KOKSMA'S INEQUALITY AND GROUP EXTENSIONS OF KRONECKER TRANSFORMATIONS

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ABSTRACT. We consider methods of establishing ergodicity of group extensions, proving that a class of cylinder flows are ergodic, coalescent and non-squashable. A new Koksma-type inequality is also obtained.

Introduction

We study locally compact group extensions of Kronecker transformations.

Let X be a compact monothetic group with Haar probability measure $m = m_X$, and G a locally compact metric group with Haar measure m_G. Let T be an ergodic translation on X, (called a *Kronecker* transformation) and set $\mu = m \times m_G$.

For $\varphi : X \longrightarrow G$ measurable (called a *cocycle*), consider the *skew product* (or *G-extension*) which is the measure preserving transformation $T_\varphi : (X \times G, \mu) \longrightarrow (X \times G, \mu)$ defined by

$$T_\varphi(x, g) = (Tx, \varphi(x)g).$$

Recall from [Aa1] that a measure preserving transformation $\tau : (Y, \nu) \longrightarrow (Y, \nu)$ is called *squashable* if $\exists Q \ni \quad Q\tau = \tau Q \quad$ and $\nu Q^{-1} = c\nu$ for certain $c \neq 1$. It follows from [Aa2, Th3.4] that if the group G is countable, and has no arbitrarily large finite normal subgroups (*e.g.* $G = \mathbb{Z}^k \times \mathbb{Q}^l$) then no ergodic G-extension is squashable.

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Most of the results in this paper are for the case $G = \mathbb{R}$. It is an open problem to decide if there is a conservative, ergodic, squashable $\mathbb{R}$-extension of a Kronecker transformation. Almost all of our results are in the other direction, showing that certain $\mathbb{R}$-extensions are nonsquashable.

We consider product-type cocycles for odometers in §1, obtaining conditions for ergodicity, nonsquashability, and coalescence (q.v.) Essentially the same ideas can be used in the context of [Kw-Le-Ru2] to obtain analytic *cylinder flows* (i.e. $\mathbb{R}$-extensions of rotations of the circle) which are ergodic, nonsquashable, and coalescent (see §4). We show in §5 that if $\varphi : \mathbb{T} \longrightarrow \mathbb{R}$ is $C^{1+\delta}$ then for a residual set of irrational rotations T, the cocycle is conservative and ergodic. We improve some recent results by D. Pask (in §6) [Pa1],[Pa2] on the ergodicity of cylinder flows also proving the non-squashability in this case.

One of our tools is a new Koksma-type inequality in $L^2(\mathbb{T})$ for functions whose Fourier coefficients are of order $O(1/n)$ (see §2) with possible speeds of convergence for smooth functions and irrational rotations admitting a speed of approximations by rationals (see §3).

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§1 Coalescence of group extensions, and ergodicity of product type cocycles

A non-singular transformation is called *coalescent* if all nonsingular commuting with it transformations are invertible. To begin this section, we study the form of non-singular transformations commuting with an ergodic, group extension of a Kronecker transformation.

Suppose that T is an ergodic measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$; let $(G, \mathcal{T})$ be an abelian, locally compact, second countable, topological group ($\mathcal{T} = \mathcal{T}(G)$ denotes the family of open sets in the topological space G), and let $\varphi : X \to G$ be a cocycle.

Let $T_\varphi : (X \times G, \mu) \longrightarrow (X \times G, \mu)$,

$$T_\varphi(x, g) = (Tx, \varphi(x)g)$$

be ergodic (this implies that G has to be amenable [Zim]), where T is a Kronecker transformation on X, and $\varphi : X \longrightarrow G$ is a cocycle.

Proposition 1.1. *Suppose that $Q : X \times G \longrightarrow X \times G$ is non-singular and $QT_\varphi = T_\varphi Q$. Then there exist a translation S of X, and a continuous group homomorphism $w : G \longrightarrow G$ which is non-singular in the sense that $m_G \circ w^{-1} \sim m_G$ and a measurable map $f : X \longrightarrow G$ such that*

$$Q(x, h) = (Sx, f(x)w(h)) \quad \textit{for each } x \in X,\ h \in G.$$

Proof. Write $Q = (S, F)$, where $S : X \times G \longrightarrow X$ and $F : X \times G \longrightarrow G$. We have

$$S \circ T_\varphi = T \circ S \ \ \&\ \ F \circ T_\varphi = (\varphi \circ S) \cdot F.$$

Let $U : X \times G \longrightarrow X$ be defined by $U(x,h) = x^{-1}S(x,h)$, then $U \circ T_\varphi = U$, hence by ergodicity of T_φ, $U(x,h) = x_1$, and $S(x,g) = Sx = xx_1 = x_1x$. Therefore

$$FT_\varphi(x,h) = \varphi(Sx)F(x,h).$$

Denote $\sigma_g(x,h) = (x,hg)$ and note that for each $g \in G$, $\sigma_g T_\varphi = T_\varphi \sigma_g$. Hence

$$\begin{aligned}\Big(F^{-1}\cdot(F\circ\sigma_g)\Big)\circ T_\varphi(x,h) &= F(T_\varphi(x,h))^{-1}F(T_\varphi(x,hg))\\ &= \Big(\varphi(Sx)F(x,h)\Big)^{-1}\varphi(Sx)F(x,hg)\\ &= \Big(F^{-1}F\circ\sigma_g\Big)(x,h),\end{aligned}$$

whence there exists $w : G \longrightarrow G$ such that $F^{-1}(F\circ\sigma_g) = w(g)$ for each $g \in G$. It follows that w is a measurable homomorphism (and hence continuous).

Set $\phi(x,h) = F(x,h)w(h)^{-1}$. By the above, $\phi\circ\sigma_g = \phi$ for each $g \in G$ whence there exists a measurable $f : X \longrightarrow G$ such that $\phi(x,h) = f(x)$ a.e., and

$$Q(x,g) = (Sx, f(x)w(g)).$$

To see that $w : G \to G$ is non-singular, note that $\mu\circ S_f^{-1} = \mu$, and since $QT_\varphi = T_\varphi Q$, $\exists\, c > 0$ such that $\mu\circ Q^{-1} = c\mu$. Moreover

$$\tilde{w} := \mathrm{Id}\times w = S_f^{-1}\circ Q$$

whence $\mu\circ\tilde{w}^{-1} = c\mu$, and $m\circ w^{-1} = cm$. □

Remarks

(1) If T is an invertible, ergodic probability preserving transformation and φ an ergodic cocycle, and $Q(x,g) = (Sx, F(x,g))$ is non-singular, and commutes with T_φ, then Q has the above form.

(2) If $w : G \to G$ is non-singular and measurable, then w is continuous, and onto. To see this, note that $w(G)$ is a m_G-measurable subgroup of G, whence

$$\begin{aligned}\exists\, x \notin w(G) \quad &\Rightarrow \quad xw(G) \subset G\setminus w(G)\\ &\Rightarrow \quad m(w(G)) = m(xw(G)) \le m(G\setminus w(G)) = 0.\end{aligned}$$

(3) If G is such that any continuous group non-singular homomorphism is 1-1 (e.g. $G = \mathbb{Z}^k\times\mathbb{Q}^l\times\mathbb{R}^m$) then any ergodic G-extension of a Kronecker transformation is coalescent. For coalescence of other group extensions, see theorem 1.5 below.

(4) In case $G = \mathbb{R}$ a skew product T_φ is squashable iff it commutes with a Q of form $Q(x,t) = (Sx, ct + \psi(x))$, where $|c| \neq 1$, or, in other words, $c\varphi - \varphi\circ S$ is a coboundary for some $|c| \neq 1$ and S a translation of X.

Next, we turn to methods of proving ergodicity of group extensions.

As in [Sch], the *essential values* of φ are defined as those group elements $a \in G$ with the property that

$$\forall\, A \in \mathcal{B}_+,\ U \in \mathcal{T}(G) \text{ with } a \in U;\ \exists\, n \geq 1\ \ni\ m(A \cap T^{-n}A \cap [\varphi^{(n)} \in U]) > 0$$

where $\varphi^{(n)}(x) = \varphi(T^{n-1}x) \cdot \ldots \cdot \varphi(x)$, $n \geq 1$.

The collection of essential values of φ is denoted by $E(\varphi)$. It is shown in [Sch] that $E(\varphi)$ is a closed subgroup of G, and is the collection of *periods* for T_φ-invariant functions:

$$E(\varphi) = \{a \in G : f(x, y + a) = f(x, y) \text{ a.e. } \forall\, f \circ T_\varphi = f \text{ measurable}\}.$$

In particular, T_φ is ergodic iff $E(\varphi) = G$. Also,

Lemma 1.2 [Sch]. *For any compact set K which is disjoint from $E(\varphi)$ there is a Borel set B, $\mu(B) > 0$, such that for each integer $m > 0$ we have*

$$\mu(B \cap T^{-m}B \cap [\varphi^{(m)} \in K]) = 0.$$

Definition. A sequence $q_n \in \mathbb{N}$ $(n \geq 1)$, $q_n \uparrow \infty$ is called a *rigidity time* for the probability preserving transformation T if $T^{q_n} \xrightarrow{\mathcal{U}(L^2(m))} \mathrm{Id}$. Here $\mathcal{U}(L^2(m)$ denotes the collection of unitary operators on $L^2(m)$. Note that if T is a translation on the compact group X with Haar measure m then $T^{q_n} \xrightarrow{\mathcal{U}(L^2(m))} \mathrm{Id}$ iff $T^{q_n} \xrightarrow{X} \mathrm{Id}$.

Lemma 1.3. *Suppose that $K \subset \mathbb{R}$ is compact, and that $\{q_n\}$ is a rigidity time for T such that*

$$\forall\, A \in \mathcal{B}_+,\quad \liminf_{n\to\infty} m(A \cap [\varphi^{(q_n)} \in K]) > 0,$$

then

$$K \cap E(\varphi) \neq \emptyset.$$

Proof. Follows immediately from Lemma 1.2. □

Let

$$D(\varphi) = \{a \in G : \exists\, q_n \to \infty,\ T^{q_n} \xrightarrow{\mathcal{U}(L^2(m))} \mathrm{Id} \ni \forall\ n_k \to \infty,\ a \in \{\varphi^{(q_{n_k})}\}'_{k\geq 1} \text{ a.e.}\}.$$

See also proofs of ergodicity in [Aa2, §4].

Proposition 1.4

$$D(\varphi) \subset E(\varphi).$$

Proof. Suppose that $y \in D$, and $T^{q_n} \to \mathrm{Id}$, $y \in \{\varphi^{(q_{n_k})} : k \geq 1\}'$ a.e. $\forall\, n_k \to \infty$, then

$$\forall\, A \in \mathcal{B}_+\ y \in U \in \mathcal{T}(G),\ \exists\, \delta > 0\ \ni\ \liminf_{n\to\infty} m(A \cap [\varphi^{(q_n)} \in U]) \geq \delta,$$

because if there were no such $\delta > 0$ we could choose $y \in U \in \mathcal{T}(G)$, and a subsequence q_{n_k}, $(k \geq 1)$ satisfying $m(A \cap [\varphi^{(q_{n_k})} \in U]) < 1/2^n$ and use the Borel-Cantelli lemma to get a contradiction to the definition of $y \in D(\varphi)$. Hence, since $T^{q_n} \longrightarrow \mathrm{Id}$, $\liminf_{n\to\infty} m(A \cap T^{-q_n} A \cap [\varphi^{(q_n)} \in U]) > \frac{\delta}{2}\ \forall\, n$ large, and therefore $y \in E(\varphi)$. $\square$

Set

$$\widetilde{D}(\varphi) = \{a \in G : \exists\, q_n \ni T^{q_n} \overset{\mathcal{U}(L^2(m))}{\longrightarrow} \mathrm{Id},\ \&\ \varphi^{(q_n)} \to a \text{ a.e.}\}.$$

Clearly $\widetilde{D}(\varphi) \subset D(\varphi)$.

Theorem 1.5. *Assume that T is an ergodic translation. If $Gp(\widetilde{D}(\varphi))$ is dense in G, then T_φ is ergodic, and*

$$Q : X \times G \to X \times G \text{ nonsingular, } QT_\varphi = T_\varphi Q \quad \Rightarrow \quad Q(x,g) = (Sx, g + f(x))$$

where $ST = TS$ and $f : X \to G$ is measurable.

In particular, such a T_φ is coalescent, and non-squashable.

Proof. By the previous proposition, T_φ is ergodic. We know from proposition 1.1 that

$$Q : X \times G \to X \times G \text{ nonsingular, } QT_\varphi = T_\varphi Q \quad \Rightarrow \quad Q(x,g) = (Sx, w(g) + f(x))$$

where $ST = TS$, $f : X \to G$ is measurable, and $w : G \to G$ is a continuous nonsingular homomorphism. It follows that

$$w(\varphi) - \varphi \circ S = f - f \circ T,$$

whence

$$\widetilde{D}(w(\varphi) - \varphi \circ S) = \{0\}.$$

However, if $a \in \widetilde{D}(\varphi)$, and

$$q_n \to \infty,\ T^{q_n} \overset{\mathcal{U}(L^2(m))}{\longrightarrow} \mathrm{Id},\ \&\ \varphi^{(q_n)} \to a \text{ a.e.},$$

then

$$w(\varphi^{(q_n)}) - \varphi^{(q_n)} \circ S \to w(a) - a \text{ a.e.}$$

whence $w(a) - a \in \widetilde{D}(w(\varphi) - \varphi \circ S) = \{0\}$ and $w(a) = a\ \forall\, a \in \widetilde{D}(\varphi)$ and hence $\forall\, a \in G$.
$\square$

Set

$$C(\varphi) = \{a \in G : \liminf_{T^q \overset{\mathcal{U}(L^2(m))}{\longrightarrow} \mathrm{Id},\ q \neq 0} 1_U(\varphi^{(q)}) = 1 \text{ a.e.} \forall\, a \in U \in \mathcal{T}(G)\}.$$

It is not hard to show that (for T Kronecker)

$$E(\varphi) \subset C(\varphi) \subset \widetilde{E}(\varphi)$$

where $\widetilde{E}(\varphi) :=$

$$\{a \in G : \forall\, I \in \mathcal{T}(X),\ a \in U \in \mathcal{T}(G)\ \exists\, n \geq 1 \ni m(I \cap T^{-n} I \cap [\varphi^{(n)} \in U]) > 0\}.$$

A popular misconception in the subject for the case $G = \mathbb{R}$ ([Con, proposition 1] [He-Lal, lemma 3]) seems to have been that $C(\varphi) \subset E(\varphi)$.

This latter claim is wrong. A counterexample for a Kronecker transformation is given in example 1.7 (below). An analogous example for the case $G = \mathbb{T}$ was given in [Furst]. See [Or, proposition 1] for a related method of proving ergodicity not based on the above.

The rest of this section is devoted to

Cocycles of product type for an odometer

For $a_n \in \mathbb{N}$, $(n \in \mathbb{N})$, set $\Omega := \prod_{n=1}^{\infty}\{0, \dots, a_n - 1\}$ equipped with the addition

$$(\omega + \omega')_n = \omega_n + \omega'_n + \epsilon_n \mod a_n$$

where $\epsilon_1 = 0$ and

$$\epsilon_{n+1} = \begin{cases} 0 & \omega_n + \omega'_n + \epsilon_n < a_n \\ 1 & \omega_n + \omega'_n + \epsilon_n \geq a_n. \end{cases}$$

Clearly, Ω equipped with the product discrete topology, is a compact Abelian topological group (called an *odometer group*), with Haar measure

$$m = \prod_{n=1}^{\infty} (\frac{1}{a_n}, \dots, \frac{1}{a_n}).$$

Also if $\tau = (1, 0, \dots)$ then $\Omega = \overline{\{n\tau\}}_{n \in \mathbb{Z}}$ whence $x \mapsto Tx (:= \tau + x)$ (called an *odometer transformation*) is ergodic.

A cocycle of *product type* is a measurable function $\varphi : \Omega \to G$ (where G is an Abelian topological group) of form

$$\varphi(\omega) = \sum_{n=1}^{\infty}(b_n(T\omega) - b_n(\omega))$$

where $b_n(\omega) = \beta_n(\omega_n)$, where $\beta_n : \{0, \dots, q_n - 1\} \longrightarrow G$ (notice that $T\omega$ differs from ω only in finitely many places whenever $\omega \neq -\tau$, so φ is well-defined except for one point).

Set $q_1 = 1$, $q_{n+1} = \prod_{k=1}^{n} a_k$, then

$$(q_n\tau)_k = \begin{cases} 1 & k = n \\ 0 & k \neq n, \end{cases}$$

whence

$$T^{q_n}\omega = (\omega_1, \dots, \omega_{n-1}, \tilde{\tau}_n + (\omega_n, \dots))$$

where

$$\tilde{\tau}_n = (1, 0, \dots) \in \prod_{k=n}^{\infty}\{0, \dots, a_k - 1\}.$$

Note that

$$\varphi^{(k)}(\omega) := \sum_{j=0}^{k-1} \varphi(T^j\omega) \overset{!}{=} \sum_{n=1}^{\infty}[b_n(T^k\omega) - b_n(\omega)],$$

whence

$$\begin{aligned}
\varphi^{(q_k)}(\omega) &= \sum_{n=1}^{\infty}[b_n(T^{q_k}\omega) - b_n(\omega)] \\
&= \sum_{n=0}^{\ell_k(\omega)-1} [\beta_{k+n}(0) - \beta_{k+n}(a_{k+n} - 1)] \\
&+ \beta_{k+\ell_k(\omega)}(\omega_{k+\ell_k(\omega)} + 1) - \beta_{k+\ell_k(\omega)}(\omega_{k+\ell_k(\omega)}),
\end{aligned}$$

where

$$\ell_k(\omega) = \min\{n \geq 0 : \omega_{k+n} < a_{k+n} - 1\}.$$

We begin by considering cocycles of form

$$\beta_n(k) = k\lambda_n(:= \underbrace{\lambda_n + \cdots + \lambda_n}_{k \text{ times}}), \quad \text{for } 0 \leq k \leq a_n - 1, \quad \text{where } \lambda_n \in G.$$

Proposition 1.6. *If $r_n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{r_n}{a_n} < \infty$, then*

$$\{k\lambda_n : n \geq 1,\ 1 \leq k \leq r_n\}' \subset \widetilde{D}(\varphi).$$

Proof. From the condition on $\{r_n\}_{n\in\mathbb{N}}$, for a.e. $\omega \in \Omega$

$$\exists\ N_\omega \in \mathbb{N}\ \ni\ \omega_n < a_n - r_n - 1\ \forall\ n > N_\omega,$$

whence $\forall\ n \geq N_\omega,\ 0 \leq k \leq r_n$,

$$\begin{aligned}
\varphi^{(kq_n)}(\omega) &= \sum_{j=1}^{k} \varphi^{(q_n)}(T^{(j-1)q_n}\omega) \\
&= \sum_{j=0}^{k-1}\Big(\beta_n(\omega_n + j + 1) - \beta_n(\omega_n + j)\Big) \quad (\because k < r_n) \\
&= k\lambda_n
\end{aligned}$$

and if $k_\nu\lambda_{n_\nu} \to a$, then for a.e. $\omega \in \Omega$,

$$\varphi^{(k_\nu \cdot q_{n_\nu})} \approx k_\nu\lambda_{n_\nu} \to a \text{ a.e,}$$

and $a \in \widetilde{D}(\varphi)$. □

Theorem 1.5, and Proposition 1.6 facilitate easy constructions of conservative, ergodic, coalescent, non-squashable G-extensions of odometers.

Example 1.7. There is a continuous $\mathbb{R}$-valued cocycle of product type which is a coboundary, and satisfies

$$\overline{\mathrm{Gp}}(C(\varphi)) = \mathbb{R}.$$

Proof. Assume that $\sum_{n=1}^{\infty} \frac{1}{a_n} < +\infty$, $a_n \geq 3$. Let

$$\varphi(\omega) = \sum_{n=1}^{\infty}(b_n(T\omega) - b_n(\omega))$$

where, as before, $b_n(\omega) = \beta_n(\omega_n)$. Set $\beta_{2n+1} \equiv 0$, and

$$\beta_{2n}(k) = \begin{cases} \frac{1}{n} & k = 1, \\ 0 & \text{else.} \end{cases}$$

By Borel-Cantelli lemma, since $\mu\{\omega : \omega_{2n} = 1\} = \frac{1}{a_n}$, $\varphi = \psi \circ T - \psi$ with

$$\psi = \sum_{n=1}^{\infty} b_n.$$

Note that $\varphi(-\tau) = 0$ (where $-\tau = (a_1 - 1, a_2 - 1, \ldots)$). For $\omega \neq -\tau$, $\ell(\omega) < \infty$

$$\begin{aligned} \varphi(\omega) &= \sum_{n=0}^{\ell(\omega)-1} [\beta_n(0) - \beta_n(a_n - 1)] \\ &+ \beta_{\ell(\omega)}(\omega_{\ell(\omega)} + 1) - \beta_{\ell(\omega)}(\omega_{\ell(\omega)}) \\ &= \beta_{\ell(\omega)}(\omega_{\ell(\omega)} + 1) - \beta_{\ell(\omega)}(\omega_{\ell(\omega)}), \end{aligned}$$

since $\beta_n(0) - \beta_n(a_n - 1) = 0$, whence

$$|\varphi(\omega)| \leq \frac{2}{\ell(\omega)}$$

and the continuity of φ is ensured.

For a.e. $\omega \in \Omega$, $\exists\, n_\omega$ such that $2 < \omega_n < a_n - 2 \;\forall\, n > n_\omega$. Set

$$\kappa_n(\omega) = a_{2n} - \omega_{2n}$$

for $n > \frac{n_\omega}{2}$. Clearly $\kappa_n(\omega) q_{2n} \tau \xrightarrow{\Omega} 0$.

Moreover, for $n > \frac{n_\omega}{2}$,

$$(T^{jq_{2n}}\omega)_{2n} = \begin{cases} \omega_{2n} + j & 0 \leq j \leq \kappa_n(\omega) - 1, \\ 0 & j = \kappa_n(\omega), \end{cases}$$

$$(T^{jq_{2n}}\omega)_{2n+1} = \begin{cases} \omega_{2n+1} & 0 \leq j \leq \kappa_n(\omega) - 1, \\ \omega_{2n+1} + 1 & j = \kappa_n(\omega), \end{cases}$$

and

$$(T^{jq_{2n}}\omega)_k = \omega_k \quad \forall\ 0 \leq j \leq \kappa_n(\omega),\ k \neq 2n, 2n+1;$$

whence

$$\begin{aligned} \varphi^{((\kappa_n(\omega)+1)q_{2n})}(\omega) &= \sum_{k=1}^{\infty} \Big(b_k(T^{(\kappa_n(\omega)+1)q_{2n}}\omega) - b_k(\omega) \Big) \\ & \sum_{k=1}^{\infty} \Big(\beta_k((T^{(\kappa_n(\omega)+1)q_{2n}}\omega)_k) - \beta_k(\omega_k) \Big) \\ &= \beta_{2n}((T^{(\kappa_n(\omega)+1)q_{2n}}\omega)_{2n}) - \beta_{2n}(\omega_{2n}) \\ &= \beta_{2n}(1) = \frac{1}{n}. \end{aligned}$$

We use the fact that

$$\forall\, y > 0,\ N \geq 1,\ \exists\ N < n_k(N) \uparrow \infty\ \ni\ \sum_{k=1}^{\infty} \frac{1}{n_k(N)} = y.$$

Now, for fixed ω, y, and $N > \frac{n_\omega}{2}$ choose m_N such that

$$|\sum_{k=1}^{m_N} \frac{1}{n_k(N)} - y| < \frac{1}{N}$$

and set

$$Q_m^{(N)}(\omega) = \sum_{k=1}^{m} (\kappa_{n_k(N)} + 1)(\omega) q_{2n_k(N)},\ \&\ Q_N = Q_N(\omega) := Q_{m_N}^{(N)}(\omega).$$

It follows that $Q_{NT} \xrightarrow{\Omega} 0$ whence $T^{Q_N} \overset{\mathcal{U}(L^2(m))}{\longrightarrow} \mathrm{Id}$. On the other hand,

$$\varphi^{(Q_N)}(\omega) = \sum_{k=1}^{m_N} \varphi^{((\kappa_{n_k}+1)q_{2n_k})}(T^{Q_{k-1}(N)}\omega) = \sum_{k=1}^{m_N} \frac{1}{n_k(N)} \longrightarrow y.$$

Thus $C(\varphi) \supset \mathbb{R}_+$. With some minor adjustments, $C(\varphi) = \mathbb{R}$ can be arranged. □

§2 Homogeneous Banach spaces and Koksma inequalities

Definition. By a *pseudo-homogeneous* Banach space on $\mathbb{T}$ we mean a Banach space $(B, \|\cdot\|_B)$ satisfying

(1) $B \subseteq L^1(\mathbb{T})$, and $\|\cdot\|_B \geq \|\cdot\|_1$,

(2) if $f \in B$ and $t \in \mathbb{T}$ then $f_t \in B$, and $\|f_t\|_B = \|f\|_B$, where $f_t(x) = f(x-t), x \in \mathbb{T}$.

A pseudo-homogeneous Banach space on $\mathbb{T}$ is called *homogeneous* if $t \mapsto f_t$ is continuous $\mathbb{T} \longrightarrow B$, $\forall\, f \in B$.

The following properties of pseudo-homogeneous Banach spaces are either contained in, or can be easily deduced from [Katzn, chapter 1]:

(1) there exists the largest homogeneous Banach subspace B_h contained in B defined by

$$B_h = \{f \in B : t \mapsto f_t \text{ is continuous } \mathbb{T} \to B\};$$

(2) the space B_h is the closure of trigonometric polynomials belonging to B (this is because B_h is homogeneous and hence if $f \in B_h$ and $g \in C(\mathbb{T})$ then the convolution of these two functions is an element of B_h);

(3) if $f \in B$ then $f \in B_h$ iff for each $n \in \mathbb{Z}$ such that $\hat{f}(n) \neq 0$ there exists $g \in B_h$ such that $\hat{g}(n) \neq 0$.

Suppose now that B is a Banach space and T is an isometry on it. Assume also that zero is the only fixed point of T. We say that for an $x \in B$ *the ergodic theorem holds* if

$$B - \lim_{n \longrightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j x = 0.$$

The set of all elements of B for which the ergodic theorem holds is denoted by $\mathrm{ET}(B,T)$. An element $x \in B$ is said to be a *(B-)coboundary* if $x = y - Ty$ for some $y \in B$ (called a *transfer* element). The following theorem is a version of the Mean Ergodic Theorem:

Theorem 2.1 (von Neumann). *An element $x \in ET(B,T)$ iff x belongs to the closure of the subspace of B-coboundaries.*

Suppose now that B is a pseudo-homogeneous Banach space on $\mathbb{T}$ (only functions with zero mean are considered). Let T denote an irrational translation by α, then T acts as an isometry on B. Note that if P is a trigonometric polynomial from B then P is a coboundary, in fact we have $P = Q - Q \circ T$, where Q is another trigonometric polynomial, hence $P, Q \in B_h$. This proves

Corollary 2.2

$$B_h \subset ET(B,T).$$

Let

$$\alpha = [0; a_1, a_2, \ldots]$$

be the continued fraction expansion of α. The positive integers a_n are called the *partial quotients* of α. Put

$$q_0 = 1,\ q_1 = a_1,\ q_{n+1} = a_{n+1} q_n + q_{n-1} \quad p_0 = 0,\ p_1 = 1,\ p_{n+1} = a_{n+1} p_n + p_{n-1}.$$

The rationals p_n/q_n are called the *convergents* of α and the inequality

$$|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}$$

holds. A *denominator* q_n is said to be a *Legendre* denominator if $|\alpha - \frac{p_n}{q_n}| < \frac{1}{2q_n^2}$. We'll sometimes denote the set of Legendre denominators of α by $\mathcal{L}(\alpha)$.

Note that if $q \in \mathcal{L}(\alpha)$ is a Legendre denominator then

$$\|j\alpha - j'\alpha\| > \frac{1}{2q} \quad \text{whenever} \quad 0 \leq j \neq j' \leq q-1. \tag{2.1}$$

Here, for $t \in \mathbb{R}$,

$$\|t\| = d(t, \mathbb{Z}) = \min_{n \in \mathbb{Z}} |n - t|.$$

We recall that one of any two consecutive denominators of an irrational α must be a Legendre denominator i.e. $(\forall\ \alpha \notin \mathbb{Q},\ n \geq 1)$, $\{q_n, q_{n+1}\} \cap \mathcal{L}(\alpha) \neq \emptyset$.

Let B be a pseudo-homogeneous Banach space on $\mathbb{T}$. We say that *Koksma's inequality* holds for the pair (B,T) provided that there exists a positive sequence

$\tilde{D}_N = \tilde{D}_N(\alpha)$, $N \geq 1$, satisfying $\tilde{D}_{q_n} = O(1/q_n)$ where $\{q_n\}$ is the sequence of denominators of α and

$$(2.2) \qquad \|\frac{1}{N}f^{(N)}(\cdot) - \int_0^1 f(t)dt\|_{L^1} \leq \|f\|_B \tilde{D}_N(\alpha) \ \ \forall\, f \in B,$$

where $f^{(N)}(x) = \sum_{j=0}^{N-1} f(T^j x)$, $x \in \mathbb{T}$. For the classical cases where Koksma inequality is satisfied for functions with bounded variation or Lipschitz continuous functions we refer to [Ku-Ni], chapter 2.

The proposition below (essentially due to M. Herman, [He], p.189) will play a role in the proofs of ergodicity of certain cylinder flows.

Proposition 2.3. *If Koksma's inequality is satisfied for the pair* (B,T) *then for each* $f \in B_h$ *with* $\int_0^1 f(t)\,dt = 0$ *we have*

$$\lim_{n \longrightarrow \infty} f^{(q_n)} = 0 \quad in \ \ L^1(\mathbb{T}).$$

Proof. Denote by B_0 the subspace of B consisting of functions with zero mean. Then define a map $S : B_0 \longrightarrow l^\infty$ by

$$Sg = (\|g^{(q_n)}\|_{L^1})_{n \geq 1}.$$

Note that by the Koksma inequality, S is well-defined and continuous. Hence, the set $S^{-1}(c_0)$ is closed as c_0 is a closed subspace of l^∞. Each coboundary $f = h - hT$, $h \in B$ is in $S^{-1}(c_0)$ since for each function $u \in L^1(\mathbb{T})$ we have

$$(2.3) \qquad uT^{q_n} \longrightarrow u \ \text{ in } \ L^1(\mathbb{T}).$$

It follows from this, theorem 2.1 and corollary 2.2, that

$$B_h \subset ET(B,T) = \overline{\{h - h \circ T : h \in B\}} \subset S^{-1}(c_0).$$

□

We will now pass to a proof of Koksma's inequality in the space $B = O(1/n)$ (of functions whose Fourier coefficients are of order $O(1/n)$), where the norm is defined as $\|f\|_B = \|f\|_{L^1} + \sup_{n \neq 0} |n\hat{f}(n)|$. If $\{x_1, \ldots, x_N\}$ is a finite set of points from $[0,1)$ then by *discrepancy* $D_N = D_N(x_1, \ldots, x_N)$ we mean

$$D_N = \sup_{x<y}\{|\frac{\#\{1 \leq j \leq N\ x_j \in [x,y)\}}{N} - (y-x)|\}.$$

Lemma 2.4.

$$\sup_x \#\{1 \leq j \leq N\ x_j \in [x, x+\frac{1}{N})\} \leq ND_N + 1.$$

Proof. For an arbitrary $x \in [0,1)$,

$$|\frac{\#\{1 \leq j \leq N\ x_j \in [x, x+\frac{1}{N})\}}{N} - (x + \frac{1}{N} - x)| \leq D_N,$$

whence the assertions follows immediately. □

Lemma 2.5. *There exists* $C > 0$ *such that*
$(\forall m \geq 1)(\forall a \geq 1)(\forall x_1, \ldots, x_{m-1} \in [0,1))$ *if in each interval of length* $\frac{1}{m}$*: there are at most* a *points of the form* x_i *then* $\sum_{\{i: x_i \in (\frac{1}{2m}, 1-\frac{1}{2m})\}} \frac{1}{\|x_i\|^2} \leq Cam^2$.

Proof. Denote by I the set of those $1 \leq i \leq m-1$ so that $x_i \in (\frac{1}{2m}, 1-\frac{1}{2m})$. Then define a map $i \mapsto j(i)$, $i \in I, 1 \leq j(i) \leq m-1$, by

$$|x_i - \frac{j(i)}{m}| \leq \frac{1}{2m}. \tag{2.4}$$

Since $\|x_i\| > \frac{1}{2m}$,

$$\frac{1}{2} \leq \frac{\|x_i\|}{\|\frac{j(i)}{m}\|} \leq 2. \tag{2.5}$$

Note that if k is in the image of the function j then

$$\# j^{-1}(k) \leq a$$

by our assumption and (2.4). Hence by (2.5)

$$\sum_{i \in I} \frac{1}{\|x_i\|^2} \leq 2a \sum_{k \in \operatorname{Im} j} \frac{1}{\|k/m\|^2} \leq 4a \sum_{i=1}^{m-1} \frac{1}{(i/m)^2} = Cam^2.$$

□

Combining this with Lemma 2.4, we obtain

Corollary 2.6. *Under the conditions of lemma 2.5,*

$$\sum_{i \in I} \frac{1}{\|x_i\|^2} \leq C(mD_m + 1)m^2,$$

where I is the same as in the proof of Lemma 2.5.

Now, suppose that $f \in \mathrm{O}(\frac{1}{n})$,

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi i k x}.$$

We have

$$f^{(m)}(x) = \sum_{i=0}^{m-1} f(x + i\alpha) = f^{(m)}(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k \frac{e^{2\pi i k m \alpha} - 1}{e^{2\pi i k \alpha} - 1} e^{2\pi i k x}.$$

Theorem 2.7 (Koksma's Inequality in $\mathrm{O}(\frac{1}{n})$). *There is a constant* $K > 0$ *such that if we denote*

$$\tilde{D}_m = \sqrt{K(\sum_{k \in A_m} \frac{1}{k^2} + (mD_m + 1)(\|m\alpha\|^2 + \frac{1}{m^2}))}$$

then $\forall\ f \in\ O(\frac{1}{n})$,

$$\|\frac{1}{m}\sum_{i=0}^{m-1} f(\cdot+i\alpha)-\int_0^1 f(t)dt\|_{L^2}^2 \leq \|f\|^2{}_{O(\frac{1}{n})}\tilde{D}_m,$$

where
$D_m = D_m(0,\alpha,2\alpha,\ldots,(m-1)\alpha)$, *and* $A_m = \{0 \leq j \leq m-1 : 0 < \|j\alpha\| < \frac{1}{2m}\}$.
Moreover,

$$\tilde{D}_{q_n} = O(1/q_n).$$

Proof. Without loss of generality we will assume that $\int_0^1 f(t)\,dt = 0$ and it is enough to prove that

$$(2.6)\quad \|f^{(m)}\|_{L^2}^2 \leq C_2\|f\|^2{}_{O(\frac{1}{n})}(m^2\sum_{k\in A_m}\frac{1}{k^2}+C(mD_m+1)m^2\|m\alpha\|^2+C_3(mD_m+1)),$$

where C_2, C, C_3 are some absolute constants. Since f is real,

$$\|f^{(m)}\|_{L^2}^2 \leq 2C_1\sum_{k=1}^{\infty}|\hat{f}_k|^2\frac{\|km\alpha\|^2}{\|k\alpha\|^2} = C_2(S_1+S_2),$$

where

$$S_1 = \sum_{k=1}^{m-1}\frac{|\hat{f}_k|^2\|km\alpha\|^2}{\|k\alpha\|^2},\quad S_2 = \sum_{k=m}^{\infty}\frac{|\hat{f}_k|^2\|km\alpha\|^2}{\|k\alpha\|^2}.$$

Now,

$$S_1 = \sum_{k=1}^{m-1}\frac{|\hat{f}_k k|^2\|km\alpha\|^2}{k^2\|k\alpha\|^2} \leq \|f\|^2{}_{O(\frac{1}{n})}\sum_{k=1}^{m-1}\frac{\|km\alpha\|^2}{k^2\|k\alpha\|^2} = \|f\|^2{}_{O(\frac{1}{n})}(S_{11}+S_{12}),$$

where

$$S_{11} = \sum_{k\in A_m}\frac{\|km\alpha\|^2}{k^2\|k\alpha\|^2},\quad S_{12} = \sum_{k\notin A_m}\frac{\|km\alpha\|^2}{k^2\|k\alpha\|^2}.$$

We have, $S_{11} \leq m^2\sum_{k\in A_m}\frac{1}{k^2}$, and $S_{12} \leq \|m\alpha\|^2\sum_{k\notin A_m}\frac{1}{\|k\alpha\|^2}$.
By Corollary 2.6,

$$S_{12} \leq \|m\alpha\|^2 C(mD_m+1)m^2.$$

We pass now to estimate S_2. We have

$$S_2 = \sum_{k=m}^{\infty}\frac{|\hat{f}_k|^2\|km\alpha\|^2}{\|k\alpha\|^2} = \sum_{p=1}^{\infty}\sum_{r=0}^{m-1}\frac{|\hat{f}_{pm+r}|^2\|(pm+r)m\alpha\|^2}{\|(pm+r)\alpha\|^2} \leq$$

$$\|f\|^2{}_{O(\frac{1}{n})}\sum_{p=1}^{\infty}\frac{1}{p^2}\sum_{r=0}^{m-1}\frac{\|(pm+r)m\alpha\|^2}{m^2\|(pm+r)\alpha\|^2} \leq$$

$$\frac{1}{m^2}\|f\|^2{}_{O(\frac{1}{n})}\sum_{p=1}^{\infty}\frac{1}{p^2}\sum_{r=0}^{m-1}\min(m^2,\frac{1}{\|pm\alpha+r\alpha\|^2}).$$

Denote $x = pm\alpha$. In the interval $(-\frac{1}{2m}, \frac{1}{2m}) = [0, \frac{1}{2m}) \cup [1 - \frac{1}{2m}, 1)$ (mod 1) we have at most mD_m+1 points of the form $x+r\alpha$ because $D_m = D_m(x, x+\alpha, \ldots, x+(m-1)\alpha)$. By Corollary 2.6 we thus have

$$S_2 \le \frac{1}{m^2}\|f\|^2_{O(\frac{1}{n})} \sum_{p=1}^{\infty} \frac{1}{p^2}((mD_m+1)m^2 + C(mD_m+1)m^2) \le C_3\|f\|^2_{O(\frac{1}{n})}(mD_m+1).$$

To complete the proof we have to show that the sequence $\{q_n \tilde{D}_{q_n}\}$ is bounded. But classically, $D_{q_n} = O(1/q_n)$ and also $q_n\|q_n\alpha\|$ is bounded. Now, note that in the interval $M_n = [0, \frac{1}{2q_n}) \cup [1 - \frac{1}{2q_n}, 1)$ we can have at most one point of the form $j\alpha$, where $j = 1, \ldots, q_n - 1$. Moreover, $|j\alpha - j\frac{p_n}{q_n}| < \frac{j}{q_n q_{n+1}}$, so if $j\alpha \in M_n$ then we must have $\frac{j}{q_n q_{n+1}} > \frac{1}{2q_n}$. In particular, $j > q_n/2$, so $\sum_{k\in A_{q_n}} \frac{1}{k^2} = O(1/q_n^2)$.

□

Now, proceeding as in the proof of Proposition 2.3, we obtain the following extension of the main result from [Le-Ma]

Corollary 2.8. *If* $f \in o(\frac{1}{n})$, $\int_0^1 f(t)\,dt = 0$ *and* $\{q_n\}$ *is the sequence of all denominators of* α *then*

$$\|f^{(q_n)}\|_{L^2} \longrightarrow 0.$$

§3 Speed of approximation in Koksma's Inequality for spaces $O(1/a(n))$

Assume that $a : \mathbb{N} \longrightarrow \mathbb{R}^+$ satisfies

$$a(k) \ge k, \tag{3.1}$$

$$a(pm + r) \ge a(p)a(m), \quad \text{for arbitrary} \quad p, m \ge 1, r = 0, \ldots, m-1. \tag{3.2}$$

We will now concentrate on a pseudo-homogeneous Banach space $B = O(1/a(n))$ of functions

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi i k x},$$

with $\hat{f}_k = O(1/a(k))$. The norm is defined as

$$\|f\|_{O(1/a(n))} = \|f\|_{L^1} + \sup_{n\neq 0} |a(n)\hat{f}_n|.$$

Notice that in this case $B_h = o(1/a(n))$ the subspace of functions whose Fourier coefficients are of order $o(1/a(n))$. Keeping the notation from the proof of Theorem 2.7 and proceeding as before we obtain that

$$S_1 \le \|f\|^2_{O(1/a(n))}(S_{11} + S_{12}),$$

where

$$S_{11} = m^2 \sum_{k\in A_m} \frac{1}{a(k)^2},$$

and by (3.1)

$$S_{12} \leq \|m\alpha\|^2 \sum_{k\notin A_m} \frac{k^2}{a(k)^2}\frac{1}{\|k\alpha\|^2} \leq \|m\alpha\|^2 m^2(D_m m+1)\cdot C.$$

In view of (3.2),

$$S_2 \leq \|f\|^2_{\mathrm{O}(1/a(n))} \sum_{p=1}^{\infty}\frac{1}{a(p)^2}\sum_{r=0}^{m-1}\frac{\|(pm+r)m\alpha\|^2}{a(m)^2\|(pm+r)\alpha\|^2} \leq$$

$$\frac{1}{a(m)^2}\|f\|^2_{\mathrm{O}(1/a(n))} m^2 C_4(mD_m+1)\sum_{p=1}^{\infty}\frac{1}{a(p)^2} \leq (\frac{m}{a(m)})^2\|f\|^2_{\mathrm{O}(1/a(n))}(mD_m+1)C_5.$$

For a function $a(\cdot)$ satisfying (3.1) and (3.2) denote

$$I(a) = \{\alpha \in [0,1)\setminus \mathbb{Q} : \liminf_{q\to\infty,\ q\in\mathcal{L}(\alpha)} a(q)\|q\alpha\| < \infty\}.$$

Lemma 3.1. *If $f = gT - g$, $g \in O(1/a(n))$, $\alpha \in I(a)$ and $q_{n_k} \in \mathcal{L}(\alpha)$ with $a(q_{n_k})\|q_{n_k}\alpha\| = O(1)$, then*

$$\|f^{(q_{n_k})}\|_{L^2} = o(\frac{q_{n_k}}{a(q_{n_k})}).$$

Proof. All we need to show is that $\sum_{s=1}^{\infty}|\hat{g}_s|^2\|q_{n_k}s\alpha\|^2 = \mathrm{o}((\frac{q_{n_k}}{a(q_{n_k})})^2)$. We have

$$\sum_{s=1}^{\infty}|\hat{g}_s|^2\|q_{n_k}s\alpha\|^2 \leq \|g\|^2_{\mathrm{O}(1/a(n))}(\sum_{s=1}^{q_{n_k}-1}\frac{\|q_{n_k}s\alpha\|^2}{a(s)^2} + \sum_{s=q_{n_k}}^{\infty}\frac{\|q_{n_k}s\alpha\|^2}{a(s)^2}) \leq$$

$$\|g\|^2_{\mathrm{O}(1/a(n))}(q_{n_k}\|q_{n_k}\alpha\|^2 + q_{n_k}\sum_{p=1}^{\infty}\frac{1}{(a(p)a(q_{n_k}))^2}) =$$

$$\|g\|^2_{\mathrm{O}(1/a(n))}(\frac{q_{n_k}}{a(q_{n_k})^2}a(q_{n_k})^2\|q_{n_k}\alpha\|^2 + \frac{q_{n_k}}{a(q_{n_k})^2}\sum_{p=1}^{\infty}\frac{1}{a(p)^2}) = \mathrm{o}((\frac{q_{n_k}}{a(q_{n_k})})^2).$$

□

Corollary 3.2. *If $f \in O(1/a(n))$, $\int_0^1 f(t)\,dt = 0$ and $\alpha \in I(a)$ and $q_{n_k} \in \mathcal{L}(\alpha)$ with $a(q_{n_k})\|q_{n_k}\alpha\| = O(1)$, then*

$$\|f^{(q_{n_k})}\|_{L^2} \leq \text{const.}\|f\|_{O(1/a(n))}\frac{q_{n_k}}{a(q_{n_k})}.$$

Moreover, if in addition $f \in o(\frac{1}{a(n)})$ then

$$\|f^{(q_{n_k})}\|_{L^2} = o(\frac{q_{n_k}}{a(q_{n_k})}). \tag{3.3}$$

Proof. Since (3.3) is satisfied for all coboundaries by Lemma 3.1, the mechanism described in the proof of Proposition 2.3 works well. The map S is defined as $Sf = (\frac{a(q_{n_k})}{q_{n_k}}\|f^{(q_{n_k})}\|_{L^2})_{k\geq 1}$.

□

Suppose now that $a(n) = \frac{1}{n^t}$ for certain natural number $t \geq 1$. Hence $I(a) =: I(t)$ is the set of those irrationals α for which $(q_{n_k}^t \|q_{n_k}\alpha\|)$ is bounded for certain subsequence of Legendre denominators of α.

Corollary 3.3. *If* $f \in o(\frac{1}{n^t})$, $\int_0^1 f d\lambda = 0$ *then for an arbitrary* $\alpha \in I(t)$ *and* $q_{n_k} \in \mathcal{L}(\alpha)$ *with* $q_{n_k}^t \|q_{n_k}\alpha\| = O(1)$, *we have*

(i) $\|f^{(q_{n_k})}\|_{L^2} = o(\frac{1}{q_{n_k}^{t-1}})$,

(ii) the sequence $(q_{n_k}^t)$ *is a rigidity time for* α *and*

$$\lim_{k\to\infty} f^{(q_{n_k}^t)} = 0 \quad in \quad L^2(\mathbb{T}).$$

Proof. It is enough to notice that $f^{(q_{n_k}^t)} = f^{(q_{n_k}q_{n_k}^{t-1})}$ and that $\|f^{(q_{n_k}q_{n_k}^{t-1})}\|_{L^2} \leq q_{n_k}^{t-1}\|f^{(q_{n_k})}\|_{L^2}$. □

§4 Constructions of ergodic analytic cylinder flows

Constructions which are known of ergodic cylinder flows are rather based on some irregularities in the smoothness of the cocycle (e.g. [He-La1], [He-La2], [Pa1], [Pa2], [Ba-Me1], [Ba-Me2]). Below, we will show a new method coming from [Kw-Le-Ru2] for constructing analytic cylinder flows which are ergodic.

Assume that $Tx = x + \alpha$, where $\alpha = [0; a_1, a_2, \ldots]$. From the continued fraction expansion of α we obtain, for each n, two Rokhlin towers $\xi_n, \overline{\xi}_n$ whose union coincides with the whole circle. For n even

$$\xi_n = \{[0, \{q_n\alpha\}), T[0, \{q_n\alpha\}), \ldots, T^{(a_{n+1}q_n+q_{n-1})-1}[0, \{q_n\alpha\})\},$$

$$\overline{\xi}_n = \{[\{q_{n+1}\alpha\}, 1), \ldots, T^{q_n-1}[\{q_{n+1}\alpha\}, 1)\}.$$

Given a subsequence $\{n_k\}$ of natural numbers we will denote

$$I_k = [0, \{a_{2n_k+1}q_{2n_k}\alpha\}), \quad J_t^k = T^{(t-1)q_{2n_k}}(0, \{q_{2n_k}\alpha\}],$$

$t = 1, \ldots, a_{2n_k+1}$. Notice that

$$I_k = \bigcup_{t=1}^{a_{2n_k+1}} J_t^k,$$

and

$$|J_1^k| < \frac{1}{a_{2n_k+1}q_{2n_k}}. \tag{4.1}$$

We will recall here a notion of an a.a.c.c.p. (almost analytic cocycle construction procedure) from [Kw-Le-Ru2] which is to construct a real 1-periodic cocycle $\tilde{\varphi} : \mathbb{R} \longrightarrow \mathbb{R}$ such that in its $\mathbb{R}$–cohomology class (for certain α) there is an analytic cocycle.

An a.a.c.c.p. is given by a collection of parameters as follows. We are given a sequence $\{M_k\}$ of natural numbers and an array $\{(d_{k,1}, \ldots, d_{k,M_k})\}$, $d_{k,i} \in \mathbb{R}$ satisfying for each k

$$\sum_{i=1}^{M_k} d_{k,i} = 0.$$

Denote $D_k = \max_{1\le i\le M_k} |d_{k,i}|$. Choose a sequence $\{\varepsilon_k\}$ of positive real numbers satisfying

$$\sum_{k=1}^{\infty} \sqrt{\varepsilon_k} M_k < +\infty,$$

$$\sum_{k=1}^{\infty} \varepsilon_k < 1,$$

$$\varepsilon_k < \frac{1}{D_k^2}, \quad k = 1, 2, \ldots.$$

Finally, we are given $A > 1$ completing the parameters of the a.a.c.c.p.

We say that this a.a.c.c.p. *is realized over an irrational number* α with continued fraction expansion $[0; a_1, a_2, \ldots]$ and convergents $p_n/q_n, n \ge 1$ if there exists a strictly increasing sequence $\{n_k\}$ of natural numbers such that

$$A^{N_k} \frac{D_k M_k \|P_k\|_{\mathcal{F}}}{a_{2n_k+1} q_{2n_k}} < \frac{1}{2^k}$$

and

$$\frac{D_k \|P_k'\|_\infty}{a_{2n_k+1} q_{2n_k}} < \sqrt{\varepsilon_k},$$

where $\{P_k\}$ is a sequence of "bump" real trigonometric polynomials, *i.e.*

(*i*) $\int_0^1 P_k(t)dt = 1$,
(*ii*) $P_k \ge 0$,
(*iii*) $P_k(t) < \varepsilon_k$ for each $t \in (\eta_k/2, 1)$,

where the η_k's are chosen in such a way that

$$4M_k\eta_k < \frac{\varepsilon_k}{q_{2n_k}} \tag{4.2}$$

and N_k is the degree of P_k. Finally, $a_{2n_k+1} > 1$ and

$$\frac{1}{a_{2n_k+1} q_{2n_k}} < \frac{1}{2}\eta_k \tag{4.3}$$

Using the above parameters define a cocycle

$$\varphi = \sum_{k=1}^{\infty} \varphi(k)$$

as follows. In view of (4.2),(4.3) (and (4.1)), in the interval $I_k = [0, \{a_{2n_k+1} q_{2n_k} \alpha\})$ we can choose $w_{k,1}, \ldots, w_{k,M_k}$ to be consecutive pairwise disjoint intervals of the same length contained between η_k and $2\eta_k$ such that each $w_{k,i}$ consists of say e_k consecutive subintervals J_t^k, where e_k is an odd number. Let $J_{s_{k,i}}^k$ be the central subinterval in $w_{k,i}$ and now define

$$\varphi(k)(x) = \begin{cases} d_{k,i} & \text{if } x \in J_{s_{k,i}}^k \\ 0 & \text{otherwise.} \end{cases}$$

Note that the $\varphi(k)$'s have disjoint supports so φ is well defined.

As proved in [Kw-Le-Ru2]
(A) The set of α's over which an a.a.c.c.p. is realized is a G_δ and dense subset of the circle.
(B) If an a.a.c.c.p. is realized over α then there exists an analytic cocycle $f : \mathbb{T} \longrightarrow \mathbb{R}$ which is α–cohomologous to φ.

We will need an additional property of an a.a.c.c.p. which is not explicitly formulated in [Kw-Le-Ru2]. Namely,

$$\varphi|_{T^s I_k} \text{ is constant for } s = 1, \dots, q_{2n_k} - 1, \ \& \ \sum_{s=1}^{q_{2n_k}-1} \varphi|_{T^s I_k} = 0 \tag{4.4}$$

which is Lemma 3 from [Kw-Le-Ru1].

Example 4.1. There is an a.a.c.c.p. with $\mathrm{Gp}(\widetilde{D}(\varphi)) = E(\varphi) = \mathbb{Z}\lambda$.

Proof. Assume that $\lambda \in \mathbb{R}$ is given. We will assume that an a.a.c.c.p. satisfies the following additional requirements:

$$a_{2n_k+1} = M_k r_k + N_k,$$

with $0 \le N_k < r_k$ and both M_k, r_k tending to infinity. We put $d_{k,1} = 0, d_{k,i} = \lambda$ for $i = 2, \dots, M_k - 1$ and $d_{k,M_k} = -(M_k - 1)\lambda$. In the definition of φ_k we require that $\varphi_k | J^k_{ir_k+1} = d_{k,i}$ for $i = 0, \dots, M_k - 1$ and zero for all others subintervals J^k_t, $k \ge 1$.

Notice that $E(\varphi) \subset \mathbb{Z}\lambda$ because the values of φ are from the group $\mathbb{Z}\lambda$. It is then enough to show that $\lambda \in \widetilde{D}(\varphi)$. Define

$$X_k = \bigcup_{s=0}^{q_{2n_k}-1} \bigcup_{t=r_k+1}^{(M_k-1)r_k} T^s J^k_t.$$

By our definition of φ and a basic property of an a.a.c.c.p. (see (4.4)) we have $\varphi^{(M_k r_k)}(x) = \lambda$ for all $x \in X_k$. It is clear also that $M_k r_k$ is a rigidity time for T. Therefore $\lambda \in \widetilde{D}(\varphi)$. □

Example 4.2. An a.a.c.c.p. with $\overline{\mathrm{Gp}(\widetilde{D}(\varphi))} = \mathbb{R}$.

This is an obvious modification of the previous construction. We divide the sequence $\{n_k\}$ into two disjoint subsequences say $\{n^i_k\}_k$ $(i = 1, 2)$ and repeat the previous construction for rationally independent $\lambda_1, \lambda_2 \in \mathbb{R}$, with the sequences $\{n^i_k\}$, $i = 1, 2$. From the previous arguments we find $\lambda_1, \lambda_2 \in \widetilde{D}(\varphi)$. The group generated by λ_1, λ_2 is dense in $\mathbb{R}$ and the advertised condition is attained.

Remark It follows from proposition 1.5 that the cocycles of example 4.2 are ergodic, coalescent, and nonsquashable.

§5 Ergodicity of smooth cylinder flows. Generic point of view

Suppose that $f : \mathbb{T} \to \mathbb{R}$ is smooth. We shall prove that under certain assumptions, the set of those irrational translations for which the corresponding cylinder flow is ergodic is residual. For similar results see [Kr], [Ka].

Assume that $f(x)=\sum_{n=-\infty}^{\infty} b_n e^{2\pi i n x}$ with zero mean is in $A(\mathbb{T})$, that is its Fourier transform is absolutely summable. Put $f_m(x)=f(x)+f(x+\frac{1}{m})+\ldots+f(x+\frac{m-1}{m})=m\sum_{l=-\infty}^{\infty} b_{lm}e^{2\pi i l m x}$, $m=1,\ldots$.

Theorem 5.1. *Suppose that there exist an infinite subsequence $\{q_n\}$ and a constant $C>0$ such that*

(1) $q_n\sum_{l=-\infty}^{\infty}|b_{lq_n}|\le C\|f_{q_n}\|_{L^2},\ \ n=1,2,\ldots,$

(2) $0<\|f_{q_n}\|_{L^2}\to 0,$

then there exists a dense G_δ set of irrational numbers α such that the corresponding cylinder flow T_f, $Tx=x+\alpha$ is ergodic.

Proof. We will need the following

Lemma 5.2. *Given $C>0$ there exist positive numbers K,L,M such that $0<K<1<L$, $0<M<1$ and for each $h\in L^4(\mathbb{T})$ if $\|h\|_4\le C\|h\|_2$, then*

$$\mu\{x\in\mathbb{T}:\ K\|h\|_2\le|h(x)|\le L\|h\|_2\}>M$$

We will prove the lemma later. Denote

$$g_n(x)=q_n\sum_{l=-\infty}^{\infty} b_{lq_n}e^{2\pi i l x}.$$

In view of (1) we have that

$$g_n(q_n x)=f_{q_n}(x),\ x\in\mathbb{T} \tag{5.1}$$

and

$$\|g_n\|_{L^\infty}\le q_n\sum|b_{lq_n}|\le C\|g_n\|_{L^2},$$

in particular, $\|g_n\|_4\le C\|g_n\|_2$. Hence by Lemma 5.2

$$\mu\{x\in\mathbb{T}:\ K\|g_n\|_2\le|g_n(x)|\le L\|g_n\|_2\}>M.$$

By (2) we have $\|g_n\|_2=\|f_{q_n}\|_2\to 0$.

Let $\{D_n\}$ be a family of pairwise disjoint closed intervals, $D_n=[c_n,d_n]$, with

$$d_n/c_n=100\frac{L}{K}\quad\text{and}\quad d_n\to 0.$$

Assume that $\{D'_n\}$ is a sequence of the above intervals with the property that each D_n repeats infinitely many times in $\{D'_n\}$.

Now, fix n, that is we have the interval D'_n. Choose a natural number k_n so that for some natural s_n

$$[s_nK\|g_{k_n}\|_{L^2},s_nL\|g_{k_n}\|_{L^2}]\subset\tilde{D}'_n,$$

where $\tilde{D}'_n$ is a strict subinterval of D'_n. This gives us a subsequence $\{k_n\}$. For it we have that

$$\mu\{x\in\mathbb{T}:\ |s_ng_{k_n}(x)|\in\tilde{D}'_n\}\ge M.$$

From this and (5.1) we obtain that for each interval I of length being a multiple of $\frac{1}{q_{k_n}}$

$$\mu\{x \in I: \ |s_n f_{q_{k_n}}(x)| \in \tilde{D}'_n\} \geq M|I|. \tag{5.2}$$

We will also use the following lemma whose proof is contained in [Kw-Le-Ru2].

Lemma 5.3. *Given an infinite set $\{Q_n\}$ of natural numbers and a positive real valued function $\delta = \delta(Q_n)$ the set*

$$\mathcal{A} = \{\alpha \in [0,1) : \#\{n : \exists\ P_n \ \ni \ \frac{P_n}{Q_n} \text{ a convergent of } \alpha, \ \& \ |\alpha - \frac{P_n}{Q_n}| < \delta(Q_n)\} = \infty\}$$

is a dense G_δ.

Let us fix r. So we have infinitely many $n = n(r)$ with $D'_n = D_r$. Consider now those α which are approximated by $\frac{p_{k_{n(r)}}}{q_{k_{n(r)}}}$ so well to have

$$\|s_{n(r)} q_{k_{n(r)}} \alpha\| \to 0$$

and

$$\mu\{x \in I: \ |f^{(s_{n(r)} q_{k_{n(r)}})}(x)| \in D'_{n(r)}\} \geq \frac{M}{2}|I| \tag{5.3}$$

for each interval I with $|I| = \frac{t}{q_{k_{n(r)}}}, t = 1, \ldots, q_{k_{n(r)}}$ (remember that we know the modulus of continuity of f and that

$$\sum_{i=0}^{s-1}\left(\sum_{j=0}^{q-1} f(x + \frac{j}{q}) - \sum_{k=0}^{q-1} f(x + iq\alpha + k\alpha)\right) =$$

$$\sum_{i=0}^{s-1}\left(\sum_{k=0}^{q-1} f(x + k\frac{p}{q}) - f(x + iq\alpha + k\alpha)\right) \leq \sum_{i=0}^{s-1}\sum_{k=0}^{q-1} \omega(f, iq\alpha + k(\alpha - \frac{p}{q})),$$

where $\gcd(p,q) = 1$, $p = p_{k_{n(r)}}, q = q_{k_{n(r)}}$ and $\omega(f,h)$ stands for the modulus of the continuity of f; now given s, q the size of the above quantity depends on the distance between α and $\frac{p}{q}$.)

In view of Lemma 5.3 we have a G_δ and dense subset of α, say Y_r, for which (5.3) holds true for an infinite subsequence of $\{q_{k_{n(r)}}\}$. Finally take

$$Y = \bigcap_{r=1}^{\infty} Y_r$$

which is G_δ and dense. If we take $\alpha \in Y$ then for each r we have an infinite subsequence $n(\alpha)$ such that

$$\mu\{x \in I: \ |f^{(s_{n(\alpha)} q_{k_{n(\alpha)}})}(x)| \in D'_{n(\alpha)}\} \geq \frac{M}{2}|I|$$

for each interval I with $|I| = \frac{t}{q_{k_{n(r)}}}$ and $D'_{n(\alpha)} = D_r$.

It remains to prove that if $Tx = x + \alpha$, where $\alpha \in Y$ then the cylinder flow T_f is ergodic. Suppose that $E(f) = \lambda \mathbb{Z}$. Choose r so big to have that the compact set $K_r := D_r \cup (-D_r)$ is disjoint with $\lambda\mathbb{Z}$. By Lemma 1.2 there exists a Borel set B, with $\mu(B) > 0$ such that for all $m \geq 1$

$$\mu(B \cap T^{-m}B \cap \{x \in \mathbb{T} : f^{(m)}(x) \in K_r\}) = 0. \tag{5.4}$$

If $m = s_n q_{k_n}$, $n = n(\alpha)$, then $\mu(B \triangle T^{s_n q_{k_n}} B) \to 0$ since $s_n q_{k_n}$ is a rigidity time for T. If y is a density point of B then for an interval I of length t/q_{k_n} containing y we will have $\mu(B \cap I) > (1 - \frac{M}{4})|I|$. Hence a subset A_n of B of measure at least $\frac{M}{4}\mu(B)$ has the property that $f^{(s_n q_{k_n})}(x) \in K_r$ whenever $x \in A_n$. This contradicts (5.4). □

Proof of Lemma 5.2. It is enough to consider the case $\|h\|_2 = 1$. Take two real numbers K, L satisfying $0 < K < 1 < L$. From Tchebycheff inequality we have

$$\mu\{|h| \leq L\} \geq \mu\{||h|^2 - 1| \leq L^2 - 1\} \geq 1 - \operatorname{Var}(|h|^2)(L^2-1)^{-2} \geq 1 - (C^4 - 1)(L^2 - 1)^{-2}.$$

On the other hand, from Cauchy-Schwartz inequality

$$1 = \int_{\{|h|>K\}} h^2 + \int_{\{|h|\leq K\}} h^2 \leq (\int h^4)^{1/2}(\mu\{|h| > K\})^{1/2} + K^2;$$

whence $\mu\{|h| > K\} \geq (1 - K^2)^2/C^4$. Now to have the conclusion of the lemma it is enough to choose $\varepsilon > 0$, put $M = 1/C^4 - 2\varepsilon$, then find K small enough to have $(1-K^2)^2/C^4 > M+\varepsilon$ and finally select L sufficiently big to have $(C^4-1)(L^2-1)^{-2} < \varepsilon$. □

Remarks

As shown in [Kw-Le-Ru2], the assumptions of Theorem 5.1 are satisfied for each zero mean function $f \in C^{1+\delta}(\mathbb{T})$, $\delta > 0$ which is not a trigonometric polynomial. Recall that a subset $E \subset \mathbb{Z}$ is called *of type* $\Lambda(2)$ if for every $q \geq 2$ there exists a constant $C = C(q, E)$ such that for every function $h \in L^q(\mathbb{T})$ we have $\|h\|_q \leq C\|h\|_2$ whenever $\operatorname{supp}(\hat{h}) \subset E$. For instance, each lacunary subset is of that type ([Katzn], Chapter 5.). Now, if $f \in L^2(\mathbb{T})$ with the absolutely summable Fourier transform has the property that the support of its Fourier transform is an *infinite* $\Lambda(2)$ type set and moreover that $\hat{f}(n) = o(1/n)$ then the assumptions of Theorem 5.1 are also satisfied.

§6 Ergodicity of a class of cylinder flows

This section will be devoted to a generalization of a result of Pask [Pa1].

A function $f : \mathbb{T} \to \mathbb{R}$ is called *piecewise linear (piecewise absolutely continuous)* if there are points $x_1 < x_2 < \ldots < x_K$ such that f restricted to $[x_j, x_{j+1})$ is linear (absolutely continuous), $j = 1, 2, \ldots$ (mod K). Denote by d_j the jump of the values of f at x_j. It is clear that if f is piecewise absolutely continuous then

$$\int_0^1 f'(t)\, dt = \sum_{j=1}^{K} d_j.$$

Lemma 6.1. *Suppose that* $f : \mathbb{T} \to \mathbb{R}$, $\int_0^1 f(t)\,dt = 0$ *is piecewise linear, and* $\sum_{j=1}^{K} d_j \neq 0$, *then for each irrational number* α *the corresponding cylinder flow* T_f *is ergodic.*

Proof. There is no loss of generality in assuming that $\sum_{j=1}^{K} d_j > 0$. Since f' is Riemann integrable, the ergodic theorem holds uniformly, so

$$\frac{1}{q}\sum_{j=0}^{q-1} f'(x+j\alpha) \to \int_0^1 f'(t)\,dt > 0$$

uniformly in x. Hence, we can find two constants $0 < C_1 < C_2$ such that for all q sufficiently large,

$$C_1 q \leq f^{(q)\prime}(x) \leq C_2 q \;\forall\, x \in \mathbb{T}. \tag{6.1}$$

On the other hand, $f^{(q)}$ is still piecewise linear with the discontinuity points of the form $x_i + j\alpha$, with the jump at it equal to d_i, where $i = 1, \ldots, K$, $j = 0, \ldots, q-1$. Substitute from now on $q = q_n$ a Legendre denominator of α. Take the division of the circle given by the points of the form $x_i + j\alpha$. It may happen that for $i \neq i'$ we will have for some $j \neq j'$ that $x_i + j\alpha = x_{i'} + j'\alpha$. This gives rise to a partition, say ξ_n, of the circle into closed-open subintervals. Consequently the number of atoms in ξ_n is not bigger than Kq_n. Note that no subinterval in ξ_n can be longer than $1/q_n$, so ξ_n is tending to the point partition. Let us call a subinterval in ξ_n *long* if its length is at least $\frac{1}{100Kq_n}$. Hence there must exist a constant $D = D(K) > 0$ such that for all $n \geq 1$ the number of long subintervals is at least Dq_n. Finally, by the classical Koksma inequality, we have

$$|f^{(q_n)}(x) - f^{(q_n)}(y)| \leq \mathrm{Var}(f) \quad \text{for all } x, y \in \mathbb{T}.$$

Suppose now that $E(f) = \mathbb{Z}\lambda$. Choose a very small $\varepsilon = \varepsilon(\lambda,\ \mathrm{Var}(f), C_1, C_2, D) > 0$ and let

$$K = \{r \in [-2\,\mathrm{Var}(f), 2\,\mathrm{Var}(f)] : \mathrm{dist}(r, \mathbb{Z}\lambda) \geq \varepsilon\}.$$

It is clear that K is compact. If ε is small enough, in view of (6.1) and (6.2), there exists a constant $F > 0$ such that for each long subinterval of ξ_n there exists a subset with measure at least $F\frac{1}{q_n}$ such that for each x from this subset we have $f^{(q_n)}(x) \in K$. It is now sufficient to apply Lemma 1.3 to obtain an obvious contradiction to $K \cap E(f) = \emptyset$. □

It is clear that the arguments from the above proof persist if instead of a piecewise continuous function we consider a function $g = f + h$, where f is piecewise linear with $\int_0^1 f'(t)\,dt \neq 0$, h is integrable, $\int_0^1 f\,dt = \int_0^1 h\,dt = 0$ and $h^{(q_n)}$ is tending to zero in measure along the sequence of Legendre denominators of α. In particular, because of Proposition 2.3, we have proved the following

Theorem 6.2. *Let* B *be a homogeneous Banach space on* $\mathbb{T}$ *and* T *an irrational translation. If for the pair* (B, T) *the Koksma inequality holds true then for each cocycle* $g = f + h$, *where* f *is piecewise linear with* $\int_0^1 f'(t)\,dt \neq 0$, $h \in B_h$, $\int_0^1 f\,dt = \int_0^1 h\,dt = 0$ *the corresponding cylinder flow* T_f *is ergodic.*

In particular (see Corollary 2.8)

Corollary 6.3. *Suppose that* $g = f + h$ *where* f *is piecewise linear with* $\int_0^1 f'(t)\,dt \neq 0$, *and* $\hat{h}(n) = o(1/n)$, $\int_0^1 f dt = \int_0^1 h dt = 0$ *then for each irrational translation* T *the corresponding cylinder flow* T_f *is ergodic.*

Remarks 1. Assume as in [Pa1] that $g : \mathbb{T} \to \mathbb{R}$ is piecewise absolutely continuous, with $\int_0^1 g'(t)\,dt \neq 0$ and $\int_0^1 g(t)\,dt = 0$. Denote by $x_1, \ldots, x_K$ the discontinuity points and let d_j be the jump at x_j. Take any piecewise linear function f with the same discontinuity points and the same jumps as g; in particular $\int_0^1 f'(t)\,dt \neq 0$. By adding a constant if necessary we can assume that $\int_0^1 f(t)\,dt = 0$. Define $h = g - f$. We have that h has zero mean and is absolutely continuous. Now, the result from [Pa1] directly follows from Corollary 6.3.

2. Notice that if g is of the form as in Corollary 6.3 then for each $\beta \in \mathbb{T}, c \neq 1$ the cocycle $g(\cdot + \beta) - cg(\cdot)$ is still of the same form, hence ergodic. We have proved that all ergodic cocycles from Corollary 6.3 are not squashable. In particular, piecewise absolutely continuous cocycles with a nonzero sum of the jumps are not squashable.

3. Using our result on the speed in Koksma's inequality (see Corollary 3.3) and the technique from [Pa2], we can slightly improve the main result of that paper by requiring that the functions from this paper can be modified by those whose Fourier coefficients are of order $o(\frac{1}{n^t})$ with an additionally remark that all those cocycles are not squashable.

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