

On non-mixing of interval exchange
transformations

1 Interval exchange transformations (IET) and inducing

Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$ be a probability vector ($\alpha_i > 0$, $i = 1, \dots, m$) and put

$$\beta_0 = 0, \quad \beta_i = \sum_{j=1}^i \alpha_j, \quad \Delta_i = [\beta_{i-1}, \beta_i)$$

for $i = 1, \dots, m$. Then assume that $\tau \in \mathcal{S}_m$ (irreducible) permutation of $\{1, \dots, m\}$ and let

$$\underline{\alpha}^\tau = (\alpha_{\tau^{-1}(1)}, \dots, \alpha_{\tau^{-1}(m)}), \quad \beta_0^\tau = 0, \quad \beta_i^\tau = \sum_{j=1}^i \alpha_j^\tau.$$

If we now put $\Delta_i^\tau = [\beta_{i-1}^\tau, \beta_i^\tau)$ then the map

$$T(x) = T_{\underline{\alpha}, \tau}(x) = x - \beta_{i-1} + \beta_{\tau(i)-1}^\tau \quad \text{for } x \in \Delta_i$$

($i = 1, \dots, m$) is an IET or m -IET (it sends linearly Δ_i into $\Delta_{\tau(i)}^\tau$).

FROM NOW ON T IS ASSUMED TO BE MINIMAL.

Facts

- If $[a, b) \subset [0, 1)$ then the induced transformation $T_{[a,b)}$ is also an IET of at most $m + 2$ intervals (consider the “first” preimages of points $a, b, \beta_2, \dots, \beta_{m-1}$ which comes back to $[a, b)$ and then “first returns” of at most $m + 2$ subintervals to $[a, b)$).
- For each $b > 0$ inducing on $[0, b)$ yields a partition of $[0, 1)$ into at most $m + 1$ towers (levels are intervals) whose bases are obtained from the previous step.

Proposition 1 (Katok, 1980). *There exist two sequences $\Xi_n = \{T^i I_n\}_{0 \leq i < p_n}$, $\Xi'_n = \{T^i J_n\}_{0 \leq i < q_n}$ of towers of intervals for T such that*

- (1) $|I_n| \rightarrow 0$ as $n \rightarrow \infty$
- (2) $|\Xi_n| \geq \frac{1}{m+1}$
- (3) $J_n \subset I_n$ and $T^{q_n} J_n \subset I_n$
- (4) $\bigcup_{0 \leq i < p_n} T^i I_n \cap \bigcup_{p_n \leq i < q_n} T^i J_n = \emptyset$
- (5) $|J_n| \geq \frac{1}{m+2} |I_n|$.

Proof. Take $0 < b_n \rightarrow 0$, induce on $[0, b_n)$. Choose $I_n \subset [0, b_n)$ so that the corresponding tower (Fact 2 above) fulfills at least $\frac{1}{m+1}$ of the space, and induce once more on I_n choosing relevant $J_n \subset I_n$ (the base of one of at most $m+2$ subintervals), so that $|J_n| \geq \frac{1}{m+2} |I_n|$. \square

Put

$$C_n = \bigcup_{0 \leq i < p_n} T^i J_n.$$

Notice that $d(T^{q_n} x, x) \rightarrow 0$ uniformly for $x \in C_n$. By (5) we obtain that $|C_n| \geq 1/(m+1)(m+2)$. Moreover, from (1) we have $|C_n \Delta T^{-1} C_n| \leq 2|I_n| \rightarrow 0$ as $n \rightarrow \infty$. Set

$$a_n := \frac{1}{|I_n|} \int_{\bigcup_{0 \leq i < p_n} T^i I_n} f(t) dt + \frac{1}{|J_n|} \int_{\bigcup_{p_n \leq i < q_n} T^i J_n} f(t) dt.$$

Proposition 2 (Katok, 1980; “distributional” D.-K. inequality). *If $f \in BV([0, 1])$ then*

$$\left| f^{(q_n)}(x) - a_n \right| \leq \text{Var}_{[0,1]}(f) \text{ for } x \in C_n,$$

where $f^{(k)}(x) = \sum_{j=0}^{k-1} f(T^j x)$ for $k \in \mathbb{N}$.

Proof. Suppose that $x \in C_n$, say $x \in T^k J_n$, where $0 \leq k < p_n$. Then

$$T^i x \in \begin{cases} T^{i+k} I_n & \text{if } 0 \leq i < p_n - k \\ T^{i+k} J_n & \text{if } p_n - k \leq i < q_n - k \\ T^{i+k-q_n} I_n & \text{if } q_n - k \leq i \leq q_n, \end{cases}$$

by (3). In particular $x, T^{q_n} x \in T^k I_n$. Then for $x \in T^k J_n$ we have

$$\begin{aligned} |f^{(q_n)}(x) - a_n| &\leq \sum_{k \leq i < p_n} \frac{1}{|I_n|} \int_{T^i I_n} |f(T^{i-k} x) - f(t)| dt \\ &\quad + \sum_{p_n \leq i < q_n} \frac{1}{|J_n|} \int_{T^i J_n} |f(T^{i-k} x) - f(t)| dt \\ &\quad + \sum_{0 \leq i < k} \frac{1}{|I_n|} \int_{T^i I_n} |f(T^{q_n-k+i} x) - f(t)| dt \\ &\leq \sum_{0 \leq i < p_n} \text{Var}_{T^i I_n} f + \sum_{p_n \leq i < q_n} \text{Var}_{T^i J_n} f \leq \text{Var}_{[0,1]} f, \end{aligned}$$

by (4). □

Recall the classical Denjoy-Koksma inequality on the circle for $Tx = x + \alpha$ (irrational rotation): if f is of bounded variation,

$$\left| f^{(N)}(x) - \int_0^1 f d\lambda \right| \leq \text{Var}(f) \cdot ND_N(x)$$

where $D_N(x)$ stands for the discrepancy of $x, x + \alpha, \dots, x + (N-1)\alpha$; in particular for $N = q_n = n$ -th denominator of α we obtain that

$$\left| f^{(q_n)}(x) - \int_0^1 f d\lambda \right| \leq \text{Var}(f).$$

Coming back to T being m -IET...

Inducing first on $|I_n| \rightarrow 0$ and obtaining at most $m+2$ subintervals $I_{n,j}$ and then inducing on EACH of them we obtain at most $(m+2)^2$ towers (fulfilling the whole space) which leads to

a conclusion (Katok) that there are subsequences $(p_n^{j,k})_{n \geq 1}$ where $1 \leq j, k \leq m + 2$ such that for some constant $\kappa > 0$

$$\liminf_{n \rightarrow \infty} \lambda(A \cap \bigcup_{j,k=1}^{m+2} T^{p_n^{j,k}} B) \geq \kappa \lambda(A \cap B).$$

Corollary 3 (Katok, 1980). *No factor of IET is mixing.*

Proof. Suppose for some set A we have mixing... Taking $A = B$ we have $\lambda(A \cap T^{p_n^{j,k}} A) \approx \lambda(A)^2$, so

$$\kappa \lambda(A) \leq \lambda(A \cap \bigcup_{j,k=1}^{m+2} T^{p_n^{j,k}} A) \leq (\approx)(m + 2)^2 \lambda(A)^2$$

which for $\lambda(A) > 0$ sufficiently small gives rise to a contradiction. □

2 Special flows, special flows over IET under BV functions

Let T be an ergodic automorphism of a standard probability space (X, \mathcal{B}, μ) . Denote by λ Lebesgue measure on \mathbb{R} . Assume that $f \in L^1(X, \mathcal{B}, \mu)$ is a positive function. The *special flow* $T^f = \{(T^f)_t\}_{t \in \mathbb{R}}$ built from T and f is defined on the space $X^f = \{(x, t) \in X \times \mathbb{R} : 0 \leq t < f(x)\}$ (considered with \mathcal{B}^f the restriction of the product σ -algebra and μ^f the restriction of the product measure $\mu \otimes \lambda$ to $X \times \mathbb{R}$). Under the action of the special flow each point in X^f moves vertically at unit speed, and we identify the point $(x, f(x))$ with $(Tx, 0)$. Given $m \in \mathbb{Z}$ we put

$$f^{(m)}(x) = \begin{cases} f(x) + f(Tx) + \dots + f(T^{m-1}x) & \text{if } m > 0 \\ 0 & \text{if } m = 0 \\ -(f(T^m x) + \dots + f(T^{-1}x)) & \text{if } m < 0. \end{cases}$$

Proposition 4 (Kochergin, 1972). *If T is an irrational rotation and f of bounded variation then T^f is not mixing.*

Proof. Assume that $\int_0^1 f d\mu = 1$. Notice that

$$\begin{aligned} T_{q_n}^f(x, 0) &= T_{q_n - f^{(q_n)}(x)}^f \circ T_{f^{(q_n)}(x)}^f(x, 0) \\ &= T_{q_n - f^{(q_n)}(x)}^f(T^{q_n}x, 0) \approx T_{q_n - f^{(q_n)}(x)}^f(x, 0) \end{aligned}$$

and since

$$\left| q_n - f^{(q_n)}(x) \right| \leq \text{Var}(f),$$

little rectangles can not be well spread out over the whole space X^f . \square

A. Katok extended this **non-mixing result to flows over IETs** (f with bounded variation).

Motivations to a come back to these results: smooth flows on surfaces (Arnold), billiards in rational polygons (Katok-Zemlyakov): spectral questions, Ratner's property,...

3 Joinings, disjointness and Markov operators

Given ergodic systems T, S on standard probability spaces $(X, \mathcal{B}, \mu), (Y, \mathcal{C}, \nu)$ respectively the set of joinings between T, S is denoted by

$$J(T, S) = \begin{array}{l} \text{set of } T \times S\text{-invariant measures} \\ \text{on } X \times Y \text{ with projections } \mu \text{ and } \nu \text{ respectively.} \end{array}$$

$$J(T, S) \supset J^e(T, S) = \text{subset of ergodic joinings.}$$

Disjointness in the sense of Furstenberg (1967):

$$T \perp S \text{ if } J(T, S) = \{\mu \otimes \nu\}.$$

Factors and joinings – “analogy” with natural numbers:

Dynamical system = number; factors of a dynamical system = divisors of a natural number; joinings of dynamical systems = common multiples of natural numbers.

Two natural numbers are coprime if (by definition) they do not have common factors (divisors) iff each common multiple of them has their product as a factor. IS THE SAME TRUE ON THE LEVEL OF DYNAMICAL SYSTEMS? One way it works, but in general systems without common factors may be non-disjoint (Rudolph 1977, and others).

Conjecture: Two systems are non-disjoint iff there are their distal extensions with common factors.

If $\lambda \in J(T, S)$ then it determines a Markov operator $\Phi_\lambda : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(Y, \mathcal{C}, \nu)$,

$$(6) \quad \int_Y \Phi_\lambda(f)g \, d\nu = \int_{X \times Y} f \otimes g \, d\lambda, \quad \Phi_\lambda \circ T = S \circ \Phi_\lambda$$

(here T or S stand for the associated Koopman operators acting on the underlying L^2 -spaces, $f \mapsto f \circ T, g \mapsto g \circ S$ – we will also

use notation U_T, U_S). That is Φ_λ is doubly stochastic: $\Phi_\lambda(1) = \Phi_\lambda^*(1) = 1$, and the image via Φ_λ of a non-negative function is non-negative. Moreover (up to some abuse of notation)

$$\Phi_\lambda(f) = E^\lambda(f|Y).$$

In fact (6) establishes a 1-1 correspondence between joinings and (intertwining) Markov operators. Recall that Markov operators form a compact subset in the weak operator topology and notice that U_T itself is a Markov operator. In particular, each operator Φ obtained as a weak limit of powers of U_T^n will be a Markov operator (hence a self-joining of a given system).

Notation for spectral measures $\sigma_{T,f}$ or σ_f if T is understood (recall that $\widehat{\sigma}(n) = \int_X f \circ T^n \cdot \overline{f} d\mu$ for $n \in \mathbb{Z}$).

4 Special flows over IETs with roof function of bounded variation

When studying weak convergences of time- t automorphisms of flows we have to study expressions of the form

$$\mu^f ((T^f)_t A \cap B) = \sum_{k \in \mathbb{Z}} \mu \otimes \lambda ((S_{-f})^k \sigma_t A \cap B),$$

where $S_{-f}, \sigma_t : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$,

$$S_{-f}(x, u) = (Tx, -f(x) + u), \quad \sigma_t(x, u) = (x, u + t)$$

or in case $A, B \subset X \times \mathbb{R}(\cap X^f)$ are measurable rectangles of the form $A = A_1 \times A_2, B = B_1 \times B_2$

$$\mu^f ((T^f)_t A \cap B) = \sum_{k \in \mathbb{Z}} \int_{T^k A_1 \cap B_1} \lambda \left((A_2 + f^{(-k)}(x)) \cap B_2 \right) d\mu(x).$$

Now a role played by the Denjoy-Koksma type inequalities can be seen in the convergence of distributions of type $(f_0^{(q_n)})_*$, where $f_0 = f - \int_X f d\mu$. For example if T is irrational rotation $Tx = x + \alpha$ then $|f_*^{(q_n)}|$ ((q_n) stands for the sequence of denominators of α) is uniformly bounded, so the limit of each (weakly) converging subsequence of distributions $(f_*^{(q_{n_k})})$ has a compact support. In fact, we have the following general result.

Proposition 5 (Frączek - L., 2003). *Let $\{q_n\}$ be a rigidity sequence for an ergodic automorphism T . Suppose that a measurable function $f : X \rightarrow \mathbb{R}$ is positive, bounded and $\int_X f(x) d\mu(x) = 1$. Assume that $\sup_{n \in \mathbb{N}} \|f_0^{(q_n)}\|_\infty \leq r$ and $(f_0^{(q_n)})_* \mu \rightarrow P$ weakly in $\mathcal{P}(\mathbb{R})$. Then*

$$(T^f)_{q_n} \rightarrow \int_{\mathbb{R}} (T^f)_{-t} dP(t).$$

Corollary 6. *All Koçhergin's flows with the roof function of bounded variation are spectrally disjoint from all mixing flows.*

Proof. Take $h \in L_0^2(X, \mathcal{B}, \mu)$ such that σ_h is a Rajchman measure (i.e. the Fourier transform of σ_h vanishes at ∞). We then have

$$0 = \lim_{n \rightarrow \infty} \langle h \circ (T^f)_{q_n}, h \rangle = \int_{\mathbb{R}} \langle h \circ (T^f)_s, h \rangle dP(s).$$

Therefore,

$$\int_{\mathbb{R}} \hat{P}(u) d\sigma_h(u) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{isu} d\sigma_h(u) \right) dP(s) = 0.$$

By taking any g in the cyclic space of h and repeating the above reasoning we obtain that $\int_{\mathbb{R}} \hat{P}(u) d\nu = 0$ for every finite measure ν absolutely continuous with respect to σ_f . Therefore, $\hat{P}(\cdot) = 0$, σ_h -a.e. On the other hand the measure P has a compact topological support and then its Fourier transform extends to an analytic function on the whole complex plane. Since the measure σ_h is continuous we deduce that σ_h must be zero measure, so h is constant equal zero and the proof is complete. \square

Remark Special flows over irrational rotations with roof functions can be mildly mixing (Frączek-L. 2007, Frączek-L.-Lesigne 2007) – through so called Ratner’s property. IETs are examples of systems with a LINEAR BLOCK GROWTH (M. Boshernitzan), however time- t automorphisms of BV flows over them need not have the same property; it would be nice to decide whether a time- t automorphism of a special flow with a piecewise constant roof function can have a linear block growth.

Coming back to “local” D.-K. inequality...

Theorem 7 (Frączek - L., 2005). *Let $T : (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$ be an ergodic automorphism and $f \in L^1(X, \mu)$ a positive function for which there is $c > 0$ such that $0 < c \leq f(x)$ for a.a. $x \in X$. Suppose that $\{C_n\}$ is a sequence of Borel subsets, $\{q_n\}$ is an increasing sequence of natural numbers, and $\{a_n\}$ is a sequence of real numbers such that*

- $\mu(C_n) \rightarrow \alpha > 0$ as $n \rightarrow \infty$,
- $\mu(C_n \Delta T^{-1}C_n) \rightarrow 0$ as $n \rightarrow \infty$,
- $\sup_{x \in C_n} d(x, T^{q_n}x) \rightarrow 0$ as $n \rightarrow \infty$,
- the sequence $\{\int_{C_n} |f_n(x)|^2 d\mu(x)\}$ is bounded, where $f_n := f^{(q_n)} - a_n$, and
- $\frac{1}{\mu(C_n)}(f_n|_{C_n})_*(\mu|_{C_n}) \rightarrow P$ weakly in $\mathcal{P}(\mathbb{R})$,
- the sequence $\{(T^f)_{a_n}\}$ converges in the weak operator topology.

Then $\{(T^f)_{a_n}\}$ converges weakly to the operator

$$\alpha \int_{\mathbb{R}} (T^f)_{-t} dP(t) + (1 - \alpha)J,$$

where $J \in J(T^f)$.

A measure-theoretic consequence of this fact is that special flows over IETs for roof function with bounded variation **are disjoint from all mixing flows; however spectral consequences remain unclear** (see below).

Open question Can we have an IET T and a BV function such that the Koopman representation associated to T^f has a Lebesgue component in the spectrum?

Recall however that it is unknown whether on a surface we can have a “smooth” flow with partly Lebesgue spectrum.

What about IETs themselves? This seems to be slightly easier. In particular Katok’s result gives rise to the following.

Proposition 8. *If T is an IET then it is partially rigid, i.e.*

$$T^{q_n} \rightarrow \kappa Id + (1 - \kappa)J$$

for some $0 < \kappa < 1$.

The partial rigidity property holds for systems with linear block growth (S. Ferenczi). If $\kappa > 1/2$ then it implies singularity of the spectrum, however for $\kappa = 1/2$ we can have a Lebesgue component (Rudin-Shapiro substitution and other examples). It is unknown whether IETs have always singular spectra (the constant κ in case of IET is smaller than $\frac{1}{2}$).

Open question: Can Rudin-Shapiro system be obtained from an IET?

5 Parreau's Theorem

We will now describe a recent surprising result of F. Parreau concerning non-mixing transformations.

A factor determined by a joining

Suppose that T (S) is an automorphism of (X, \mathcal{B}, μ) ((Y, \mathcal{C}, ν)) and let $\rho \in J(T, S)$. Then $\Phi_\rho : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(Y, \mathcal{C}, \nu)$ and also $\Phi_\rho : L^\infty(X, \mathcal{B}, \mu) \rightarrow L^\infty(Y, \mathcal{C}, \nu)$. Let

$$F(\rho) = \overline{\text{span}}\{\Phi_\rho(f_1) \cdot \dots \cdot \Phi_\rho(f_k) : f_i \in L^\infty(X, \mathcal{B}, \mu), k \geq 1\}.$$

Then $F(\rho) \subset L^\infty(X, \mathcal{B}, \mu)$ is an algebra of functions.

Lemma 9. *There exists an S -invariant σ -algebra $\mathcal{C}_\rho \subset \mathcal{C}$ (a factor of S) such that*

$$\overline{F(\rho)}^{L^2} = L^2(\mathcal{C}_\rho).$$

Theorem 10 (L.-Parreau-Thouvenot (2000)). *The factor $S|_{\mathcal{C}_\rho}$ is also a factor, say $\mathcal{B}_\infty(\rho)$, of an infinite self-joining $\rho^\infty|_{X^\infty}$ of T .*

Parreau's Theorem on non-mixing systems

Denote by $M(\mathbb{T})$ the Banach space of complex measures on \mathbb{T} and let $M^+(\mathbb{T})$ stand for the subset of positive measures. Let

$M_0 \subset M^+(\mathbb{T})$ be an ideal ($\nu * \kappa \in M_0$ whenever $\nu \in M_0$ and $\kappa \in M^+(\mathbb{T})$).

Examples: Rajchman measures, absolutely continuous measures.

Assume that T is an ergodic automorphism of (X, \mathcal{B}, μ) and let $\rho \in J_2(T)$. Notice that $\overline{\text{Im } \Phi_\rho} \subset L^2(X, \mathcal{B}, \mu)$ is a closed T -invariant subspace.

Theorem 11 (Parreau's Theorem). *If the maximal spectral type of T on $\overline{\text{Im } \Phi_\rho}$ is singular with respect to all measures from M_0 then for each ergodic automorphism S (of (Y, \mathcal{C}, ν)) whose maximal spectral type is in M_0 we have*

$$T|_{\mathcal{B}_\rho} \perp S.$$

Let us pass to some consequences...

Lemma 12. *Let U be a unitary operator on a separable Hilbert space H . Suppose that $n_k \rightarrow \infty$ and $U^{n_k} \rightarrow V$ weakly in $L(H)$. Then the maximal spectral type of U on $\overline{\text{Im}(V)}$ is singular with respect to any Rajchman measure.*

Finally just recall that T is **mixing if and only if** $U_T^n \rightarrow 0$ **weakly on the space** $L_0^2(X, \mathcal{B}, \mu)$. Therefore if T is not mixing, in the weak closure of powers of T (seen as Markov operators) we will always find a non-trivial Markov operator.

Corollary 13 (Parreau). *Assume that T is an ergodic automorphism of (X, \mathcal{B}, μ) . Let $n_k \rightarrow \infty$ and $U_T^{n_k} \rightarrow \Phi_\rho$ in the space of Markov operators. Then for every mixing automorphism S we have*

$$T|_{\mathcal{B}_\infty(\rho)} \perp S.$$

In particular, if T is not mixing then it has a non-trivial factor disjoint from all mixing transformations.

Corollary 14. *Assume that T is an ergodic automorphism of (X, \mathcal{B}, μ) . Assume that there exists a non-trivial automorphism S (defined on (Y, \mathcal{C}, ν)), which has a singular spectrum and such that $T \not\sim S$. Then T has a non-trivial factor, which is disjoint with any automorphism having absolutely continuous spectrum.*

Proof. Let $\Psi : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(Y, \mathcal{C}, \nu)$ be a non-trivial Markov operator, $\Psi \circ U_T = U_S \circ \Psi$. Then $\Phi = \Psi^* \circ \Psi$ is still non-trivial and moreover $\sigma_{\Psi^* \circ \Psi(f)} \ll \sigma_{\Psi(f)}$, whence the spectral measure $\sigma_{\Phi(f)}$ is singular. It follows that spectral measures of all functions belonging to $\overline{Im(\Phi)}$ are singular. \square

Automorphisms with “big factors”

Given an ergodic automorphism T of (X, \mathcal{B}, μ) , for the purpose of this lecture, let us call a factor T_1 of it a “big” factor if for every S

$$T_1 \perp S \Rightarrow T \perp S.$$

We are interested in T so that all its non-trivial factors are “big”: **prime automorphisms** (those without any non-trivial proper factors), in particular **MSJ-automorphisms** (Rudolph); but also those T for which all non-trivial factors are relatively distal: **simple automorphisms** (Veech, del Junco-Rudolph), **automorphisms with Ratner’s property** (Ratner – horocycle automorphisms, Thouvenot, Ryzhikov, Frączek-L.), **distally simple automorphisms** (Ryzhikov-Thouvenot, del Junco-L.).

For us – **Veech’s Question**: Are almost all IETs simple?

If this is true then they have only “big” factors! Recall that del Junco (1983) gave an example of a 3-IET with the MSJ property, and Ferenczi-Holton-Zamboni (2003) gave examples of rigid simple 3-IETs. Veech’s question is open even in the class of 3-IETs.

Corollary 15. *Assume that T has only “big” non-trivial factors.*

(i) *Either T is mixing or $T \perp \text{Mixing}$.*

(ii) *Either $T \perp \text{singular spectrum}$ or $T \perp \text{Lebesgue spectrum}$.*

Remark 16. It seems to be an open question whether one can find a non-trivial T which is at the same time disjoint from all automorphisms with singular spectrum and disjoint from all automorphisms with Lebesgue spectrum.

Multipliers and Parreau's factor

If \mathcal{S} is a certain family of ergodic automorphisms of a fixed standard probability Borel space (X, \mathcal{B}, μ) (automorphisms are considered up to isomorphism). A *multiplier* of \mathcal{S} is any ergodic automorphism T such that for each $S \in \mathcal{S}$,

$$(T \times S, \rho) \in \mathcal{S}$$

for an arbitrary $\rho \in J^e(T, S)$. The family of all multipliers of \mathcal{S} will be denoted by $\mathcal{M}(\mathcal{S})$. If the one-point system is in \mathcal{S} then $\mathcal{M}(\mathcal{S}) \subset \mathcal{S}$. In what follows \mathcal{S} will be of the form A^\perp : i.e. $T \in \mathcal{M}(A^\perp)$ if for each R disjoint from all members of A

$$(T \times R, \rho) \perp S$$

for arbitrary $\rho \in J^e(T, R)$ and $S \in A$. Clearly, $\mathcal{M}(A^\perp) \subset A^\perp$.

Recall that spectral disjointness implies disjointness. Here is a simple spectral criterion of being a multiplier.

Proposition 17. *Assume that T is an ergodic automorphism of (X, \mathcal{B}, μ) . Assume moreover that $M_0 \subset M^+(\mathbb{T})$ is an ideal. If the maximal spectral type of U_T on $L_0^2(X, \mathcal{B}, \mu)$ is singular with respect to any measure in M_0 then for an arbitrary ergodic automorphism S of (Y, \mathcal{C}, ν) whose maximal spectral type (on $L_0^2(Y, \mathcal{C}, \nu)$) belongs to M_0 ,*

$$T \in \mathcal{M}(\{S\}^\perp).$$

Corollary 18. *Any automorphism with singular spectrum is a multiplier of the class Lebesgue Spectrum $^\perp$.*

Remark 19. Notice that by a result of Smorodinsky-Thouvenot there are automorphisms T with Lebesgue spectrum such that $T \vee T$ has a non-trivial factor with singular spectrum. However, given T with singular spectrum and considering ergodic self-joinings $T \vee T$ we cannot produce a factor with a Lebesgue spectrum.

With a little bit more effort one can prove in Parreau's Theorem the following assertion (without changing the assumption)

$$T|_{\mathcal{B}_\rho} \in \mathcal{M}(S^\perp),$$

that is, if $R \perp S$ then any ergodic joining

$$T|_{\mathcal{B}_\rho} \vee R \perp S.$$

Now, given an ergodic automorphism T and a certain family $\mathcal{S} = \mathcal{A}^\perp$ consider

$$\{\mathcal{A} \subset \mathcal{B}; T^{-1}\mathcal{A} = \mathcal{A} \text{ and } T|_{\mathcal{A}} \in \mathcal{M}(\mathcal{S})\}.$$

This family is closed under taking inverse limits, it is then closed under taking arbitrary joinings.

Corollary 20. *For each ergodic automorphis T there exists a biggest factor $\mathcal{A} \subset \mathcal{B}$ such that $T|_{\mathcal{A}} \in \mathcal{M}(A^\perp)$.*

Corollary 21. *If T is ergodic but not mixing then the biggest factor of it being a multiplier of M^\perp is non-trivial.*

For non-mixing transformations such a factor I call the Parreau factor.

Parreau factor for IETs and BV roof function special flows over IETs

We recall that if the Parreau factor of a system T is the whole system then it means (it is a known general measure-theoretic fact) that all self-joinings systems of T are disjoint from all mixing transformations – the strongest possible version of measure-theoretic disjointness (Furstenberg called this property *double disjointness*).

Corollary 22. *It T is an IET and the roof function f is of bounded variation then the Parreau factor of T^f is T^f . The Parreau factor of T itself is T .*

Proof. Assume that $T_{t_n} \rightarrow a \int T_t dP(t) + (1-a)J$ ($T = (T_t)$) acts on (X, \mathcal{B}, μ) . Assume that $S = (S_t)$ acting on (Y, \mathcal{C}, ν) is mixing and let $R = (R_t)$ (acting on (Z, \mathcal{D}, κ)) be disjoint with the mixing class. Take an ergodic joining

$$\rho \in J^e(T, R, S)$$

and we need to show that $\rho = \rho|_{X \times Z} \otimes \nu$. With no loss of generality we can assume that

$$R_{t_n} \rightarrow H.$$

We look at the joining ρ as a joining of T with $R \times S$, the latter is the genuine product as $R \perp \text{Mixing}$. Hence

$$\begin{aligned} \Phi = \Phi_\rho : L^2(X, \mathcal{B}, \mu) &\rightarrow L^2(Z \times Y, \kappa \otimes \nu), \\ \Phi \circ T_t &= (R_t \times S_t) \circ \Phi. \end{aligned}$$

By passing to the limits for $t_n, n \rightarrow \infty$ we obtain that

$$\Phi \circ \left(a \int T_t dP + (1-a)J \right) = (H \otimes \Pi_Y) \circ \Phi =: \Psi.$$

Notice that

$$\text{Im } \Psi \subset \text{Im}(H \otimes \Pi_Y) \subset L^2(Z, \kappa) \otimes 1_Y.$$

It follows that if ρ' is the joining corresponding to Ψ then

$$\int_{X \times Z \times Y} f(x)h(z)g(y) d\rho'(x, z, y) = \int_Z \Psi(f)(z)h(z)d\kappa(z) \int_Y g d\nu.$$

Thus

$$\rho' = \lambda \otimes \nu$$

where $\lambda \in J(T, R)$ is given by Ψ viewed as a Markov operator between $L^2(X, \mu)$ and $L^2(Z, \kappa)$. Since ν is weakly mixing (even mixing) the ergodic decomposition of ρ' is of the form

$$\rho' = \left(\int_{J^e(T, R)} \gamma d\Theta(\gamma) \right) \otimes \nu = \int_{J^e(T, R)} \gamma \otimes \nu d\Theta(\gamma).$$

On the other hand

$$\Psi = a \int \Phi \circ T_t dP(t) + (1 - a)\Phi \circ J$$

where the operators $\Phi \circ T_t$ correspond to ergodic joinings: as such they have to appear in the ergodic decomposition of ρ' . It follows that on a set of positive Θ -measure the Markov operator corresponding to $\gamma \otimes \nu$ is of the form $\Phi \circ T_t$. It follows that $Im(\Phi \circ T_t) \subset L^2(Z, \kappa) \otimes 1_Y$ which implies that $Im \Phi \subset L^2(Z, \kappa)$ and as we have already seen this implies that $\rho = \rho|_{X \times Z} \otimes \nu$. \square