcal{L}_\varphi(T)$, and $w \in \text{End}\,(G)$, then

$$w = \pi_\varphi(S) \quad \Leftrightarrow \quad \varphi \circ S \overset{T}{\sim} w \circ \varphi.$$

We prove the following version of theorem 1:

Theorem 1'.
Suppose that T is an ergodic probability preserving transformation, $S_1, \dots, S_d \in C(T)$ $(d \le \infty)$ are such that $(T, S_1, \dots, S_d)$ generate a free $\mathbb{Z}^{d+1}$ action of probability preserving transformations of X.

If $w_1, \dots, w_d \in End\,(G)$ commute (i.e. $w_i \circ w_j = w_j \circ w_i \ \forall\ 1 \le i, j \le d$), then there is a measurable function $\varphi : X \to G$ such that T_φ is ergodic, and

$$\varphi \circ S_i \overset{T}{\sim} w_i \circ \varphi \quad (1 \le i \le d).$$

Lemma 4.1. *Let $\phi : X \to G$ be a T-coboundary, let $S_1, \dots, S_d$ be probability preserving transformations generating a free Z^{d+1} action together with T, and let $w_1, \dots, w_d \in End\,(G)$, $w_i \circ w_j = w_j \circ w_i$. If α is a finite, measurable partition of X, and $\varepsilon > 0$, then there is a measurable function $f : X \to G$ such that*

$$m([f \circ T - f \ne 0]) < \varepsilon, \tag{1}$$

$$m([f \circ S_j \ne w_j \circ f]) < \varepsilon, \ (1 \le j \le d) \tag{2}$$

and such that

$$\phi + f - f \circ T \textit{ satisfies } EVC_T(N(\gamma, \varepsilon), \varepsilon, \alpha). \tag{3}$$

Proof. Write $\phi = H - H \circ T$. Possibly refining α, we may assume that for $\frac{\varepsilon}{2}$-a.e. $a \in \alpha$, the oscillation of H on a is less than $\frac{\varepsilon}{2}$.

For $\underline{i} = (i_1, \dots, i_d) \in \mathbb{Z}_+^d$, we'll write

$$S_{\underline{i}} := S_1^{i_1} \circ \dots \circ S_d^{i_d}, \ \& \ w_{\underline{i}} := w_1^{i_1} \circ \dots \circ w_d^{i_d}.$$

Then

$$S_{\underline{i}+\underline{j}} = S_{\underline{i}} \circ S_{\underline{j}}, \ \& \ w_{\underline{i}+\underline{j}} = w_{\underline{i}} \circ w_{\underline{j}}$$

since $S_i \circ S_j = S_j \circ S_i$ and $w_i \circ w_j = w_j \circ w_i$.

Fix $k > \frac{10}{\varepsilon}$. There is an ergodic cocycle $\varphi : X \to G$ such that

$$m([\varphi \ne 0]) < \frac{\varepsilon}{3k^d}.$$

It follows that $w_{\underline{i}}\circ\varphi\circ S_{-\underline{i}}$ is ergodic for $\underline{i}\geq\underline{0}$ (as $w_{\underline{i}}$ is surjective, and $S_{-\underline{i}}$ commutes with T for $\underline{i}\geq\underline{0}$), whence $\phi+w_{\underline{i}}\circ\varphi\circ S_{-\underline{i}}$ is ergodic for $\underline{i}\geq\underline{0}$ (as ϕ is a coboundary), and so satisfies $\mathrm{EVC}_T(N(\gamma,\frac{\varepsilon}{4}),\frac{\varepsilon}{4k^d},\alpha)$. Therefore (by propositions 3.3 and 3.4),

$$\exists\ M\in I\!N \text{ such that } \phi+w_{\underline{i}}\circ\varphi\circ S_{-\underline{i}} \text{ satisfies } \mathrm{EVC}^T(N(\gamma,\frac{\varepsilon}{4}),\frac{\varepsilon}{4k^d},\alpha,M)$$

for $\underline{0}\leq\underline{i}\leq\underline{k}$ where $\underline{k}:=(\underbrace{k,\ldots,k}_{d\text{-times}})$, and $\underline{0}\leq\underline{i}<\underline{k}$ means $0\leq i_j<k_j\ \forall\ 1\leq j\leq d$.

Now choose $N\geq 1$ such that

$$\frac{M}{N}<\frac{\varepsilon\eta_\alpha}{5}$$

where $\eta_\alpha:=\min\{m(a):a\in\alpha\}$. By the Katznelson-Weiss Rohlin lemma ([K-W]), $\exists\ F\in\mathcal{B}(X)$ such that

$$\{T^jS_{\underline{i}}F:\ 0\leq j\leq N-1,\ \underline{0}\leq\underline{i}<\underline{k}\}\text{ are disjoint,}$$

and

$$m\Big(X\setminus\bigcup_{0\leq j\leq N-1,\ \underline{0}\leq\underline{i}<\underline{k}}T^jS_{\underline{i}}F\Big)<\frac{\varepsilon\eta_\alpha}{6}.$$

Let

$$C=\bigcup_{j=0}^{N-1}T^jF,\quad \widetilde{C}=\bigcup_{j=0}^{N-M}T^jF,\quad \mathcal{T}=\bigcup_{\underline{0}\leq\underline{i}<\underline{k}}S_{\underline{i}}C,\quad \widetilde{\mathcal{T}}=\bigcup_{\underline{0}\leq\underline{i}<\underline{k}}S_{\underline{i}}\widetilde{C}.$$

There is a measurable function $f_0:X\to G$ such that

$$\varphi=f_0-f_0\circ T\text{ on }\mathcal{T}.$$

Set $\varphi'=f_0-f_0\circ T$, then $m([\varphi\neq\varphi'])<\frac{\varepsilon\eta_\alpha}{6}$.

Now define $f:\mathcal{T}\to G$ by

$$f=\begin{cases} w_{\underline{i}}\circ f_0\circ S_{-\underline{i}} & \text{on } S_{\underline{i}}C\quad(\underline{0}\leq\underline{i}\leq\underline{k})\\ 0\quad\text{else,}\end{cases}$$

and define

$$\psi=f-f\circ T.$$

To establish (1),

$$\begin{aligned}m([\psi\neq 0])&<m([\psi\neq 0]\cap\widetilde{\mathcal{T}})+m(X\setminus\widetilde{\mathcal{T}})\\&\leq k^d m([\varphi\neq 0]\cap\widetilde{C})+m(X\setminus\widetilde{\mathcal{T}})\\&<\frac{\varepsilon}{3}+\frac{M}{N}\\&<\varepsilon.\end{aligned}$$

Next, to prove (2), suppose that $\underline{0} \le \underline{i} < \underline{k},\ 1 \le j \le d$ and $i_j < k-1$. If $x \in S_{\underline{i}}C$, then

$$\begin{aligned} f(S_j x) &= w_{\underline{i}+\underline{e}_j} \circ f_0 \circ S_{-(\underline{i}+\underline{e}_j)}(S_j x) \\ &= w_j \circ w_{\underline{i}} \circ f_0 \circ S_{-\underline{i}}(x) \\ &= w_j \circ f(x) \end{aligned}$$

whence

$$\begin{aligned} m([f \circ S_j \ne w_j \circ f]) &< m\Bigg(\bigcup_{\underline{0} \le \underline{i} < \underline{k},\ i_j = k-1} S_{\underline{i}}C\Bigg) + m(X \setminus \mathcal{T}) \\ &< \frac{1}{k} + \frac{\varepsilon \eta_\alpha}{6} < \varepsilon. \end{aligned}$$

To complete the proof, we show (3).
We know that $\phi + w_{\underline{i}} \circ \varphi \circ S_{-\underline{i}}$ satisfies $\mathrm{EVC}^T(N(\gamma, \frac{\varepsilon}{4}), \frac{\varepsilon}{4k^d}, \alpha, M)\ \forall\ \underline{i}$, whence $\phi + w_{\underline{i}} \circ \varphi' \circ S_{-\underline{i}}$ satisfies $\mathrm{EVC}^T(N(\gamma, \frac{\varepsilon}{4}), \frac{\varepsilon}{3k^d}, \alpha, M)\ \forall\ \underline{i}$.
It follows that for $\frac{\varepsilon}{3}$-a.e. $a \in \alpha$, and for each $\underline{0} \le \underline{i} < \underline{k}$,
$\exists\ R_{\underline{i}} = R_{a,\underline{i}} \in [R]_+$ such that $\mathcal{D}(R_{\underline{i}}),\ \Im(R_{\underline{i}}) \subset a$,
$m(a \setminus \mathcal{D}(R_{\underline{i}})) < \frac{\varepsilon}{3k^d} m(a)$, and
$(\phi + w_{\underline{i}} \circ \varphi \circ S_{-\underline{i}})_{R_{\underline{i}}} \in N(\gamma, \frac{\varepsilon}{4})$ on $\mathcal{D}(R_{\underline{i}})$.
Define $R = R_a \in [T]_+$ by

$$\mathcal{D}(R) = \bigcup_{\underline{0} \le \underline{i} < \underline{k}} \mathcal{D}(R_{\underline{i}}) \cap S_{\underline{i}}\widetilde{C},$$

and

$$R = R_{\underline{i}} \text{ on } S_{\underline{i}}\widetilde{C}, \quad (\underline{0} \le \underline{i} < \underline{k}).$$

For $x \in \mathcal{D}(R),\ \exists\ \underline{i} = \underline{i}(x)$ such that $x \in \mathcal{D}(R_{\underline{i}}) \cap S_{\underline{i}}\widetilde{C}$, and we have that

$$\begin{aligned} (\phi + \psi)_R(x) &= (\phi + w_{\underline{i}} \circ \varphi' \circ S_{-\underline{i}})_{R_{\underline{i}}}(x) \\ &\in N(\gamma, \frac{\varepsilon}{4}). \end{aligned}$$

Lastly,

$$\begin{aligned} m(a \setminus \mathcal{D}(R)) &= \sum_{0 \le \underline{i} < \underline{k}} m((a \setminus \mathcal{D}(R)) \cap S_{\underline{i}}\widetilde{C}) + m(\mathcal{T} \setminus \widetilde{\mathcal{T}}) + m(X \setminus \mathcal{T}) \\ &< \sum_{0 \le \underline{i} < \underline{k}} m(a \cap S_{\underline{i}}\widetilde{C} \setminus \mathcal{D}(R_{\underline{i}})) + \frac{M}{N} + m(X \setminus \mathcal{T}) \\ &\le \sum_{0 \le \underline{i} < \underline{k}} m(a \setminus \mathcal{D}(R_{\underline{i}})) + \frac{\varepsilon}{5}\eta_\alpha + \frac{\varepsilon}{6}\eta_\alpha \\ &\le \varepsilon m(a). \end{aligned}$$

□

Proof of theorem 1' in case d finite.

Choose a countable, dense subset Γ of G. Let $(\gamma_1, \gamma_2, \dots) \in \Gamma^{\mathbb{N}}$ satisfy

$$\{\gamma_k : k \geq 1\} = \Gamma, \ \& \ \forall \ \gamma \in \Gamma, \ \gamma_k = \gamma \text{ for infinitely many } k,$$

let the partitions $\{\alpha_j\}$ approximately generate $\mathcal{B}$, and let $\varepsilon_k = 2^{-k^2}$.

Construct (sequentially) using lemma 4.1, a sequence of coboundaries

$$\phi_k = f_k - f_k \circ T$$

such that

$$m([f_k \circ S_j \neq E \circ f_k]) \leq \varepsilon_k \ (1 \leq j \leq d),$$

$\tilde{\phi}_k := \sum_{j=1}^{k} \phi_j$ satisfies $\mathrm{EVC}^T(N(\gamma_k, \varepsilon_k), \varepsilon_k, \alpha_k, N_k)$ where $N_k \in \mathbb{N}, \ N_k \uparrow$, and

$$m([\phi_k \neq 0]) \leq \frac{\varepsilon_k}{N_{k-1}}.$$

Clearly $\phi := \sum_{k=1}^{\infty} \phi_k$ converges a.e.. Also

$$\psi_j := \sum_{k=1}^{\infty} (f_k \circ S_j - w_j \circ f_k) \ (1 \leq j \leq d)$$

converges a.e., whence

$$\phi \circ S_j - w_j \circ \phi = \psi_j - \psi_j \circ T \ (1 \leq j \leq d).$$

Theorem 3.6 now shows that $\Gamma \subset E(\phi)$, and the ergodicity of ϕ is established. □

We prove the following version of theorem 2.

Theorem 2'. *Suppose that T is an ergodic probability preserving transformation, $\{S_t : t \in \mathbb{R}\} \subset C(T)$ are such that T and $\{S_t : t \in \mathbb{R}\}$ generate a free $\mathbb{Z} \times \mathbb{R}$ action of probability preserving transformations of X.*

There is a measurable function $\varphi : X \to \mathbb{R}$ such that
T_φ is ergodic; and
$\exists \ g : \mathbb{R} \times X \to \mathbb{R}$ measurable (with respect to $m_{\mathbb{R}} \times m$) such that

$$\varphi \circ S_t(x) - e^t \varphi(x) = g(t, Tx) - g(t, x), \tag{1}$$

and

$$g(t + u, x) = g(t, S_u x) + e^t g(u, x). \tag{2}$$

If $Q_t(x, y) := (S_t x, e^t y + g(t, x)$, then (1) implies that $Q_t \in C(T_\varphi) \ \forall \ t \in \mathbb{R}$, and by (2), $\{Q_t : t \in \mathbb{R}\}$ is a flow, whence T_φ is a Maharam transformation.

Lemma 4.2. *Let $\phi : X \to \mathbb{R}$ be a T-coboundary, let $\{S_t : t \in \mathbb{R}\}$ be probability preserving transformations generating a free $\mathbb{Z} \times \mathbb{R}$ action together with T.*

If α is a finite, measurable partition of X, $\varepsilon > 0$, and $J \subset \mathbb{R}_+$ is an open interval, then there is a measurable function $f : X \to \mathbb{R}$ such that

$$m([|f \circ T - f| \geq \varepsilon]) < \varepsilon, \tag{1}$$

$$m([f \circ S_t \neq e^t f]) < \varepsilon, \ (0 \leq t \leq 1) \tag{2}$$

and such that

$$\phi + f - f \circ T \text{ satisfies } EVC_T(J, \varepsilon, \alpha). \tag{3}$$

Proof of lemma 4.2. Write $J = ((1-\delta)b, (1+\delta)b)$ where $b,\ \delta > 0$. We'll sometimes use the notation $x = (1 \pm \delta)b$ which means $x \in J$.

Write $\phi = \psi \circ T - \psi$ where $\psi : X \to \mathbb{R}$ is measurable.

Choose a refinement α_1 of α with the property that

$$\forall\ a \in \alpha_1,\ \exists\ y_a \in \mathbb{R}\ \ni\ |\psi - y_a| < \frac{b\delta}{2} \text{ a.e. on } a,$$

and set $\eta_\alpha := \min\{m(a) : a \in \alpha\}$.

Fix $K = \frac{10}{\varepsilon}$, and $0 = t_0 < t_1 < \cdots < t_M = K$ such that $e^{t_{i+1}} < (1 + \frac{\delta}{3})e^{t_i}$.

By lemma 3.1, $\exists\ p, q \in \mathbb{N}$ such that $\frac{be^K}{pq} < \varepsilon$, and

$$\forall\ a \in \alpha_1,\ \ 0 \leq k \leq M-1,\ \exists\ R_{a,k} \in [T]_+$$

such that

$$\mathcal{D}(R_{a,k}),\ \Im(R_{a,k}) \subset a,\ m(a \setminus \mathcal{D}(R_{a,k})) < \frac{\varepsilon}{7M} m(a), \text{ and } \phi^{(R_{a,k})} = e^{-t_k} pq(1 \pm \frac{\delta}{9}).$$

Now choose $N \geq 1$ such that

$$\frac{e^K pq}{N} < \frac{\varepsilon \eta_\alpha}{5}.$$

By the Rokhlin theorem for continuous groups ([Lin], [O-W])

$$\exists\ F \in \mathcal{B}(X) \text{ such that } T^k S_t F \text{ are disjoint for } 0 \leq k \leq N,\ 0 \leq t \leq K,$$

and

$$m\Big(X \setminus \bigcup_{0 \leq k \leq N-1,\ 0 \leq t \leq K} T^k S_t F\Big) < \frac{\varepsilon \eta_\alpha}{6}.$$

Let

$$C = \bigcup_{j=0}^{N-1} T^j F,\ \ \widetilde{C} = \bigcup_{j=0}^{N-2} T^j F,\ \ \mathcal{T} = \bigcup_{0 \leq t \leq K} S_t C,\ \ \widetilde{\mathcal{T}} = \bigcup_{0 \leq t \leq K} S_t \widetilde{C}.$$

There is a measurable function $f : \mathcal{T} \to I\!R$ such that

$$f \circ T - f = \frac{b}{pq} e^t \text{ on } S_t \widetilde{C}.$$

Complete the definition of $f : X \to I\!R$ by setting $f = 0$ on $\mathcal{T}^c$.

It is immediate from this construction that f satisfies (1) and (2). We establish (3) by showing that $f \circ T - f$ satisfies $\text{EVC}_T(J, \varepsilon, \alpha_1)$. Let

$$\widehat{C} = \bigcup_{j=0}^{N-pq} T^j F, \quad \widehat{\mathcal{T}} = \bigcup_{0 \le t \le K} S_t \widehat{C}.$$

Let, for $0 \le k \le M-1$,

$$\widehat{\mathcal{T}}_k = \bigcup_{t_k \le t < t_{k+1}} S_t \widehat{C}.$$

Fix $a \in \alpha_1$, and define $R'_a \in [T]_+$ by

$$R'_a = R_{a,k} \text{ on } \mathcal{D}(R_{a,k}) \cap \widehat{\mathcal{T}}_k.$$

It follows that $\mathcal{D}(R'_a),\ \Im(R'_a) \subset a$,

$$\begin{aligned} m(a \setminus \mathcal{D}(R'_a)) &= \sum_{k=0}^{M-1} m(\widehat{\mathcal{T}}_k \cap [a \setminus \mathcal{D}(R_{a,k})]) \\ &\le \sum_{k=0}^{M-1} m(a \setminus \mathcal{D}(R_{a,k})) \\ &\le \frac{\varepsilon}{7} m(a); \end{aligned}$$

and, on $\mathcal{D}(R'_a) \cap \widehat{\mathcal{T}}_k$:

$$|\psi \circ R'_a - \psi| < \frac{b\delta}{2},$$

whence, on $S_t \widetilde{C},\ t \in [t_k, t_{k+1}]$,

$$\begin{aligned} \varphi_{R'_a} &= \frac{e^t b}{pq} \phi^{(R'_a)} \pm \frac{b\delta}{2} \\ &= e^{t-t_k} b(1 \pm \frac{\delta}{9}) \pm \frac{b\delta}{2} \\ &= b(1 \pm \frac{\delta}{9})(1 \pm \frac{\delta}{3})(1 \pm \frac{\delta}{2}) \in J. \end{aligned}$$

□

Proof of theorem 2'. Fix $(g_1, g_2, \dots) = (1, \sqrt{2}, 1, \sqrt{2}, \dots)$.
Construct using lemma 4.2, a sequence of coboundaries

$$f_k \circ T - f_k$$

such that

$$m([f_k \circ S_t \neq e^t f_k]) \leq \frac{1}{2^k} \quad (0 \leq t \leq 1),$$

$$\phi_k := \sum_{j=1}^{k} (f_j \circ T - f_j \text{ satisfies } \mathrm{EVC}^T((\gamma_k - \frac{1}{2^k}, \gamma_k + \frac{1}{2^k}), \varepsilon_k, \alpha_k, N_k)$$

where $N_k \in \mathbb{N}$, $N_k \uparrow$, and

$$m([|f_k \circ T - f_k| \geq \frac{1}{2^k N_{k-1}}]) \leq \frac{1}{2^k N_{k-1}}.$$

The ergodicity of

$$\sum_{k=1}^{\infty} (f_k \circ T - f_k)$$

follows from

$$1, \ \sqrt{2} \ \in \ E\Big(\sum_{k=1}^{\infty} (f_k \circ T - f_k)\Big)$$

which follows from theorem 3.6 □

§5 Maharam transformations

In this section, we give conditions for a conservative, ergodic, measure preserving transformation to be isomorphic to a Maharam transformation. The first proposition shows that the transformations constructed in theorem 2 are Maharam transformations, and the second (a converse of theorem 2 for Kronecker transformations) will be used to construct completely squashable, ergodic $\mathbb{R}$-extensions which are not isomorphic to any Maharam transformation.

Proposition 5.1. *A conservative, ergodic, measure preserving transformation T of the standard, non atomic, σ-finite measure space $(X, \mathcal{B}, m)$ is isomorphic to a Maharam transformation if, and only if there is a flow $\{Q_t : t \in \mathbb{R}\} \subset C(T)$ such that $D(Q_t) = e^t \ \forall \ t \in \mathbb{R}$.*

Proof.
Suppose first that T is a Maharam transformation, i.e. $T : X = \Omega \times \mathbb{R} \to X$ is defined by

$$T(x, y) = (Rx, y - \log \frac{dp \circ R}{dp})$$

and preserves the measure $dm(x, y) := dp(x)e^y dy$, where R is a non-singular conservative, ergodic transformation of the standard probability space $(\Omega, \mathcal{A}, p)$. Set $Q_t(x, y) = (x, y + t)$, then $\{Q_t : t \in \mathbb{R}\} \subset C(T)$ is a flow, and $D(Q_t) = e^t$.

Conversely, suppose that there is a flow $\{Q_t : t \in \mathbb{R}\} \subset C(T)$ such that $D(Q_t) = e^t \ \forall \ t \in \mathbb{R}$. The flow $\{Q_t : t \in \mathbb{R}\}$ is dissipative on X. It is well known that up to measure theoretic isomorphism, $X = \Omega \times \mathbb{R}$ where Ω is some probability space, $Q_t(x, y) = (x, y + t)$, and $dm(x, y) = e^y dp(x)dy$ where p is the probability on Ω.

Since $\{Q_t : t \in \mathbb{R}\} \subset C(T)$, $\exists$ a non-singular transformation $R : \Omega \to \Omega$ such that

$$T(x, y) = (Rx, Y(x, y)).$$

A calculation shows that indeed $Y(x,y) = y - \log R'(x)$ where $R' = \frac{d\lambda\circ R}{d\lambda}$, i.e. T is the Maharam transformation of R. The ergodicity of T implies that Ω is non-atomic, and hence standard. $\square$

Remarks.
1) By proposition 5.1, the skew products constructed in theorem 2 are isomorphic to Maharam transformations.
2) Let T be a Bernoulli transformation. We claim that there is a $\mathbb{Z}$-extension of T which is isomorphic to a Maharam transformation.

Indeed, by theorem 2 and the above remark, there is such an $\mathbb{R}$-extension of T. By proposition 2.5, this $\mathbb{R}$-extension of T is isomorphic to a $\mathbb{Z}$-extension of T.

Proposition 5.2.
Let T be a Kronecker transformation of the compact, metric, Abelian group X.

If there is an ergodic $\mathbb{R}$-extension of T which is isomorphic to some Maharam transformation, then there is a continuous, injective group homomorphism $\mathbb{R} \to X$.

Proof. Let T_φ be an ergodic $\mathbb{R}$-extension of T which is isomorphic to some Maharam transformation. By proposition 5.1, there is a flow $\{Q_t : t \in \mathbb{R}\} \subset C(T_\varphi)$ such that $D(Q_t) = e^t \ \forall\ t \in \mathbb{R}$. It follows from [A-L-M-N] that Q_t has form $(*)\ \forall\ t \in \mathbb{R}$, i.e.

$$Q_t(x,y) = (S_t x, e^t y + g_t(x)).$$

Clearly, the map $t \mapsto S_t$ is a measurable homomorphism from $\mathbb{R} \to C(T) \cong X$, whence by Banach's theorem, continuous. To see that $t \mapsto S_t$ is injective, suppose otherwise, that $S_a =$Id for some $a \neq 0$. Then $Q_a(x,y) = (x, e^a y + g_a(x))$, whence

$$\varphi = G - G\circ T \text{ where } G = \frac{g_a}{e^a - 1}$$

contradicting ergodicity of T_φ. $\square$

§6 COMPLETELY SQUASHABLE $\mathbb{R}$-EXTENSIONS OF ODOMETERS

For $a_n \in \mathbb{N},\ (n \in \mathbb{N})$, set

$$X := \prod_{n=1}^{\infty}\{0,\dots,a_n - 1\}$$

equipped with the addition

$$(x + x')_n = x_n + x'_n + \varepsilon_n \qquad \text{mod } a_n$$

where

$$\varepsilon_1 = 0,\ \&\ \varepsilon_{n+1} = \begin{cases} 0 & x_n + x'_n + \varepsilon_n < a_n \\ 1 & x_n + x'_n + \varepsilon_n \geq a_n. \end{cases}$$

Clearly, X equipped with the product discrete topology, is a compact Abelian topological group, with Haar measure

$$m = \prod_{n=1}^{\infty}(\frac{1}{a_n},\dots,\frac{1}{a_n}).$$

Also if $\tau = (1,0,\dots)$ then $X = \overline{\{n\tau\}}_{n\in\mathbb{Z}}$ whence $x \mapsto Tx(:= x+\tau)$ is ergodic.

Set $q_1 = 1,\ q_{n+1} = \prod_{k=1}^n a_k$, then

$$(q_n\tau)_k = \begin{cases} 1 & k = n \\ 0 & k \neq n, \end{cases}$$

whence

$$T^{q_n}x = (x_1, \dots, x_{n-1}, \tilde{T}_n(x_n, \dots))$$

where $\tilde{T}_n : \prod_{k=n}^\infty \{0,\dots,a_k-1\} \to \prod_{k=n}^\infty \{0,\dots,a_k-1\}$ is defined by $\tilde{T}_n(x) = x + \tilde{\tau}_n$ where $\tilde{\tau}_n = (1,0,\dots)$.

The transformation $T \cong (X,T)$ is called the *odometer with digits* $\{a_n : n \in \mathbb{N}\}$. Let G be a second countable LCA group, and let (X,T) be an odometer.

We consider cocycles $\varphi : X \to G$ of form

$$\varphi(x) := \sum_{n=1}^\infty [\beta_n((Tx)_k) - \beta_n(x_k)],$$

the sum being a finite sum. Cocycles of this form are called *of product type*. We'll call the functions $\{\beta_k : k \in \mathbb{N}\}$ the *partial transfer functions* of φ. Clearly if the sum of the partial transfer functions converges, then indeed the limit is a transfer function for the cocycle, which is a coboundary.

We prove

Theorem 6.1. *There is an odometer (X,T), and an ergodic cocycle $\varphi : X \to \mathbb{R}$ of product type such that T_φ is completely squashable, indeed*
$\forall\ c > 0,\ \exists$ a measurable function $\psi_c : X \to G$, and a translation $S_c : X \to X$ satisfying

$$\varphi \circ S_c = c\varphi + \psi_c \circ T - \psi_c.$$

We claim that T_φ is not isomorphic to a Maharam transformation.

Otherwise, by proposition 5.2, there would be a continuous, injective group homomorphism $\mathbb{R} \to X$, whose existence is prevented by the disconnectedness of X.

We prove ergodicity of φ using rigid essential value conditions.

Proposition 6.2. *Let $\varphi : X \to G$ be a cocycle, and let $\gamma \in G$. If $\forall\ \varepsilon > 0,\ \ \exists\ \delta_k \to 0$, and a sequence of partitions α_k which approximately generate $\mathcal{B}$, such that*

$$m\Big(\bigcup_{a\in\alpha_k} a\Big) > 1 - \delta_k,$$

and for every $k \geq 1$, for δ_k-a.e. $a \in \alpha_k$, $\exists\ n = n(a)$ such that

$$m(a\Delta T^{-n}a) < \delta_k m(a),\ \ \textit{and}\ m(a \cap [\varphi_n \in N(\gamma,\varepsilon)]) > \frac{m(a)}{25};$$

then

$$\gamma \in E(\varphi).$$

Proof. This is a special case of lemma 3.1. □

The functions β_k are defined by means of blocks. To $\underline{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{R}^m$, associate a *canonical difference block* $B_{\underline{\gamma}} = (b(0), b(1), \ldots, b(2^m - 1)) \in \mathbb{R}^{2^m}$ defined by

$$b\Big(\sum_{k=1}^m \varepsilon_k 2^{k-1}\Big) = \sum_{k=1}^m \varepsilon_k \gamma_k \quad (\varepsilon_1, \ldots \varepsilon_m) \in \{0,1\}^m.$$

It is evident that for $0 \le \nu \le 2^m - 1,\ 1 \le j \le m$ with $\varepsilon_j(\nu) = 0$, we have $\nu + 2^{j-1} \le 2^m - 1$, and $b(\nu + 2^{j-1}) - b(\nu) = \gamma_j$. It follows that $\forall\ 1 \le j \le m,\ \exists\ 1 \le n = n(j) \le 2^m$ such that

$$\#\{1 \le \nu \le 2^m - 1 : \nu + n \le 2^m - 1,\ b(\nu + n) - b(\nu) = \gamma_j\} \ge \frac{2^m}{2}. \tag{1}$$

We'll need some control over the size of $|b(j)|$, $(0 \le j \le 2^m - 1)$ and to obtain this, we need the
balanced canonical difference block associated to $\underline{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{R}^m$, defined by

$$B = (b(0), b(1), , \ldots, b(4^m - 1)) \in \mathbb{R}^{4^m}$$

where

$$b\Big(\sum_{k=1}^m \varepsilon_k 2^{k-1} + \sum_{\ell=1}^m \delta_\ell 2^{m+\ell-1}\Big) = \sum_{k=1}^m (\varepsilon_k - \delta_k)\gamma_k, \quad (\underline{\varepsilon}, \underline{\delta} \in \{0,1\}^m).$$

Let B be the balanced canonical difference block associated to $(\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m$. Since B is also the canonical difference block associated to
$(\gamma_1, \ldots, \gamma_m, -\gamma_1, \ldots, -\gamma_m) \in \mathbb{R}^{2m}$, we have by (1) that

$$\#\{1 \le \nu \le 4^m - 1 : \nu + n \le 4^m - 1,\ b(\nu + n) - b(\nu) = \gamma_j\} \ge \frac{4^m}{2}. \tag{2}$$

Also, we claim that

$$\#\{0 \le \nu \le 4^m - 1 : b(\nu)| \ge m^{\frac{3}{4}}\} \le \max_{1 \le j \le m} |\gamma_j|^2 \frac{4^m}{\sqrt{m}}. \tag{3}$$

To see this

$$\begin{aligned} |\{0 \le \nu \le 4^m - 1 : |b(\nu)| \ge m^{\frac{3}{4}}\}| &\le \frac{1}{m^{\frac{3}{2}}} \sum_{\underline{\varepsilon}, \underline{\delta} \in \{0,1\}^m} \Big(\sum_{k=1}^m (\varepsilon_k - \delta_k)\gamma_k\Big)^2 \\ &= \frac{4^m}{m^{\frac{3}{2}}} \sum_{k=1}^m \frac{\gamma_k^2}{2} \\ &\le \frac{4^m}{\sqrt{m}} \max_{1 \le j \le m} |\gamma_j|^2. \end{aligned}$$

We now construct the odometer and cocycle. We construct our cocycle φ to have $1, \frac{1}{\sqrt{2}} \in E(\varphi)$ thus ensuring ergodicity. Let $g_{2n} = 1$, and $g_{2n+1} = \frac{1}{\sqrt{2}}$.

For $k \geq 1$, choose natural numbers ν_k and μ_k satisfying

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k} < \infty, \tag{3}$$

$$\sum_{k=1}^{\infty} \frac{e^{2\mu_k}}{\sqrt{\mu_k \nu_k}} < \infty, \tag{4}$$

and

$$\sum_{k=1}^{\infty} \frac{\mu_k}{\nu_k^{\frac{1}{4}}} < \infty. \tag{5}$$

For example:

$$\mu_k = k^2, \quad \text{and } \nu_k = k^2 3^{4k^2}.$$

Set $m_k = \mu_k \nu_k$, and let

$$B_k = (b_k(0), b_k(1), \ldots, b_k(4^{m_k} - 1))$$

be the balanced canonical difference block associated to $(\gamma_k(1), \ldots, \gamma_k(m_k))$ where

$$\gamma_k(j) = g_k e^{-\frac{j-1}{\nu_k}}.$$

Now let

$$a_k = m_k 4^{m_k}$$

and let (X, T) be the odometer with digits $\{a_n : n \in \mathbb{N}\}$.

We specify a cocycle $\varphi : X \to \mathbb{R}$ of product type, defining it's partial transfer functions $\beta_k : \{0, \ldots, a_k - 1\} \to \mathbb{R}$ by

$$\beta_k(j4^{m_k} + \nu) = e^{\frac{j}{\nu_k}} b_k(\nu) \quad (0 \leq j \leq m_k - 1,\ 0 \leq \nu \leq 4^{m_k} - 1).$$

Note that for $0 \leq j \leq m_k - 1$,

$$\begin{aligned} &|\{j4^{m_k} \leq \nu < (j+1)4^{m_k} : \beta_k(\nu + n(j,k)) - \beta_k(\nu) = g_k\}| \\ &\geq |\{0 \leq \nu < 4^{m_k} : b_k(\nu + n(j,k)) - \beta_k(\nu) = \gamma_k(j)\}| \\ &\geq \frac{4^{m_k}}{2}. \end{aligned} \tag{6}$$

To check ergodicity of φ, we show, using the essential value condition that $1, \frac{1}{\sqrt{2}} \in E(\varphi)$. For $k \geq 1$, we let

$$\alpha_k = \{A((u_1, \dots, u_{k-1}), j) : 0 \leq u_\nu < a_\nu,\ 1 \leq \nu \leq k-1,\ 0 \leq j \leq m_k - 2\}$$

where

$$A((u_1, \dots, u_{k-1}), j) = \{x \in X : x_\nu = u_\nu,\ 1 \leq \nu \leq k-1,\ \&\ jN_k \leq u_k < (j+1)N_k\}.$$

It follows that

$$m\Big(\bigcup_{a\in\alpha_k} a\Big) = 1 - \frac{1}{m_k}.$$

Also, if $n = n(j,k)q_k$ where $n(j,k)$ is as in (6), then

$$m(A(\underline{u}, j)\Delta T^{-n} A(\underline{u}, j)) < \frac{1}{m_k} m(A(\underline{u}, j)),$$

and, by (6)

$$m(A(\underline{u}, j) \cap [\varphi_{n(j,k)} = \gamma_k]) \geq \frac{m(A(\underline{u}, j))}{2}.$$

The essential value condition now shows that

$$1, \frac{1}{\sqrt{2}} \in E(\varphi).$$

To conclude, we show that $\forall\ c \in (1, e),\ \exists S : X \to X$ such that $\varphi \circ S - c\varphi$ is a coboundary. Fix $c \in (1, e)$ and let

$$r_k = [\nu_k \log c] \leq \nu_k.$$

Let

$$S = (r_1 4^{m_1}, r_2 4^{m_2}, \dots).$$

We claim that $\forall\ k \geq 1$,

$$m(\{x \in X : |\beta_k(x_k)| \geq m_k^{\frac{3}{4}}\}) \leq \frac{e^{2\mu_k}}{\sqrt{m_k}}.$$

To see this

$$\begin{aligned}
&m(\{x \in X : |\beta_k(x_k)| \geq m_k^{\frac{3}{4}}\}) \\
&= \frac{1}{a_k} \#\{0 \leq \nu \leq a_k - 1 : |\beta_k(\nu)| \geq m_k^{\frac{3}{4}}\} \\
&= \frac{1}{a_k} \sum_{j=0}^{m_k-1} \#\{0 \leq \nu \leq 4^{m_k} - 1 : |\beta_k(j4^{m_k} + \nu)| \geq m_k^{\frac{3}{4}}\} \\
&= \frac{1}{a_k} \sum_{j=0}^{m_k-1} \#\{0 \leq \nu \leq 4^{m_k} - 1 : e^{\frac{j}{\nu_k}} |b_k(\nu)| \geq m_k^{\frac{3}{4}}\} \\
&\leq \frac{1}{4^{m_k}} \#\{0 \leq \nu \leq 4^{m_k} - 1 : e^{\mu_k} |b_k(\nu)| \geq m_k^{\frac{3}{4}}\} \\
&\leq \frac{e^{2\mu_k}}{\sqrt{m_k}}.
\end{aligned}$$

It now follows from (4), and the Borel-Cantelli lemma that for a.e. $x \in X, \exists\, k_x \geq 1$ such that $\beta_k(x_k) \leq m_k^{\frac{3}{4}}\ \forall\, k \geq k_x$.

It follows from (3) that

$$\sum_{k=1}^{\infty} m(\{x \in X : x_k \geq a_k - 2r_k 4^{m_k}\}) < \infty,$$

whence by the Borel-Cantelli lemma, for a.e. $x \in X, \exists\, K_x \geq 1$ such that $(Sx)_k = x_k + r_k N_k\ \forall\, k \geq K_x$.

It follows that a.e. $x \in X$, for $k \geq k_x, K_x$,

$$\begin{aligned}|\beta_k((Sx)_k) - c\beta_k(x_k)| &= (c - e^{\frac{r_k}{\nu_k}})|\beta_k(x_k)| \\ &\leq \frac{cm_k^{\frac{3}{4}}}{\nu_k} = \frac{c\mu_k^{\frac{3}{4}}}{\nu_k^{\frac{1}{4}}}\end{aligned}$$

whence by (5),

$$\sum_{k=1}^{\infty} |\beta_k((Sx)_k) - c\beta_k(x_k)| < \infty \text{ for a.e. } x \in X$$

and $\varphi \circ S - c\varphi$ is a coboundary, being a product type cocycle, the sum of whose partial transfer functions converges.

§7 Smooth completely squashable $I\!R$-extensions

In this section, we construct smooth completely squashable $I\!R$-extensions of rotations of the circle. Ergodicity is established again using proposition 6.2.

Theorem 7.1. *Let $T:\ x \mapsto x+\alpha$ mod 1 be an irrational circle rotation (the circle is represented as the unit interval $[0,1)$). Let α have unbounded partial quotients. Then there exists a real smooth ergodic cocycle F such that for every $c \neq 0$ there exists a rotation S for which*

$$f \circ S - c \cdot F \tag{1}$$

is a (T-)coboundary:

1. If $1 \leq p < \infty$ is an integer and $\limsup_{n\to\infty} q_{n+1}/q_n^p = \infty$, the cocycle F can be found in $\mathcal{C}^p$. (For $p = 1$ the condition means unbounded partial quotients.)

2. If $\limsup_{n\to\infty} q_{n+1}/q_n^p = \infty$ for all positive integers p, F can be found in $\mathcal{C}^\infty$.

It is known (cf. [Ba-Me]) that if $\limsup_{n\to\infty} q_{n+1}/q_n^p < \infty$, every $F \in \mathcal{C}_0^p$ is a coboundary.

Proof. We shall prove Statement 1; then, the other one can be derived rather easily.

Let a_n be the partial quotients, q_n the convergents. We define
$c_k = k^3 e^k$,
$r_k = k^4 e^k$, $k = 1, 2, \ldots$.
Since $\limsup_{n\to\infty} q_{n+1}/q_n^p = \infty$, there exist sequences of positive integers
$\ell_k = 2[(a_{n_k} - 1)/2r_k]$ (where $[x]$ denotes the integer part of x),

$\ell'_k = \ell_k/2$,
$d_k = e^k k^6/q_{n_k}$,
$\bar{d}_k = d_k(r_k q_{n_k-1})^p = e^{k(p+1)}k^{4p+6}q^p_{n_k-1}/q_{n_k}$,
where

$$\sum_{k=1}^{\infty} \bar{d}_k e^k < \infty. \tag{2}$$

Let us suppose that $n = n_k$ is odd (the case with n_k even is similar).
From the continued fraction expansion we get two Rohlin towers:
$[\{j\alpha\}, \{(q_{n-1}+j)\alpha\})$, $j = 0, \ldots, q_n - 1$ and
$[\{q_n\alpha\}, 1), [\{(j+q_n)\alpha\}, |[j\alpha\})$, $j = 1, \ldots, q_{n-1} - 1$
(for $0 \le x < 1 - \|q_{n-1}\alpha\|$ we thus have $T^{q_{n-1}}x = x + \|q_{n-1}\alpha\|$). Let us denote
$I_0 = [0, \|q_{n-1}\alpha\|)$ $I_i = T^i I_0$, $i = 1, \ldots, q_n - 1$.
For $j = 0, \ldots, q_{n-1} - 1$, the intervals
$I_{j+q_{n-1}}, I_{j+2q_{n-1}}, \ldots I_{j+a_n q_{n-1}}$
are adjacent.
For $j = 0, \ldots, r_k - 1$ and $u = 0, \ldots, q_{n_k-1} - 1$ we define

$$J'_{0,0} = \bigcup_{i=0}^{\ell'_k - 1} I_{i\cdot q_{n_k-1}}, \qquad J'_{0,j} = T^{j\cdot\ell_k\cdot q_{n_k-1}} J'_{0,0},$$
$$J''_{0,j} = T^{\ell'_k\cdot q_{n_k-1}} J'_{0,j},$$
$$J'_{u,j} = T^u J'_{0,j},$$
$$J''_{u,j} = T^u J''_{0,j},$$
$$J_{u,j} = J'_{u,j} \cup J''_{u,j},$$
$$J_u = \bigcup_{j=0}^{r_k-1} J_{u,j}.$$

Notice that for every u the sets $J_{u,j}$ are adjacent intervals composing the interval $J_u = [\{u\alpha\}, r_k \cdot \ell_k \cdot \|q_{n_k-1}\alpha\|)$; each $J_{u,j}$ is cut in the middle into $J'_{u,j}$ and $J''_{u,j}$.
Let $\bar{F}_k$ be a $\mathcal{C}^\infty$ function on $[0,1)$ which is
- zero out of $J'_{0,0}$,
- d_k on the middle half of $J'_{0,0}$
(i.e. on the interval $[\ell'_k\|q_{n_k-1}\alpha\|/4, 3\ell'_k\|q_{n_k-1}\alpha\|/4]$),
- has values between 0 and d_k on the rest of $J'_{0,0}$,
$F^{(i)}(0) = 0 = F^{(i)}(\ell'_k\|q_{n_k-1}\alpha\|)$ for $i = 1, \ldots, p$.
Moreover, the functions $\bar{F}_k$ can be found such that there exists a positive constant C,

$$\|\bar{F}_k\|_{\mathcal{C}^p} < C\bar{d}_k$$

for all k. Let us show this for $p = 1$:
Let $\bar{f}$ be a function which is zero on
$[0,1) \setminus ((0, \ell'_k\|q_{n_k-1}\alpha\|/4) \cup (3\ell'_k\|q_{n_k-1}\alpha\|/4, \ell'_k\|q_{n_k-1}\alpha\|))$,
on the interval
$(0, \ell'_k\|q_{n_k-1}\alpha\|/4)$
it is a tent-like function, and on the interval

$(3\ell'_k\|q_{n_k-1}\alpha\|/4, \ell'_k\|q_{n_k-1}\alpha\|)$
it is a reversed tent-like function, both of height $(8d_k/(\ell'_k\|q_{n_k-1}\alpha\|))$. The indefinite integral $\bar{F}_k(t) = \int_0^t \bar{f}(x)\,dx$ will thus be zero on $[0,1)\setminus J'_{0,0}$, d_k on the middle half of $J'_{0,0}$ and monotone on the remaining part of $J'_{0,0}$; there exists a constant C such that $\|\bar{F}_k\|_{\mathcal{C}^1}/\bar{d}_k < C$ for every k.
The case of larger exponents p is left to the reader (it can be done in a recursive way; the constant C depends on p).

We define $\tilde{F}_k = \bar{F}_k - \bar{F}_k \circ T^{-\ell'_k\cdot q_{n_k-1}}$ (i.e. $\tilde{F}_k = \bar{F}_k$ on $J'_{0,0}$; on $J''_{0,0}$, $\tilde{F}_k$ is got by shifting $\bar{F}_k$ by $\ell'_k\|q_{n-1}\alpha\|$ and changing the sign);

$$F_k = (-1)^j(1+\frac{1}{c_k})^j\tilde{F}_k\circ T^{-(u+j\cdot\ell_k\cdot q_{n_k-1})} \quad \text{on } J_{u,j}, \quad j=0,\dots,r_k-1, \quad u=0,\dots,q_{n_k-1}-1$$

$$F_k = 0 \qquad \text{otherwise.}$$

We have

$$|F_k| \le d_k\cdot(1+\frac{1}{c_k})^{r_k} \le d_k\cdot e^k,$$
$$\|F_k\|_{\mathcal{C}^p} \le C\cdot\bar{d}_k\cdot e^k.$$

By (2) we have $\|F_k\|_{\mathcal{C}^p} < \infty$, hence for each subset K of $\mathbb{N}$ there exists a $\mathcal{C}^p_0$ function

$$F_{(K)} = \sum_{k\in K} F_k.$$

Let $0 \le j \le r_k - 1$ and $x \in I_{j\cdot\ell_k\cdot q_{n_k-1}}$.
For $i = u + p\cdot q_{n_k-1}$ where $0 \le u \le q_{n_k-1}-1$, $0 \le p \le \ell'_k - 1$ we have $T^ix \in J'_{u,j}$, and for $i = u + p\cdot q_{n_k-1} + \ell'_k\cdot q_{n_k-1}$ $(0 \le u \le q_{n_k-1}-1,\ 0 \le p \le \ell'_k - 1)$ we have $T^ix \in J''_{u,j}$.
¿From the definition of F_k we thus get

$$\sum_{i=0}^{\ell'_k q_{n_k-1}-1} F_k(T^ix) = -\sum_{i=\ell'_k q_{n_k-1}}^{\ell_k q_{n_k-1}-1} F_k(T^ix) \qquad (x\in I_{j\cdot\ell_k\cdot q_{n_k-1}}), \tag{3}$$

hence for every $0 \le j \le r_k - 2$ and $x \in I_0$,

$$\sum_{i=j\cdot\ell_k\cdot q_{n_k-1}}^{(j+1)\cdot\ell_k\cdot q_{n_k-1}-1} F_k(T^ix) = 0 \tag{3'}$$

and

$$F_k = G_k - G_k\circ T \tag{4}$$

where

$$G_k(T^ux) = -\sum_{i=0}^{u-1} F_k(T^ix) \quad \text{for } x\in I_0, \quad u=0,\dots,q_{n_k-1}$$

$$G_k(x) = 0 \qquad \text{for } x \in [0,1)\setminus\bigcup_{i=0}^{q_{n_k}-1} I_i.$$

Therefore, F_k is a coboundary with the transfer function $G_k \in \mathcal{C}^p$.
Let us compute $\sup |G_k|$. We have

$$\sup |G_k| = \sup_{x\in I_0} \max\{|\sum_{i=0}^{u-1} F_k(T^i x)| : 0 \le u \le q_{n_k} - 1\};$$

by (3'), the partial sums are zero for every $u = j \cdot \ell_k \cdot q_{n_k-1}$, $1 \le j \le r_k-$. From this and from (3) we get

$$\begin{aligned} |G_k| &\le \sup\{|\sum_{i=0}^{u-1} F_k \circ T^i(x)| : x \in X,\ 1 \le u \le \ell_k \cdot q_{n_k-1}\} \le \\ &\ell_k \cdot q_{n_k-1} \sup |F_k| = \\ &\ell_k \cdot q_{n_k-1} \cdot C \cdot d_k \cdot (1 + \frac{1}{c_k})^{r_k} \le C \cdot e^k \cdot k^2. \end{aligned} \tag{5}$$

As $T^{q_{n_k}-1}$ is the shift (mod 1) by $\|q_{n_k-1}\alpha\|$, $T^{j\cdot q_{n_k-1}}$, $j = 1, \dots, [\ell'_k/k]$, $k = 1, 2, \dots,$ is a rigid time. For any fixed positive integer p we thus have

$$\lim_{k\to\infty} \max_{j=1,\dots,[\ell'_k/k]} |S_{j\cdot q_{n_k-1}}(\sum_{i=0}^{p} F_i)| = 0.$$

¿From this and from $|F_k| \le d_k \cdot e^k \to 0$ (cf. (2)) follows that there exists an infinite subset $K \subset \mathbb{N}$ s.t.

$$\lim_{k\in K, k\to\infty} \max_{j=1,\dots,[\ell'_k/k]} |S_{j\cdot q_{n_k-1}}(F_{(K)} - F_k)| = 0 \tag{6}$$

where $F_{(K)} = \sum_{k\in K} F_k$. The set K can be chosen such that

$$\sum_{k\in K} \frac{1}{k} < \infty. \tag{7}$$

Let $\mathcal{A}_k$ be the partition of $[0, 1)$ into the sets $J_{u,j}$ and the complement of their union; as $a_{n_k} \to \infty$, $\mathcal{A}_k \nearrow \mathcal{A}$ (for a subsequence of the numbers k).

Let $0 \le u \le q_{n_k-1} - 1$, $0 \le j \le r_k - 1$ be fixed, E be the middle third of the interval $J'_{u,j}$. If $1 \le i \le \ell'_k/k$ and $k \ge 12$,

$$S_{i\cdot q_{n_k-1}}(F_k) = i \cdot q_{n_k-1} \cdot (-1)^j (1 + \frac{1}{c_k})^j d_k.$$

on E.
Let a be a fixed number of the same sign as $(-1)^j$ and let $\epsilon > 0$. Without loss of generality we can suppose that j is even, $a \ge 0$. By (2) we have

$$q_{n_k-1} d_k (1 + \frac{1}{c_k})^{r_k} \le q_{n_k-1} d_k e^k \to 0$$

and for k sufficiently big,

$$d_k q_{n_k-1} [\ell'_k/k] \ge \frac{e^k k^6}{q_{n_k}} \cdot q_{n_k-1} (\frac{a_{n_k}}{2k^4 e^k} - 1) \frac{1}{k} \ge k/3,$$

hence if k is bigger than some constant $k(a,\epsilon)$, then there exists $1 \le i \le \ell'_k/k$ for which $S_{i\cdot q_{n_k-1}}(F_k) \in \mathcal{U}_\epsilon(a)$ on E.
The rotation $T^{q_{n_k-1}}$ is the shift by $\|q_{n_k-1}\alpha\|$ (mod 1), hence, if $k \ge 30$, we have

$$\lambda(E \cap T^{-i\cdot q_{n_k-1}}E) > 0,9\,\lambda(E)$$

for every $1 \le i \le \ell'_k/k$.
¿From this and from $\lambda(E) \ge \lambda(J_{u,j})/6$ we get the rigid EVC for F_k. ¿From the rigid EVC for F_k and (6) follows that for any infinite subset K' of K, $F_{(K')} = \sum_{k\in K'} F_k$ satisfies the essential value condition.
This way we have found an uncountable set of ergodic cocycles $F = F_{(K')} \in \mathcal{C}_0^p$.

It remains to prove that the set K can be chosen so that for every $c \neq 0$ there exists a rotation S for which

$$(1) \qquad F\circ S - c\cdot F \qquad \text{is a } T-\text{coboundary.}$$

If $F\circ S' - c'\cdot F$ and $F\circ S'' - c''\cdot F$ are coboundaries, then $F\circ S'\circ S'' - c'\cdot c''\cdot F = F\circ S'\circ S'' - c''\cdot(F\circ S') + c''\cdot(F\circ S' - c')$ is a coboundary, too, hence the set of numbers c for which (1) holds true is a group. It thus suffices to find S for $|c| > 1$. First we shall show the proof for $c > 1$. Let us suppose that the $c > 1$ is fixed.

Let $j(k)$ be the greatest positive even integer for which

$$(1 + \frac{1}{c_k})^{j(k)} < c.$$

For nonnegative integers k, v let us define a number
$\sigma(k,v) = \{(v(k) + j(k)\cdot \ell_k q_{n_k-1})\alpha\}$
and a rotation
$\sigma(k,v) = \{(v(k) + j(k)\cdot \ell_k q_{n_k-1})\alpha\}$
(we denote both by the same symbol).
We'll recursively define $K' = \{k_0 < k_1 < \dots\} \subset K$, nonnegative integers $\{v(k) : k \in K'\}$, $k = k_0, k_1, \dots$, numbers and rotations $\{\sigma(k) : k \in K'\}$ (denoted by the same symbol):
For k_0 we choose the smallest element of K and define
$v(k_0) = 0$, $\sigma(k_0) = \sigma(k_0, v(k_0))$.
If k_i, $v(k_i)$ have been defined for $i = 0,\dots,m$, we define k_{m+1} as the smallest $k \in K$ such that:

1. $k > k_m$.
2. There exists an integer $0 \le v = v(k) < q_{n_k-1}/k$ such that

$$|\sigma(k_m) - \sigma(k,v)| < 1/2^m,$$
$$\sup|G_j\circ\sigma(k_m) - G_j\circ\sigma(k,v)| < 1/2^m \quad \text{for } j = k_0,\dots,k_m.$$

Set $\sigma(k_{m+1}) = \sigma(k_{m+1}, v(k_{m+1}))$.
The numbers $\sigma(k_m)$ then for $m \to \infty$ converge to a limit σ.
By S we denote the rotation $x \mapsto x + \sigma$ mod 1, $K' = \{k_0, k_1, \dots\}$.
For $k = k_m \in K'$

$$\|G_k\circ S - G_k\circ\sigma(k)\|_\infty \le \sum_{i=0}^{\infty}\|G_k\circ\sigma(k_{m+i}) - G_k\circ\sigma(k_{m+i+1})\|_\infty < \sum_{i=0}^{\infty} 1/2^{m+i} = 1/2^{m-1},$$

hence

$$\sum_{k\in K'} \|G_k\circ S - G_k\circ\sigma(k)\|_\infty$$

converges.

We have

$$F\circ S - c\cdot F = \sum_{k\in K'} (F_k\circ S - c\cdot F_k) =$$

$$\sum_{k\in K'} \big((F_k\circ S - F_k\circ\sigma(k)) + (F_k\circ\sigma(k) - (1+\frac{1}{c_k})^{j(k)}F_k) + ((1+\frac{1}{c_k})^{j(k)} - c)F_k\big).$$

By (4), each of the functions F_k is a coboundary, hence all summands in the last sum are coboundaries, too. For proving (1) it suffices to show that the sum of the corresponding transfer functions converges:

1. $\sum_{k\in K'}(F_k\circ S - F_k\circ\sigma(k))$.

Every F_k is a coboundary with a transfer function G_k. We have shown that $\sum_{k\in K'}(G_k\circ S - G_k\circ\sigma(k))$ converges; it is a transfer function of $\sum_{k\in K'}(F_k\circ S - F_k\circ\sigma(k))$.

2. $\sum_{k\in K'}(F_k\circ\sigma(k) - (1+\frac{1}{c_k})^{j(k)}F_k)$.

The function $F_k\circ\sigma(k) - (1+\frac{1}{c_k})^{j(k)}F_k$ is a coboundary with a transfer function

$$\tilde{G_k} = G_k\circ\sigma(k) - (1+\frac{1}{c_k})^{j(k)}G_k.$$

¿From the definitions of F_k, G_k, and $\sigma(k)$ follows that for $x\in I_i$, $i = u + t\cdot\ell_k\cdot q_{n_k-1} + p\cdot q_{n_k-1}$, $0\le u$, $u+v(k)\le q_{n_k-1}$, $0\le t$, $t+j(k)\le r_k-1$, and $0\le p\le \ell_k-1$, we have

$$\tilde{G_k} = G_k\circ\sigma(k) - (1+\frac{1}{c_k})^{j(k)}G_k = 0.$$

¿From $v(k)\le q_{n_k-1}/k$ and $j(k)\le c_k\cdot(\log c+1) = (\log c+1)\cdot r_k/k$ (for k sufficiently big) we get that

$$\lambda(\tilde{G_k}\ne 0) < (3+\log c)/k$$

for k big enough. From this and (7) follows that the sum $\sum_{k\in K'}\tilde{G_k}$ converges almost surely.

3. $\sum_{k\in K'}((1+\frac{1}{c_k})^{j(k)} - c)F_k$.

By (4), F_k is a coboundary with a transfer function G_k and by (5), $|G_k|$ is bounded by $(1/2)e^k k^2$. We can easily see that $c - (1+\frac{1}{c_k})^{j(k)} \le 2c/c_k$, hence

$$\sum_{k\in K'} |((1+\frac{1}{c_k})^{j(k)} - c)G_k| \le \sum_{k\in K'} c\cdot e^k k^2/c_k \le \sum_{k\in K'} 1/k$$

where the last sum is finite by (7).

If we define $j(k)$ as the biggest odd number for which

$$(1+\frac{1}{c_k})^{j(k)} < c,$$

we get the rotation S for $c < -1$, hence (1) holds true for all $c\ne 0$.

All the cocycles F_k which we defined are from $\mathcal{C}_0^\infty$. If $\limsup_{n_k\to\infty} q_{n_k+1}/q_n^p = \infty$, we can find the cocycles F_k such that $\sum_{k=1}^\infty F_k$ converges in every $\mathcal{C}^p$, $1\le p<\infty$, hence $F = \sum_{k=1}^\infty F_k \in \mathcal{C}_0^\infty$. This proves the second statement of the theorem. □

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