RIGIDITY AND NON-RECURRENCE ALONG SEQUENCES

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Abstract. Two properties of a dynamical system, rigidity and non-recurrence, are examined in detail. The ultimate aim is to characterize the sequences along which these properties do or do not occur for different classes of transformations. The main focus in this article is to characterize explicitly the structural properties of sequences which can be rigidity sequences or non-recurrent sequences for some weakly mixing dynamical system. For ergodic transformations generally and for weakly mixing transformations in particular there are both parallels and distinctions between the class of rigid sequences and the class of non-recurrent sequences. A variety of classes of sequences with various properties are considered showing the complicated and rich structure of rigid and non-recurrent sequences.

1. Introduction

Let $(X, \mathcal{B}, p, T)$ be a dynamical system: that is, we have a non-atomic probability space $(X, \mathcal{B}, p)$ and an invertible measure-preserving transformation T of $(X, \mathcal{B}, p)$. We consider here two properties of the dynamical system $(X, \mathcal{B}, p, T)$, rigidity and non-recurrence. Ultimately we would like to characterize the sequences along which these properties do, or do not occur, for different classes of transformations. The main focus here is to characterize which subsequences (n_m) in $\mathbb{Z}^+$ can be a sequence for rigidity, and which can be a sequence for non-recurrence, for some weakly mixing dynamical system. In the process of doing this, we will see that there are parallels and distinctions between the class of rigid sequences and the class of non-recurrent sequences, both for ergodic transformations in general and for weakly mixing transformations in particular.

The properties of rigidity and non-recurrence along a given sequence (n_m) are opposites of one another. By *rigidity along the sequence* (n_m) we mean that the powers (T^{n_m}) are converging in the strong operator topology to the identity; that is, $\|f \circ T^{n_m} - f\|_2 \to 0$ as $m \to \infty$, for all $f \in L_2(X, p)$. So rigidity along (n_m) means that $p(T^{n_m}A \cap A) \to p(A)$ as $m \to \infty$ for all $A \in \mathcal{B}$. On the other hand, *non-recurrence along the sequence* (n_m) means that for some $A \in \mathcal{B}$ with $p(A) > 0$, we have $p(T^{n_m}A \cap A) = 0$ for all $m \geq 1$. Nonetheless, there are structural parallels between these two properties of a sequence (n_m). For example, neither property can occur for an ergodic transformation unless the sequence is sparse. Also, these two properties cannot occur without the sequence (n_m) having (or avoiding) various combinatorial or algebraic structures. These properties can occur simultaneously for a given transformation if the sequences are disjoint. For example, we are able to use Baire category results to show that the generic transformation T is weak mixing and rigid along some sequence (n_m) such that it is also non-recurrent along $(n_m - 1)$. In proving this, one sees a connection between rigidity and non-recurrence. The non-recurrence along $(n_m - 1)$ is created by first using rigidity to take a rigid sequence (n_m) for T and a set $A, p(A) > 0$, such that $\sum\limits_{m=1}^{\infty} p(T^{n_m}A\Delta A) \leq \frac{1}{100}p(A)$. This allows one to prove that T is non-recurrent along $(n_m - 1)$ for some subset C of TA. One can extend this argument somewhat and show that for every whole number K, there is a weakly

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mixing transformation T that is rigid along a sequence (n_m), such that also for some set C, $p(C) > 0$, T is non-recurrent for C along $(n_m + k)$ for all $k \neq 0, |k| \leq K$.

First, in Section 2, we discuss some generalities about rigidity and weak mixing. We also consider the more restrictive property of IP-rigidity. We will see that both rigidity and IP-rigidity can be viewed as a spectral property and therefore characterized in terms of the behavior of the Fourier transforms $\widehat{\nu}$ of the positive Borel measures ν on $\mathbb{T}$ that are the spectral measures of the dynamical system. We will see that rigidity sequences must be sparse, but later in Section 3.1.3, it is made clear that they are not necessarily very sparse. In addition, we show that rigidity sequences cannot have certain types of algebraic structure for rigidity to occur even for an ergodic transformations, let alone a weakly mixing one. For example, a sequence cannot be a good sequence to average along for norm convergence theorems, and still be a rigidity sequence for an ergodic transformation.

After this in Section 3, we prove a variety of results about rigidity that serve to demonstrate how rich and complex is the structure of rigid sequences. Here is a sample of what we prove:

a) In Proposition 3.5 we show that if $\lim\limits_{m\to\infty} \frac{n_{m+1}}{n_m} = \infty$, then (n_m) is a rigidity sequence for some weakly mixing transformation T. This result uses the Gaussian measure space construction. Also, by a cutting and stacking construction, we construct an infinite measure preserving rank one transformation S for which (n_m) is a rigidity sequence. Under some additional assumptions on (n_m), we can use the cutting and stacking construction to produce a weakly mixing rank one transformation T on a probability space for which (n_m) is a rigidity sequence. See specifically Proposition 3.10 and generally Section 3.1.2.

b) In contrast, we show that sequences like $(a^m : m \geq 1)$, and $a \in \mathbb{N}, a \neq 1$, are also rigidity sequences for weakly mixing transformations. However, perturbations of them, like $(a^m + p(m) : m \geq 1)$ with $p \in \mathbb{Z}[x], p \neq 0$, are never rigidity sequences for ergodic transformations, let alone weakly mixing transformations. See Proposition 3.27 and Remark 2.23 c).

c) We prove a number of results in Section 3.1.3 that show that rigidity sequences do not necessarily have to grow quickly, but rather can have their density decreasing to zero infinitely often slower than any given rate. One consequence may illustrate what this tells us: we show that there are rigidity sequences for weakly mixing transformations that are not Sidon sets. See Proposition 3.16 and Corollary 3.21.

d) In Section 3.3, we show that there is no universal rigid sequence. That is, we show that given a weakly mixing transformation T that is rigid along some sequence, there is another weakly mixing transformation S which is rigid along some other sequence such that $T \times S$ is not rigid along any sequence.

e) In Section 3.4, we show how cocycle construction can be used to construct rigidity sequences for weakly mixing transformations. One particular result is Corollary 3.49: if $(\frac{p_n}{q_n} : n \geq 1)$ are the convergents associated with the continued fraction expansion of an irrational number, then (q_n) is a rigidity sequence for a weakly mixing transformation.

We then consider non-recurrence in Section 4. We show that the sequences exhibiting non-recurrence must be sparse and cannot have certain types of algebraic structure for there to be non-recurrence even for ergodic transformation, let alone a weakly mixing one. We conjecture that any lacunary sequence is a sequence of non-recurrence for some weakly mixing transformation, but we have not been able to prove this result at this time. Here are some specific results on non-recurrence that we prove:

a) It is well-known that the generic transformation T is weakly mixing and rigid. We show that in addition, there is a rigidity sequence (n_m) for such a generic T, so that for any whole number K, each $(n_m + k), 0 < |k| \leq K$, is a non-recurrent sequence for T. See Proposition 4.4 and Remark 4.5.

b) We observe in Proposition 4.6 that some weakly mixing transformations, like Chacon's transformation, are non-recurrent along a lacunary sequence with bounded ratios.

c) We also show that for any increasing sequence (n_m) with $\sum_{m=1}^{\infty} \frac{n_m}{n_{m+1}} < \infty$, and a whole number K, there is a weakly mixing transformation T and a set $C, p(C) > 0$, such that (n_m) is a rigidity sequence for T and T non-recurrent for C along $(n_m + k)$ for all $k, 0 < |k| \leq K$. See Proposition 4.8 and Remark 4.9.

When considering both rigidity and non-recurrence of measure-preserving transformations, there are often unitary versions of the results that are either almost identical in statement and proof, or worth more consideration. When possible, we will take note of this. See Krengel [32] for a general reference on this and other aspects of ergodic theory used in this article.

There is also a larger issue of considering both rigidity and non-recurrence for general groups of invertible measure-preserving transformations. This will require a careful look at general spectral issues, including the irreducible representations of the groups. We plan to pursue this in a later paper.

2. **Generalities on Rigidity and Weak Mixing**

Suppose we consider a dynamical system $(X, \mathcal{B}, p, T)$. Unless it is noted otherwise, we will be assuming that $(X, \mathcal{B}, p)$ is a standard Lebesgue probability space i.e. it is measure theoretically isomorphic to $[0, 1]$ in Lebesgue measure. In particular it is non-atomic and $L_2(X, p)$ has a countable dense subset in the norm topology. We say that the dynamical system is separable in this case. We also assume that T is an invertible measure-preserving transformation $(X, \mathcal{B}, p)$.

This section provides the background information needed in this article. First, in Section 2.1 we deal with the basic properties of rigidity and weak mixing in order to give a general version of the well-known fact that the generic transformation is both weakly mixing and rigid along some sequence. Second, in Section 2.2 we look at weak mixing and aspects of it that are important to this article. Third, in Section 2.3 we consider rigidity itself in somewhat more detail. See Furstenberg and Weiss [19] and Queffelec [51], especially Section 3.2.2, for background information about rigidity as we consider it, and other types of rigidity that have been considered by other authors.

2.1. **Rigidity and Weak Mixing in General.** Given an increasing sequence (n_m) of integers we consider the family

$$\mathcal{A}(n_m) = \{A \in \mathcal{B} : \ p\left(T^{n_m} A \triangle A\right) \to 0\}.$$

We now recall some basic and well-known facts about $\mathcal{A}(n_m)$. See Walters [62] for the following result.

Proposition 2.1. *$\mathcal{A}(n_m) \subset \mathcal{B}$ is a sub-σ-algebra which is also T-invariant. $\mathcal{A}(n_m)$ is the maximal σ-algebra $\mathcal{A} \subset \mathcal{B}$ such that*

$$T^{n_m}|_{\mathcal{A}} \to Id|_{\mathcal{A}} \ \text{ as } \ m \to \infty.$$

Moreover

$$L_2(X, \mathcal{A}(n_m), p) = \{f \in L_2(X, \mathcal{B}, p) : \ f \circ T^{n_m} \to f \ \text{ in } \ L_2(X, \mathcal{B}, p)\}. \tag{2.1}$$

Remark 2.2. An approach to the above result different than in [62] begins by observing that $\{f \in L_\infty(X,p) : \|f \circ T^{n_m} - f\|_2 \to 0\}$ is an algebra. So there is a corresponding factor map of $(X, \mathcal{B}, p, T)$ for which there is an associated T invariant sub-σ-algebra, namely $\mathcal{A}(n_m) \subset \mathcal{B}$.

If $\mathcal{A}(n_m) = \mathcal{B}$ then one says that (n_m) is a *rigidity sequence* for $(X, \mathcal{B}, p, T)$. Systems possessing rigidity sequences are called *rigid*. The fact that (n_m) is a rigidity sequence for T is a *spectral property*; that is, it is a unitary invariant of the associated Koopman operator U_T on $L_2(X,p)$ given by the formula $U_T(f) = f \circ T$. The following discussion should make this clear.

First, recall some basic notions of spectral theory (see e.g. [8], [29], [48]). For each $f \in L_2(X,p)$, the function $\rho(n) = \langle f \circ T^n, f\rangle$ is a positive-definite function and hence, by the Herglotz Theorem, is the Fourier transform of a positive Borel measure on the circle $\mathbb{T}$. So for each $f \in L_2(X,p)$, there is a unique positive Borel measure ν_f^T on $\mathbb{T}$, called the *spectral measure* for T corresponding to f which is determined by $\widehat{\nu_f^T}(n) = \langle f \circ T^n, f\rangle$ for all $n \in \mathbb{Z}$. Spectral measures are non-negative and have $\nu_f^T(\mathbb{T}) = \|f\|_2^2$. We will also need to use the adjoint ν^* given by $\nu^*(E) = \overline{\nu(E^{-1})}$ for all Borel sets $E \subset \mathbb{T}$. The adjoint has $\widehat{\nu^*}(n) = \overline{\widehat{\nu}(-n)}$ for all $n \in \mathbb{Z}$.

Absolute continuity of measures is important here: given two positive Borel measures ν_1 and ν_2 on $\mathbb{T}$, we say ν_1 is absolutely continuous with respect to ν_2, denoted by $\nu_1 \ll \nu_2$, if $\nu_1(E) = 0$ for all Borel sets E such that $\nu_2(E) = 0$. Now, among all spectral measures there exist measures ν_F^T such that all other spectral measures are absolutely continuous with respect to ν_F^T. Any one of these is called a *maximal spectral measure* of T. These measures are all mutually absolutely continuous with respect to one another. The equivalence class of the maximal spectral measures is denoted by ν^T. By abuse of notation, we refer to ν^T as a measure too. Recall that the type of a finite positive measure (e.g. whether the measure is singular, absolutely continuous with respect to Lebesgue measure, etc.) is a property of the equivalence class of all finite positive measures ω such that $\omega \ll \nu$ and $\nu \ll \omega$. The type of ν^T (that is, of a maximal spectral measure ν_F^T) has a special role in the structure of the transformation. For this reason the type of ν^T is called the *maximal spectral type* of T. For example, rigid transformations must have singular maximal spectral type; see Remark 2.8. Also, Bernoulli transformations must have Lebesgue type i.e. their maximal spectral measures are equivalent to Lebesgue measure. In general, a strongly mixing transformation does not need to be of Lebesgue type. It could be of singular type (this occurs when every maximal spectral measure is singular but yet has the Fourier transform tending to zero at infinity).

For a given sequence (n_m), a transformation T and a function $f \in L_2(X,p)$, we say (n_m) *is a rigidity sequence of* T *for* f if $f \circ T^{n_m} \to f$ in L_2-norm. Recall that $\widehat{\nu_f^T}(n_m) = \langle f \circ T^{n_m}, f\rangle$.

Proposition 2.3. *Fix the transformation T. The following are equivalent for $f \in L_2(X,p)$:*

1. *The sequence (n_m) is a rigidity sequence for the function f.*
2. $\langle f \circ T^{n_m}, f\rangle = \int_X f \circ T^{n_m} \cdot \overline{f}\, dp \to \|f\|_2^2$.
3. $\widehat{\nu_f^T}(n_m) \to \|f\|_2^2$.
4. $z^{n_m} \to 1$ *in* $L_2(\mathbb{T}, \nu_f^T)$.
5. $z^{n_m} \to 1$ *in measure with respect to* ν_f^T.

Proof. We have $\|f \circ T^{n_m} - f\|_2^2 = 2\|f\|_2^2 - 2\mathrm{Re}\langle f \circ T^{n_m}, f\rangle$. Since $|\langle f \circ T^{n_m}, f\rangle| \le \|f\|_2^2$, we see that (1) is equivalent to (2). Now (2) is equivalent to (3) by the definition of the spectral measure ν_f^T. We also have $\int |z^{n_m} - 1|^2\, d\nu_f^T(z) = 2\|f\|_2^2 - 2\mathrm{Re}(\widehat{\nu_f^T}(n_m))$. Since $|\widehat{\nu_f^T}(n_m)| \le \|f\|_2^2$, we see that (3) is equivalent to (4). It is clear that (4) is equivalent to (5) because $|1 - z^{n_m}| \le 2$ and ν_f^T is a finite, positive measure. □

Remark 2.4. This result is really a fact about a unitary operator U on a Hilbert space H. That is, a sequence (n_m) and vector $v \in H$ satisfy $\lim\limits_{m\to\infty} ||U^{n_m}v - v\|_H = 0$ if and only if the spectral measure ν_v^U determined by $\widehat{\nu_v^U}(k) = \langle U^k v, v\rangle$ for all $k \in \mathbb{Z}$ has the property that $\lim\limits_{m\to\infty} \widehat{\nu_v^U}(n_m) = \|v\|_H^2$.

Proposition 2.3 shows that if (n_m) is a rigidity sequence for T for a given function F, then for any spectral measure $\nu_f^T \ll \nu_F^T$, we would also have $z^{n_m} \to 1$ in measure with respect to ν_f^T. Hence, (n_m) would be a rigidity sequence for T for the function f too. It follows then easily that T is rigid and has a rigidity sequence (n_m) if and only if (n_m) is a rigidity sequence for F where ν_F^T is a maximal spectral measure for T.

Corollary 2.5. *T is rigid if and only if for each function $f \in L_2(X,p)$ there exists $(n_m) = (n_m(f))$ such that $f \circ T^{n_m} \to f$ in $L_2(X,p)$.*

Remark 2.6. It is clear that an argument like this works equally well for a unitary transformation U of a separable Hilbert space H. That is, there is one sequence (n_m) such that for all $v \in H$, $\|U^{n_m}v - v\|_H \to 0$ as $m \to \infty$ if and only if for every vector $v \in H$, there exists a sequence (n_m) such that $\|U^{n_m}v - v\|_H \to 0$ as $m \to \infty$

Remark 2.7. J.-P. Thouvenot was the first to observe that T is rigid if and only if for each $f \in L_2(X,p)$ (or just for each characteristic function $f = 1_A, A \in \mathcal{B}$), there exists (n_m) depending on f such that $\|f \circ T^{n_m} - f\|_2 \to 0$ as $m \to \infty$. There are a number of different ways to prove this. We have given one such argument above. Another argument would use the characterization up to isomorphism of unitary operators as multiplication operators. Here is an interesting approach via Krieger's Generator Theorem; see Krieger [34]. It is sufficient to prove rigidity holds assuming that one has the weaker condition of there being rigidity sequences for each characteristic function. Suppose that an automorphism T has the property that for each set $A \in \mathcal{B}$ there exists $(n_m) = (n_m(A))$ such that $p\,(T^{-n_m}A \triangle A) \to 0$. Then all spectral measures of functions of the form 1_A, $A \in \mathcal{B}$ are singular, and since the family of such functions is linearly dense, the maximal spectral type of T is singular. It follows that T has zero entropy; see Remark 2.8 for an explanation of this point. Hence, by Krieger's Generator Theorem, there exists a two element partition $P = \{A, A^c\}$ which generates $\mathcal{B}$. Now, let $(n_m) = (n_m(A))$ and notice that for each $k \ge 1$ and for each $B \in \bigvee_{i=0}^{k-1} T^i P$ we have $p(T^{n_m}B \triangle B) \to 0$. Hence by approximating the $L_2(X,p)$ functions by simple functions, (n_m) is a rigidity sequence for T.

Remark 2.8. From Proposition 2.3, we see that a maximal spectral measure ν^T of a rigid transformation is a *Dirichlet measure*. This means that for some increasing sequence (n_m), we have $\gamma^{n_m} \to 1$ in measure with respect to ν^T as $m \to \infty$. Hence, as in Proposition 2.3, we have $\widehat{\nu^T}(n_m) \to \nu^T(\mathbb{T})$ as $m \to \infty$. A measure with this property is also sometimes called a *rigid measure*. Note that a measure absolutely continuous with respect to a Dirichlet measure is a Dirichlet measure. So by the Riemann-Lebesgue Lemma, there is no non-zero positive measure ν which is absolutely continuous with respect to Lebesgue measure such that $\nu \ll \nu_f^T$ for a non-zero spectral measure ν_f^T of a rigid transformation. Therefore, for a rigid transformation, all spectral measures, and ν^T itself, are Dirichlet measures and hence singular measures. So T has singular maximal spectral type. Rokhlin shows in his classical paper [53] that if T has positive entropy, then for every maximal spectral measure ν_F^T, there is a non-zero spectral measure

$\nu_f^T \ll \nu_F^T$ that is equivalent to (mutually absolutely continuous with respect to) Lebesgue measure. Therefore, all rigid transformations have zero entropy.

One can often use Baire category arguments to distinguish the behavior of transformations. For this we use the Polish group $Aut(X, \mathcal{B}, p)$ of invertible measure-preserving transformations on $(X, \mathcal{B}, p)$, with the topology of strong operator convergence. That is, a sequence (S_n) in $Aut(X, \mathcal{B}, p)$ converges to $S \in Aut(X, \mathcal{B}, p)$ if and only if $\|f \circ S_n - f \circ S\|_2 \to 0$ as $n \to \infty$ for all $f \in L_2(x, p)$. By a *generic property*, we mean that the property holds on at least a dense G_δ subset of $Aut(X, \mathcal{B}, p)$, and any set containing a dense G_δ set is called a *generic set*. So a generic property is one that holds on a set whose complement is first category. For example, it is well-known that the generic dynamical system is weakly mixing. See Halmos [22] where this was used to give a Baire category argument for the existence of weakly mixing transformations that are not strongly mixing. Also, the generic transformation has a rigidity sequence. See Katok and Stepin [30] and Walters [62]. Hence, the generic transformation is weakly mixing, rigid, and has zero entropy (see Remark 2.8). We will show this in a slightly more general setting.

First, to see that a generic transformation is rigid we state the following stronger result. This result is well-known so we do not provide a proof for brevity's sake.

Proposition 2.9. *Given an increasing sequence (n_m) of natural numbers, let $\mathcal{G}_{(n_m)}$ be the set consisting of all $S \in Aut(X, \mathcal{B}, p)$ such that $S^{n_{m_k}} \to Id$ in the strong operator topology for some subsequence (n_{m_k}) of (n_m). Then $\mathcal{G}_{(n_m)}$ is a generic subset of $Aut(X, \mathcal{B}, p)$.*

2.2. **Weak Mixing Specifically.** Now we consider weakly mixing transformations. Recall that T is *weakly mixing* if and only if for all $A, B \in \mathcal{B}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |p(T^n A \cap B) - p(A)p(B)| = 0.$$

So T is weakly mixing if and only if for all mean-zero $f \in L_2(X, p)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle f \circ T^n, f\rangle| = 0.$$

Now recall Wiener's Lemma: given a positive Borel measure ν on $\mathbb{T}$ we have

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\widehat{\nu}(n)|^2 = \sum_{\gamma\in\mathbb{T}} \nu^2(\{\gamma\}).$$

It follows that ν is continuous (i.e. has no point masses) if and only if $\lim\limits_{N\to\infty} \frac{1}{2N+1} \sum\limits_{n=-N}^{N} |\widehat{\nu}(n)|^2 = 0$. The latter condition is well-known to be equivalent to the fact that $\widehat{\nu}(n)$ tends to zero along a subsequence of density 1. Here we say that a set $S \subset \mathbb{N}$ of density one if

$$\lim_{N\to\infty} \frac{1}{N} \#(S \cap \{1, 2, \dots, N\}) = 1.$$

So a transformation T is weakly mixing if and only if ν_f^T is continuous for each $f \in L_2(X, p)$ which is mean-zero, that is (by Wiener's Lemma) we have $\langle f \circ T^n, f\rangle$ tends to zero along a sequence of density one. Denote a given such density one sequence by $\mathcal{N}_f^T$. As f changes, this sequence generally might need to change. However, it is a well-known fact that because

$(X, \mathcal{B}, p)$ is separable, we can choose a subsequence of density 1 which works for all L_2-functions. See Petersen [49] and also Jones [27].

Proposition 2.10. *Assume that T is weakly mixing. Then there is a sequence (n_m) in $\mathbb{Z}^+$ of density one such that for all mean-zero $f \in L_2(X,p)$, one has $\lim_{m\to\infty} \langle f \circ T^{n_m}, f\rangle = 0$.*

Remark 2.11. This result also holds for a unitary operator U on a separable Hilbert space H. That is, if all the spectral measures ν_v^U for $v \in H$ are continuous, then there exists a sequence (n_m) of density one such that $\widehat{\nu_v^U}(n_m) \to 0$ as $m \to \infty$.

Let $L_{2,0}(X,p)$ denote the mean-zero functions in $L_2(X,p)$. We can rewrite the assertion of Proposition 2.10 as

$$U_T^{n_m} \to 0 \text{ weakly in the space } L_{2,0}(X,p). \tag{2.2}$$

Each sequence (n_m) (not necessarily of density 1) of integers for which (2.2) holds is called a *mixing sequence* for T. Any transformation possessing a mixing sequence is weakly mixing.

The following result about mixing subsequences is also well-known.

Proposition 2.12. *Given an increasing sequence (n_m) of natural numbers, consider the set $\mathcal{M}_{(n_m)}$ that consists of all $S \in Aut(X,\mathcal{B},p)$ such that $S^{n_{m_k}} \to 0$ weakly in $L_{2,0}(X,p)$, for some subsequence (n_{m_k}) of (n_m). Then $\mathcal{M}_{(n_m)}$ is a generic subset of $Aut(X,\mathcal{B},p)$.*

Remark 2.13. It is easy to see that Proposition 2.12 shows that weakly mixing transformations $\mathcal{W}$ are generic set because they can be characterized as having only the trivial eigenvalue 1 with the eigenvectors being the constant functions. Now taking $n_m = m$ for all m, we have any transformation with a non-trivial eigenvalue must be in $\mathcal{M}(n_m)^c$. So $\mathcal{W}^c \subset \mathcal{M}(n_m)^c$, and $\mathcal{M}(n_m) \subset \mathcal{W}$. Actually, it is also well-known that $\mathcal{W}$ itself is a G_δ set.

Combining our two basic category results, Proposition 2.9 and Proposition 2.12, gives the following.

Proposition 2.14. *Given an increasing sequence (n_m) of natural numbers, consider the set $\mathcal{B}_{(n_m)}$ that consists of all $S \in Aut(X,\mathcal{B},p)$ such that $S^{n_{m_k(1)}} \to Id$ weakly in $L_{2,0}(X,p)$, for some subsequence $(n_{m_k(1)})$ of (n_m) and such that $S^{n_{m_k(2)}} \to 0$ weakly in $L_{2,0}(X,p)$, for some subsequence $(n_{m_k(2)})$ of (n_m). Then $\mathcal{B}_{(n_m)}$ is a generic subset of $Aut(X,\mathcal{B},p)$.*

Remark 2.15. The category result in Proposition 2.14 also holds if we ask for the stronger property that $S^\sigma \to Id$ weakly in $L_{2,0}(X,p)$, as $\sigma \to \infty(IP)$ for the IP set generated by some subsequence (n_{m_k}) of (n_m). See Proposition 2.31 and the discussion before it for the definition and basic characterization of IP rigidity.

Remark 2.16. We can also formulate unitary versions of Proposition 2.9, Proposition 2.12, and Proposition 2.14.

2.3. **Rigidity Specifically.** In addition to the examples given inherently by Proposition 2.14, each ergodic transformation with discrete spectrum is rigid. One can see this in several ways. One way is to note that T is rigid for each eigenfunction f since if $\gamma \in \mathbb{T}$ there is a sequence (n_m) such that $\gamma^{n_m} \to 1$ in $\mathbb{T}$. Then use the principle of Corollary 2.5. Alternatively, in order to see this via the Halmos-von Neumann Theorem, consider an ergodic rotation $Tx = x + x_0$ where X is a compact metric monothetic group, x_0 is its topological cyclic generator, and p stands for Haar measure of X. Take any increasing sequence (n_t) of integers, and consider $(n_t \cdot x_0)$. By passing to a subsequence if necessary, we can assume that $n_t x_0 \to y \in X$. This is equivalent to saying that $T^{n_t} \to S$, where $Sx = x + y$. Because the convergence is taking part in the strong operator topology, it is not hard to see that we will obtain

$$T^{n_{t_{k+1}} - n_{t_k}} \to S \circ S^{-1} = Id,$$

and therefore T is rigid (indeed, $n_{t_{k+1}} - n_{t_k} \to \infty$ by Proposition 2.18 below). These arguments show that each purely atomic measure is a Dirichlet measure. Moreover, we have also shown that in the discrete spectrum case the closure of $\{T^n : n \in \mathbb{Z}\}$ in the strong operator topology is compact. The converse is also true. See for example Bergelson and Rosenblatt [6] and Kušhnirenko [36]. It also is not difficult to see that the centralizer of T in $Aut(X, \mathcal{B}, p)$, denoted by $C(T)$, can be identified with this closure and so is compact in the strong operator topology. The converse of this is also true (see again, e.g. [36]). Moreover, T is isomorphic to the translation by T on $C(T)$ considered with Haar measure. We will see later that ergodic transformations with discrete spectrum are completely determined by their rigidity sequences (see Corollary 3.41 below). There we will be using the information summarized here.

A positive finite Borel measure ν on $\mathbb{T}$ is called a *Rajchman measure* if its Fourier transform vanishes at infinity, that is

$$\widehat{\nu}(n) \to 0 \text{ when } |n| \to \infty. \tag{2.3}$$

So the spectral measures of a strongly mixing transformation are Rajchman measures, and T is strongly mixing if and only if the maximal spectral type ν^T is a Rajchman measure. Moreover, by the *Gaussian measure space construction* (GMC) discussed in Remark 2.26, any Rajchman measure is one of the spectral measures for some strongly mixing transformation. It is not hard to see that a measure absolutely continuous with respect to a Rajchman measure is Rajchman. Also, Rajchman measures and Dirichlet measures are mutually singular.

Remark 2.17. It would be interesting to characterize the sets $\mathcal{N} = \{n_m\}$ of density one that occur in Proposition 2.10. This means characterizing sets $\mathcal{L}$ of density zero that are the complements of such sets. Characterizing rigidity sequences for weakly mixing transformations means characterizing certain types of sets $\mathcal{L}$. However, this may not capture all sets in $\mathcal{N}$. For example, it may be possible for a set $\mathcal{N}$ to fail to have a rigidity sequence in its complement, but contain a set of the form $\{n \geq 1 : |\widehat{\mu}(n)| \geq \delta\}$ for some $\delta > 0$, e.g. with μ that is a spectral measure for a mildly mixing, not strongly mixing, transformation.

Proposition 2.10 certainly shows that rigidity sequences for weakly mixing transformations are density zero. Proposition 2.18 below shows also that more than this is true without the assumption that T is weakly mixing. There is a general principle in play here, but the argument has to be different when there are eigenfunctions. If the system is not ergodic, or if some power T^n is not ergodic, then there can exist a non-zero, mean-zero function $f \in L_2(X, p)$ and a periodic sequence (n_m) such that $f \circ T^{n_m} = f$ for all $m \geq 1$. Otherwise, the only way a sequence can exhibit rigidity for a function, or for the whole dynamical system, is when the sequence has gaps tending to ∞, and hence is certainly of density zero. We recall that our probability spaces are standard Lebesgue spaces and so have no atoms. This is important in the next result where the Rokhlin Lemma is used.

Proposition 2.18. *Let (n_m) be an increasing sequence of integers.*

a) Let T be totally ergodic. If $\|f_0 \circ T^{n_m} - f_0\|_2 \to 0$ as $m \to \infty$ for some non-zero, mean-zero $f_0 \in L_2(X, p)$, then the sequence (n_m) has gaps tending to ∞ and hence has zero density.

b) Suppose T is ergodic. If $\|f \circ T^{n_m} - f\|_2 \to 0$ as $m \to \infty$ for all $f \in L_2(X, p)$, then (n_m) has gaps tending to ∞ and hence has zero density.

Proof. In a) we claim that $n_{m+1} - n_m \to \infty$ as $m \to \infty$. Otherwise, there would be a value $d \geq 1$ such that $d = n_{m+1} - n_m$ infinitely often. It follows that $f_0 \circ T^d = f_0$. This is not possible since f_0 is non-zero and mean-zero, and T is totally ergodic. To prove b), one again argues that $n_{m+1} - n_m \to \infty$ as $m \to \infty$ since otherwise there exists $d \geq 1$ such that $d = n_{m+1} - n_m$ for infinitely many m, and hence $f \circ T^d = f$ for all $f \in L_2(X, p)$. But this is impossible since our system is ergodic. Indeed, for any d_0, using the Rokhlin Lemma, there is a set B of positive measure such that $T^j B$ are pairwise disjoint for all $j, 1 \leq j \leq d_0$. Take f_0 supported on B that is non-zero and mean-zero. Then $f_0 \circ T^j \neq f_0$ for all $j, 1 \leq j \leq d_0$. Hence, once $d_0 > d$, we cannot have $f_0 \circ T^d = f_0$. □

Remark 2.19. Consider part b) above in the case of suitable unitary operators. Since we used the Rokhlin Lemma, we would need a different proof to show that a rigidity sequence for a unitary operator has gaps tending to infinity. This can be seen by the above if the operator has an infinite discrete spectrum. An additional argument is needed in case all the non-trivial spectral measures are continuous. Then using the GMC (see Remark 2.26) and the result in part a) gives the result in this case too.

We will be constructing various examples of rigidity sequences in Section 3. To have some contrast with these constructions, it is worthwhile to make some remarks now about sequences that cannot be rigidity sequences. We have seen from the above, that rigidity sequences must have gaps growing to infinity. But much more structural information is needed to guarantee that the sequence can be a rigidity sequence.

For example, we have the following basic result. First, recall that a sequence (n_m) is *norm good* if for all measure-preserving transformations T, the averages $\frac{1}{M}\sum_{m=1}^{M} f \circ T^{n_m}$ converge in L_2-norm for all $f \in L_2(X, p)$. See Rosenblatt [54] for some background about norm good averaging methods. In particular, Theorem 1 there shows that (n_m) is norm good if and only if for each $\gamma \in \mathbb{T}$, the limit $\lim\limits_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \gamma^{n_m}$ exists. This is also true if the sequence is just norm good for all ergodic transformations. These well-known observations show that the following result covers a class of sequences that includes norm good sequences.

Proposition 2.20. *Suppose (n_m) is an increasing sequence such that $\lim\limits_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \gamma^{n_m}$ exists for all but countably many values. Then (n_m) cannot be a rigidity sequence for an ergodic transformation.*

Proof. Let $h(\gamma)$ denote $\lim\limits_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \gamma^{n_m}$ if it exists. We know that $h(\gamma) = 0$ on a dense set because $(\gamma^{n_m} : m \geq 1)$ is uniformly distributed a.e. with respect to Lebesgue measure; see Kuipers and Niederreiter [35]. Suppose (n_m) is a rigidity sequence for an ergodic transformation T. Suppose first that T does not have discrete spectrum; so there is some non-zero continuous spectral measure ν_f^T with $\|f\|_2 = 1$. Because h exists except for countably many values, we have $\int\limits_{\mathbb{T}} h(\gamma)\, d\nu_f^T(\gamma) = \lim\limits_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \int\limits_{\mathbb{T}} \gamma^{n_m}\, d\nu_f^T(\gamma) = 1$. But then since $|h| \leq 1$ a.e. with respect to ν_f^T, which is a probability measure, it must be that $h(\gamma) = 1$ a.e. with respect to ν_f^T. Thus, there exists some element γ_0 in the support of ν_f^T of infinite order such that $h(\gamma_0) = 1$. Then the subgroup H generated by γ_0 is dense in $\mathbb{T}$, and for any $\gamma \in H$, $h(\gamma) = \lim\limits_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \gamma^{n_m} = 1$, So $h = 1$ on a dense set in $\mathbb{T}$. Apply Proposition 27 in Rosenblatt [54] to the functions $\phi_M(\gamma) = \frac{1}{M}\sum_{m=1}^{M}\gamma^{n_m}$. It follows that h fails to exist on a dense G_δ set, contradicting our assumption that it exists except for a countable set. If there is no non-trivial spectral measure, then T has discrete spectrum and its eigenvalues form a dense subgroup H in $\mathbb{T}$. Because (n_m) is a rigidity sequence for T, for any $\gamma \in H$, we would have $\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\gamma^{n_m} = 1$. Hence again, $h = 1$ on a dense set in $\mathbb{T}$. This is impossible for the same reason given above. □

Remark 2.21. a) There are many examples for which this result applies. For example, if the sequence (n_m) has $(n_m x \mod 1)$ uniformly distributed for all but countably many values, then of course this results applies. This holds for prime numbers (p_m) in increasing order because then $(p_m x \mod 1)$ is uniformly distributed for all irrational numbers x. So the prime numbers cannot be a rigidity sequence for an ergodic transformation. Also, the result above applies to any polynomial sequence $(p(m))$, with p a non-zero polynomial with integer coefficients. Again, the fact used here is that $(p(m)x \mod 1)$ uniformly distributed for all irrational numbers x. Another interesting source of examples provided by tempered functions as in Lemma 5.12 in Bergelson and Håland [4]. See Kuipers and Niederreiter [35] for these examples and general information on uniform distribution of sequences. But note that we can reach the same conclusion for polynomial sequences $(p(m))$ by a simpler argument using successive differences. See Remark 2.23 b) below.

b) Both of the examples in a) are actually norm good sequences. There is a large literature on norm good sequences. To cite just two references, see Lemańczyk et al [41] for examples that are constructed randomly, and see Bergelson and Leibman [5] for examples given by generalized polynomials.

The proof that certain sequences, like the prime numbers, satisfy the hypothesis in Proposition 2.20 is linked to another property that prohibits rigidity. First, we consider what happens when a linear form on the sequence has bounded values. Recall that our underlying probability space is a standard Lebesgue space.

Proposition 2.22. *Suppose $F(x_1,\dots,x_K) = \sum_{k=1}^{K} c_k x_k$ where $c_1,\dots,c_K \in \mathbb{Z}\backslash\{0\}$. Suppose (n_m) is a sequence of whole numbers. Assume that for some non-zero $d \in \mathbb{Z}$, we know that for any $M \ge 1$, there are $m_k \ge M$ for all $k = 1,\dots,K$, such that $F(n_{m_1},\dots,n_{m_K}) = d$. Then (n_m) is not a rigidity sequence for an ergodic transformation.*

Proof. Suppose ν is a Borel probability measure on $\mathbb{T}$ such that $\widehat{\nu}(n_m) \to 1$ as $m \to \infty$. Hence, $\gamma^{n_m} \to 1$ in measure with respect to ν as $m \to \infty$. Then also $\gamma^d = \gamma^{F(n_{m_1},\dots,n_{m_K})} \to 1$ in measure with respect to ν as $m_k \to \infty$, for all $k = 1,\dots,K$. That is, ν is supported on the d-th roots of unity. However, if T is ergodic, we cannot have all the spectral measures supported in the d-th roots of unity. □

Remark 2.23. a) The proof is showing that given the hypothesis of Proposition 2.22, (n_m) cannot be a rigidity sequence for a transformation T and a specific function $f \in L_2(X,p)$ unless $f \circ T^d = f$. So if T is totally ergodic, (n_m) cannot be a rigidity sequence for any non-trivial function, let alone a rigidity sequence for all functions.

b) It is easy to see that polynomial sequences $(p(n) : n \ge 1)$ satisfy the hypothesis of Proposition 2.22 and therefore cannot be rigidity sequences for weakly mixing transformations. The easiest way to see this is to note that the difference $q(n) = p(n+1) - p(n)$ is a polynomial of less degree than p. So successive differences will eventually lead to a constant. For example, if $p(n) = n^2$, then let $q(n) = p(n+1) - p(n)$. Then $q(n+1) - q(n) = 2$. So $p(n+2) - 2p(n+1) + p(n) = p(n+2) - p(n+1) - (p(n+1) - p(n)) = q(n+1) - q(n) = 2$. So let $F(x,y,z) = x - 2y + z$. Then for all (large) n, $F(p(n+2), p(n+1), p(n)) = 2$. Hence, (n^2) cannot be a rigidity sequence for a weakly mixing transformation.

c) Here are other simple examples of the above. Suppose $n_m = 2^m + 1$. Then $2n_m - n_{m+1} = 1$. So if $F(x,y) = 2x - y$, then $F(n_m, n_{m+1}) = 1$ for all m. Also, suppose $n_m = 2^m + m$. Then $2n_m - n_{m+1} = m - 1$. Hence, $2n_{m+1} - n_{m+2} - (2n_m - n_{m+1}) = 1$ for all m. So with $F(x,y,z) = 2y - z - (2x - y) = 3y - z - 2x$, we have $F(n_m, n_{m+1}, n_{m+2}) = 1$ for all m. Therefore, both $(2^n + 1)$ and $(2^n + n)$ are not rigidity sequences for a weakly mixing transformation. It is not hard to see that a calculation of this sort can also be carried out for any sequence $(2^n + p(n))$ where p is a non-zero polynomial with integer coefficients. Hence, such sequences are generally not rigidity sequences for weakly mixing transformations. See Proposition 3.27 which shows that (2^m) itself is a rigidity sequence for a weakly mixing transformation.

d) Another example will give some idea of other issues that can arise in using Proposition 2.22. Take the sequence $(2^n + p_n)$ where (p_n) is the prime numbers in increasing order. If this is a rigidity sequence, then so is $(a_n) = (p_{n+1} - 2p_n)$ since $a_n = 2^{n+1} + p_{n+1} - 2(2^n + p_n)$. It is not clear what holds for this resulting sequence. For example, does this (a_n) satisfy the hypothesis in Proposition 2.20?

e) Proposition 2.22 can also be used in a positive way. For example, start with the fact that (2^n) is a rigidity sequence for a weakly mixing transformation proved in Proposition 3.27. It follows that for any $m_1, \dots, m_K$ and $N_1, \dots, N_K$, the sequence $(n_m) = (\sum_{k=1}^{K} m_k 2^{N_k + m})$ is also a rigidity sequence.

Proposition 2.22 in turn can be used to prove the following result. This result was suggested by the fact that certain sequences, like the prime numbers, were originally seen to satisfy the hypothesis of Proposition 2.20 by using analytic number theory arguments which also gave the hypothesis of Proposition 2.24. We will use in the next proposition the notion of upper density: given a set $A = \{a_n : n \ge 1\}$ of integers, the upper density of A is

$$\limsup_{N\to\infty} \frac{\#(\{a_n : n \ge 1\} \cap \{-N, \dots, N\})}{2N+1}.$$

Proposition 2.24. *Suppose that* **a** *is a sequence of whole numbers. Assume that the set* $A = \{a_n : n \ge 1\}$ *has the property that for some integers* $c_1, \dots, c_L$ *the set of sums* $c_1 A + \ldots + c_L A = \{\sum_{l=1}^{L} c_l a_{n_l} : n_1, \dots, n_L \in A\}$ *has positive upper density. Then* **a** *cannot be a rigidity sequence for an ergodic transformation.*

Proof. Suppose that the condition on the set of sums holds and L is the smallest possible value for which this holds for some $c_1, \dots, c_L$. Let $B = c_1 A + \ldots + c_L A$ and assume it has upper density $D > 0$. Consider the set of sums $B_N = \{\sum_{l=1}^{L} c_l a_{n(l)} : n(1), \dots, n(L) \ge N\}$. B_N can be obtained from B by deleting a finite set and a finite number of translates of sets of sums of the form $c_{i_1} A + \ldots + c_{i_{L'}} A$ with $L' < L$. Since these sets of sums are assumed to all be of upper density zero, B_N also has upper density D. Then there are infinitely many pairs $\sigma_1 < \sigma_2$ with $\sigma_1, \sigma_2 \in B_N$ and $\sigma_2 = d + \sigma_1$ for some non-zero $d \le 2\frac{1}{D}$. So let $K = 2L$ and

$F(x_1,\dots,x_K) = \sum_{k=1}^{L} c_k x_k - \sum_{k=L+1}^{K} c_{k-L} x_k$. With this linear form F, we have shown that A satisfies the hypothesis of Proposition 2.22. $\square$

Remark 2.25. Proposition 2.24 clearly applies to squares $S = \{n^2 : n \geq 0\}$. Indeed, $(n+1)^2 - n^2 = 2n+1$, so the odd numbers are a subset of $S - S$. It also applies to the prime numbers P, because of the well-known fact that for some whole number K, the sum of K copies of P contains all of the whole numbers $n \geq 3$.

By using spectral measures and the *Gaussian measure space construction* (denoted here by GMC), we can see that our basic desire to characterize rigidity sequences is equivalent to a fact about Fourier transforms of measures.

Remark 2.26. The GMC is a standard method of creating a weakly mixing transformation G_ν such that one of its spectral measures is a given continuous measure ν. The transformation G_ν itself is a coordinate shift on an infinite product space, but the probability measure on the product space that it leaves invariant must be constructed specifically with ν in mind. See Cornfeld, Fomin, and Sinai [8] for details about the GMC. We will use the notation G_ν for the transformation obtained by applying GMC to the positive Borel measure ν. One can actually see that this method applies to all measures by using complex scalars, but it is traditional to apply it to symmetric measures whose Fourier transform is real-valued so that GMC gives a real centered stationary Gaussian process. In all of our applications, we can symmetrize the measures ν that we construct by replacing them with $\nu_s = \nu \star \nu^*$, or with $\nu_s = \nu + \nu^*$. This will allow us to use the GMC in its traditional form, while still preserving the properties that we need the GMC to give us.

Remark 2.27. We will also have occasion to use another general method, that of Poisson suspensions. See Cornfeld, Fomin, and Sinai [8], Kingman [31], and Neretin [47] for information about Poisson suspensions. Here is briefly the idea. Let T be a transformation of a standard Lebesgue space $(X,\mathcal{B},\mu)$, where μ is a σ-finite, infinite positive measure. We define a probability space $(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu})$. The points of the configuration space $\widetilde{X}$ are infinite countable subsets $\widetilde{x} = \{x_n : n \geq 1\}$ of X. Given a set $A \in \mathcal{B}$ of finite measure we define $N_A : \widetilde{X} \to \mathbb{N} \cup \{\infty\}$ by setting

$$N_A(\widetilde{x}) = \#\{n \in \mathbb{N} : x_n \in A\}.$$

Then $\widetilde{\mathcal{B}}$ is defined as the smallest σ-algebra of subsets of $\widetilde{X}$ making all variables N_A, $\mu(A) < +\infty$, measurable. The measure $\widetilde{\mu}$ is the only probability measure (see [31] for details) such that

- the variables N_A satisfy the Poisson law with parameter $\mu(A)$;
- for each family $A_1,\dots,A_k$ of pairwise disjoint subsets of X of finite measure the corresponding variables $N_{A_1},\dots,N_{A_k}$ are independent.

The space $(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu})$ is a standard Lebesgue probability space. Then, we define $\widetilde{T}$ on $\widetilde{X}$ by setting

$$\widetilde{T}(\{x_n\}) = (\{Tx_n\})$$

and obtain a transformation of $(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu})$ which is called the Poisson suspension of T. Then $\widetilde{T}$ is ergodic if and only if T has no non-trivial invariant sets of finite measure. In this case, $\widetilde{T}$ turns out to be weakly mixing and moreover $\widetilde{T}$ is spectrally isomorphic to the GMC transformation G given by the unitary operator U_T acting on $L_2(X,\mathcal{B},\mu)$, i.e. the unitary operators $U_{\widetilde{T}}$ and U_G are equivalent.

Proposition 2.28. *The sequence (n_m) is a rigidity sequence for some weakly mixing dynamical system if and only if there is a continuous Borel probability measure ν on $\mathbb{T}$ such that $\lim\limits_{m\to\infty} \widehat{\nu}(n_m) = 1$.*

Proof. First, if $f \in L_2(X, p)$ is norm one, then rigidity along the sequence (n_m) for f means that we will have $\lim\limits_{m\to\infty} \widehat{\nu_f^T}(n_m) = 1$. So when T is weakly mixing and f is mean-zero, then ν_f^T is continuous.

Conversely, if there is a continuous Borel probability measure ν on $\mathbb{T}$ such that $\widehat{\nu}(n_m) \to 1$ as $m \to \infty$, then the GMC gives us a weakly mixing dynamical system $(X, \mathcal{B}, p, T)$, with $T = G_\nu$, and a mean-zero function $f \in L_2(X, p)$ with $\|f\|_2^2 = 1$ and $\widehat{\nu}(n) = \langle f \circ T^n, f\rangle$ for all n. Indeed, it is not hard to see that this construction gives us a weakly mixing dynamical system such that T is rigid along the sequences (n_m). $\square$

Remark 2.29. The second part of this proof is saying that the GMC preserves rigidity. In particular, if $\nu = \nu^S$ for some $S \in Aut(Y, \mathcal{B}_Y, p_Y)$ then S and $T := G_\nu$ have the same rigidity sequences.

Remark 2.30. The unitary version of this result is that there is a continuous Borel probability measure ν on $\mathbb{T}$ such that $\lim\limits_{m\to\infty} \widehat{\nu}(n_m) = 1$ if and only if there is a unitary operator U on a Hilbert space H such that 1) for no non-zero $v \in H$ is the orbit $\{U^k v : k\mathbb{Z}\}$ precompact in the strong operator topology, and 2) for all $v \in H$, $\lim\limits_{m\to\infty} \|U^{n_m} v - v\|_H = 0$. The first property here 1) is saying that U is called a *weakly mixing unitary operator*. See Bergelson and Rosenblatt [6].

The result in Proposition 2.28 holds with appropriate changes if we ask for the stronger property that we have rigidity along the IP set generated by a sequence. Some notation is useful for understanding this. The *IP set generated by a sequence* consists of all finite sums of elements with distinct indices in the sequence. So the notation $\Sigma = FS(n_m)$ for this IP set makes sense. Here given $m_1 < \ldots < m_k$, let $\sigma = \sigma(n_{m_1}, \ldots, n_{m_k}) = n_{m_1} + \ldots + n_{m_k}$. We say $\sigma \to \infty(IP)$ if $\sigma = m_1$ tends to ∞. Also, given the dynamical system, we say that T is *IP-rigid* along Σ if $T^\sigma \to Id$ in the strong operator topology as $\sigma \to \infty(IP)$. With this understood, it is easy to see the following.

Proposition 2.31. *There is a weakly mixing dynamical system that is IP-rigid along Σ if and only if there is a continuous Borel probability measure ν on $\mathbb{T}$ such that $\widehat{\nu}(\sigma) \to 1$ as $\sigma \in FS(n_m : m \geq 1)$ tends to ∞ (IP).*

There is another way of phrasing the Fourier transform condition above for rigidity, and IP-rigidity, that is very useful. We have already pointed this out in terms of spectral measures in Proposition 2.3. We leave the routine proof to the reader.

Proposition 2.32. *Given a sequence (n_m) and a positive Borel measure on $\mathbb{T}$, we have $\widehat{\nu}(n_m)$ tends to $\nu(\mathbb{T})$ if and only if $\gamma^{n_m} \to 1$ in measure with respect to ν as $m \to \infty$. Also, we have $\widehat{\nu}(\sigma)$ tends to 1 as $\sigma \to \infty$ (IP) if and only if $\gamma^\sigma \to 1$ in measure with respect to ν as $\sigma \to \infty$ (IP).*

Remark 2.33. a) The Fourier transform characterizations of rigidity sequences above suggests that we might be able to take this further by finding the correct growth/sparsity condition on a strictly increasing sequence (n_m) to be a rigidity sequence or IP-rigidity sequence for a weakly mixing dynamical system. First, consider the property of IP-rigidity. It is easy to see, from the spectral measure characterization of IP-rigidity above, that it is sufficient to have a criterion that guarantees there is an uncountable Borel set of points $K \subset \mathbb{T}$ such that for all $\gamma \in K$, we have $\gamma^\sigma \to 1$ as $\sigma \to \infty(IP)$. See the beginning of the proof of Proposition 3.2 where the same point is made. Let us consider this in the parametrization of $\mathbb{T}$ where $\gamma = \exp(2\pi i x)$ for $x \in [0,1)$. We also denote by K the set of x corresponding to $\exp(2\pi i x) \in K$. Our pointwise criterion then means that for all $x \in K$, we have both $\cos(2\pi\sigma x) \to 1$ and $\sin(2\pi\sigma x) \to 0$ as $\sigma \to \infty\,(IP)$. It is enough to just have $\sin(2\pi\sigma x) \to 0$ as $\sigma \to \infty\,(IP)$. Indeed, with the usual notation that $\{z\} = z - \lfloor z \rfloor$ is the fractional part of a real number z, the $\{y = \{2x\} : x \in K\}$ will give an uncountable set of values y such that both $\cos(2\pi\sigma y) \to 1$ and $\sin(2\pi\sigma y) \to 0$ as $\sigma \to \infty\,(IP)$

b) The pointwise spectral property above is equivalent to having rigidity along Σ for ALL functions whose spectral measure in a given dynamical system is supported in K. This is stronger than what is needed for rigidity along Σ for some weakly mixing dynamical system. This weaker notion is equivalent to having a continuous positive measure ν supported on K such that $\exp(2\pi i\sigma x) \to 1$ in measure with respect to ν as $\sigma \to \infty\,(IP)$.

3. Constructions of Rigid Sequences for Weakly Mixing Transformations

We give a number of different approaches here for constructing weakly mixing transformations that have a specific type of sequence as a rigidity sequence. These methods are sometimes overlapping, but the different approaches give us insights into the issues nonetheless. Also, there are a variety of number theoretic and harmonic analysis connections with some of these methods; these are also explored in this section.

When this paper was in the final draft, we learned of the work of Eisner and Grivaux [10]. Our papers are largely complementary, although we do cover some of the same basic issues. We will cite their work more in place later in this section.

Besides the question of the structure of rigidity sequences in general, we would like to be able to answer the following questions:

Questions: Which rigidity sequences of an ergodic transformation with discrete spectrum can be rigidity sequences for weakly mixing transformation? Which rigidity sequences for weakly mixing transformations can be rigidity sequences for an ergodic transformation with discrete spectrum?

Remark 3.1. a) At this time, we do not know if there is a counterexample to either of the questions above.

b) As discussed later in this section, one viewpoint to answering these questions is to consider, for fixed (n_m), the group $\mathcal{R}(n_m) = \{\gamma \in \mathbb{T} : \lim_{m\to\infty} \gamma^{n_m} = 1\}$. We will see that if $\mathcal{R}(n_m)$ is uncountable, then (n_m) is a rigidity sequence for an ergodic rotation of $\mathbb{T}$ and for some weakly mixing transformation. If $\mathcal{R}(n_m)$ is countably infinite, then there is an ergodic transformation T with discrete spectrum such that (n_m) is a rigidity sequence for T, but we do not know in general if there is a weakly mixing transformation with (n_m) as a rigidity sequence. For example, sequences like (2^n) cannot be a rigidity sequence for an ergodic rotation of the circle, but can be a rigidity sequence for a weakly mixing transformation. This does not make this sequence a counterexample to the second question above because this sequence is a rigidity sequence for the ergodic generator of another compact abelian group G. Just take G to be the inverse limit of the finite groups $\mathcal{R}_n = \{\gamma \in \mathbb{T} : \gamma^{2^n} = 1\}$.

c) Our results, in particular Proposition 3.27 or Proposition 3.54, and the structure of subgroups of $\mathbb{T}$ in the discrete topology, show that all rigidity sequences for transformations with discrete spectrum are rigidity sequences for some weakly mixing transformation if the following is true:

given an element $\gamma \in \mathbb{T}$ of infinite order and a sequence (n_m) such that $\gamma^{n_m} \to 1$ as $m \to \infty$, the sequence (n_m) is a rigidity sequence for some weakly mixing transformations.

d) If $\mathcal{R}(n_m)$ is finite, then no ergodic transformation with discrete spectrum has this sequence as a rigidity sequence, but it might be possible to construct weakly mixing transformations with (n_m) as a rigidity sequence. This is not at all clear yet. A good example of a candidate sequence for this case is $n_m = 2^m + 3^m$ for which $\mathcal{R}(n_m) = \{1\}$. To see this, note that $n_{m+1} - 2n_m = 3^m$ and $3n_m - n_{m+1} = 2^m$. So any $\gamma \in \mathcal{R}(n_m)$ must have both γ^{3^m} and γ^{2^m} tending to 1 as $m \to \infty$. Hence, γ is simultaneously a root of unity for a power of 2 and a power of 3, and hence $\gamma = 1$.

A sequence being just density zero is clearly not enough for rigidity for an ergodic transformation because a sequence can be density zero and have infinitely many pairs of terms (n_m, n_{m+1}) with, say, $n_{m+1} - n_m \leq 10$. More sparsity is needed than just density zero. In this direction, lacunary sequences might seem to be good candidates to be rigidity sequences for some weakly mixing transformations because they certainly have the necessary sparseness. We take lacunarity here to be as usual: (n_m) is lacunary if and only if there exists $\rho > 1$ such that $n_{m+1}/n_m \geq \rho$ for all $m \geq 1$. However, even in this class of sequences the situation is not clear as there are lacunary sequences which cannot even be rigidity sequences for any ergodic transformations. For example, (2^n+1) is lacunary but cannot be a rigidity sequence even for an ergodic transformation with discrete spectrum. See Remark 2.23 c). This answers the question in Eisner and Grivaux [10], p. 5.

3.1. **Diophantine approach.**

3.1.1. **Existence of Supports.** Denote by $[x]$ the nearest integer to $x \in \mathbb{R}$, choosing $\lfloor x \rfloor$ if $\{x\} = 1/2$. So $|x - [x]|$ would be the distance of x to $\mathbb{Z}$. We will denote this by $\|x\|$. The distance $\|x\| = \{x\}$ if $\{x\} \leq \frac{1}{2}$ and $\|x\| = 1 - \{x\}$ if $\frac{1}{2} \leq \{x\}$. We have the following result.

Proposition 3.2. *The following are equivalent:*
a) there exists some infinite perfect compact set $K \subset [0,1)$*, such that* $\sin(2\pi\sigma x) \to 0$ *as* $\sigma \to \infty (IP)$ *for all* $x \in K$*,*
b) for some uncountable set of x *values, we have*

$$\sum_{m=1}^{\infty} \|n_m x\| < \infty. \tag{3.1}$$

Proof. The condition $\sum\limits_{m=1}^{\infty} \|n_m x\| < \infty$ describes a Borel set of values x. Hence, the set theoretic aspects of this proposition work because any uncountable Borel set in $[0,1)$ contains an infinite perfect compact subset. See Sierpiński [61], p. 228.

Assume that $\sum\limits_{m=1}^{\infty} \|n_m x\| < \infty$ for an uncountable set of x values. Consider separately the values of $\|n_m x\|$ where it is $\{n_m x\}$ or it is $1 - \{n_m x\}$. In the first case, as $m \to \infty$, $0 \leq \sin(2\pi n_m x) \sim 2\pi\{n_m x\} = 2\pi\|n_m x\|$. In the second case, as $m \to \infty$, $0 \geq \sin(2\pi n_m x) \sim -2\pi(1 - \{n_m x\}) = -2\pi\|n_m x\|$. So Equation (3.1) implies that $\sum\limits_{m=1}^{\infty} \sin(2\pi n_m x)$ converges absolutely for an uncountable set. Using the formula

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

repeatedly, we see that $\sin(2\pi x\sigma) = \sum_{j=1}^{k} f_j \sin(2\pi x n_{m_j})$ with coefficients f_j that suitable products of cosines. Here $|f_j| \leq 1$. So the convergence in Equation (3.1) tells us that for an uncountable set we have $\sin(2\pi\sigma x) \to 0$ as $\sigma \to \infty$ (IP).

Conversely, suppose we have for an uncountable set of x such that $\sin(2\pi\sigma x) \to 0$ as $\sigma \to \infty\,(IP)$. By doubling the x values, we see that this means that for an uncountable set K, if $x \in K$, and $\epsilon > 0$, we can choose $M_\epsilon \geq 1$ such that for all finite sets F of whole numbers, all no smaller than M_ϵ, we have $\|\sum_{m\in F} n_m x\| \leq \epsilon$. In particular, for all $x \in K$, we have $\|n_m x\| \to 0$ as $m \to \infty$. It follows that there is an uncountable set $K_0 \subset K$ and some M_0 such that for all $x \in K_0$ and all finite sets F of whole numbers all no smaller than M_0, we have $\|\sum_{m\in F} n_m x\| \leq \frac{1}{100}$. In particular, if $m \geq M_0$, $\|n_m x_0\| \leq \frac{1}{100}$. Now suppose $\sum_{m=1}^{\infty} \|n_m x_0\| = \infty$ for some $x_0 \in K_0$. Then also $\sum_{m=M_0}^{\infty} \|n_m x_0\| = \infty$. Each $\|n_m x_0\|$ is either $\{n_m x_0\}$ or it is $1 - \{n_m x_0\}$. Say I_1 is the set of $m \geq M_0$ where the first formula holds, and I_2 is the set of $m \geq M_0$ where the second formula holds. Then either $\sum_{m\in I_1} \|n_m x_0\| = \infty$ or $\sum_{m\in I_2} \|n_m x_0\| = \infty$. Assume it is the first case. Then by an upcrossing argument, we can choose a finite set $I \subset I_1$ such that $\sum_{m\in I} \|n_m x_0\| \in (\frac{1}{8}, \frac{3}{8})$. But then $\sin(2\pi \sum_{m\in I} n_m x_0) = \sin(2\pi \sum_{m\in I} \|n_m x_0\|) \geq \frac{1}{\sqrt{2}}$. This is not possible for $x \in K_0$ because we know that $\|\sum_{m\in I} n_m x_0\|$ is small and so $|\sin(2\pi \sum_{m\in I} n_m x_0)| = |\sin(2\pi \|\sum_{m\in I} n_m x_0\|)| \leq 2\pi \|\sum_{m\in I} n_m x_0\| \leq 2\pi \frac{1}{100}$. In the second case, again by an upcrossing argument, we can choose a finite set $I \subset I_2$ such that $\sum_{m\in I} \|n_m x_0\| \in (\frac{1}{8}, \frac{3}{8})$. So then $\sin(2\pi \sum_{m\in I} n_m x_0) = -\sin(2\pi \sum_{m\in I} \|n_m x_0\|) \leq -\frac{1}{\sqrt{2}}$. Again this is not possible for $x \in K_0$ because we know that $\|\sum_{m\in I} n_m x_0\|$ is small and so $|\sin(2\pi \sum_{m\in I} n_m x_0)| = |\sin(2\pi \|\sum_{m\in I} n_m x_0\|)| \leq 2\pi \|\sum_{m\in I} n_m x_0\| \leq 2\pi \frac{1}{100}$. □

This result, and Remark 2.33 a) at the end of Section 2, give this basic result.

Proposition 3.3. *If there exists an uncountable set of x values such that $\sum_{m=1}^{\infty} \|n_m x\| < \infty$, then there is a weakly mixing transformation that is rigid along the IP set generated by (n_m).*

Remark 3.4. It is not clear what growth property for (n_m) corresponds to Equation (3.1) holding for an uncountable set of points. At least this analysis shows that we can use results from Erdős and Taylor [12]. In particular, they show that a sufficient condition for the absolute convergence on an uncountable set that we need is that $\sum_{m=1}^{\infty} n_m/n_{m+1}$ converges. See the beginning of the proof of Proposition 4.8. Sometimes much less of a growth condition is needed. For example, if one knows n_{m+1}/n_m tends to infinity and n_{m+1}/n_m is eventually a whole number, then we have $\sum_{m=1}^{\infty} \|n_m x\|$ converging for an uncountable set of x values. However, Erdős and Taylor [12] also give an example where $\lim_{m\to\infty} n_{m+1}/n_m = \infty$, but the ratio is infinitely often not a whole number, and yet $\sum_{m=1}^{\infty} \|n_m x\|$ converges only for a countable set of values.

Erdős and Taylor [12] observe that an earlier result of Eggleston [11] is relevant here. Eggleston [11] showed that it is necessary to have some hypothesis on (n_m) that prevents n_{m+1}/n_m from being bounded because if these ratios are bounded then one has $\exp(2\pi i n_m x) \to 1$ as $m \to \infty$ for at most a countable set of values x. This fact is related to the weaker version of the result above that is worth observing here, the pointwise criterion needed for rigidity along the sequence itself. The question is: what growth condition on a strictly increasing sequence (n_m) is needed for the sequence to admit a weakly mixing dynamical system for which there is a non-trivial rigid function along (n_m)? The same method as above, using the GMC, shows that it is sufficient to know when there is an uncountable set of points $K \subset [0,1]$ such that for all $x \in K$, we have $\exp(2\pi i n_m x) \to 1$ as $m \to \infty$. Hence, this property can be characterized by having

$$\lim_{m\to\infty} \|n_m x\| = 0 \tag{3.2}$$

for an uncountable set of x values. As with convergence along an IP set, this property is stronger than what is needed to produce just one weakly mixing dynamical system with rigidity along (n_m). This property guarantees that any dynamical system, weakly mixing or not, whose non-trivial spectral measures are supported on a subset of K, would have (n_m) as a rigidity sequence.

These characterizations, Equation (3.2) and Equation (3.1), show the difference between having rigidity along a sequence versus having rigidity along the IP set that the sequence generates for all spectral measures supported in the set. Eggleston shows the following in [11].

Proposition 3.5. *If* $\lim\limits_{m\to\infty} n_{m+1}/n_m = \infty$*, then Equation (3.2) holds for an uncountable set. Hence, if* $\lim\limits_{m\to\infty} n_{m+1}/n_m = \infty$*, then* (n_m) *is a rigidity sequence for some weakly mixing transformation.*

Remark 3.6. It is not necessary to have $\lim\limits_{m\to\infty} n_{m+1}/n_m = \infty$ for Equation (3.2) to hold on an uncountable set. Indeed, it is not hard to construct examples of strictly increasing sequences (n_m) with Equation (3.2) holding on an uncountable set, and yet there are arbitrarily long pairwise disjoint blocks $B_k \subset \mathbb{N}$ such that $n_{m+1}/n_m = 2$ for all $m \in B_k$ and all k. However, as commented above, Eggleston also shows in [11] that for Equation (3.2) to hold on an uncountable set, it is certainly necessary to know that the ratios n_{m+1}/n_m are not uniformly bounded. Also, Eisner and Grivaux [10] show in Proposition 3.8 that one can weaken the hypothesis of Proposition 3.5 to just $\limsup\limits_{m\to\infty} \frac{n_{m+1}}{n_m} = \infty$ if $\frac{n_{m+1}}{n_m}$ is always a whole number. This also follows from using Proposition 3.27.

Remark 3.7. a) Proposition 3.5 allows us to give interesting examples of disjoint weakly mixing dynamical systems with common rigidity sequences. Here disjointness is the standard disjointness from Furstenberg [17], their product is their only non-trivial joining. This property means they also do not have any common factors, and so of course are not isomorphic. We construct weakly mixing T and S as follows. Let $(a_m) = (2^{m^2})$, and let (b_m) be the sequence which is 2^{m^2} for even m and $2^{m^2}+1$ for odd m. Since $a_{m+1}/a_m \to \infty$ and $b_{m+1}/b_m \to \infty$ as $m \to \infty$, by Proposition 3.5 there exists T which is weakly mixing and rigid along (a_m) and S which is weakly mixing and rigid along (b_m). We see that T and S are rigid along the sequence $(2^{(2m)^2})$. Now take ν^T and ν^S to be the maximal spectral types of T and S on $L_{2,0}(X,p)$. If we show that ν^T and ν^S are mutually singular, then by Hahn and Parry [21], T and S are disjoint. But if $\omega \ll \nu^T$ and $\omega \ll \nu^S$, we have ω rigid along both $(2^{(2m+1)^2})$ and $(2^{(2m+1)^2}+1)$. It follows that ω would have to be concentrated at $\{1\}$, which means $\omega = 0$ because $\nu^T(\{1\}) = \nu^S(\{1\}) = 0$.

b) Given ergodic transformations T and S, with T rigid along (a_m) and S rigid along (b_m), such that $b_m = a_m + p(m)$ for a non-zero polynomial, one can argue in the style above, by taking successive differences, that the only spectral overlap of T and S can be with eigenvalues that are d-th roots of unity where $d = p(0)$. So if either T or S is totally ergodic (or even say weakly mixing), then T and S are disjoint.

3.1.2. **Rank One Constructions.** In this section, given an increasing sequence (n_m) such that either $\frac{n_{m+1}}{n_m} \to \infty$ or $\frac{n_{m+1}}{n_m}$ is a whole number for each m, we will explicitly construct an infinite measure-preserving rank one map T such that $T^{n_m} \to Id$ in the strong operator topology. As observed in Remark 2.27, the Poisson suspension gives an example of a weakly mixing finite measure-preserving transformation S such that $S^{n_m} \to Id$ in the strong operator topology. The Poisson suspension is an appealingly natural construction in that no spectral measure intervenes. However, it is also worth noting that by the following lemma U_T automatically has continuous spectrum so that one may apply the GMC to its maximal spectral type to obtain the desired S. We note that any rank one T is necessarily ergodic.

Lemma 3.8. *If T is an ergodic measure-preserving automorphism of an infinite measure space $(X, \mathcal{B}, \mu)$ then U_T has continuous spectrum.*

Proof. Suppose $f \in L_2(X, \mu)$ and $f \circ T = \lambda f$. Then $|\lambda| = 1$ so $|f|$ is T-invariant and it follows that $|f|$ is constant. Since μ is infinite it follows that $f = 0$. $\square$

We use below the notation $a := b$ or $b =: a$ to mean that a is defined to be b. We assume that the reader has some familiarity with rank one constructions but the following is a quick refresher. For more details see [46] or [15]. Suppose T is a rank one map preserving a finite or infinite measure μ and $\{\tau_N\}$ is a refining sequence of rank one towers for T. This means that τ_{N+1} may be viewed as constructed from τ_N by cutting τ_N into columns of equal width and stacking them above each other, with the possible addition of spacer levels between the columns. We will refer to these columns of τ_N as copies of τ_N. The crucial condition that makes T rank one is that the towers $\{\tau_N\}$ are required to converge to the full sigma-algebra of the space in the sense that for any measurable set E of finite measure and $\epsilon > 0$ there is an E' which is a union of levels of some τ_N (and hence of all τ_N for N sufficiently large) such that $\mu(E \triangle E') < \epsilon$. We let X_N denote the union of the levels of τ_N.

Any such T may be realized concretely as a map of an interval $I \subset \mathbb{R}$ as follows. We take X_0 to be a finite sub-interval of $\mathbb{R}$ and let τ_0 be the tower of height 1 consisting of the single level X_0. Now suppose that $\tau_1, \ldots, \tau_N$ have been constructed, each τ_i a tower whose levels are intervals and the union of the levels of each τ_i is an interval X_i. At this point T is partially defined on X_N by mapping each level of τ_N to the level directly above it by the appropriate translation, except for the top level, where T remains undefined as yet. Divide the base of τ_N into q subintervals of equal width w and denote the columns of τ_N over these by $C_1, \ldots, C_q$. Let $r \geq 0$, take S to be an interval of width rw adjacent to the interval X_N, divide S into r spacer intervals of width w, stack $C_1, \ldots, C_q$ in order above each other and interleave the r spacer intervals in any way between, below and above the columns $C_1, \ldots, C_q$. We then define T partially on X_{N+1} using τ_{N+1} in the same way it was defined on X_N and this is evidently consistent with the definition of T on X_N. Thus, in the limit T is almost everywhere defined on $I = \bigcup_{N=1}^{\infty} X_N$ and is evidently rank one. Note that in the concrete model the convergence of τ_N to the Borel σ-algebra of I is automatic.

We let $S_N = X_{N+1} \backslash X_N$ and $\epsilon_N = \frac{\mu(S_N)}{\mu(X_{N+1})}$, the fraction of the levels of τ_{N+1} which are not contained in a level of τ_N; that is, they are spacers added at stage n of the construction. We observe that μ is infinite precisely when $\sum_{N=1}^{\infty} \epsilon_N = \infty$.

For a fixed τ_N we will say a time $N > 0$ is ϵ-rigid for τ_N if for each level E of τ_N we have $\mu(T^N E \triangle E) < \epsilon\mu(E)$. Note that one then has the same inequality for any E which is a union of levels of τ_N. Consequently if N is ϵ-rigid for τ_N then it is also ϵ-rigid for any τ_M, $M \leq N$. We will say that the sequence (n_m) is *rigid for a set E of finite measure* if $\mu(T^{n_m} E \triangle E) \to 0$; and that (n_m) *is rigid for τ_N, N fixed* if (n_m) is rigid for each level of τ_N (equivalently, for the base of τ_N). Finally note that if (n_m) is rigid for every τ_N then T is rigid along (n_m).

Proposition 3.9. *Suppose $\frac{n_{m+1}}{n_m} \to \infty$ as $m \to \infty$ or $\frac{n_{m+1}}{n_m}$ is a whole number, $\frac{n_{m+1}}{n_m} \geq 2$ for all m. Then there exists an infinite measure-preserving, weakly mixing, rank one transformation T such that T is rigid along (n_m).*

Proof. Suppose first that $\frac{n_{m+1}}{n_m} \to \infty$ as $m \to \infty$. We introduce the notation $h_m := n_m$ as, in this case, these will be the heights of the rank one towers we construct. Write $h_{m+1} = q_m h_m + r_m$, $0 \leq r_m < h_m$. Define $p_m < q_m$ to be the least integer $l \geq 0$ such that, $\frac{r_m + l h_m}{h_{m+1}} > \frac{1}{m}$ (p_m may be zero) and let $\epsilon_m = \frac{r_m + p_m h_m}{h_{m+1}}$. Thus we have $\sum_{m=1}^{\infty} \epsilon_m = \infty$. Moreover $\epsilon_m - \frac{1}{m} < \frac{h_m}{h_{m+1}} \to 0$ so $\epsilon_m \to 0$.

Construct T as follows. Start with a tower τ_1 of height h_1 and suppose the towers $\tau_1, \ldots, \tau_m$ have been constructed. Form τ_{m+1} by slicing τ_m into $s_m := q_m - p_m$ columns, stacking these directly above each other and then following them by $h_{m+1} - s_m h_m = r_m + p_m h_m$ spacers to create a tower τ_{m+1} of height h_{m+1}. Thus $\frac{\mu(S_m)}{\mu(X_{m+1})} = \epsilon_m$, and since $\sum_{m=1}^{\infty} \epsilon_m = \infty$ we see that the measure of the space we have constructed is infinite. By Lemma 3.8, T is weakly mixing.

We now check that T is rigid along (h_m). If E is a level of τ_m and $E_{s_1}, \ldots, E_{s_m}$ are its pieces in τ_{m+1} then $T^{h_m} E_i = E_{i+1}$, except for $i = s_m$. It follows that $\mu(T^{h_m} E \backslash E) < \frac{1}{s_m} \mu(E) =: \delta_m \mu(E)/2$ so h_m is δ_m-rigid for τ_m. Since this holds for every m it follows that, for any fixed n, h_m is δ_m rigid for τ_n, for each $m > n$. Since $\delta_m \to 0$, it follows that for each fixed m, (h_m) is rigid for τ_m. So T is rigid along (h_m). This concludes the argument in case $\frac{n_{m+1}}{n_m} \to \infty$.

Now suppose that $\frac{n_{m+1}}{n_m}$ is a whole number as large as 2 for all m. For simplicity we will consider only the case $n_m = 2^m$. The general case is no more difficult. Let $h_m = 2^{m^2}$ so $q_m := h_{m+1}/h_m = 2^{2m+1} \to \infty$. Let $p_m \geq 0$ be the least integer r so that $\frac{r h_m}{h_{m+1}} \geq \frac{1}{m}$ and let $\epsilon_m = \frac{p_m h_m}{h_{m+1}}$. As before we have $\sum_{m=1}^{\infty} \epsilon_m = \infty$ and $\epsilon_m \to 0$.

We construct the towers τ_m for T as before, by concatenating $s_m := q_m - p_m$ copies of τ_m and adding $p_m h_m$ spacers to get the tower τ_{m+1} of height h_{m+1}. As before the space on which T acts has infinite measure and we need only check the rigidity of the sequence $(n_m) = (2^m)$.

Now suppose E is a level of τ_m and $E_1, \ldots, E_l$, $l = s_m s_{m+1}$, are its pieces in τ_{m+2}. These occur with period h_m in τ_{m+2}, except for gaps corresponding to the spacers in S_m and S_{m+1}. More precisely, let us divide τ_{m+2} into $q_m q_{m+1}$ blocks of length h_m and also into q_{m+1} blocks of length h_{m+1} and refer to these as m-blocks and $(m+1)$-blocks respectively. Each m-block is contained in either X_m, S_m or S_{m+1}. Call these three types X_m-blocks, S_m-blocks and S_{m+1}-blocks and let the numbers of the three types be $a = s_m s_{m+1}, b = p_m s_{m+1}$ and $c = q_m p_{m+1}$.

Now suppose that $M > 0$ and let $m = m_M > 0$ be the integer such that $m^2 \leq M < (m+1)^2$. Since there is at least one $(m+1)$-block at the top of τ_{m+2} which is contained in S_{m+1} we see that for each i, $1 \le i \le l$, $T^{2^M} E_i$ is still a level of τ_{m+2}. Thus, if it is not contained in E it must lie in an S_m-block or an S_{m+1}-block. It follows that

$$\frac{\mu(T^{2^M}E\backslash E)}{\mu(E)} \le \frac{b+c}{a} = \frac{p_m s_{m+1} + q_m p_{m+1}}{s_m s_{m+1}}$$
$$= \frac{p_m}{s_m} + (\frac{q_m}{s_m})(\frac{p_{m+1}}{s_{m+1}}) = \epsilon_m + \frac{1}{1-\epsilon_m}\epsilon_{m+1} =: \delta_m/2.$$

This shows 2^M is δ_{m_M}-rigid for τ_{m_M}. Fixing any $k \le n_M$, it follows that 2^M is δ_{m_M}-rigid for τ_k. Letting $M \to \infty$ we have $m_M \to \infty$ and $\delta_{m_M} \to 0$. So we see that the sequence $\{2^M\}$ is rigid for τ_k, and since k is arbitrary it follows that T is rigid along (2^M), as desired.

□

We also note that there is a special case of Proposition 3.9 where we can provide a direct construction of a finite measure-preserving rank one T such that $T^{n_m} \to Id$.

Proposition 3.10. *Suppose that $n_{m+1} = q_m n_m + r_m$, $0 \le r_m < n_m$, $q_m \to \infty$, $\sum_{n}^{\infty} \frac{r_m}{n_{m+1}} < \infty$ and $r_m \ne 0$ infinitely often. Then there is a finite measure-preserving, weakly mixing, rank one transformation T that is rigid along (n_m).*

Proof. We construct the rank one towers τ_m of height $h_m := n_m$ for T as follows. Start with a tower of height h_1. When $r_m = 0$, we construct τ_{m+1} from τ_m by simply concatenating q_m copies of τ_m to obtain the tower τ_{m+1} of height h_{m+1}. When $r_m \ne 0$, we place $a_m := [q_m/3]$ consecutive copies of τ_m followed by one spacer, followed by $q_m - a_m$ consecutive copies of τ_m, followed by $r_m - 1$ spacers, again giving τ_{m+1} of height h_{m+1}. The resulting T is finite measure-preserving because we have assumed $\sum_{m=1}^{\infty} \frac{r_m}{h_{m+1}} < \infty$ and it is very easy see that T is rigid along (h_m).

We now check that T is weakly mixing. Suppose that $f \in L_2(X,\mu)$ and $f \circ T = \lambda f$. Without loss of generality $|f| = 1$. Given $\epsilon > 0$, find $n = n_m$ such that $r_m \ne 0$, and f' which is a linear combination of the characteristic functions of the levels of τ_m such that $\|f - f'\|_2 < \epsilon$. In addition we may assume that $|f(x)| = 1$ for all $x \in X_m$. We agree to write $g \overset{\delta}{\sim} h$ whenever $g, h \in L_2(X,\mu)$ and $\|g - h\|_2 < \delta$. Note that for any k we have

$$f' \circ T^k \overset{\epsilon}{\sim} f \circ T^k = \lambda^k f \overset{\epsilon}{\sim} \lambda^k f'.$$

Let E_1 denote the union of the first $s_m := a_m h_m$ levels of τ_{m+1} and E_2 the union of the s_m levels after the first spacer in τ_{m+1}. We observe that

$$f'|_{E_1} = (f' \circ T^{s_m+1})|_{E_1} \overset{2\epsilon}{\sim} \lambda^{s_m+1} f'|_{E_1}.$$

It follows that $\lambda^{s_m+1} \overset{2\epsilon/\|f'|_{E_1}\|_2}{\sim} 1$. By taking m sufficiently large we may assume that $\mu(E_1) \ge \frac{1}{4}$ and so

$$\|f'|_{E_1}\|_2 = \sqrt{\mu(E_1)} > \frac{1}{2}.$$

Thus, $\lambda^{s_m+1} \overset{4\epsilon}{\sim} 1$. A similar argument with E_2 replacing E_1 shows that $\lambda^{s_m} \overset{4\epsilon}{\sim} 1$ so we get $\lambda^{s_m+1} \overset{8\epsilon}{\sim} \lambda^{s_m}$. Since $|\lambda| = 1$ it follows that $\lambda \overset{8\epsilon}{\sim} 1$ and since $\epsilon > 0$ is arbitrary we conclude that $\lambda = 1$.

□

3.1.3. **Rates of Growth.** Here is some information on the question of rates of growth of the gaps in a rigidity sequence. These results show in various ways that although rigidity sequences (n_m) have the gaps $n_{m+1} - n_m$ tending to infinity, they do not need to have these gaps growing quickly. Indeed, there is no rate, no matter how slow, that these gaps must grow for either ergodic rotations of the circle or weakly mixing transformations.

Suppose we have an increasing sequence $\mathbf{n} = (n_m)$. We let

$$D(N, \mathbf{n}) = \frac{\#(\{n_m : m \geq 1\} \cap \{1, \dots, N\})}{N}.$$

We say that $\mathbf{n}$ has *density zero* if $D(N, \mathbf{n}) \to 0$ as $N \to \infty$. The following result can be improved, see Corollary 3.24 below. We prove this here because it gives insight into the ideas in Proposition 3.16.

Proposition 3.11. *Given any sequence $(d_N : N \geq 1)$ such that $d_N \to 0$ as $N \to \infty$, and any ergodic rotation T of $\mathbb{T}$, there exists a rigidity sequence $\mathbf{n} = (n_m)$ for T such that $D(N, \mathbf{n}) > d_N$ for infinitely many $N \geq 1$.*

Proof. We have some $\gamma \in \mathbb{T}$ of infinite order such that $T(\alpha) = \gamma\alpha$ for all $\alpha \in \mathbb{T}$. Because $d_N \to 0$ as $N \to \infty$, we can choose an increasing sequence (N_k) such that for all $N \geq N_k$, we have $d_N \leq \frac{1}{4^k}$. Now we construct a suitable (n_m) that is rigid for T. First, we can inductively choose an increasing sequence (M_k) so that we have $\#\{n \in [1, M_k] : |\gamma^n - 1| \leq \frac{1}{2^k}\} \geq \frac{M_k}{2^k}$. This is possible because the Lebesgue measure of the arc $\{\alpha : |\alpha - 1| \leq \frac{1}{2^k}\}$ is $\frac{2}{2^k}$ and $(\gamma^n : n \geq 1)$ is uniformly distributed in $\mathbb{T}$. In the process of choosing (M_k), there is no obstruction to taking each $M_k \geq N_k$. Now let (n_m) be the increasing sequence whose terms are $\bigcup\limits_{k=1}^{\infty} \{n \in [1, M_k] : |\gamma^n - 1| \leq \frac{1}{2^k}\}$. By the construction, (n_m) is rigid for T. Now we claim that $D(N, \mathbf{n}) \geq d_N$ for infinitely many N. Indeed, $D(M_k, \mathbf{n}) \geq \frac{1}{2^k}$ by the choice of M_k and the definition of $\mathbf{n}$. However, since $M_k \geq N_k$, we have $d_{M_k} \leq \frac{1}{4^k}$. ☐

Corollary 3.12. *Given any sequence $G(m)$ tending to infinity and any ergodic rotation T of $\mathbb{T}$, there exists a rigidity sequence (n_m) for T such that $\limsup\limits_{m\to\infty} \frac{G(m)}{n_{m+1} - n_m} = \infty$.*

Proof. We have some $\gamma \in \mathbb{T}$ of infinite order such that $T(\alpha) = \gamma\alpha$ for all $\alpha \in \mathbb{T}$. Take $g(m)$ tending to infinity.

Claim: There is a sequence (d_N) tending to zero, determined by g alone, such that for any sequence $\mathbf{n} = (n_m)$ that has $n_{m+1} - n_m \geq g(m)$ for all $m \geq 1$, we would have $D(N, \mathbf{n}) \leq d_N$ for all $N \geq 1$.

Proof of Claim: Observe that among sequences with $n_{m+1} - n_m \geq g(m)$ for all $m \geq 1$, $\#\{n_m \leq N\}$ is largest in the case that we take the explicit sequence $n_1 = 1$ and $n_{m+1} = n_m + \lceil g(m) \rceil$ for all $m \geq 1$. So, take this as our sequence. Let $g(0) = 0$. Then

$$\#\{n_m \leq N\} \leq \sup\{m \geq 1 : 1 + \sum_{k=0}^{m-1} \lceil g(k) \rceil \leq N\}.$$

Thus, let $d_N = \frac{\sup\{m : 1 + \sum\limits_{k=0}^{m-1} \lceil g(k) \rceil \leq N\}}{N}$, which tends to zero as $N \to \infty$ because $g(m) \to \infty$ as $m \to \infty$. We have $D(N, \mathbf{n}) \leq d_N$ for all $N \geq 1$.

Continuing now with our proof, for any sequence $\mathbf{n} = (n_m)$ that has $n_{m+1} - n_m \geq g(m)$ for all $m \geq M$, then there exists N_M such that $D(N, \mathbf{n}) \leq 2d_N$ for all $N \geq N_M$. Using $(2d_N)$ in place of (d_N), Proposition 3.11 and Claim 3.12 above how that we can construct a rigidity sequence (n_m) for T such that $n_{m+1} - n_m < g(m)$ for infinitely many m. Now, for any $G(m)$ increasing to ∞, we can construct $g(m)$ tending to ∞ so that $\lim\limits_{m\to\infty} \frac{G(m)}{g(m)} = \infty$. Using this g above, we have shown that there is a rigid sequence (n_m) for T such that $\limsup\limits_{m\to\infty} \frac{G(m)}{n_{m+1}-n_m} = \infty$. □

Corollary 3.13. *For any ergodic rotation T of $\mathbb{T}$, there is a rigidity sequence (n_m) such that $\liminf\limits_{m\to\infty} \frac{n_{m+1}}{n_m} = 1$.*

Proof. Let $G(m) = \sqrt{m}$. Then use Corollary 3.12 to construct a rigidity sequence for T such that $n_{m+1} - n_m \leq \sqrt{m}$ infinitely often. Since $n_m \geq m$, we have $\liminf\limits_{m\to\infty} \frac{n_{m+1}}{n_m} = 1$. □

Our next result, and some that follow, show that although rigidity sequences are sparse sets, they are not always *thin sets* in certain senses that are commonly used in harmonic analysis. See Lopez and Ross [43] for background information on thin sets in harmonic analysis. In particular, we will see that rigidity sequences are not always *Sidon sets*. By a Sidon set here we mean a subset $\mathcal{S}$ of the integers such that given any bounded complex-valued function ψ on $\mathcal{S}$, there exists a complex-valued Borel measure ν on $\mathbb{T}$ such that $\widehat{\nu} = \psi$ on $\mathcal{S}$. Originally, this property was observed for lacunary sets, and finite unions of lacunary sets, but the general notion of Sidon sets gives a larger class of sets to work with that in a general sense will have similar harmonic analysis properties.

Corollary 3.14. *Given any ergodic rotation T of $\mathbb{T}$, there exists a rigidity sequence for T which is not a Sidon set, and so is not the union of a finite number of lacunary sequences.*

Proof. It is a standard fact that finite unions of lacunary sequences are Sidon sets. See for example [43]. Also in [43], Corollary 6.11, is the proof that if $\mathbf{n} = (n_m)$ is a Sidon set then there is a constant C such that $D(N, \mathbf{n}) \leq \frac{C \log N}{N}$. By the argument above, there exists some $d_n \to 0$ as $n \to \infty$ such that for all C, eventually $d_N \geq C \frac{\log N}{N}$. Using this (d_N), construct a rigidity sequence $\mathbf{n}$ for T as in Proposition 3.11. This choice of (d_N) shows that $\mathbf{n}$ is not a Sidon set. □

Remark 3.15. We did not need it here, but sometimes when dealing with classes of sequences, it is good to have a result as follows. Suppose we have a sequence of sequences $(d_N(s) : N \geq 1)$ where $d_N(s) \to 0$ as $N \to \infty$ for every s. Then there exists a sequence (d_N) which also has $d_N \to 0$ as $N \to \infty$, but also for all s, $d_N \geq d_N(s)$ for large enough N. This is a standard result. First, let $d_N^*(k) = \max(d_N(1), \ldots, d_N(k))$. Then for all k, again $d_N^* \to 0$ as $N \to \infty$. Choose an increasing sequence (N_k) such that $d_N^*(k) \leq \frac{1}{2^k}$ all $N \geq N_k$. Let $d_N = 1$ for all $1 \leq N < N_1$, and for $k \geq 1$, let $d_N = d_N^*(k)$ for $N_k \leq N < N_{k+1}$. Then $d_N \leq \frac{1}{2^k}$ for $N \geq N_k$. Also, for all j, $d_N \geq d_N^*(k) \geq d_N(j)$ for any $N \geq N_k$ with $k \geq j$.

Now we extend the construction above to give weakly mixing transformations with rigidity sequences that satisfy similar slow decay properties.

Proposition 3.16. *Given any sequence $(d_N : N \geq 1)$ such that $d_N \to 0$ as $N \to \infty$, there exists a weakly mixing transformation and a rigidity sequence $\mathbf{n} = (n_m)$ for T such that $D(N, \mathbf{n}) > d_N$ for infinitely many N.*

Proof. To carry out this construction, we start with a closed perfect set $\mathcal{K}$ in $\mathbb{T}$ such that every finite set $F \subset \mathcal{K}$ generates a free abelian group of order $\#F$. Except that this is in the circle, it is the same as constructing a closed perfect set $\mathcal{K}$ in $[0, 1]$ of rationally independent real numbers modulo 1 (i.e. $\mathcal{K} \cup \{1\}$ is rationally independent in the real numbers). See

Rudin [57]. The rational independence tells us that for any $\gamma_1,\dots,\gamma_L \in \mathcal{K}$, the sequence of powers $(\gamma_1^n,\dots,\gamma_L^n), n \ge 1$ is uniformly distributed in $\mathbb{T}^L$.

Because $d_N \to 0$ as $N \to \infty$, we can choose an increasing sequence (N_k) such that for all $N \ge N_k$, we have $d_N \le (\frac{1}{4^k})^{4^k}$.

We now inductively construct K, a closed subset of $\mathcal{K}$, and (M_k) with certain properties. This will be a Cantor set type of construction. First, choose distinct $\gamma(i_1), i_1 = 1,2$, in $\mathcal{K}$. Then choose $M_1 \ge N_1$ so that $\#\{n \in [1,M_1] : \text{for}\, i_1 = 1,2,\ |\gamma(i_1)^n - 1| \le \frac{1}{3}\} \ge (\frac{2}{3})^2 \frac{M_1}{2}$. This is possible because the set in $\mathbb{T}^2$ consisting of $(\alpha_1(1),\alpha_1(2))$ with $|\alpha_1(i)-1| \le \frac{1}{3}$ for both $i = 1,2$ has Lebesgue measure $(\frac{2}{3})^2$, and $((\gamma(1),\gamma(2))^n : n \ge 1)$ is uniformly distributed in $\mathbb{T}^2$. We then choose two disjoint closed arcs $B(i_1), i_1 = 1,2$ with $\gamma(i_1) \in \operatorname{int}(B(i_1))$ for $i_1 = 1,2$ and such that for any $\omega(i_1) \in B(i_1)$, and $n \in [1,M_1]$ such that $|\gamma(i_1)^n - 1| \le \frac{1}{3}$ for $i_1 = 1,2$, we have the somewhat weaker inequality $|\omega(i_1)^n - 1| \le \frac{2}{3}$. Let $A(i_1) = B(i_1) \cap \mathcal{K}$ for $i_1 = 1,2$. Let $A(i_0) = K_0 = \mathcal{K}$.

We have to continue this inductively. Suppose for fixed $k \ge 1$, we have constructed K_l and $M_l \ge N_l$ for all $l = 1,\dots,k$ with the following properties. Each $K_l \subset K_{l-1}$ and each K_l is a union of a finite number of pairwise disjoint, non-empty, closed perfect sets $A(i_1,\dots,i_l) = B(i_1,\dots,i_l) \cap \mathcal{K}$, given by $i_j = 1,2$ for all $j = 1,\dots,l$. Here the sets $B(i_1,\dots,i_l)$ are closed arcs , with $\operatorname{int}(B(i_1,\dots,i_l)) \cap \mathcal{K}$ not empty, for all $i_j = 1,2, j = 1,\dots,l$. For each $(i_1,\dots,i_l)$, we have $A(i_1,\dots,i_l) \subset A(i_1,\dots,i_{l-1})$. In addition, consider the set E_l of $n \in [1,M_l]$ such that for all $\omega(i_1,\dots,i_l) \in B(i_1,\dots,i_l)$, we have $|\omega(i_1,\dots,i_l)^n - 1| \le \frac{2}{3^l}$. We assume inductively that $\#E_l \ge (\frac{2}{3^l})^{2^l} \frac{M_l}{2}$.

Now, for each $(i_1,\dots,i_k)$, choose two distinct $\gamma(i_1,\dots,i_k,i_{k+1}) \in A(i_1,\dots,i_k)$ where $i_{k+1} = 1,2$. We can choose $\gamma(i_1,\dots,i_k,i_{k+1}) \in \operatorname{int}(B(i_1,\dots,i_k))$. Then the point p in $\mathbb{T}^{2^{k+1}}$ with coordinates $\gamma(i_1,\dots,i_k,i_{k+1})$, listed in any order, has $(p^n : n \ge 1)$ uniformly distributed in $\mathbb{T}^{2^{k+1}}$. So, there exists $M_{k+1} \ge N_{k+1}$ such that $\#E_{k+1} \ge (\frac{2}{3^{k+1}})^{2^{k+1}} \frac{M_{k+1}}{2}$ where E_{k+1} is the set of $n \in [1,M_{k+1}]$ such that for all $(i_1,\dots,i_{k+1})$ we have $|\gamma(i_1,\dots,i_{k+1})^n - 1| \le \frac{1}{3^{k+1}}$. Choose pairwise disjoint closed arcs $B(i_1,\dots,i_{k+1})$ with $\gamma(i_1,\dots,i_{k+1}) \in \operatorname{int}(B(i_1,\dots,i_{k+1}))$ such that we have the following. Consider any $\omega(i_1,\dots,i_{k+1}) \in B(i_1,\dots,i_{k+1})$, and any $n \in [1,M_{k+1}]$ such that $|\gamma(i_1,\dots,i_{k+1})^n - 1| \le \frac{1}{3^{k+1}}$. Then we have the somewhat weaker inequality $|\omega(i_1,\dots,i_{k+1})^n - 1| \le \frac{2}{3^{k+1}}$. There is no difficulty in also having $B(i_1,\dots,i_{k+1}) \subset B(i_1,\dots,i_k)$ because we chose $\gamma(i_1,\dots,i_k,i_{k+1}) \in \operatorname{int}(B(i_1,\dots,i_k))$. We now let

$$A(i_1,\dots,i_{k+1}) = B(i_1,\dots,i_{k+1}) \cap \mathcal{K}.$$

Since $B(i_1,\dots,i_{k+1}) \subset B(i_1,\dots,i_k)$, we have $A(i_1,\dots,i_{k+1}) \subset A(i_1,\dots,i_k)$. Let $K_{k+1} = \bigcup\limits_{(i_1,\dots,i_{k+1})} A(i_1,\dots,i_{k+1})$. This completes the inductive step.

To finish this construction, let $K = \bigcap\limits_{k=1}^{\infty} K_k$. Then K is a closed perfect subset of $\mathcal{K}$. Let $\mathbf{n} = (n_m)$ be the increasing sequence whose terms are all values of n such that $n \in [1,M_k]$ for some k, and for all $\omega(i_1,\dots,i_k) \in A(i_1,\dots,i_k)$, we have $|\omega(i_1,\dots,i_k)^n - 1| \le \frac{2}{3^k}$. By the construction, for all $\omega \in K$, we have $\omega^{n_m} \to 1$ as $m \to \infty$. Indeed, given $m \ge 1$ choose k_m to be the largest k so that $n_m > M_k$. Notice that $k_m \to \infty$ when $m \to \infty$, and the term n_m was chosen from $[1,M_{s_m}]$ with $s_m > k_m$ and satisfying $|\omega(i_1,\dots,i_{s_m})^{n_m} - 1| < \frac{2}{3^{s_m}}$ for all $\omega(i_1,\dots,i_{s_m}) \in B(i_1,\dots,i_{s_m})$. In particular, if $\omega \in \mathcal{K}$, then $|\omega^{n_m} - 1| \le \frac{2}{3^{s_m}} \le \frac{2}{3^{k_m}}$ because $\omega \in B(i_1,\dots,i_{s_m})$ for some choice of $(i_1,\dots,i_{s_m})$.

Hence, for any Borel probability measure ν supported in K, we have $\widehat{\nu}(n_m) \to 1$ as $m \to \infty$. Since K is a closed perfect set, there are continuous Borel probability measures ν supported in K. Take the symmetrization of any such measure and use the GMC to construct the corresponding weakly mixing transformation T. Then $\mathbf{n}$ is a rigidity sequence for T. But also we claim that $D(N,\mathbf{n}) > d_N$ for infinitely many N. Indeed, our construction guarantees that $D(M_k,\mathbf{n}) = \#\{m : n_m \in [1,M_k]\} \geq \frac{1}{2}(\frac{2}{3^k})^{2^k}$ for all k. But also, because we have chosen $M_k \geq N_k$ for all k, we have $d_{M_k} \leq (\frac{1}{4^k})^{4^k}$. Therefore, $d_{M_k} < D(M_k,\mathbf{n})$. □

Corollary 3.17. *Given any sequence $G(m)$ tending to infinity, there exists a weakly mixing transformation and a rigidity sequence (n_m) for T such that* $\limsup\limits_{m\to\infty} \frac{G(m)}{n_{m+1}-n_m} = \infty$.

Proof. The proof proceeds in the same manner as Corollary 3.12. □

Using the same argument as given in Corollary 3.13, one can see from Corollary 3.17 that there is a weakly mixing transformation T and a rigidity sequence (n_m) for T such that $\liminf\limits_{m\to\infty} \frac{n_{m+1}}{n_m} = 1$. See also Remark 3.50 for another construction of this type. However, this is actually a pervasive principle.

Proposition 3.18. *Given any rigid weakly mixing transformation T, there is a rigidity sequence (n_m) for T such that* $\liminf\limits_{m\to\infty} \frac{n_{m+1}}{n_m} = 1$.

Proof. Take a rigidity sequence (N_m) for T. We can replace this by a subsequence so that the IP set it generates when written in increasing order is also a rigidity sequence (n_m) for T. We can also arrange that this IP set is sufficiently rarified (by excluding more terms from (N_m) if necessary) so that (n_m) has $\liminf\limits_{m\to\infty} \frac{n_{m+1}}{n_m} = 1$. □

Remark 3.19. The syndetic nature of recurrence noted at the beginning of the proof of Proposition 3.23 below shows that the above can be modified to give a rigidity sequence (n_m) for T, either in the ergodic rotation case, or the weakly mixing case, for which the ratios $\frac{n_{m+1}}{n_m}$ are near one for arbitrarily long blocks of values m, infinitely often.

Remark 3.20. In Example 3.18, Eisner and Grivaux [10] construct a weakly mixing transformation T and a rigidity sequence (n_m) for T such that $\lim\limits_{m\to\infty} \frac{n_{m+1}}{n_m} = 1$. However, their example does not necessarily give the type of upper density rate results of Proposition 3.16 and Proposition 3.17.

The techniques used in the above constructions give the following important consequence in Corollary 3.21. If one looks at all of the other constructions of rigidity sequences given in this article, one might expect the opposite of what Corollary 3.21 gives us. Also, given this type of example of a rigidity sequence for a weakly mixing transformation, and the others in this article, it seems that it may be very difficult to characterize these sequences in any simple structural fashion. See also Remark 2.23, c) for a different viewpoint on the issue of characterizing rigidity sequences.

Corollary 3.21. *There is a weakly mixing transformation T and a rigidity sequence (n_m) for T such that (n_m) is not a Sidon set, and so is not the union of a finite number of lacunary sequences.*

Proof. The proof is just like the proof of Corollary 3.14, only here we use Proposition 3.16 instead of Proposition 3.11. □

Remark 3.22. a) Aaronson [1] gives constructions of rigidity sequence which are as dense as possible given the constraint of having gaps tending to infinity. His technique is different than the ones used here. But his results are certainly related to the ones here. In particular, it follows

from his result that there are rigidity sequences for some weakly mixing transformations that are not Sidon sets.
b) Again, Example 3.18 in Eisner and Grivaux [10] gives a rigidity sequence for a weakly mixing transformation such that $\lim\limits_{m\to\infty} \frac{n_{m+1}}{n_m} = 1$. Such a sequence cannot be a Sidon set because its density $D(N, \mathbf{n})$ is not bounded by $C \log N/N$ for any constant C.

There are other types of sequences that our technique here could apply to, and show that they cannot characterize rigidity sequences. For example, consider the sequences studied by Erdős and Turán [13], which they call Sidon sets but are now given a different name (they are called $\mathcal{B}_2$ sequences) because of the current use of the term Sidon sets mentioned above. They show there sets have density at most $C\frac{N^{1/4}}{N}$. So again, we can construct rigidity sequences that cannot be a finite union of such $\mathcal{B}_2$ sequences.

Another direction we can seek for constructing special rigidity sequences (n_m) is to try and construct them with $n_m \le \Psi(m)$ where $\Psi(m)$ is growing slowly, or to prove instead that this would force $\Psi(m)$ to grow quickly. First, we have this result.

Proposition 3.23. *Suppose $\Psi(m) \ge m$ for all $m \ge 1$, and $\lim\limits_{m\to\infty} \frac{\Psi(m)}{m} = \infty$. Then for any ergodic rotation T of $\mathbb{T}$, there is a rigidity sequence (n_m) and a constant C such that $n_m \le C\Psi(m)$ for all $m \ge 1$.*

Proof. First choose $\gamma \in \mathbb{T}$ of infinity order so that $T(\alpha) = \gamma\alpha$ for all $\alpha \in \mathbb{T}$. Fix open arcs A_s centered on 1 with $\lambda_{\mathbb{T}}(A_s) = \epsilon_s$, where (ϵ_s) is a sequence decreasing to 0 as $s \to \infty$. For each s, the sequence $(n \ge 1 : \gamma^n \in A_s)$ is syndetic. That is, there is some $N_s \ge 1$ such that for all M, there exists $n \in [M+1, \ldots, M+N_s]$ such that $\gamma^n \in A_s$. The syndetic property here is automatic from minimality. But it is easy to see this explicitly in this case. Indeed, fix an open arc B_s centered at 1 with $\lambda_{\mathbb{T}}(B_s) = \epsilon_s/4$. Then $\{\gamma^n B_s : n \ge 1\}$ covers $\mathbb{T}$ and so by compactness there exists $(\gamma^{n_1} B_s, \ldots, \gamma^{n_K} B_s\}$ which also covers $\mathbb{T}$. But then, for all M, there exists some n_k such that $\gamma^{n_k} B_s \cap \gamma^{-M} B_s \ne \emptyset$. Hence, $\gamma^{n_k+M} \in B_s B_s^{-1} \subset A_s$. So if we let $N_s = \max\{n_1, \ldots, n_K\}$, then there exists $n \in [M+1, \ldots, M+N_s]$ such that $\gamma^n \in A_s$, i.e. $|\gamma^n - 1| < \epsilon_s/2$.

By replacing Ψ by a more slowly growing function, we can arrange without loss of generality for the additional property that $\Psi(m)/m$ is increasing. We write $\Psi(m) = m\theta(m)$. Let $C \ge N_1$ and take an increasing sequence of whole numbers $(K_L : L \ge 1)$ such that $\theta(K_L) \ge N_{L+1}$ for all $L \ge 1$. Let $K_0 = N_0 = 0$. Consider the blocks in the integers, for $L \ge 1$, of the form

$$B(L,j) = [K_0N_0 + \ldots + K_{L-1}N_{L-1} + jN_L + 1, K_0N_0 + \ldots + K_{L-1}N_{L-1} + (j+1)N_L]$$

where $j = 0, \ldots, K_L - 1$. Then we have a sequence of K_L blocks of length N_L. So we can choose $n(L,j) \in B(L,j)$ such that $|\gamma^{n(L,j)} - 1| < \epsilon_L/2$. Let (n_m) be the increasing sequence consisting of all such choices $n(L,j)$ where $L \ge 1$ and $j = 0, \ldots, K_L - 1$. By construction, (n_m) is a rigidity sequence for T.

We want to show that $n_m \le Cm\theta(m)$ for all m. But the values of m here are of the form $K_0 + \ldots + K_{L-1} + j + 1$. For each such m, the corresponding n_m is being chosen in the interval $B(L,j)$. So, it is sufficient for us to prove that

$$K_0N_0 + \ldots + K_{L-1}N_{L-1} + (j+1)N_L \le C\Psi(K_0 + \ldots + K_{L-1} + j + 1).$$

For $L = 1$, we need to have $(j+1)N_1 \le C\Psi(j+1)$. But this follows since $\Psi(m) \ge m$ for all $m \ge 1$, and $C \ge N_1$. For $L \ge 2$, we see that the inequality is

$$K_1N_1 + \ldots + K_{L-1}N_{L-1} + (j+1)N_L \le C(K_1 + \ldots + K_{L-1} + j + 1)\theta(K_1 + \ldots + K_{L-1} + j + 1).$$

But we have

$$C(K_1 + \ldots + K_{L-1} + j + 1)\theta(K_1 + \ldots + K_{L-1} + j + 1) \geq$$
$$(K_1 + \ldots + K_{L-1} + j + 1)\theta(K_{L-1}) \geq (K_1 + \ldots + K_{L-1} + j + 1)N_L$$
$$\geq K_1 N_1 + \ldots + K_L N_{L-1} + (j + 1)N_L.$$

□

Corollary 3.24. *Given any sequence $(d_N : N \geq 1)$ such that $d_N \to 0$ as $N \to \infty$, and any ergodic rotation T of $\mathbb{T}$, there exists a rigidity sequence $\mathbf{n} = (n_m)$ for T and a constant $c > 0$ such that $D(N, \mathbf{n}) > c d_N$ all $N \geq 1$.*

Proof. We assume without loss of generality that (d_N) is decreasing. Take the sequence (n_m) constructed in Proposition 3.23 with $\Psi(m) = \frac{m}{d_{m-1}}$ for $m \geq 2$. Fix N with $n_m \leq N < n_{m+1}$. Then

$$\begin{aligned}\frac{\#\{n_k : k \geq 1\} \cap \{1, \ldots, N\}}{N} &= \frac{m}{N} \geq \frac{m}{n_{m+1}} \\ &\geq \frac{m}{C\Psi(m+1)} \geq \frac{d_m}{2C} \\ &\geq \frac{d_{n_m}}{2C} \geq \frac{d_N}{2C}.\end{aligned}$$

□

Remark 3.25. The arguments above for an ergodic rotation of the circle can easily be generalized to an ergodic rotation of any compact metric abelian group G. The details of this are a straightforward generalization of the arguments given here. Indeed, an ergodic rotation T of G is given by $T(g) = g_0 g$ for some $g_0 \in G$ that generates a dense subgroup of G. But then also for any U, an open neighborhood of the identity in G, we have $\{n \geq 1 : g_0^n \in U\}$ is a syndetic sequence. Again, the syndetic property here is automatic from minimality. But it is easy to see this explicitly in this general case by a proof similar to the one given at the beginning of Proposition 3.23. Another approach to Proposition 3.23 and Corollary 3.24, and their generalizations to compact metric abelian groups, would be to use discrepancy estimates and the fact the $\{g_0^n : n \geq 1\}$ is uniformly distributed in G. See Kuipers and Niederreiter [35] for background information about uniform distribution in compact abelian groups.

Remark 3.26. It has not yet been possible (and may not be) to carry out a construction as in Proposition 3.23 or Corollary 3.24 for some weakly mixing transformation.

3.2. **Symbolic approach.** In this section, we use shifts on products of finite spaces, a standard model that appears in symbolic dynamics. We look at a construction of rigid sequences for weakly mixing transformations that is given by taking the usual coordinate shift on the product of certain finite sets, and giving a careful construction of a measure on this product space. This will allow us to show that certain types of sequences (n_m) can be rigidity sequences for weakly mixing transformations, even though the sequences do not have the pointwise behavior of the sequences in Section 3.1 i.e. the set $\mathcal{R}(n_m)$ contains no elements at all in $\mathbb{T}$ of infinite order, let alone an infinite perfect set of points.

We have seen that if n_{m+1}/n_m is bounded then a pointwise approach to the rigidity question will not work. So when this happens, the next result is giving us weakly mixing transformations that are rigid along (n_m) even though $\mathcal{R}(n_m) = \{\gamma \in \mathbb{T} : \lim_{m\to\infty} \gamma^{n_m} = 1\}$ is countable.

Proposition 3.27. *Given an increasing sequence (n_m) such that n_{m+1}/n_m is always a whole number, there is a weakly mixing dynamical system for which there is rigidity along (n_m).*

Proof. let $a_1 = n_1$ and $a_{m+1} = n_{m+1}/n_m$ for all $m \geq 1$. So $n_m = \prod_{k=1}^{m} a_k$ and $\frac{n_m}{n_M} = \prod_{k=M+1}^{m} a_k$ for all $m \geq M+1$. Consider the series representations for $x \in [0,1)$ of the form $x = \sum_{m=1}^{\infty} \frac{b_m}{n_m}$ where b_m is a whole number such that $0 \leq b_m < a_m$. Except for a set of Lebesgue measure zero, such series are uniquely determined by x, and vice versa. We will write $b_m(x)$ to indicate the dependence of (b_m) on x. Let $\Pi = \prod_{m=1}^{\infty} \{0,\dots,a_m-1\}$ and let $\pi = \prod_{m=1}^{\infty} \pi_m$ where π_m is the uniform counting measure on $\{0,\dots,a_m-1\}$. It is easy to see that there is a one-to-one, onto Borel mapping Φ of $[0,1)$ to Π such that $\pi \circ \Phi^{-1}$ is Lebesgue measure.

Now we use a block construction to build a positive continuous Borel measure ν such that $\|n_m x\| \to 0$ in measure with respect to ν as $m \to \infty$. Consider disjoint intervals $I_k = [N_k+1,\dots,N_{k+1}]$, with $0 = N_0 < N_1 < N_2 < \dots$, and $N_{k+1} - N_k = |I_k|$, the length of I_k, increasing to ∞. Fix (ϵ_k) with $0 < \epsilon_k \leq \frac{1}{2}$ such that $\sum_{k=1}^{\infty} \epsilon_k = \infty$ and $\lim_{k\to\infty} \epsilon_k = 0$. Define ν_k on Π as follows. Let $\overline{0_k}$ be the element in $\prod_{I_k}\{0,\dots,a_m-1\}$ whose entries are all 0. Let $\nu_k(\overline{0_k}) = 1-\epsilon_k$ and ν_k uniformly distributed over the points in $\prod_{I_k}\{0,\dots,a_m-1\}\backslash\{\overline{0_k}\}$ with the total mass $\nu_k(\prod_{I_k}\{0,\dots,a_m-1\}\backslash\{\overline{0_k}\}) = \epsilon_k$. Then let $\nu = \prod_{k=1}^{\infty} \nu_k$ and consider the measure $\nu\circ\Phi$ on $[0,1)$.

We know that ν and $\nu\circ\Phi$ are regular Borel probability measures because all finite Borel measures are regular in this situation. Moreover, ν and $\nu\circ\Phi$ are continuous, i.e. they have no point masses. Indeed, $\nu(\{\overline{0}\}) = \prod_{k=1}^{\infty}(1-\epsilon_k) = 0$ because $\sum_{k=1}^{\infty}\epsilon_k = \infty$, and by the definition of ν for every other point $\overline{x} \in \Pi$, we have $\nu(\{\overline{x}\}) \leq \nu(\{\overline{0}\})$.

We claim that $\|n_M x\| \to 0$ in measure with respect to $\nu\circ\Phi$ as $M \to \infty$. We see that $\|n_M x\| = \sum_{m=M+1}^{\infty} \frac{b_m}{n_m/n_M} = \sum_{m=M+1}^{\infty} \frac{b_m}{a_{M+1}\cdots a_m}$. For any $M \geq 1$, we have $M \in I_k$ for a unique $k = k(M)$. Clearly, as $M \to \infty$, we have $k(M) \to \infty$. Consider the set D_k of vectors $\overline{b} = (b_m) \in \Pi$ such that $b_m = 0$ for $m \in I_k \cup I_{k+1}$. We have $\nu(D_k) = (1-\epsilon_k)(1-\epsilon_{k+1}) \to 1$ as $k \to \infty$. But also for $\overline{b} \in D_{k(M)}$, we have $b_m = 0$ for all $m, N_{k(M)} \leq m \leq N_{k(M)+2}$. Hence, if $\Phi(x) \in D_{k(M)}$, then $\|n_M x\| = \sum_{m=N_{k(M)+2}+1}^{\infty} \frac{b_m}{a_{M+1}\cdots a_m} \leq \sum_{m=N_{k(M)+2}+1}^{\infty} \frac{1}{a_{M+1}\cdots a_{m-1}} \leq \sum_{m=N_{k(M)+2}+1} \frac{1}{2^{m-M-1}} = 2^{M+1}\frac{1}{2^{N_{k(M)+2}}}$. But because we chose $|I_k|$ increasing to ∞, we have $2^{M+1}\frac{1}{2^{N_{k(M)+2}}} \to 0$ as $M \to \infty$. This means that, as $M \to \infty$, we have $\nu\circ\Phi(\Phi^{-1}D_{k(M)}) \to 1$ and for $x \in \Phi^{-1}D_{k(M)}$, we have $\|n_M x\| \to 0$. □

This result leads to the following important special case.

Corollary 3.28. *Given any whole number $a \geq 2$, there is a weakly mixing dynamical system for which there is rigidity along $(a^m : m \geq 1)$. However, there is never rigidity along $\sigma \in FS(a^m : m \geq 1)$ as $\sigma \to \infty(IP)$.*

Proof. The first statement follows from Proposition 3.27. For the second part, see Proposition 3.55. But here is at least the idea. First consider the sequence $(2^j : j \geq 1)$. When we take all finite sums $2^{j_1} + \dots + 2^{j_k}$ with $m \leq j_1 < \dots < j_k$, then we get all whole numbers $2^m s, s \geq 1$. Hence, to have rigidity for $f \circ T^\sigma$ as $\sigma \to \infty(IP)$, with $\sigma \in \Sigma = FS(2^j : j \geq 1)$ would imply that $\|f \circ T^{2^m s} - f\|_2 \to 0$ as $m \to \infty$, independently of $s \geq 1$. Assume that f is mean-zero and not zero. Then we fix m such that $\|f \circ T^{2^m s} - f\|_2 \leq \frac{1}{2}$ for all $s \geq 1$. But if T is weakly mixing, so is T^{2^m} and hence

$$\lim_{s\to\infty} \langle f \circ T^{2^m s}, f\rangle = 0.$$

This is not possible. The same argument works with any sequence $(a^j : j \geq 1)$ in place of (2^j) with $a \in \mathbb{Z}^+, a \geq 2$. The only difference is that we would need to use some fixed number of finite sums of elements in Σ when $a \geq 3$. □

Remark 3.29. a) See Proposition 3.55 for a different approach to a version of the above result.

b) The contrast of this result with the examples in Remark 2.23 c) is clear. While (2^n) is a rigidity sequence for a weakly mixing transformation, certain simple perturbations of it, that are still lacunary sequences, like $(2^n + 1)$ are not rigidity sequences for weakly mixing transformations. This leads to the obvious question: if we assume that (n_m) is lacunary, but not necessarily as above a power of a fixed whole number $a \geq 2$, when is there still a continuous Borel probability measure on $[0, 1)$ with $\widehat{\nu}(n_m) \to 1$ as $m \to \infty$? This seems to be a difficult problem because it is not just about the growth rate inherent in lacunarity, but also about the algebraic nature of the sequence (n_m).

c) On the other hand, it is not clear what happens for the lacunary sequence $(2^m + 3^m)$. That is, can there be a continuous Borel probability measure ν on $\mathbb{T}$ such that $\widehat{\nu}(2^m + 3^m) \to 1$ as $m \to \infty$? By Proposition 2.3, if this happens, then $z^{2^m+3^m} \to 1$ in measure with respect to ν. So $z^{2(2^m+3^m)} \to 1$ and $z^{2^{m+1}+3^{m+1}} \to 1$ in measure with respect to ν. Taking the ratio gives $z^{3^m} \to 1$ in measure with respect to ν, and so again taking the appropriate ratio both $z^{2^m} \to 1$ and $z^{3^m} \to 1$ in measure with respect to ν. But the converse is also true by taking the product of these. So Proposition 2.3 shows that $\widehat{\nu}(2^m + 3^m) \to 1$ as $m \to \infty$ if and only if both $\widehat{\nu}(2^m) \to 1$ and $\widehat{\nu}(3^m) \to 1$ as $m \to \infty$. Therefore, using the GMC, what we are seeking is a weakly mixing transformation which has both (2^m) and (3^m) as rigidity sequences.

d) The example of Eisner and Grivaux [10] of an increasing sequence (n_m) such that $\lim\limits_{m\to\infty} \frac{n_{m+1}}{n_m} = 1$, for which there is rigidity along (n_m) for some weakly mixing dynamical system, is one that is constructed inductively. Concretely, is there rigidity along (n_m) for some weakly mixing dynamical system if (n_m) is the sequence obtained by writing $\{2^k 3^l : k, l \geq 1\}$ in increasing order? Note: results of Ajtai, Havas, and Komlós [3] show that for any sequence (ϵ_m) decreasing to 0, no matter how slowly, there exists an increasing sequence (n_m) such that $\frac{n_{m+1}}{n_m} \geq 1+\epsilon_m$ but there is not rigidity along (n_m). Indeed, they give examples where $\lim\limits_{M\to\infty} \frac{1}{M} \sum\limits_{m=1}^{M} \exp(2\pi i n_m x) = 0$ for all $x \in (0, 1)$, and hence for any positive Borel measure $\nu \neq \delta_0$, $\lim\limits_{M\to\infty} \frac{1}{M} \sum\limits_{m=1}^{M} \widehat{\nu}(n_m) = 0$.

3.3. **Disjoint rigidity.** In this section, we modify a construction of LaFontaine [38] to obtain Proposition 3.34 below. We will need two procedural lemmas. First, given a finite Borel measure ω on $\mathbb{R}$, we let $FT_B(\omega)(n) = \int\limits_0^B \exp(-2\pi i n \frac{x}{B})\, d\omega(x)$. We use the notation $\widehat{\omega}$ for the Fourier transform on $\mathbb{R}$ given by $\widehat{\omega}(t) = \int \exp(-2\pi i t x)\, d\omega(x)$. Let $m = m_{\mathbb{R}}$ denote the Lebesgue measure on $\mathbb{R}$, and let $L_2(m)$ denote the Lebesgue space $L_2(\mathbb{R}, \mathcal{B}_m, m)$.

Lemma 3.30. *Suppose we have a positive Borel measure ω on $\mathbb{R}$ that has compact support. Then $\frac{d\omega}{dm} \in L_2(m)$ if and only if $\sum\limits_{n=-\infty}^{\infty} |\widehat{\omega}(n)|^2 < \infty$.*

Proof. By translating ω, we may assume without loss of generality that the support of ω is a subset $[0,B]$ for some whole number B. Then

$$\begin{aligned}\widehat{\omega}(n) &= \int \exp(-2\pi i n x)\, d\omega(x) = \sum_{k=0}^{B-1} \int_k^{k+1} \exp(-2\pi i n x)\, d\omega(x) \\ &= \sum_{k=0}^{B-1} \int_0^1 \exp(-2\pi i n(x+k))\, d\omega(x+k) \\ &= \sum_{k=0}^{B-1} \int_0^1 \exp(-2\pi i n x)\, d\omega(x+k) = \int_0^1 \exp(-2\pi i n x)\, d\Omega(x).\end{aligned}$$

where $d\Omega(x) = 1_{[0,1)}(x) \sum_{k=0}^{B-1} \omega(x+k)$ is a positive Borel measure supported on $[0,1]$. That is, $\widehat{\omega}(n) = FT_1(\Omega)(n)$ for all n. The usual classical argument shows that $\frac{d\Omega}{dm} \in L_2(m)$ if and only if $\sum_{n=-\infty}^{\infty} |FT_1(\Omega)(n)|^2 < \infty$ because Ω is supported on $[0,1)$. Therefore, $\sum_{n=-\infty}^{\infty} |\widehat{\omega}(n)|^2 < \infty$ is equivalent to knowing that $\frac{d\Omega}{dm} \in L_2([0,1),m)$.

So we conclude that if $\frac{d\Omega}{dm} \in L_2([0,1),m)$, then for each $k = 0,\dots,B-1$, the positive measure $1_{[0,1]} d\omega(x+k)$ satisfies $0 \le 1_{[0,1]} d\omega(x+k) \le d\Omega(x)$, and so it also has a density in $L_2([0,1),m)$. Hence, by translating the terms back again and adding them together, we also know that $\frac{d\omega}{dm}$ is in $L_2(\mathbb{R},m)$. Of course also conversely, if $\frac{d\omega}{dm}$ is in $L_2(\mathbb{R},m)$, then $\frac{d\Omega}{dm} \in L_2([0,1),m)$. This proves that $\frac{d\omega}{dm} \in L_2(m)$ if and only if $\sum_{n=-\infty}^{\infty} |\widehat{\omega}(n)|^2 < \infty$. □

Remark 3.31. This remark and Lemma 3.30 are related to the ideas behind Shannon sampling and the Nyquist frequency for band-limited signals. In Lemma 3.30, if we replace ω by a dilation of it, then one can see that more generally for a positive Borel measure ω on $\mathbb{R}$ with compact support, we have $\frac{d\omega}{dm} \in L_2(m)$ if and only if for some $b > 0$ (or for all $b > 0$), we have $\sum_{n=-\infty}^{\infty} |\widehat{\omega}(bn)|^2 < \infty$.

The assumption that ω is positive is necessary here. Indeed, suppose we have a compactly supported complex-valued Borel measure ω on $\mathbb{R}$. Suppose that the support of ω is a subset of $[0,B]$. Then by the usual classical argument, we know that $\frac{d\omega}{dm} \in L_2(m)$ if and only if $\sum_{n=-\infty}^{\infty} |FT_B(\omega)(n)|^2 < \infty$. But $\widehat{\omega}(\frac{n}{B}) = FT_B(\omega)(n)$ because ω is supported on $[0,B]$. So we have in this case, $\frac{d\omega}{dm} \in L_2(m)$ if and only if $\sum_{n=-\infty}^{\infty} |\widehat{\omega}(\frac{n}{B})|^2 < \infty$. Here, B can be replaced by any larger value, but not necessarily by a smaller value. For example, if we take ω_0 supported on $[0,1)$ and define $d\omega(x) = d\omega_0(x) - d\omega_0(x-1)$, then our value of $B = 2$, but $\widehat{\omega}(n) = 0$ for all n. However, the measure ω_0 could be singular to m and hence ω might not have an $L_2(\mathbb{R},m)$-density.

We also want to make a few observations about the differences between convolving measures on $\mathbb{T}$ and convolving their associated measures on $\mathbb{R}$. Given a Borel measure ω on $\mathbb{T}$, let $\omega_{\mathbb{R}}$ denote the Borel measure on $\mathbb{R}$ given by $\omega_{\mathbb{R}} = \omega \circ E$ where $E : [0,1) \to \mathbb{T}$ by $E(x) = \exp(2\pi i x)$. Let $m_{\mathbb{T}}$ denote the usual Lebesgue measure on $\mathbb{T}$ i.e. $m_{\mathbb{T}} \circ E = 1_{[0,1)} m_{\mathbb{R}}$.

Lemma 3.32. *If μ and ν are Borel measures on $\mathbb{T}$, then $\mu * \nu$ is absolutely continuous with respect to $m_{\mathbb{T}}$ if $\mu_{\mathbb{R}} * \nu_{\mathbb{R}}$ is absolutely continuous with respect to $m_{\mathbb{R}}$.*

Proof. For $f \in C(\mathbb{T})$, we have

$$\begin{aligned}\int f(\gamma)d(\mu*\nu)(\gamma) &= \int\int f(\gamma_1\gamma_2)d\mu(\gamma_1)d\nu(\gamma_2)\\ &= \int\int f(E(x)E(y))d\mu_{\mathbb{R}}(x)d\nu_{\mathbb{R}}(y)\\ &= \int f\circ E(x+y)d\mu_{\mathbb{R}}(x)d\nu_{\mathbb{R}}(y)\\ &= \int f\circ E(z)d(\mu_{\mathbb{R}}*\nu_{\mathbb{R}})(z).\end{aligned}$$

Hence, it is clear that if $\mu_{\mathbb{R}} * \nu_{\mathbb{R}}$ is absolutely continuous with respect to $m_{\mathbb{R}}$ with density F, then $\mu * \nu$ is absolutely continuous with respect to $m_{\mathbb{T}}$ with density $F \circ E^{-1}$. □

Remark 3.33. The converse statement to Lemma 3.32 is not true without additional assumption, for example the assumption that both measures are positive. For example, take a non-zero measure ω_0 on $\mathbb{T}$ supported in the arc $E([0,1/2))$. Let $\omega = \omega_0 - \omega_0 * \delta_{-1}$. Then let $\nu = \delta_1 + \delta_{-1}$, a positive discrete measure on $\mathbb{T}$. We have

$$\begin{aligned}\omega*\nu &= \omega_0 - \omega_0*\delta_{-1} + \omega_0*\delta_{-1} - \omega_0*\delta_1\\ &= \omega_0 - \omega*\delta_1\\ &= 0.\end{aligned}$$

However, $\omega_{\mathbb{R}} = (\omega_0)_{\mathbb{R}} - (\omega_0)_{\mathbb{R}} * \delta_{1/2}$ and $\nu_{\mathbb{R}} = \delta_0 + \delta_{1/2}$. So

$$\begin{aligned}\omega_{\mathbb{R}}*\nu_{\mathbb{R}} &= (\omega_0)_{\mathbb{R}} - (\omega_0)_{\mathbb{R}}*\delta_{1/2} + (\omega_0)_{\mathbb{R}}*\delta_{1/2} - (\omega_0)_{\mathbb{R}}*\delta_1\\ &= (\omega_0)_{\mathbb{R}} - (\omega_0)_{\mathbb{R}}*\delta_1.\end{aligned}$$

This measure is not zero as a measure on $\mathbb{R}$. If also ω_0 is singular to $m_{\mathbb{T}}$, then we have $\omega * \nu$ absolutely continuous with respect to $m_{\mathbb{T}}$, since it is 0, while $\omega_{\mathbb{R}} * \nu_{\mathbb{R}}$ is not absolutely continuous with respect to $m_{\mathbb{R}}$.

These two lemmas will help in proving the following.

Proposition 3.34. *Assume that $(X, \mathcal{B}_X, p_X, T)$ is a rigid, weakly mixing dynamical system. Then there is a weakly mixing, rigid dynamical system $(Y, \mathcal{B}_Y, p_Y, S)$ such that the maximal spectral type of $U_{T\times S}$ in the orthocomplement of $F := L_2(X, p_X) \otimes 1_Y \oplus 1_X \otimes L_2(Y, p_Y)$ is Rajchman. In other words, for all $f \in L_2(X \times Y, p_X \otimes p_Y)$ that are orthogonal to both the X-measurable functions and the Y-measurable functions, we have $\langle f \circ (T \times S)^n, f\rangle \to 0$ as $n \to \infty$. In fact, $U_{T\times S}$ has absolutely continuous spectrum on $F^{\perp}$.*

Proof. From the spectral point of view we want to show that given a continuous Dirichlet measure μ, there is a continuous Dirichlet probability measure ν on $\mathbb{T}$ such that $\mu * \nu$ is an absolutely continuous (hence a Rajchman measure). Indeed, we then can take $\mu = \nu^T$ and we let $(Y, \mathcal{B}_Y, p_Y, S)$ be given by $S = G_\nu$. As $\mu * \nu^{*k} = (\mu * \nu) * \nu^{*(k-1)}$ we easily check that $\mu * \sum_{k=1}^{\infty} \frac{1}{2^k}\nu^{*k}$ is still absolutely continuous. Given the references we are using, it is better to carry out our construction in $\mathbb{R}$. So consider first the measure $\mu \circ E$ on $\mathbb{R}$. Lemma 3.32 shows that, to get our result, it will be enough to construct a suitable positive Borel measure $\nu \circ E$

on $\mathbb{R}$, with support in $[0,1)$, such that $(\mu \circ E) * (\nu \circ E)$ is absolutely continuous. For notational convenience we will denote $\mu \circ E$ and $\nu \circ E$ by μ and ν in the rest of this proof.

Hence, suppose we have a continuous positive Borel measure μ on $\mathbb{R}$, which is compactly supported. Consider the dilation μ_λ, $\lambda > 0$, given by $\mu_\lambda(E) = \mu(\frac{1}{\lambda}E)$ for all Borel sets $E \subset \mathbb{R}$. Then $\widehat{\mu_\lambda}(t) = \int \exp(-2\pi itx)\, d\mu_\lambda(x) = \int \exp(-2\pi it\lambda x)\, d\mu(x) = \widehat{\mu}(\lambda t)$. We would like to construct a suitable continuous positive Borel measure ν supported in $[0,1]$ such that $J(\lambda) = \sum\limits_{n=-\infty}^{\infty} |\widehat{\mu_\lambda}(n)|^2 |\widehat{\nu}(n)|^2$ is finite. But

$$\int\limits_0^1 J(\lambda)\, dm(\lambda) = \sum_{n=-\infty}^{\infty} |\widehat{\nu}(n)|^2 \left(\int\limits_0^1 |\widehat{\mu}(\lambda n)|^2\, dm(\lambda) \right).$$

Using Wiener's lemma for measures on $\mathbb{R}$ (e.g. [28], Chapter VI.2) and the fact that μ is continuous

$$a_n := \int\limits_0^1 |\widehat{\mu}(\lambda n)|^2\, dm(\lambda) = \frac{1}{n} \int_0^n |\widehat{\mu}(t)|^2\, dt \to 0$$

as $|n| \to \infty$. So, we are seeking a suitable ν such that

$$\sum_{n=-\infty}^{\infty} |\widehat{\nu}(n)|^2 a_n < +\infty. \tag{3.3}$$

Clearly, if ν were actually absolutely continuous with respect to m with a square integrable density, then we would have this condition. But ν could not be rigid in this situation. However, as LaFontaine points out, the two articles of Salem [58, 59] give a construction of a Borel probability measure ν with support in $[0,1)$ with this property, and which is also continuous and rigid. See LaFontaine [38] and Salem [58, 59] for the details. It follows that there exists a continuous probability measure ν supported on $[0,1)$ such that for some increasing sequence (n_m) of integers

$$\widehat{\nu}(n_m) \to 1 \tag{3.4}$$

and (3.3) holds.

It follows from (3.3) that for m-a.e. $\lambda \in [0,1]$

$$\sum_{n=-\infty}^{\infty} |\widehat{\mu_\lambda * \nu}(n)|^2 = \sum_{n=-\infty}^{\infty} |\widehat{\mu_\lambda}(n)|^2 |\widehat{\nu}(n)|^2 = J(\lambda) < \infty. \tag{3.5}$$

The measure $\mu_\lambda * \nu$ is supported in $[0, 1+\frac{1}{\lambda}]$ and in view of Equation (3.5) and Lemma 3.30, it must be absolutely continuous with respect to m, with $\frac{d(\mu_\lambda * \nu)}{dm} \in L_2(m)$. Thus, if we choose any value of λ satisfying (3.5), we have

$$\int\limits_{-\infty}^{\infty} |\widehat{\mu}(\lambda t)|^2 |\widehat{\nu}(t)|^2\, dm(t) = \frac{1}{\lambda} \int\limits_{-\infty}^{\infty} |\widehat{\mu}(t)|^2 |\widehat{\nu}(\frac{t}{\lambda})|^2\, dm(t)$$

is finite. It follows that the measure $\mu * \nu_{1/\lambda}$ is compactly supported and absolutely continuous. In view of (3.4), $\widehat{\nu_{1/\lambda}}(\lambda n_m)$ converges to 1 as $m \to \infty$. Moreover, with respect to m, for almost every λ, we would know that $(n_k\lambda : k \ge 1)$ is uniformly distributed modulo 1. Hence, for some subsequence (n_{k_j}) and some sequence of integers (m_j), we have $\lim\limits_{j\to\infty} \left(n_{k_j}\lambda - m_j\right) = 0$. Then by the uniform continuity of $\widehat{\nu_{1/\lambda}}$, we would also have $\widehat{\nu_{1/\lambda}}(m_j) \to 1$ as $j \to \infty$. Hence, with respect to m, almost every choice of $0 < \lambda < 1$ gives $\nu_{1/\lambda}$ supported on $[0, \lambda] \subset [0, 1]$, which is a continuous rigid Borel probability measure, such that $\mu * \nu_{1/\lambda}$ is absolutely continuous. $\square$

Remark 3.35. The transformations T and S in Proposition 3.34 must be disjoint in the sense that their only joining is their product. See Example 3.7 a). As a first step, assume that $T^{n_m} \to Id$ in the strong operator topology. By passing to a subsequence if necessary we can assume that $S^{n_m} \to \Phi$ in the weak operator topology, where $\Phi : L_2(Y, p_Y) \to L_2(Y, p_Y)$ is a Markov operator. We claim that $\Phi(g) \neq g$ for each non-zero $g \in L_{2,0}(Y, p_Y)$. Indeed suppose for some non-zero g, $\Phi(g) = g$. Take any non-zero $f \in L_{2,0}(X, p_X)$. Then $\langle (T\times S)^{n_m}(f\otimes g), f\otimes g\rangle \to \langle f \otimes g, f \otimes g\rangle$. So the spectral measure of $f \otimes g$ is a Dirichlet measure, contrary to construction in Proposition 3.34. Now as a second step, take a joining J of T and S. On the operator level, this means that we have a Markov operator $W = W_J$ corresponding to J such that $W : L_2(X, p_X) \to L_2(Y, p_Y)$ and $WT = SW$. Then $WT^{n_m} = S^{n_m}W$, and by passing to limits we obtain $W = \Phi W$. So by our first step, we have $W(L_{2,0}(Y, p_Y)) = \{0\}$. But this means that W is the Markov operator for the product joining, and so J is the product measure $p_T \otimes p_S$. Thus, T and S are disjoint. For background, see Glasner [20], Chapter 6, Section 2.

Remark 3.36. We can argue differently that if a continuous probability measure ρ supported on $[0, 1] \subset \mathbb{R}$ satisfies $\widehat{\rho}(r_k) \to 1$ for some sequence of reals $r_k \to \infty$, then, as a circle measure, ρ is Dirichlet. Indeed, consider the flow $V_t(f)(x) = e^{itx}f(x)$ on $L_2(\mathbb{R}, \rho)$. Our assumption says that $V_{r_k} \to Id$. Consider then $V_{\{r_k\}}$, $k \geq 1$, which replaces r_k by its fractional part $\{r_k\}$. By passing to a subsequence if necessary and using the continuity of the unitary representation $\mathbb{R} \ni t \mapsto V_t$, we have $V_{\{r_k\}} \to V_s$ for some $s \in [0, 1]$. It follows that $V_{[r_k]} \to Id \circ (V_s)^{-1} = V_{-s}$. Then, $([r_{k+1}] - [r_k])$ is a rigidity sequence for ρ.

Remark 3.37. Clearly, there is an IP version of Proposition 3.34 that follows by passing to a subsequence of the rigidity sequence for S.

This result shows that whenever we have a weakly mixing rigid transformation T, then there is a weakly mixing rigid transformation S such that $T \times S$ is not rigid for any sequence. More generally, we can prove the following result. We again use the closure of $\{T^n : n \in \mathbb{Z}\}$ in the strong operator topology, which can be identified with the centralizer of T in case T has discrete spectrum.

Corollary 3.38. *Assume that $(X, \mathcal{B}, p, T)$ is an ergodic dynamical system. Then T has discrete spectrum if and only if for each weakly mixing rigid system $(Y, \mathcal{B}_Y, p_Y, S)$ the Cartesian product system $T \times S$ remains rigid.*

Proof. Assume that T is an ergodic rotation and let S be weakly mixing, $S^{n_m} \to Id$. By passing to a subsequence if necessary, $T^{n_m} \to R \in C(T)$, and so $T^{n_{m+1}-n_m} \to Id$ and still $S^{n_{m+1}-n_m} \to Id$; thus $T \times S$ is rigid (see also Proposition 2.18).

To prove the converse, suppose that T does not have discrete spectrum, but $T \times S$ is rigid for each S which is rigid and weakly mixing. Then there is some continuous ν with $\nu \ll \nu^T$. It follows that ν is a Dirichlet measure, and if $\widehat{\nu^T}(n_m) \to \nu^T(\mathbb{T})$ then $\widehat{\nu}(n_m) \to \nu(\mathbb{T})$. Consider the Gaussian system G_ν given by ν. Then G_ν is weakly mixing and each sequence which is a rigid sequence for T is also a rigid sequence for G_ν. It follows that $G_\nu \times S$ is rigid for each weakly mixing rigid S which is in conflict with from Proposition 3.34. $\square$

Remark 3.39. Using the viewpoint in Bergelson and Rosenblatt [6], it is clear that Corollary 3.38 has a unitary version. That is, a unitary operator U on a Hilbert space H_U has discrete spectrum if and only if for every weakly mixing rigid unitary operator V on a Hilbert space H_V, the product $U \times V$ on $H_U \times H_V$ is a rigid unitary operator.

The following is a folklore result.

Proposition 3.40. *Assume that T and S are ergodic transformations with discrete spectrums. Then they are isomorphic if and only if they have the same rigidity sequences.*

Proof. Consider $C(T)$ and $C(S)$ respectively. Both are given as weak closure of powers. Take the map:

$$F : C(T) \to C(S), \quad F(T^n) = S^n, \ n \in \mathbb{Z}.$$

We easily show that this extends to a homeomorphism equivariant with rotation by T and S respectively. Now refer back to the comments about centralizers at the beginning of Section 2.3. We know that T is isomorphic to the translation by T on $C(T)$ considered with Haar measure and S is isomorphic to the translation by S on $C(S)$ considered with Haar measure. Hence, T and S are isomorphic. □

Corollary 3.41. *Assume that $(X, \mathcal{B}_X, p_X, T)$ is an ergodic transformation with a discrete spectrum, and $(Y, \mathcal{B}_Y, p_Y, S)$ is ergodic and has the same rigidity sequences as T. Then S is isomorphic to T.*

Proof. It follows directly from Corollary 3.38 that S has discrete spectrum. The result follows from Proposition 3.40. □

3.4. **Cocycle Methods.**

3.4.1. **Tools.** We will now describe tools using cocycles over rotations to produce weakly mixing transformations with a prescribed sequence as rigidity sequences. We will start with a transformation T having discrete spectrum and its sequence of rigidity. (In fact, for applications, we will consider one dimensional rotations by irrational α and the sequence given the denominators of α.) Then we will consider cocycles over T with values in locally compact abelian groups. We will then pass to the associated unitary operators (weighted operators) and we will try to "lift" some rigidity sequences for the rotation to the weighted operator. Once such an operator has continuous spectrum we apply the GMC which preserves rigidity. Another option to obtain "good" weakly mixing transformations will be to pass to Poisson suspensions (in case we extend by a locally compact and not compact group) - which in a sense will be even easier as ergodicity of Poisson suspension is closely related to the fact that the cocycles are not coboundaries. See Remark 2.27, and also [8], [9] and [55], for details concerning ergodic properties of Poisson suspensions. Note: at times in this section $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ with multiplicative notation, and at times it will mean $[0, 1)$ with addition modulo one. The reader will be able to distinguish which model for the circle is being used by the context of the discussion.

3.4.2. **Compact Group Extensions and Weighted Operators.** Assume that T is an ergodic transformation acting on a standard Lebesgue space $(X, \mathcal{B}, p)$. Let G be a compact metric abelian group with Haar measure λ_G. We take the λ_G measurable sets, denoted by $\mathcal{G}$ as our measurable sets for $(G, \mathcal{B}_G, \lambda_G)$. A measurable map $\varphi : X \to G$ generates a *cocycle* $\varphi^{(\cdot)}(\cdot)$ which is given by $\varphi^{(n)}(\cdot) : X \to G$, $n \in \mathbb{Z}$, by the formula, for $x \in X$,

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{n-1}x) & \text{if } \ n > 0, \\ 0 & \text{if } \ n = 0, \\ -(\varphi(T^{-n}x) + \ldots + \varphi(T^{-1}x)) & \text{if } \ n < 0. \end{cases} \tag{3.6}$$

Using T and φ we define a *compact group extension* T_φ of T which acts on the space $(X \times G, \mathcal{B} \otimes \mathcal{B}_G, p \otimes \lambda_G)$ by the formula

$$T_\varphi(x,g) = (Tx, \varphi(x) + g) \text{ for } (x,g) \in X \times G. \tag{3.7}$$

Notice that for each $n \in \mathbb{Z}$ and $(x,g) \in X \times G$

$$(T_\varphi)^n(x,g) = (T^n x, \varphi^{(n)}(x) + g). \tag{3.8}$$

The natural decomposition of $L_2(G, \lambda_G)$ using the character group $\widehat{G}$ yields the decomposition

$$L_2(X \times G, p \otimes \lambda_G) = \bigoplus_{\chi \in \widehat{G}} L_2(X,p) \otimes (\mathbb{C}\chi). \tag{3.9}$$

Here $\mathbb{C}\chi$ is the one-dimensional subspace spanned by the character χ. To understand ergodic and other mixing properties of T_φ we need to study the associated Koopman operator U_{T_φ},

$$U_{T_\varphi} F = F \circ T_\varphi, \text{ for } F \in L_2(X \times G, p \otimes \lambda_G).$$

As the (closed) subspaces $L_2(X,p) \otimes (\mathbb{C}\chi)$ in (3.9) are U_{T_φ}-invariant, we can examine those mixing properties separately on all such subspaces (notice that for $\chi = 1$ we consider the original Koopman operator U_T). It is well-known and not hard to see that the map $f \otimes \chi \mapsto f$ provides a spectral equivalence of $U_{T_\varphi}|_{L_2(X,p)\otimes(\mathbb{C}\chi)}$ and the operator $V^T_{\chi\circ\varphi}$ acting on $L_2(X, \mathcal{B}, p)$ by the formula

$$V^T_{\chi\circ\varphi} f = \chi \circ \varphi \cdot f \circ T \text{ for } f \in L_2(X,p). \tag{3.10}$$

Each such operator is an example of a *weighted operator* V^T_ξ over T, where $\xi : X \to \mathbb{T}$ is a measurable function with values in the (multiplicative) circle $\mathbb{T}$ and $V^T_\xi f = \xi \cdot f \circ T$.

Assume now that T is an ergodic transformation with discrete spectrum, i.e. without loss of generality, we can assume that X is a compact monothetic metric group with $p = \lambda_X$ Haar measure on X, and $Tx = x + x_0$ where x_0 and $\{nx_0 : n \in \mathbb{Z}\}$ is dense in X. Assume that $\xi : X \to \mathbb{T}$ is measurable. By Helson's analysis [23] (see also e.g. [26]):

Theorem 3.42 ([23])**.** *For T as above, the maximal spectral type of V^T_ξ is either discrete or continuous. If it is continuous then either it is singular or it is Lebesgue.*

We will use only the first part of this theorem. An important practical point that comes from this theorem is that once we find a function $f \in L_2(X, \mathcal{B}, p)$ such that the spectral measure $\sigma_f = \sigma_f^{V^T_\xi}$ is continuous, then V^T_ξ has purely continuous spectrum. Consider $f = 1$. Using (3.8), for each $n \in \mathbb{Z}$, we obtain

$$\langle \left(V^T_\xi\right)^n 1, 1\rangle = \int_X \xi^{(n)}(x)\, dp(x). \tag{3.11}$$

It follows that if there is a subsequence $(n_m)_{m\geq 1}$ such that

$$\int_X \xi^{(n_m)}(x)\, dp(x) \to 0 \;\Rightarrow\; V^T_\xi \text{ has continuous spectrum.} \tag{3.12}$$

It is also nice to note in passing that we have a bit stronger result:

$$\int_X \xi^{(n_m)}(x)\, dp(x) \to 0 \;\Rightarrow\; \left(V^T_\xi\right)^{n_m} \to 0 \text{ weakly,} \tag{3.13}$$

which follows directly from (3.12); indeed,

$$\int_X \xi^{(n_m)}(x) f(T^{n_m}x)\overline{f(x)}\, dp(x) \to 0$$

whenever f is a character of X.

It is well-known and easy to check that if (n_m) is a rigidity sequence for T

$$\xi^{(n_m)} \to 1 \text{ in measure} \Rightarrow \left(V_\xi^T\right)^{n_m} \to Id \text{ strongly.} \tag{3.14}$$

3.4.3. **$\mathbb{R}$-extensions, Weighted Operators and Poisson Suspensions.** We will now consider the case $G = \mathbb{R}$. We assume now that $f : X \to \mathbb{R}$ is a cocycle for T acting ergodically on a standard probability Borel space $(X, \mathcal{B}, p)$. We consider T_f

$$T_f(x, r) = (Tx, f(x) + r)$$

acting on $(X \times \mathbb{R}, p \otimes \lambda_{\mathbb{R}})$. Note that we are now on a standard Lebesgue space with a σ-finite (and not finite) measure. In particular, constants are not integrable.

To study spectrally T_f we will write it slightly differently, namely

$$T_f = T_{f,\tau}$$

where τ is the natural action of $\mathbb{R}$ on itself by translations: $\tau_t(r) = r + t$ and

$$T_{f,\tau}(x, r) = (Tx, \tau_{f(x)}(r)).$$

This transformation is a special case of so called *Rokhlin extension* (of T), see for example [39], and the spectral analysis below is similar to the one in [39]. So let us just imagine a slightly more general situation

$$T_{f,\mathcal{S}}(x, y) = (Tx, S_{f(x)}(y))$$

where $\mathcal{S} = (S_t)$ is a flow acting on $(Y, \mathcal{C}, \nu)$ (ν can be finite or infinite). We will denote the spectral measure of $a \in L_2(Y, \mathcal{C}, \nu)$ (for the Koopman representation $t \mapsto U_{S_t}$ on $L_2(Y, \mathcal{C}, \nu)$) by $\sigma_{a,\mathcal{S}}$.

The space $L_2(X \times Y, p \otimes \nu)$ is nothing but a tensor product of two Hilbert subspaces, so to understand spectral measures we only need to study spectral measures for tensors $a \otimes b$ and we have

$$\begin{aligned}
&\int_{X\times Y} (a \otimes b) \circ (T_{f,\mathcal{S}})^n \cdot \overline{a \otimes b}\, dp\, d\nu \\
&= \int_X \int_Y a(T^n x)\overline{a(x)} b(S_{f^{(n)}(x)} y)\overline{b(y)}\, dp(x)\, d\nu(y) \\
&= \int_X a(T^n x)\overline{a(x)} \left(\int_Y e^{2\pi i t f^{(n)}(x)}\, d\sigma_{b,\mathcal{S}}(t)\right) dp(x) \\
&= \int_Y \left(\int_X e^{2\pi i t f^{(n)}(x)} a(T^n x)\overline{a(x)}\, dp(x)\right) d\sigma_{b,\mathcal{S}}(t).
\end{aligned}$$

Proposition 3.43. *If $T^{n_k} \to Id$ and $f^{(n_k)} \to 0$ in measure then (n_k) is a rigidity sequence for $T_{\varphi,\mathcal{S}}$.*

Proof. Take $a \in L^\infty(X, p)$ and notice that by assumption for each $t \in \mathbb{R}$

$$\int_X e^{2\pi i t f^{(n_k)}(x)} a(T^{n_k} x)\overline{a(x)}\, dp(x) \to \int_X |a|^2\, dp.$$

By the Lebesgue Dominated theorem

$$\int_Y \left(\int_X e^{2\pi i t f^{(n_k)}(x)} a(T^{n_k} x)\overline{a(x)}\, dp(x)\right) d\sigma_{b,\mathcal{S}}(t) \to \int_Y \left(\int_X |a|^2\, dp\right) d\sigma_{b,\mathcal{S}} = \|a \otimes b\|^2_{L_2(p\otimes\lambda_{\mathbb{R}})}.$$

□

We need more information about sequences of the form

$$\int_X e^{2\pi i t f^{(n)}(x)} a(T^n x)\overline{a(x)}\, dp(x),\ \ n\in\mathbb{Z}.$$

In fact, they turn out to be again Fourier coefficients of some spectral measures. Indeed, consider V_t acting on $L_2(X,\mathcal{B},p)$ by the formula

$$(V_t a)(x) = e^{2\pi i t f(x)} a(Tx).$$

This is nothing but a weighted unitary operator and

$$\langle V_t^n a, a\rangle = \int_X e^{2\pi i t f^{(n)}(x)} a(T^n x)\overline{a(x)}\, dp(x).$$

(Notice that we came back to the finite measure-preserving case.)

Clearly, for $b\in L_2(\mathbb{R})$ with compact support and $\mathcal{S}=\tau$

$$\widehat{\sigma}_{b,\mathcal{S}}(t) = \int_{\mathbb{R}} b\circ S_t\cdot \overline{b}\, dr$$

$$= \int_{\mathbb{R}} b(r+t)\overline{b(r)}\, dr = (b*\overline{b})(-t).$$

Hence, the Fourier transform of $\sigma_{b,\mathcal{S}}$ is square summable, and therefore this spectral measure is absolutely continuous. In fact, the maximal spectral type of $\mathcal{S}$ is Lebesgue, and we can see the maximal spectral type of U_{T_f} as an integral (against "Lebesgue" measure) of the maximal spectral types of the family indexed by $t\in\mathbb{R}$ of weighted operators.

Suppose now that U_{T_f} has an eigenvalue c, $|c|=1$. Then we cannot have that all spectral measures $\sigma_{a\otimes b,T_f}$ are continuous. In fact, we must have that c appears as an eigenvalue for "many" V_t (on a set of $t\in\mathbb{R}$ of positive Lebesgue measure), and the following result is well-known (it is an exercise).

Lemma 3.44. *The scalar c is an eigenvalue of V_t if and only if we can solve the following functional equation:*

$$e^{2\pi i t f} = \frac{c\cdot \xi\circ T}{\xi}$$

in measurable functions $\xi : X\to\mathbb{T}$.

It follows that having an eigenvalue for U_{T_f} means that we can solve the above multiplicative equations on a set of positive Lebesgue measure of $t\in\mathbb{R}$. We are now in the framework of the classical Helson's problem (e.g. [45]) of passing from multiplicative coboundaries to additive coboundaries. Using known results in this area ([45], and see also the appendix in [40]) we obtain the following (remember that constant functions are not elements of $L_2(X\times\mathbb{R}, p\otimes\lambda_{\mathbb{R}})$).

Proposition 3.45. *If U_{T_f} has an eigenvalue then f is an additive quasi-coboundary, that is there exist a measurable $g: X\to\mathbb{R}$ and $r\in\mathbb{R}$ such that $f(x) = r + g(x) - g(Tx)$ for p-a.e. $x\in X$.*

Therefore, if f is a non-trivial cocycle then automatically U_{T_f} has continuous spectrum and classically the Poisson suspension over T_f is ergodic, hence weakly mixing (see Remark 2.27). Recall that from spectral point of view Poisson suspension over $(X\times\mathbb{R}, p\otimes\lambda_{\mathbb{R}}, T_f)$ will be the same as Gaussian functor over $(L_2(X\times\mathbb{R}, p\otimes\lambda_{\mathbb{R}}), U_{T_f})$. In particular, if $(T_f)^{n_t}\to Id$ on $L_2(X\times\mathbb{R}, p\otimes\lambda_{\mathbb{R}})$ then (n_t) will be a rigidity sequence for the suspension (in view of Proposition 3.43). From the above discussion, it follows that to have a weakly mixing transformation $\widetilde{T}_f$ with a rigidity sequence (N_t), we need: (i) f is not an additive coboundary, (ii) $T^{N_t} \to Id$, (iii) $f^{(N_t)} \to 0$ in measure.

3.4.4. **Denominators of α and Rigidity.** We have already seen that the sequence (2^n) is a rigidity sequence for a weakly mixing transformation. We can construct some other explicit examples of rigidity sequences by using known results from the theory of "smooth" cocycles over one-dimensional rotations. This will allow us to show that if α is irrational, and (q_n) stands for its sequence of denominators then (q_n) is also a rigidity sequence for a weakly mixing transformation. The most interesting case is of course the bounded partial quotient case (for example for the Golden Mean).

So assume that $\varphi : \mathbb{T} \to \mathbb{R}$ is a smooth mean-zero cocycle. We use the term "smooth" here in a not very precise way; it may refer to a good speed of decaying of the Fourier transform of φ.

We recall first that one of consequences of the Denjoy-Koksma Inequality for AC_0 (absolutely continuous mean-zero) cocycles is that

$$\varphi^{(q_n)} \to 0 \ \text{ uniformly} \tag{3.15}$$

for every irrational rotation by α, see [24]. Another type of Denjoy-Koksma inequality has been proved in [2] for functions φ whose Fourier transform is of order $\mathrm{O}(1/|n|)$ – as its consequence we have the following:

$$\text{If } \widehat{\varphi}(n) = \mathrm{o}(\tfrac{1}{|n|}),\ \widehat{\varphi}(0) = 0 \text{ then } \varphi^{(q_n)} \to 0 \text{ in measure} \tag{3.16}$$

for every rotation by an irrational α.

We would like also to recall another (unpublished) result by M. Herman [25]. While this may not be available, one can see also Krzyżewski [33] for generalizations of Herman's result.

Theorem 3.46. *Assume that a mean-zero $\varphi : \mathbb{T} \to \mathbb{R}$ is in $L_2(\mathbb{T}, \lambda_{\mathbb{T}})$ and its Fourier transform is concentrated on a lacunary subset of $\mathbb{Z}$. Suppose that*

$$\varphi(x) = g(x) - g(x+\alpha), \quad \lambda_{\mathbb{T}} - a.e.$$

for some irrational $\alpha \in [0,1)$. Then $g \in L_2(\mathbb{T}, \lambda_{\mathbb{T}})$.

Fix $\alpha \in [0,1)$ irrational, and let $\alpha = [0 : a_1, a_2, \ldots]$ stand for the continued fraction expansion of α. Denote by (q_n) the sequence of denominators of α: $q_0 = 1$, $q_1 = a_1$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$ for $n \geq 2$. Then

$$\frac{q_{n+2}}{q_n} = \frac{a_{n+2}q_{n+1} + q_n}{q_n} \geq a_{n+2} + 1 \geq 2.$$

It follows that

$$(q_{2n}) \text{ is lacunary.} \tag{3.17}$$

Moreover,

$$q_n \|q_n \alpha\| \leq 1 \text{ for each } n \geq 1. \tag{3.18}$$

We define $\varphi : \mathbb{T} \to \mathbb{R}$ by

$$\varphi(x) = \sum_{n=0}^{\infty} a_{q_{2n}} \cos 2\pi i q_{2n} x \tag{3.19}$$

where for $n \geq 1$

$$a_{q_{2n}} = \frac{1}{\sqrt{n}} \|q_{2n}\alpha\|. \tag{3.20}$$

We then have $\varphi:\mathbb{T}\to\mathbb{R}$, $\widehat{\varphi}(n)=\mathrm{o}(1/|n|)$ and $(a_{q_{2n}})\in l_2$ in view of (3.18). Now, suppose that

$$\varphi(x)=g(x)-g(x+\alpha) \tag{3.21}$$

for a measurable $g:\mathbb{T}\to\mathbb{R}$. In view of Theorem 3.46 and (3.17), $g\in L_2(\mathbb{T},\lambda_{\mathbb{T}})$. Hence,

$$g(x)=\sum_{k=-\infty}^{\infty}b_ke^{2\pi ikx}.$$

Furthermore, by comparing Fourier coefficients on both sides in (3.21),

$$b_k=0\ \text{ if } k\neq q_{2n} \text{ and } \ b_{q_{2n}}=a_{q_{2n}}/(1-e^{2\pi iq_{2n}\alpha})$$

with $(b_{q_{2n}})\in l_2$. However

$$|b_{q_{2n}}|=a_{q_{2n}}/\|q_{2n}\alpha\|=1/\sqrt{n}$$

which is a contradiction. We hence proved the following.

Proposition 3.47. *For each irrational $\alpha\in[0,1)$ there is a mean-zero $\varphi:\mathbb{T}\to\mathbb{R}$ such that $\widehat{\varphi}(n)=o(1/|n|)$ and φ is not an additive coboundary.*

Using (3.16) and Proposition 3.43 we hence obtain the following.

Proposition 3.48. *For φ satisfying the assertion of Proposition 3.47 the sequence (q_n) of denominators of α is a rigidity sequence of T_φ on $L_2(\mathbb{T}\times\mathbb{R},\lambda_{\mathbb{T}}\otimes\lambda_{\mathbb{R}})$ and U_{T_φ} has continuous spectrum.*

By using GMC method or by passing to the relevant Poisson suspension we obtain:

Corollary 3.49. *For each sequence (q_n) of denominators there exists a weakly mixing transformation R such that $R^{q_n}\to Id$.*

Remark 3.50. (i) We would like to emphasize that in general the sequence (q_n) of denominators of α is not lacunary. Indeed, assume that $\alpha=[0:a_1,a_2,\ldots]$ stands for the continued fraction expansion of α. Suppose that for a subsequence (n_k) we have $a_{n_k+1}=1$, $a_{n_k}\to\infty$. Then by the recurrence formula $q_{m+1}=a_{m+1}q_m+q_{m-1}$ we obtain that

$$\liminf_{n\to\infty}\frac{q_{n+1}}{q_n}=1.$$

So the sequences of denominators are another type of non-lacunary sequences which can be realized as rigidity sequences for weakly mixing transformations, besides the ones in Section 3.1.3.

(ii) As the above shows $\{q_n:\ n\geq 1\}$ is always a Sidon set (see [28],[56]); indeed,

$$\{q_n:\ n\geq 1\}=\{q_{2n}:n\geq 1\}\cup\{q_{2n+1}:\ n\geq 0\}.$$

It follows that the set of denominators is the union of two lacunary sets and hence is a Sidon set ([28],[56]).

(iii) The assertion of Theorem 3.46 is true for functions whose Fourier transform is concentrated on a Sidon set and when T is an arbitrary ergodic rotation on a compact metric abelian group (by the proof of the main result in [23] or by [33]).

(iv) It follows that to prove Proposition 3.47 we could have used all denominators, with (for example) $a_{q_n}=\frac{1}{\sqrt{n}}\|q_n\alpha\|$.

(v) Eisner and Grivaux [10] obtain some results in the direction of Corollary 3.49, but their examples are restricted to badly approximated irrational numbers. Also, Fayad [14] obtained a result as in Corollary 3.49 but his result is restricted to the denominators of Liouville numbers. It should be noted though that Fayad also gets uniform rigidity for some weakly mixing automorphism of $\mathbb{T}^2$.

There are also more complicated constructions showing that for each irrational α there is an absolutely continuous mean-zero $\varphi : \mathbb{T} \to \mathbb{R}$ which is not a coboundary – see [42]. We can then use such cocycles and (3.16) for another proof of the above corollary.

Remark 3.51. It is not clear how to characterize rigidity sequences that cannot be IP rigidity sequences. See Proposition 3.28, and its generalization Proposition 3.55 for examples of this phenomenon. In reference to the above, it would be interesting to show that $FS((q_n))$ is not a rigidity net.

Remark 3.52. The above results can be used to answer positively the following question: *Given an increasing sequence (n_m) of integers is there a weakly mixing transformation R such that $R^{n_{m_k}} \to Id$ for some subsequence (n_{m_k}) of (n_m)?* In fact, we have already answered this question (see Proposition 2.9), but we will now take a very different approach. We start with the following well-known lemma; see for example [37].

Lemma 3.53. *Given an increasing sequence (n_m) of natural numbers, consider the set of $\alpha \in [0,1)$ such that a subsequence of (n_m) is a subsequence of denominators of α. This is a generic subset of $[0,1)$.*

Now, given (n_m) choose any irrational α so that for some subsequence (n_{m_k}) we have all numbers n_{m_k} being denominators of α. Then use previous arguments to construct a weakly mixing "realization" of the whole sequence of denominators of α.

3.4.5. **Integer Lacunarity Case.** We will now give an alternative proof of Proposition 3.27 using the cocycle methods that have been developed here. Assume that $(n_m)_{m\geq 0}$ is an increasing sequence of positive integers such that $n_0 = 1$, $n_{m+1}/n_m \in \mathbb{Z}$ with

$$\rho_m := n_{m+1}/n_m \geq 2 \text{ for } m \geq 0. \tag{3.22}$$

Notice that in view of (3.22) there exists a constant $C > 0$ such that

$$\frac{1}{n_{m+1}^2} + \frac{1}{n_{m+2}^2} + \ldots \leq \frac{C}{n_m^2} \text{ for each } m \geq 0. \tag{3.23}$$

Let $X = \Pi_{m=1}^{\infty}\{0, 1, \ldots, \rho_m - 1\}$ which is a metrizable compact group when we consider the product topology and the addition is meant coordinatewise with carrying the remainder to the right. On X we consider Haar measure p_X which is the usual product measure of uniform measures. Define $Tx = x + \hat{1}$, where

$$\hat{1} = (1, 0, 0, \ldots).$$

The resulting dynamical system is called the (n_m)-*odometer*.

For each $t \geq 0$ set

$$D_0^{n_m} = \{x \in X : x_0 = x_1 = \ldots = x_{m-1} = 0\}.$$

Note that $\{D_0^{n_m}, TD_0^{n_m}, \ldots, T^{n_m-1}D_0^{n_m}\}$ is a Rokhlin tower fulfilling the whole space X and

$$\hat{1} \in TD_0^{n_m} \text{ for each } m \geq 0. \tag{3.24}$$

The character group $\widehat{X}$ of X is discrete and is isomorphic to the (discrete) group of roots of unity of degree n_m, $m \geq 0$. More precisely, for $m \geq 0$ set

$$1_{n_m}(x) = \varepsilon_{n_m}^j := e^{2\pi i j/n_m} \text{ for } x \in T^j D_0^{n_m},\ j = 0, 1, \ldots, n_m - 1.$$

Then $\widehat{X} = \{1_{n_m}^j : j = 0, 1, \ldots, n_m - 1, m \geq 0\}$.

From now on we will consider $f \in L_{2,0}(X, p_X)$ whose Fourier transform is "concentrated" on $\{1_{n_m} : m \geq 0\}$. We have

$$f(x) = \sum_{m=1}^{\infty} a_{n_m} 1_{n_m}(x), \quad \sum_{m=1}^{\infty} |a_{n_m}|^2 < +\infty. \tag{3.25}$$

A) Small divisors. Assume that f satisfies (3.25) and suppose that

$$f(x) = g(x) - g(x + \hat{1}) \text{ for } p_X\text{-a.e. } x \in X. \tag{3.26}$$

Suppose moreover that $g \in L_2(X, p_X)$. Hence $g(x) = \sum_{\chi \in \widehat{X}} b_\chi \chi(x)$ and by comparison of Fourier coefficients on both sides in (3.25) we obtain

$$b_\chi = 0 \text{ whenever } \chi \neq 1_{n_m} \text{ and } a_{n_m} = b_{n_m}(1 - 1_{n_m}(\hat{1})) \text{ for } m \geq 1.$$

Using (3.24) we obtain that $b_{n_m} = \frac{a_{n_m}}{1-\varepsilon_{n_m}}$, so

$$|b_{n_m}|^2 = \frac{|a_{n_m}|^2}{|1 - \varepsilon_{n_m}|^2} = n_m^2 |a_{n_m}|^2 \text{ for } m \geq 1.$$

We have proved the following

$$\text{(3.26) has an } L_2\text{-solution if and only if } (n_m a_{n_m})_m \in l_2. \tag{3.27}$$

B) Estimate of L_2-norms for the cocycle. Assume that $n \in \mathbb{N}$ then

$$\begin{aligned} f^{(n)}(x) &= \sum_{m=1}^{\infty} a_{n_m}(1 + 1_{n_m}(\hat{1}) + \ldots + 1_{n_m}((n-1)\hat{1})) 1_{n_m}(x) \\ &= \sum_{m=1}^{\infty} a_{n_m} \left(\sum_{j=0}^{n-1} \varepsilon_{n_m}^j \right) 1_{n_m}(x). \end{aligned}$$

Fix $n = n_{m_0}$. We then have

$$\begin{aligned} f^{(n_{m_0})}(x) &= \sum_{m=1}^{m_0} a_{n_m} \left(\sum_{j=0}^{n_{m_0}-1} \varepsilon_{n_m}^j \right) 1_{n_m}(x) + \sum_{m=m_0+1}^{\infty} a_{n_m} \left(\sum_{j=0}^{n_{m_0}-1} \varepsilon_{n_m}^j \right) 1_{n_m}(x) \\ &= \sum_{m=m_0+1}^{\infty} a_{n_m} \frac{1 - \varepsilon_{n_m}^{n_{m_0}}}{1 - \varepsilon_{n_m}} 1_{n_m}(x). \end{aligned}$$

Since $|1 - \varepsilon_{n_m}| = 1/n_m$ and $|1 - \varepsilon_{n_m}^{n_{m_0}}| \leq n_{m_0}/n_t$,

$$\|f^{(n_{m_0})}\|_{L_2(X, p_X)}^2 \leq n_{m_0}^2 \sum_{m=m_0+1}^{\infty} |a_{n_m}|^2. \tag{3.28}$$

C) Sidon sets and a "good" function. According to [56] (see Example 5.7.6 therein) every infinite subset of a discrete group contains an infinite Sidon set. Hence we can choose a subsequence (n_{m_k}) of (n_m) so that

$$\{1_{n_{m_k}} : k \geq 1\} \text{ is a Sidon subset of } \widehat{X}. \tag{3.29}$$

We set

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k} n_{m_k}} 1_{n_{m_k}}(x). \tag{3.30}$$

Suppose now that (3.26) has a measurable solution g (we should consider f real valued, so in fact we should consider $f+\overline{f}$ below). In view of (3.29) and [33], $g\in L_2(X,p_X)$. But

$$\sum_{m=1}^{\infty}|n_m a_{n_m}|^2=\sum_{k\geq 1}\frac{1}{k},$$

so by (3.27), we cannot obtain an L_2-solution. This means that f is not a measurable coboundary. According to (3.28), the definition of f and (3.22) for each $s\geq 1$

$$\|f^{(n_s)}\|^2_{L_2(X,p_X)}\leq n_s^2\sum_{m=s+1}^{\infty}|a_{n_m}|^2=n_s^2\sum_{k\geq 1:\, n_{m_k}\geq n_{s+1}}^{\infty}|a_{n_{m_k}}|^2=n_s^2\sum_{k=k_s}^{\infty}|a_{n_{m_k}}|^2$$

$$\leq\frac{n_s^2}{k_s}\sum_{j=s+1}^{\infty}\frac{1}{n_j^2}\leq\frac{C}{k_s}\to 0$$

as clearly $k_s\to\infty$ when $s\to\infty$.

Using our general method we hence proved the following.

Proposition 3.54. *Assume that (n_m) is an increasing sequence of integers with n_{m+1}/n_m being an integer at least 2. Then there exists a weakly mixing transformation R such that $R^{n_m}\to Id$.*

We will now discuss the problem of IP-rigidity along (n_m). First of all notice that (n_m) is a sequence of IP rigidity for the (n_m)-odometer (indeed, $\sum_{m=1}^{\infty}|1_{n_m}(\hat{1})-1|<+\infty$).

Let us also notice that if R is weakly mixing and $R^{n_m}\to Id$ then by passing to a subsequence, we will get $IP-R^{n_{m_k}}\to Id$. But if we then set $m_k=n_{m_k}$ then m_k divides m_{k+1} and (m_k) is a sequence of IP-rigidity for R. It means that if the sequence (n_{m+1}/n_m) is unbounded then, at least in some cases, it is a sequence of IP-rigidity for a weakly mixing transformation. On the other hand we have already seen (Corollary 3.28) that when $\rho_t=a$, $t\geq 1$ then IP-rigidity does not take place. The proposition below generalizes that result and shows that in the bounded case an IP-rigidity is excluded.

Proposition 3.55. *Assume that $n_{t+1}/n_t\in\mathbb{N}\setminus\{0,1\}$, $t\geq 0$, and $\sup_{t\geq 0}n_{t+1}/n_t=:C<+\infty$. Then for any weakly mixing transformation R for which $R^{n_t}\to Id$, the sequence (n_t) is not a sequence of IP-rigidity.*

Proof. Recall that $Tx=x+\hat{1}$ where X stands for the (n_t)-odometer. Each natural number $r\geq 1$ can be expressed in a unique manner as

$$r=\sum_{t=0}^{N}a_t n_t,\ 0\leq a_t<\rho_t=n_{t+1}/n_t.$$

Assume that $T^{r_m}\to Id_X$. Write

$$r_m=\sum_{t=0}^{N_m}a_t^{(m)}n_t,\ 0\leq a_t^{(m)}<\rho_t$$

and set $k_m=\max\{t\geq 0:\ a_0^{(m)}=\ldots=a_t^{(m)}=0\}$ (so $r_m=\sum_{t=k_m}^{N_m}a_t^{(m)}n_t$). We claim that

$$k_m\to\infty\ \text{ whenever } m\to\infty. \tag{3.31}$$

Indeed, suppose that the claim does not hold. Then without loss of generality we can assume that there exists $t_0 \geq 0$ such that $k_m = t_0$ for all $m \geq 1$, that is

$$r_m = a_{t_0}^{(m)} n_{t_0} + \sum_{t=t_0+1}^{N_m} a_t^{(m)} n_t \text{ with } 1 \leq a_{t_0}^{(m)} < \rho_{t_0}.$$

Consider the tower $\{D_0^{n_{t_0+1}}, \dots, D_{n_{t_0+1}-1}^{n_{t_0+1}}\}$ and let $A = D_0^{n_{t_0+1}}$. Notice that for each $i \geq 0$ and $j \geq 1$ we have $T^{in_{t_0+j}} A = A$. It follows that

$$T^{r_m} A = T^{a_{t_0}^{(m)} n_{t_0}} \left(T^{\sum\limits_{t=t_0+1}^{N_m} a_t^{(m)} n_t} (A) \right)$$

$$= T^{a_{t_0}^{(m)} n_{t_0}} (A) \in \{D_1^{n_{t_0+1}}, \dots, D_{n_{t_0+1}-1}^{n_{t_0+1}}\},$$

where the latter follows from the fact that $1 \leq a_{t_0}^{(m)} < \rho_{t_0}$. So $p_X(T^{r_m}(A) \triangle A) = 2/n_{t_0+1}$ and hence (r_m) is not a rigidity sequence for T, a contradiction. Thus (3.31) has been shown.

Assume now that R is a weakly mixing transformation for which (n_t) is its IP-rigidity sequence. It follows that we have a convergence along the net

$$R^{\sum\limits_{t=k}^{N} \eta_t n_t} \to Id \text{ whenever } k \to \infty \text{ and } \eta_k = 1, 0 \leq \eta_t \leq 1, t \geq k+1. \tag{3.32}$$

We claim that also

$$R^{\sum_{t=k}^{N} a_t n_t} \to Id \text{ whenever } k \to \infty \text{ and } 1 \leq a_k \leq \rho_k - 1, 0 \leq a_t \leq \rho_t - 1, t \geq k+1. \tag{3.33}$$

Indeed, we write $R^{\sum\limits_{t=k}^{N} a_t n_t}$ as the composition of at most $S_1 \circ \dots \circ S_D$ with $D \leq C = \max_t \rho_t$ automorphisms of the form $R^{\sum\limits_{t=k}^{N} \delta_t n_t}$ with $\delta_t \in \{0, 1\}$ (to define the first automorphism S_1 we put $\delta_t = 1$ as soon as $a_t \geq 1$ and $\delta_t = 0$ elsewhere, for the second automorphism S_2 we put $\delta_t = 1$ as soon as $a_t \geq 2$ and $\delta_t = 0$ elsewhere, etc.). Notice that for each $i = 1, \dots, D$

$$S_i = R^{\sum_{t=k_i}^{N} n_t} \text{ with } k_i \geq k.$$

We need to use a metric d for $Aut(X, \mathcal{B}, p)$ that is compatible with the strong topology. Take $\{A_i : i \geq 1\}$ which is a dense subset in $(\mathcal{B}, p)$. Let d be given by

$$d(R, S) = \sum_{i=1}^{\infty} \frac{1}{2^i} (p\,(RA_i \triangle SA_i) + p\left(R^{-1}A_i \triangle S^{-1}A_i\right)). \tag{3.34}$$

Let $\|T\| = d(T, Id)$. Now if we assume that in (3.32), $\|R^{\sum_{t=k}^{N} n_t}\| < \varepsilon$ for $k \geq K$, then $\|S_1 \circ \dots \circ S_D\| < D\varepsilon$ and thus (3.33) follows.

Combining (3.33) and (3.31) we see that each rigidity sequence for T is also a rigidity sequence for R. This however contradicts Corollary 3.41 (or rather to its proof). □

Question. If (ρ_t) is bounded, but not always a whole number, can it still (n_t) be a sequence of IP-rigidity for some weakly mixing transformation? How fast does (ρ_t) have to grow for (n_t) to be a sequence of IP-rigidity for some weakly mixing transformation?

4. Non-Recurrence

In the previous sections, we have seen that the characterization of which sequences (n_m) exhibit rigidity for some ergodic, or more specifically weakly mixing dynamical system will, most certainly be difficult. The only aspect that is totally clear at this time is that these sequences must have density zero because their gaps tend to infinity. Lacunary sequences are always candidates for consideration in such a situation. However, we have seen in Remark 2.23 d) that there are lacunary sequences which cannot be rigidity sequences for even ergodic transformations, let alone weakly mixing ones.

In a similar vein, we would like to characterize which increasing sequences (n_m) in $\mathbb{Z}^+$ are not recurrent for some ergodic dynamical system i.e. (n_m) has the property that for some ergodic system $(X, \mathcal{B}, p, T)$ and some set A of positive measure, the sets $T^{n_m}A$ are disjoint from A for all $m \geq 1$. So we are taking *recurrence along* (n_m) here to mean that $p(T^{n_m}A \cap A) > 0$ for some m. A central unanswered question is the following

Question: Is it the case that any lacunary sequence is a sequence of non-recurrence for a weakly mixing system?

Remark 4.1. It is not hard to see that any lacunary sequence fails to be a recurrent sequence for some ergodic dynamical system. Indeed, this happens even with ergodic rotations of $\mathbb{T}$. See Pollington [50], de Mathan [44], and Furstenberg [18], p. 220. They show that for any lacunary sequence (n_m) there is some $\gamma \in \mathbb{T}$ of infinite order, and some $\delta > 0$, such $|\gamma^{n_m} - 1| \geq \delta$ for all $m \geq 1$. The arguments there also give information about the size of the set of rotations that work for a given lacunary sequence. The constructions in these articles are made more difficult, as with a number of other results about lacunary sequences, by not knowing the degree or nature of lacunarity. But if one just wants some ergodic dynamical system to exhibit non-recurrence, then the construction is easier. This was observed by Furstenberg [18]. In short, his argument goes like this. Suppose (n_k) is lacunary, say $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k \geq 1$. Depending only on λ, we can choose K so that the subsequences $(p_{m,j} : m \geq 1)$ given by $p_{m,j} = n_{j+Km}, j = 0, \ldots, K-1$ each have lacunary constant $\inf\limits_{m\geq 1} \frac{p_{m+1,j}}{p_{m,j}} \geq 5$. Then for each j, a standard argument shows that there is a closed perfect set C_j such that for all $\gamma \in C_j$, we have $|\gamma^{p_{m,j}} - 1| \geq \frac{1}{100}$ for all $m \geq 1$. We can choose $(\gamma_0, \ldots, \gamma_{K-1})$ with each $\gamma_j \in C_j$ and such that $\gamma_0, \ldots, \gamma_{K-1}$ are independent. Then we would know that the transformation T of the K-torus $\mathbb{T}^K$ given by $T(\alpha_0, \ldots, \alpha_{K-1}) = (\gamma_0\alpha_0, \ldots, \gamma_{K-1}\alpha_{K-1})$ is ergodic Also, for a sufficiently small ϵ, the arc I of radius ϵ around 1 in $\mathbb{T}$ will give a set $C = I \times \ldots \times I \subset T^K$ such that for all n_k, we have $T^{n_k}C$ and C disjoint. Indeed, each n_k is some $p_{m,j}$ and so $T^{n_k}C$ and C are disjoint because in the j-th coordinate T^{n_k} corresponds to the rotation $\gamma_j^{p_{m,j}}I$ which is disjoint from I.

The main idea in Remark 4.1 that appears in Furstenberg [18] gives us this basic principle.

Proposition 4.2. *If a sequence* $\mathbf{n}$ *is a finite union of sequences* $\mathbf{n}_i, i = 1, \ldots, I$*, each of which is a sequence of non-recurrence for some weakly mixing transformation* T_i*, then* $\mathbf{n}$ *is also a sequence of non-recurrence for a weakly mixing transformation.*

Proof. Write $\mathbf{n}_i = (\mathbf{n}_i(j) : j \geq 1)$. We take $T = T_1 \times \ldots \times T_I$. This is a weakly mixing transformation since each T_i is weakly mixing. There is a set C_i such that $T_i^{\mathbf{n}_i(j)}C_i$ is disjoint from C_i for all j. So $C = C_1 \times \ldots \times C_i$ has the property that $T^{\mathbf{n}_i(j)}C$ is disjoint from C for all i and all j. That is, the sequence $T_1 \times \ldots \times T_I$ is not recurrent along $\mathbf{n}$ for the set C. □

Remark 4.3. a) This property of non-recurrent sequences does not hold for rigidity sequences. For example, consider $\mathbf{n}_1 = (2^{m^2})$ and $\mathbf{n}_2 = (2^{m^2} + 1)$. By Proposition 4.8 below, these are both sequences of non-recurrence for a weakly mixing transformation and hence by the above their union is too. These two sequences are also rigidity sequences for ergodic rotations and weakly mixing transformations by Proposition 3.5. But the union of these two sequences is not a rigidity sequence for an ergodic transformation because a rigidity sequence for an ergodic transformation cannot have infinitely many terms differing by 1.

b) Here is a related example that shows how rigidity sequences and non-recurrent sequences behave differently. The sequence $A = (p : p \text{ prime})$ is not recurrent but the sequence $A = (p-1 : p \text{ prime})$ is recurrent. See Sárközy [60] and apply the Furstenberg Correspondence Principle. But we see from either Proposition 2.20 or Proposition 2.24 that neither sequence is a rigidity sequence for an ergodic transformation.

We see that the non-recurrence phenomenon is both pervasive and not, depending on how one chooses the quantifiers. As usual, the measure-preserving transformations of a non-atomic separable probability space $(X, \mathcal{B}, p)$ can be given the weak topology, and become a complete pseudo-metric group $\mathcal{G}$ in this topology. By a generic transformation, we mean an element in a set that contains some dense G_δ set in $\mathcal{G}$.

Proposition 4.4. *The generic transformation is both weakly mixing and rigid, and moreover is not recurrent along some increasing sequence in* $\mathbb{Z}^+$.

Proof. Remark 2.15 pointed out that the generic transformation is weakly mixing and rigid. Fix such a transformation T and some (n_m) such that $\|f \circ T^{n_m} - f\|_2 \to 0$ for all $f \in L_2(X,p)$. Then it follows that for any set A, $\lim_{m\to\infty} p(T^{n_m} A \Delta A) = 0$. Since T is ergodic, we can choose a set A with $p(A) > 0$ and TA and A disjoint. Let $B = TA$. We have B and A disjoint, and $\lim_{m\to\infty} p(T^{n_m - 1} B \Delta A) = 0$. Hence, we can pass to a subsequence (m_s) so that $\sum_{s=1}^{\infty} p(T^{n_{m_s}-1} B \Delta A) \le \frac{1}{100} p(A) = \frac{1}{100} p(B)$. So

$$C = B \setminus \bigcup_{s=1}^{\infty} T^{-(n_{m_s}-1)}(T^{n_{m_s}-1} B \Delta A)$$

will have $p(C) > 0$. Also, $C \subset B = TA$ is disjoint from A. But at the same time

$$T^{n_{m_s}-1} C \subset T^{n_{m_s}-1} B \setminus \big(T^{n_{m_s}-1} B \triangle A\big) = T^{n_{m_s}-1} B \cap A \subset A$$

for all $s \ge 1$.

Thus, the generic transformation is weakly mixing and rigid, and additionally for some sequence of powers $T^{n_{m_s}}$ and some set C of positive measure, we have non-recurrence because C is disjoint from all $T^{n_{m_s}} C$. $\square$

Remark 4.5. a) This argument can easily be used to show that if T is rigid along (n_m), then for any K, by passing to a subsequence (n_{m_s}), we can have for each $k \neq 0, |k| \le K$, the transformation T is non-recurrent along $(n_{m_s} + k)$ for some set $C_k, p(C_k) > 0$. By taking S to be a product of T with itself $2K$ times, we can arrange that the weakly mixing transformation S is rigid along (n_m) and there is one set $C, p(C) > 0$, such that S is non-recurrent for C along each of the sequences $(n_{m_s} + k)$ with $0 < |k| \le K$.

b) One cannot restrict the sequence along which the non-recurrence is to occur. It is not hard to see that the class of transformations that is non-recurrent for some set along a fixed sequence is a meager set of transformations.

We do have some specific, interesting examples of the failure of recurrence for a weakly mixing dynamical system. See Chacon [7] for the construction of the rank one Chacon transformation. The important point here is that the non-recurrence occurs along a lacunary sequence (n_m) with ratios n_{m+1}/n_m bounded. See Remark 4.7 a) for more information about this example.

Proposition 4.6. *The Chacon transformation is not recurrent for sequence* $(n_m) = (\frac{3^{m+1}-1}{2} - 1)$.

Proof. Let $n_m = \frac{3^{m+1}-1}{2} - 1$. The transformation T that we are using here is constructed inductively as follows. Take the current stack (single tower) T_m of interval and cut it in thirds $T_{m,1}$, $T_{m,2}$, and $T_{m,3}$. Add a spacer s of the size of levels above the middle third $T_{m,2}$, and let the new stack T_{m+1} consists of $T_{m,1}$, $T_{m,2}$, s, and $T_{m,3}$ in that order from bottom to top. We take T_0 to be $[0,1)$ to start this construction. So it is easy to see that the height h_m of our m-th tower is $\frac{3^{m+1}-1}{2}$. To see the failure of recurrence, use the standard symbolic dynamics for T i.e. assign the symbol 1 to all the spacer levels and 0 to the rest of the levels. Then let B_m be the name of length h_m of a point in the base of the m-th tower T_m. Then $B_0 = 0$, $B_1 = 0010$, and $B_{m+1} = B_m\ B_m\ 1\ B_m$ in general. It is a routine check that when one shifts B_k by $n_m = h_m - 1$ for k larger than m, and compares this with B_k, then they have no common occurrences of 1. So if A is the first added spacer level, then we have $T^{n_m}A$ and A disjoint for all m. □

Remark 4.7. a) The Chacon transformation is mildly mixing so it cannot have rigidity sequences at all. But there is partial rigidity in that $T^{h_m} \to \frac{1}{2}(Id + T^{-1})$ in the strong operator topology. So (h_m) and (h_m+1) are recurrent sequences for T in a strong sense, while the above is showing that $(h_m - 1)$ is not recurrent for T.

b)The obvious question here is what other sequences, besides ones like the one above for the Chacon transformation, can be show to be sequences of non-recurrence for weakly mixing transformations via classical cutting and stacking constructions?

b) Friedman and King [16] consider a class of weakly mixing, but not strongly mixing, transformations constructed by Chacon; they prove that these, unlike the Chacon transformation above, are lightly mixing (see [16] for the definition) and so there is always recurrence for these transformations along any increasing sequence. Hence, these transformations form a meager set by the category result in Proposition 4.4.

If we have a sufficient growth rate assumed for (n_m), we can give a construction of a weakly mixing transformation which exhibits non-recurrence along the sequence. The argument here starts like the constructions in Section 3.1

Proposition 4.8. *Suppose we have a sequence (n_m) such that $\sum\limits_{k=1}^{\infty} n_m/n_{m+1} < \infty$. Then there is a weakly mixing transformation T for which (n_m) is an IP rigidity sequence and such that T is not recurrent for $(n_m - 1)$.*

Proof. Choose a non-decreasing sequence of whole numbers (h_m) with $1/h_m \geq 10(n_m/n_{m+1})$ and $\sum\limits_{m=1}^{\infty} 1/h_m < \infty$. We construct a Cantor set $\mathcal{C}$ with constituent intervals at each level that are arcs of size $1/(n_m h_m)$ around some of the n_m-th roots of unity (determined as part of the induction). Our conditions allow us to find in each such constituent interval many points j/n_{m+1} because $1/n_{m+1}$ is sufficiently smaller than $1/n_m h_m$, and then select in these constituent intervals new ones of length $1/(n_{m+1}h_{m+1})$ around some of the n_{m+1}-th roots of unity for $m \geq M$. The resulting Cantor set $\mathcal{C}$ has the property that for all points x in the set

$n_m x$ is within $1/h_m$ of an integer for $m \geq M$. Also, it follows that if we take a continuous, positive measure ν_0 on $\mathcal{C}$, with $\nu_0([0,1)) = 1$, then $|1 - \widehat{\nu_0}(n_m)| \leq 1/h_m$ for $m \geq M$.

Now we take the GMC construction corresponding to the symmetrization ω of ν_0. We can use Proposition 3.3 and the result from Erdős and Taylor [12] cited in Remark 3.4 to conclude that (n_m) is an IP rigidity sequence for the weakly mixing transformation $T = G_\omega$. This gives us a weakly mixing dynamical system $(X, \mathcal{B}, p, T)$ and a function f of norm one in $L_2(X,p)$ such that $\|f \circ T^{n_m} - f\|_2^2 \leq C/h_m$ for $m \geq M$. The function f here is the one in the GMC such that $\nu_f^T = \omega$. It follows immediately, by the fact that ω is symmetric and by the symmetric Fock space construction in the GMC, that f is a Gaussian variable from the first chaos. So f is real-valued, and being a Gaussian random variable it takes both positive and negative values. So we have a non-constant, real-valued function of norm one such that $\|f \circ T^{n_m} - f\|_2^2 \leq 4/h_m$ for $m \geq M$.

Now we claim that both the positive part f^+ and the negative part f^- of f satisfies the same inequality. To see this, write

$$\int |f \circ T^{n_m} - f|^2 \, dp = \int |f^+ \circ T^{n_m} - f^- \circ T^{n_m} - f^+ + f^-|^2 \, dp.$$

Expand this into the sixteen terms involved. Use that fact that the terms $f^+ f^-$ and $(f^+ \circ T^{n_m})(f^- \circ T^{n_m})$ are zero, and regroup terms to see that

$$\begin{aligned} \int |f \circ T^{n_m} - f|^2 \, dp &= \int |f^+ \circ T^{n_m} - f^+|^2 \, dp \\ &+ \int |f^- \circ T^{n_m} - f^-|^2 \, dp \\ &+ \int 2(f^+ \circ T^{n_m}) f^- + 2(f^- \circ T^{n_m}) f^+ \, dp. \end{aligned}$$

Because $2(f^+ \circ T^{n_m}) f^- + 2(f^- \circ T^{n_m}) f^+$ is positive, we have

$$\int |f \circ T^{n_m} - f|^2 \, dp \geq \int |f^+ \circ T^{n_m} - f^+|^2 \, dp$$

and

$$\int |f \circ T^{n_m} - f|^2 \, dp \geq \int |f^- \circ T^{n_m} - f^-|^2 \, dp.$$

In addition, the same argument above shows that for every constant L, the function $(f - L)^+$ also would satisfy this last estimate too. Hence, taking $L = 1/2$, we would have $F = f 1_{\{f \geq L\}}$ satisfies this inequality too and not being the zero function. But let $A = \{f \geq L\}$. On $T^{n_k} A \backslash A$, we would have $|F \circ T^{n_m} - F|^2 \geq 1/4$. So $p(T^{n_m} A \backslash A) \leq 64/h_m$. But similarly, on $A \backslash T^{n_m} A$, we would have $|F \circ T^{n_m} - F|^2 \geq 1/4$. So $p(A \backslash T^{n_m} A) \leq 64/h_m$. The result is that from our original GMC construction, we can infer the existence of a proper set A of positive measure, which depends on the original function f and not on m, such that $p(T^{n_m} A \Delta A) \leq 64/h_m$ for all $m \geq M$. Hence, $\sum\limits_{m=1}^{\infty} p(T^{n_m} A \Delta A) < \infty$.

We do have to also make certain that the set A here is a proper set i.e. $p(A) < 1$. But f is a Gaussian random variable and so both f^+ and f^- are non-trivial, and so it is easy to choose a value of L so that the above construction gives us a proper set A of positive measure.

Now we can use the convergence of $\sum\limits_{m=1}^{\infty} p(T^{n_m} A \Delta A)$ to construct a set of positive measure B for which $T^{n_m - 1} B$ and B are disjoint for all m. The argument is a variation on the one given in Proposition 4.4. There is the issue that TA and A are not necessarily disjoint. But there is some subset A_0 of A of positive measure such that TA_0 and A are disjoint. So if we take $B = TA_0 \setminus \bigcup_{m=M}^{\infty} T^{-(n_m-1)}(T^{n_m}A\Delta A)$, for suitably large M, then we would have $p(B) > 0$, $B \subset TA_0 \setminus A$, and $T^{n_m-1}B \subset A$ for all $m \geq M$. Hence $T^{n_m-1}B$ and B disjoint for all $m \geq M$.

Now we need to revise the result above so that we get the same disjointness for all m. But here we know that T is weakly mixing, and consequently all of its powers are ergodic. So one can inductively revise B as follows. One takes a subset B_1 of B such that $T^{n_1-1}B_1$ and B_1 are disjoint, then one takes a subset B_2 of B_1 such that $T^{n_2-1}B_2$ and B_2 are disjoint, and so on. After a finite number of steps one ends up with a subset B_{M-1} of B such that $T^{n_m-1}B_{M-1}$ and B_{M-1} are disjoint for all $m \geq 1$. Now, with B_{M-1} replacing B, we have $T^{n_m-1}B$ and B disjoint for all $m \geq 1$. So this construction gives a weakly mixing transformation that is not only IP rigid along (n_m), but such that along $(n_m - 1)$ it is not recurrent. □

Remark 4.9. With the hypothesis of Proposition 4.8, by taking S to be a product of T with itself $2K$ times, we can arrange that the weakly mixing transformation S is rigid along (n_m) and there is one set $C, p(C) > 0$, such that S is non-recurrent for C along each of the sequences $(n_m + k)$ with $0 < |k| \leq K$.

Remark 4.10. Proposition 4.2 allows us to use Proposition 4.8 to give other examples of non-recurrent sequences. Again, as in Remark 4.3 a), both (2^{n^2}) and $(2^{n^2}+1)$ satisfy the hypothesis of Proposition 4.8, so there is a weakly mixing transformation for which $\mathbf{n} = (\ldots, 2^{n^2}, 2^{n^2}+1, \ldots)$ is a sequence of non-recurrence, even though $\mathbf{n}$ does not satisfy the hypothesis of Proposition 4.8.

Remark 4.11. Using Proposition 3.10, we can construct examples of non-recurrence along $(n_m - 1)$ for T which is weakly mixing and rank one. For example, take (n_m) such that $n_{m+1}/n_m \geq 2$ is a whole number for all m and such that $\sum_{m=1}^{\infty} \frac{n_m}{n_{m+1}} < \infty$.

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