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ERGODIC COMPACT ABELIAN GROUP EXTENSIONS OF ROTATIONS

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z podziękowaniami
za wiele życzliwości
i pomocy
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To Barbara

PREFACE. The main subject of this work is a description of ergodic properties of some group extensions of rotations. The questions we have in mind are connected with the weak isomorphism problem and the Banach problem. The material presented here is based on some papers written or coauthored by the author in 1987-1988. Many problems considered here have been raised in discussions with J.P. Thouvenot to whom I express many thanks. Also, it is my great pleasure to thank J. Kwiatkowski, P. Liardet, M.K. Mentzen and D. Rudolph for many stimulating discussions and collaboration. My special thanks to A. Iwanik who read the manuscript and made several useful remarks. I would like to express my gratitude to J.P. Allouche, S. Ferenczi, P. Gabriel, G.R. Goodson, B. Host, A. del Junco, T. Kamae, B. Kamiński, M. Keane, J. King, Z. Kowalski, F. Ledrappier, M. Misiurewicz, J.F. Méla, M. Mendès-France, F. Parreau, K. Petersen, F. Przytycki, M. Queffelec, M. Urbański, A.M. Vershik. Their interest in my work was of unending encouragement.

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CONTENTS

INTRODUCTION. Let $T:(X,\mathcal{B},\mu)\circlearrowleft$ be an ergodic automorphism of a standard Borel space. Then the following theorem holds.

Structure Theorem. <u>There is an ordinal η and an increasing systems of factors</u> $\mathcal{B}_\xi \subseteq \mathcal{B}$ ($\xi \le \eta$) <u>such that</u>

(i) $\mathcal{B}_0=\{\emptyset,X\}$, $\mathcal{B}_\eta=\mathcal{B}$,

(ii) <u>for each</u> ξ, $\xi+1<\eta$, <u>the system</u> $T:(X,\mathcal{B}_{\xi+1},\mu)\circlearrowleft$ <u>is a compact extension of</u> $T:(X,\mathcal{B}_\xi,\mu)\circlearrowleft$,

(iii) <u>if</u> ξ, $\xi\le\eta$, <u>is a limit ordinal then</u> $T:(X,\mathcal{B}_\xi,\mu)\circlearrowleft$ <u>is the inverse limit of the systems</u> $T:(X,\mathcal{B}_{\xi'},\mu)\circlearrowleft$, $\xi'<\xi$,

(iv) <u>if</u> $\eta=\eta'+1$ <u>then</u> $T:(X,\mathcal{B}_\eta,\mu)\circlearrowleft$ <u>is either a relatively weakly mixing extension or a compact extension of</u> $T:(X,\mathcal{B}_{\eta'},\mu)\circlearrowleft$.

(This is the Z-action version of Theorem 6.16 [11]; see also [61], [70]). In this work we will be concerned with dynamical systems for which only compact extensions occur in Structure Theorem. If $\eta=1$ then $\mathcal{B}_\eta$ is not very interesting from our point of view since measure-theoretic problems are in fact spectral. But if we consider $\eta=2$ then we obtain isometric extensions of rotations ([61]) and the situation is quite different. Many interesting problems arise and can be solved in this latter class. From this point of view this work can be viewed as the first interesting step in the transfinite chain. All the systems obtained in this way are necessarily of zero entropy.

We will deal with ergodic compact group extensions of rotations, i.e. with the following situation. $T:(X,\mathcal{B},\mu)\circlearrowleft$ is an ergodic rotation on a compact monothetic group X and $\varphi : X \longrightarrow G$ is an ergodic cocycle, which means that the automorphism $T_\varphi : (X \times G, \tilde{\mathcal{B}}, \tilde{\mu}) \circlearrowleft$ defined by

(1) $\quad T_\varphi(x,g) = (Tx, \varphi(x)g) \quad , \quad (x \in X, g \in G)$

is ergodic. Here G is a compact metric group with Haar measure μ_G and $\tilde{\mu} = \mu \times \mu_G$. Assume that $\tau : (Y, \mathcal{C}, \nu) \circlearrowleft$ is an ergodic automorphism and $\mathcal{C}_1 \subsetneq \mathcal{C}$ is a τ-invariant sub-σ-algebra (factor). Then the following are known to be equivalent ([20], [65]).

(2)

(i) $\tau : (Y, \mathcal{C}, \nu) \circlearrowleft$ <u>is a compact group extension of</u> $\mathcal{C}_1$.

(ii) <u>The set</u> $\mathcal{H}$ <u>of all</u> $S : (Y, \mathcal{C}, \nu) \circlearrowleft$ <u>satisfying</u> $S\tau = \tau S$ <u>and</u> $S^{-1}A = A$ <u>for each</u> $A \in \mathcal{C}_1$ <u>is a compact group</u> (<u>with respect to weak topology</u>) <u>and</u> $\mathcal{C}_1 = \{A \in \mathcal{C} : S^{-1}A = A$ <u>for each</u> $S \in \mathcal{H}\}$.

(iii) <u>The relatively independent extension of the diagonal measure on</u> $\mathcal{C}_1 \otimes \mathcal{C}_1$ <u>has ergodic decomposition consisting solely of graph measures</u>.

In the first part we will work on the coalescence and the weak isomorphism problems. They seem to belong to the most complicated problems in abstract ergodic theory. The question of whether there are two ergodic automorphisms that are weakly isomorphic but not isomorphic was asked by Sinai [60] in 1963. Although some solutions are known in the weak mixing case, our approach offers some new methods in ergodic theory : the constructions in [57] are based on the minimal self-joining property and those in [62] on properties of Kronecker-Gaussian automorphisms, while our constructions exploit properties of cocycles. Also, the known constructions ([57], [62]) seem not to have the important loosely Bernoulli (LB) property. We recall that an ergodic zero entropy automorphism is LB iff it is induced from an irrational rotation ([48]). In view of [48], all the automorphisms studied in this work will be LB.

Now, we will pass to a more detailed description of the results.

In Section 1 we deal with joinings of ergodic abelian group extensions of a discrete spectrum automorphism. The problem of how two automorphisms can be joined is quite general, certainly more general than the isomorphism problem or the disjointness problem. For some results in this direction we refer to [10], [13], [20]. In our class, the ergodic joinings turn out to be rather natural: in fact every such a joining is simply the relatively independent extension of an isomorphism between two special "natural" factors. We note in passing that our method gives a new "joining" proof of the classical result that two ergodic automorphisms with pure point spectrum are isomorphic whenever they have the same group of eigenvalues.

In Section 2 we describe the structure of invariant sub-σ-algebras for $T_{\varphi}:(X\times G,\tilde{\mathcal{B}},\tilde{\mu})\circlearrowleft$. We prove that the following Veech correspondence holds. If $\mathcal{C}\subseteq\tilde{\mathcal{B}}$ is a factor of $T_{\varphi}:(X\times G,\tilde{\mu})\circlearrowleft$ then there exist a natural factor $T_{\varphi,H}:(X\times G/H,\tilde{\mu})\circlearrowleft$ and a compact subgroup $\mathcal{H}_{\varphi,H}(\mathcal{C})$ consisting of exactly those transformations in the centralizer $C(T_{\varphi,H})$ that fix every element in $\mathcal{C}$. Moreover, if a measurable set A is fixed by $\mathcal{H}_{\varphi,H}(\mathcal{C})$ then necessarily $A\in\mathcal{C}$. An unpleasent phenomenon we have to overcome is that $C(T_{\varphi})$ need not be a group. In spite of this $\mathcal{H}_{\varphi,H}(\mathcal{E})$ turns out to be a group. This differs from the situation in the class of 2-fold simple automorphisms considered in [20],[65] where the Veech correspondence holds but all automorphisms under considerations as well as their factors, are coalescent. We then pass to describe how the isomorphic factors are organized. We show that if $\mathcal{E}_1, \mathcal{E}_2 \subseteq \tilde{\mathcal{B}}$ are two isomorphic factors then there exist two natural factors $T_{\varphi,H_i}:(X \times G/H_i,\tilde{\mu})\circlearrowleft$ containing $\mathcal{E}_i$ respectively, i=1,2, such that $\mathcal{H}_{\varphi,H_1}(\mathcal{E}_1)$ and $\mathcal{H}_{\varphi,H_2}(\mathcal{E}_2)$ are conjugate. The conjugation is given by an isomorphism of these natural factors extending an original isomorphism of $\mathcal{E}_1$ and $\mathcal{E}_2$. This will allow us to prove that if $T_{\varphi}:(X \times G,\tilde{\mathcal{B}},\tilde{\mu})\circlearrowleft$ as well as all its natural factors are coalescent, then all factors of T_{φ} are coalescent. Hence, we obtain an "almost" positive answer to D. Newton question asked in 1971 [44] of whether the class of coalescent automorphisms is closed under taking factors (we cannot expect more in this direction: in Section 5 we deliver an example of an ergodic coalescent $T_{\varphi}:(X \times G,\tilde{\mu})\circlearrowleft$ with a noncoalescent natural factor). D. Newton question is in turn connected with the open problem of whether coalescence implies zero entropy.

In Section 3 we take up the coalescence problem for ergodic group extensions of rotations. If in (1) the group G is finite then we obtain coalescent automorphisms. Another case of special interest is when G is the circle. In 1970, W. Parry and P. Walters ([53]) proved the coalescence property in topological setting for continuous cocycles. The main aim of Section 3 is to disprove a conjecture that has been brought to the author's attention by B. Kamiński and J.P. Thouvenot. They conjectured that each ergodic Anzai skew product (i.e. an ergodic circle extension of an irrational rotation) was coalescent. We deliver a counterexample. This counterexample is also interesting for some other reasons because it does not involve the technique of infinite self-joinings, unlike other related constructions in [57],[62] as well as in Section 5 and 6 below.

In Section 4 we exhibit Anzai skew products which are on the coalescence side. We consider the following cases,

(i) two-valued step cocycles over an irrational rotation by α, where α is well approximated by rationals,

(ii) two-valued step cocycles over an irrational rotation by α, where α has bounded partial quotients,

(iii) uniformly Lipschitz continuous cocycles with nonzero topological degree.

We prove that all ergodic Anzai skew products given by (i)-(iii) as well as all their factors are coalescent.

Section 5 is devoted to present a new (and LB) construction of weakly isomorphic automorphisms that are not isomorphic. We briefly describe our approach. Given an ergodic rotation T we look for an ergodic Z_2-cocycle φ and $S \in C(T)$ such that all Z_2-cocycles of the form

$$(3) \begin{cases} \varphi S^{i_1} + \ldots + \varphi S^{i_k} + \varphi U, \ i_1 < \ldots < i_k, \ k \geq 2, \ U \in C(T) \\ \text{are ergodic.} \end{cases}$$

This allows us to solve the isomorphism problem for certain infinite self-joinings of T_φ. Then, we indicate those infinite ergodic self-joinings of T_φ that are weakly isomorphic but not isomorphic. Other applications are also provided. We construct an ergodic coalescent group extension of a rotation with a noncoalescent factor, hence giving negative answer to the D. Newton question [44]. We show that in general compact rank property of an ergodic automorphism does not imply that it is coalescent (J.P. Thouvenot question). Then we exhibit T,φ and S satisfying (3). It turns out that many well-known automorphisms enjoy this property. For instance, we show that if φ is a cocycle generated by a substitution ([6],[7]) then the set of S's for which (3) holds is even of Haar measure 1.

In Section 6 we consider problems similar as in Section 5, but with $G=S^1$ (the circle group) instead of $G=Z_2$. In this case, it is well-known ([28],[58]) that each cocycle can be modified by a coboundary to get a continuous one. This will allow us to extend results of Section 5 to the strictly ergodic homeomorphism case. We construct two strictly ergodic homeomorphisms on infinite dimensional torus that are topological factors of each other but are not measure-theoretically isomorphic. These examples improve a construction of a topological noncoalescent minimal skew product on infinite dimensional torus obtained in [53]. Also, we consider the question of whether each ergodic automorphism with partly continuous spectrum has a noncoalescent ergodic infinite self-joining (a specialized version of this question was raised to the author by J.P. Thouvenot). We prove that in general the answer is no even in the case of Anzai skew products.

In Part II we are mainly concerned with spectral problems of Z_2-extensions of special rotations called adding machines.

We consider these investigations a possible step towards a solution of the Banach problem. We recall that the still open Banach problem is the question of whether there exists an ergodic automorphism with Lebesgue spectrum of multiplicity 1. In 1978 H. Helson and W. Parry [17] asked whether it was possible to find an ergodic automorphism with Lebesgue component of finite multiplicity. The positive answer to this latter question has been found by several authors ([2],[22],[32], [41],[50]) for the multiplicity equal to 2. Later, J. Mathew and M.G. Nadkarni generalized their original construction of [41] to the even Lebesgue multiplicities of the form $n\varphi(n)$ where φ is the Euler function [42]. Our aim is to deliver Z_2-extensions of adding machines with Lebesgue component with an arbitrary even multiplicity.

In Section 7 we prove that if a Z_2-cocycle admits a sufficiently high speed of approximation then it is cohomologous to a Morse cocycle. This will allow us to conclude that an arbitrary ergodic Z_2-cocycle is cohomologous to the sum of two Morse cocycles. This statement gives rise to a striking conclusion that the (possibly Lebesgue) spectral type and the (possibly infinite) multiplicity function of each ergodic Z_2-cocycle must be the same as those of a factor of an ergodic joining of two Morse cocycles. These latter cocycles however are well-konown to have simple singular spectra ([25]).

In Section 8 we exhibit a relationship between the Z_2-cocycles and some classical objects considered in combinatorial ergodic theory such as Toeplitz sequences [19],[68], Morse sequences [24], [30], [31], automatic sequences and substitutions [6], [7], [18], [36], [50]. We also develop some methods how to calculate the maximal spectral type and the multiplicity function.

In Section 8 we provide examples of Z_2-extensions with Lebesgue spectrum (in the orthocomplement of the space of eigenfunctions) of an arbitrary even uniform multiplicity. Independently of the author, O.N. Ageev [2] has also constructed Z_2-extensions of a rank 1 automorphism with an arbitrary even multiplicity of the Lebesgue component.

0. BASIC DEFINITIONS, NOTATIONS AND SOME RESULTS. Let $(X, \mathcal{B})$ be a measurable space. We say that $(X, \mathcal{B})$ is <u>standard Borel</u> if there exist a compact metric space Y with Borel σ-algebra $\mathcal{C}$ and an invertible map $\iota : X \longrightarrow Y$ such that ι is $\mathcal{B}-\mathcal{C}$, measurable and ι^{-1} is $\mathcal{C}-\mathcal{B}$ measurable. Suppose that $(X, \mathcal{B})$ is equipped with a probability measure μ. By a <u>transformation</u> $T:(X, \mathcal{B}, \mu)\circlearrowleft$ we mean a measurable and μ-preserving map from X onto X. If $T:X\circlearrowleft$ is an invertible, bimeasurable transformation then T is called an <u>automorphism</u> of $(X, \mathcal{B}, \mu)$.

Remark 0.1. Wherever no confusion can arise we will abbreviate $T:(X, \mathcal{B}, \mu)\circlearrowleft$ to $T:(X,\mu)\circlearrowleft$ or to T or even to $\mathcal{B}$. All statements contain a tacit "a.e." where appropriate. In particular, all equalities between sets, functions, transformations and σ-algebras are understood modulo null sets■

Unless explicitly indicated, we will restrict our attention to <u>ergodic</u> automorphisms, this meaning that the only T-invariant sets are those of measure 0 or 1. An automorphism T will be called <u>totally ergodic</u> if T^n is ergodic for each integer $n\neq 0$. Any automorphism T induces a unitary operator U_T, $U_T f = fT$, of the Hilbert space $L^2(X,\mu)$ of all square integrable functions on X. By Sp(T), we will denote the set of all eigenvalues of U_T. Let $L^2_0(X,\mu)$ be the orthocomplement of the constants. By spectral properties of T we mean spectral properties of U_T restricted to $L^2_0(X,\mu)$. As U_T is a unitary operator we will need a spectral theorem.

Let U be a unitary operator on a separable Hilbert space H.

For any $f \in H$ we define the <u>cyclic space</u>

$$Z(f) = \text{span}\{U^n f : n \in Z\}.$$

By the <u>spectral measure</u> σ_f of f we mean a Borel measure on S^1 determined by the equalities $\int_{S^1} z^n d\,\sigma_f(z) = (U^n f, f)$, $n \in Z$.

S p e c t r a l   T h e o r e m (see for instance [29],[52]). <u>There exists a sequence</u> $f_1, f_2, \ldots$ <u>in</u> H <u>such that</u>

(4) $\sigma_{f_1} \gg \sigma_{f_2} \gg \ldots$ <u>and</u> $H = \bigoplus_{n=1}^{\infty} Z(f_n)$.

<u>Moreover, for any sequence</u> $f_1', f_2', \ldots$ <u>in</u> H <u>satisfying</u> (4) <u>we have</u> $\sigma_{f_1} \equiv \sigma_{f_1'}$, $\sigma_{f_2} \equiv \sigma_{f_2'}, \ldots$ ∎

The spectral type of σ_{f_1} (the equivalence class of measures) will be denoted by σ_U and called the <u>maximal spectral type</u> of U. By the <u>multiplicity function</u> M_U of U we mean the function $M_U : S^1 \longrightarrow N \cup \{+\infty\}$ given by $M_U(z) = \sum_{n=1}^{\infty} \chi_{A_n}(z)$ where $A_n = \{z \in S^1 : d\,\sigma_{f_n}/d\,\sigma_{f_1}(z) > 0\}$, $n \geq 1$ (M_U is defined σ_U-a.e.). The esssupM_U (with respect to σ_U) will be called the <u>maximal spectral multiplicity of</u> U. The multiplicity is <u>uniform</u> if $M_U(z) = k$ ($k \in N \cup \{+\infty\}$) σ_U-a.e. . If this is the case and if $k=1$ then we say that U has <u>simple spectrum</u>. U is said to have <u>Lebesgue spectrum</u> if $\sigma_{f_1} \equiv \lambda$, where λ is Lebesgue measure on the circle, and to have <u>singular spectrum</u> if $\sigma_{f_1} \perp \lambda$. The following theorem giving an estimation of spectral multiplicity is due to Chacon [5] (see also [29] p. 395).

S p e c t r a l   M u l t i p l i c i t y   T h e o r e m. <u>The maximal spectral multiplicity of</u> U <u>is upper bounded by the supremum of those natural numbers</u> K <u>for which we can find orthonormal vectors</u> $f_1, f_2, \ldots, f_K \in H$ <u>such that for each</u> $g \in H$ <u>and any</u> $g_i \in Z(g)$

(5) $$\sum_{i=1}^{K} \|f_i - g_i\|^2 \geq K-1$$ ∎

Two automorphisms T_i, $i=1,2$, are said to be spectrally disjoint ([13]) provided that $\sigma_{T_1} \perp \sigma_{T_2}$ (recall that $\sigma_{T_i} = \sigma_{U_{T_i}|L_0^2(X,\mu)}$) We say that T_1 and T_2 are spectrally isomorphic if there is an isometry V from $L_0^2(X_1,\mu_1)$ onto $L_0^2(X_2,\mu_2)$ such that $VU_{T_1}=U_{T_2}V$. Two automorphisms are spectrally isomorphic iff they have the same maximal spectral type and the same multiplicity function.

Given $T:(X,\mu)\circlearrowleft$ we define the centralizer $C(T)$ of T to be the set of all transformations S of (X,μ) such that $ST=TS$. Immediately from the definition it follows that $C(T)$ is a semigroup. If $C(T)$ is a group then following [13],[44] the automorphism T is called coalescent.

Proposition 0.1. ([44]) *If T is ergodic and $\operatorname{esssup} M_T < +\infty$, then T is coalescent* ■

We will define below a class of automorphisms whose maximal spectral multiplicity is finite, hence coalescent automorphisms. By a partition of X we mean any finite collection of pairwise disjoint measurable sets of X whose union covers X. We recall that a sequence of partitions $\{P_n\}$ of X is said to converge to $\mathcal{B}$ whenever for each set $A\in\mathcal{B}$ and $\varepsilon>0$ there exists n_0 such that for all $n\geq n_0$ there is a set A_n, a union of atoms of P_n, for which

(6) $$\mu(A\triangle A_n)<\varepsilon .$$

An automorphism $T:(X,\mathcal{B},\mu)\circlearrowleft$ is said to have rank at most r ([48]) whenever there exist sets F_k^j, $1\leq j\leq r$, $k=1,2,\dots$ and positive integers n_k^j, $1\leq j\leq r$, $k=1,2,\dots$ such that for a fixed k the sets $\{T^iF_k^j\}_{i=0}^{n_k^j-1}$, $1\leq j\leq r$ are pairwise disjoint and the partitions $P_k=\{T^iF_k^j:\ i=0,\dots,n_k^j-1,\ j=1,\dots,r\}\ \cup\ \{X_k\}$ where $X_k=X\setminus\bigcup_{j=1}^{r}\bigcup_{i=0}^{n_k^j-1}T^iF_k^j$, converge to $\mathcal{B}$. The <u>rank</u> of T is r provided that T has the rank at most r but not at most r-1.

P r o p o s i t i o n 0 . 2 . ([5]) *If $T:(X,\mu)\circlearrowleft$ is ergodic and has rank r then the maximal spectral multiplicity of T is not greater than r*∎

The centralizer C(T) is endowed with <u>weak topology</u> ([14]) given by

(7) $S_n\longrightarrow S$ if $\mu(S_n^{-1}A\,\Delta\,S^{-1}A)\longrightarrow 0$ for each $A\in\mathcal{B}$.

With this topology C(T) becomes a Polish metric semigroup ([14]). We say that T is <u>rigid</u> if there is a sequence $n_t\nearrow+\infty$ such that $T^{n_t}\longrightarrow Id$. If this is the case then C(T) is uncountable as then C(T) is a perfect set (and it has the Baire property).

Given two automorphisms $T_i:(X_i,\mathcal{B}_i,\mu_i)\circlearrowleft$ (i=1,2) and a measure-preserving transformation $f:(X_1,\mathcal{B}_1,\mu_1)\to(X_2,\mathcal{B}_2,\mu_2)$ satisfying $fT_1=T_2f$ we say that T_2 is a <u>factor</u> of T_1 or that T_1 is an <u>extension</u> of T_2. If, besides, T_1 is a factor of T_2 then we say that T_1 and T_2 are <u>weakly isomorphic</u> ([60]). Note that if T_1 and T_2 are weakly isomorphic then they have the same maximal spectral type and the same multiplicity function, hence T_1 and T_2 are spectrally isomorphic. If the map f is invertible then T_1 and T_2 are said to be <u>isomorphic</u>. Note that if T_1 is coalescent and T_2 is weakly isomorphic with T_1 then in fact T_2 is isomorphic with T_1.

Remark 0.2. If f establishes a factor map from T_1 to T_2 then $\mathcal{E} = f^{-1}(\mathcal{B}_2)$ is a T_1-invariant sub-σ-algebra of $\mathcal{B}_1$. Let $\bar{X}_1$ be the quotient space obtained by identification of those points of X_1 that cannot be separated by $\mathcal{E}$. By some abuse of notations we use $T_1:(\bar{X}_1, \mathcal{E}, \mu_1)\circlearrowleft$ to denote the action of T_1 on the quotient space $\bar{X}_1$. Now, $T_1:(\bar{X}_1, \mathcal{E}, \mu_1)\circlearrowleft$ is isomorphic to $T_2:(X_2, \mathcal{B}_2, \mu_2)\circlearrowleft$. Consequently, any factor of T_1 can be identified with a T_1-invariant sub-σ-algebra of $\mathcal{B}_1$ (with the action of T_1 on the quotient space)$\blacksquare$

Note that an automorphism $T:(X, \mathcal{B}, \mu)\circlearrowleft$ is coalescent iff for each proper factor $\mathcal{E} \subsetneq \mathcal{B}$ the action of T on $\mathcal{E}$ is not isomorphic to the action of T on $\mathcal{B}$. Given $T:(X, \mathcal{B}, \mu)\circlearrowleft$ and a system of factors $\mathcal{E}_n \subseteq \mathcal{B}$, n=1,2,... we say that T is the <u>inverse limit</u> of the factors $\mathcal{E}_n$ whenever $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \dots$ and $\mathcal{E}_n$ converges to $\mathcal{B}$. This latest condition, as in case of partitions, means that for each $A \in \mathcal{B}$ and $\varepsilon > 0$ we can find n_0 such that for each $n \geq n_0$ there is a set $A_n \in \mathcal{E}_n$ such that (6) holds. Following [45], a factor $\mathcal{E}$ of an ergodic automorphism $T:(X, \mathcal{B}, \mu)\circlearrowleft$ is called <u>canonical</u> if for each factor $\mathcal{E}' \subseteq \mathcal{B}$ such that the actions of T on $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic it follows that $\mathcal{E} = \mathcal{E}'$. If this is the case then the factor $\mathcal{E}$ is coalescent. An ergodic automorphism is said to be a <u>canonical system</u> ([45]) whenever it is a canonical factor of each its ergodic extension. An automorphism $T:(X, \mu)\circlearrowleft$ is said to have <u>discrete spectrum</u> (pure point spectrum) if $L_0^2(X, \mu)$ is spanned by the eigenfunctions of T. Since for the ergodic case all eigenfunctions have multiplicity 1, the following holds.

Proposition 0.3. ([45]) Each ergodic automorphism with discrete spectrum is a canonical system∎

From the classical Halmos-von Neumann results ([15], also [29] p. 328) it follows that if an automorphism has discrete spectrum and is ergodic then it is isomorphic to a rotation $T:(X,\mathcal{B},\mu)\circlearrowleft$ where X is a compact monothetic group, μ is Haar measure and $T(x)=x+x_0$, where x_0 is a topological cyclic generator of X. Then

(8) $$C(T)=\{\ \sigma_y:\ y\in X\}\quad,\ \sigma_y(x)=x+y,$$

since each transformation commuting with an ergodic rotation must also be a rotation.

We recall that $T:(X,\mathcal{B},\mu)\circlearrowleft$ is weakly mixing if for any pair of sets $A,B\in\mathcal{B}$

$$\lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1}|\mu(T^{-n}A\cap B)-\mu(A)\mu(B)|=0.$$

This is equivalent to saying that for each $f\in L_0^2(X,\mu)$ $\lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1}|(U_T^nf,f)|=0$ or what is the same that the spectral measure of f is continuous.

We now pass to the problem of a relativization of the notions of discrete spectrum and weak mixing. Let $\mathcal{E}\subseteq\mathcal{B}$ be a factor of $T:(X,\mathcal{B},\mu)\circlearrowleft$ with the factor map $f:(X,\mathcal{B},\mu)\longrightarrow(\bar{X},\mathcal{E},\mu)$. Then, following [11] p.108, there is a system of measures

(9) $$(\mu_{\bar{x}})_{\bar{x}\in\bar{X}}$$

where $\mu_{\bar{x}}$ is a probability measure on $(X,\mathcal{B})$ concentrated on $f^{-1}(\bar{x})$. Moreover, for each $g\in L^1(X,\mathcal{B},\mu)$ we have

(9i) $g\in L^1(X,\mathcal{B},\mu_{\bar{x}})$,

(9ii) $\int_X g\,d\mu=\int_{\bar{X}}(\int_X g\,d\mu_{\bar{x}})\,d\mu(\bar{x})$,

(9iii) $E(g|\mathcal{E})(\bar{x})=\int_X g\,d\mu_{\bar{x}}$.

Here $E(g|\mathcal{E})$ denotes the conditional expectation of $g\in L^1(X,\mathcal{B},\mu)$ with respect to $\mathcal{E}$. Below, we list some basic properties of the operator $g\longmapsto E(g|\mathcal{E})$.

Proposition 0.4. ([11] pp. 104-107, [66] p. 9)
If $g\in L^1(X,\mathcal{B},\mu)$ then

(i) $g\longmapsto E(g|\mathcal{E})$ is a linear operator,

(ii) If $g\geq 0$ then $E(g|\mathcal{E})\geq 0$,

(iii) If $g\in L^1(\overline{X},\mathcal{E},\mu)$ then $E(g|\mathcal{E})=g$,

(iv) If $h\in L^\infty(\overline{X},\mathcal{E},\mu)$ then $E(hg|\mathcal{E})=hE(g|\mathcal{E})$,

in particular, $\int_X g\,d\mu=\int_{\overline{X}} E(g|\mathcal{E})\,d\mu$,

(v) $E(gT|\mathcal{E})=E(g|\mathcal{E})T$ ∎

We say that $T:(X,\mathcal{B},\mu)\circlearrowleft$ has relatively discrete spectrum with respect to to a factor $\mathcal{E}\subseteq\mathcal{B}$ (or that T is a compact extension of $\mathcal{E}$), if there is a dense set $\mathcal{F}\subseteq L_0^2(X,\mu)$ such that if $g\in\mathcal{F}$, $\delta>0$ then there exist $g_1,\dots,g_K\in L_0^2(X,\mu)$ such that for each $n\in Z$

$$\min_{1\leq j\leq K}||gT^n-g_j||_{\overline{x}}<\delta \qquad \text{for a.e. } \overline{x}\in\overline{X}$$

($||\cdot||_{\overline{x}}$ denotes the $L^2(X,\mathcal{B},\mu_{\overline{x}})$ norm). For some reformulations of this notion we refer to [11] p.131, [61],[69],[70].

Let $T:(X,\mathcal{B},\mu)\circlearrowleft$ be an automorphism and $\mathcal{E}\subseteq\mathcal{B}$ be its factor. Let $f:(X,\mathcal{B},\mu)\longrightarrow(\overline{X},\mathcal{E},\mu)$ denote the factor map. A measure λ on $(X\times X,\mathcal{B}\otimes\mathcal{B})$ is called the relative product measure (with respect to $\mathcal{E}$) provided that

$$\lambda(A)=\int_{\overline{X}}(\mu_{\overline{x}}\times\mu_{\overline{x}})(A)\,d\mu(\overline{x}),$$

where $(\mu_{\overline{x}})_{\overline{x}\in\overline{X}}$ is the decomposition (9) of μ with respect to $\mathcal{E}$. Hence, by (9iii), for $A=B_1\times B_2$

$$\lambda(A)=\int_{\overline{X}}E(B_1|\mathcal{E})(\overline{x})E(B_2|\mathcal{E})(\overline{x})\,d\mu(\overline{x}).$$

The relative product measure will be denoted by $\mu\times_{\mathcal{E}}\mu$. If

$X \times_{\mathcal{E}} X = \{ (x,y) \in X \times X: f(x)=f(y) \}$ denotes the fibered product of two copies of X over $\bar{X}$ then $\mu \times_{\mathcal{E}} \mu(X \times_{\mathcal{E}} X)=1$. An automorphism $T:(X,\mathcal{B},\mu)\circlearrowleft$ is called the <u>relatively weakly mixing extension</u> (with respect to $\mathcal{E}$) if $\mu \times_{\mathcal{E}} \mu$ is a $T\times T$-ergodic measure on $(X\times X, \mathcal{B}\otimes\mathcal{B})$. For more information on these relativization concepts we refer to [11].

In some sense the notions of the centralizer, the factor and the relative product measure can be put together. This more general concept is the one of a <u>joining</u> between automorphisms. Given $T_i:(X_i,\mathcal{B}_i,\mu_i)\circlearrowleft$ ($i=1,\dots,n$, $n\in N\cup\{+\infty\}$) by $J(T_1,\dots,T_n)$ we denote the set of all joinings of $T_1,\dots,T_n$, i.e

$\lambda \in J(T_1,\dots,T_n)$ whenever λ is a $T_1\times\dots\times T_n$-invariant probability measure on $\mathcal{B}_1\otimes\dots\otimes\mathcal{B}_n$ such that

$\lambda(X_1\times\dots\times X_{i-1}\times A\times X_{i+1}\times\dots\times X_n)=\mu_i(A)$ for each $A \in \mathcal{B}_i$.

By $J^e(T_1,\dots,T_n)$ we mean the set of all ergodic joinings of $T_1,\dots,T_n$. If $\lambda \in J(T_1,\dots,T_n)$ and

$$\lambda = \int_{E(T_1,\dots,T_n)} \gamma \, d\nu(\gamma)$$

is its ergodic decomposition (see e.g. [20]) where $E(T_1,\dots,T_n)$ is the set of all $T_1\times\dots\times T_n$-ergodic measures (not necessarily with "right" marginals) then $\gamma \in J^e(T_1,\dots,T_n)$ for ν-a.a. γ . This is quite meaningful and gives rise to the conclusion that $J^e(T_1,\dots,T_n)$ is nonempty as always $\mu_1\times\dots\times\mu_n \in J(T_1,\dots,T_n)$. Following [10], T_1 and T_2 are said to be <u>disjoint</u> ($T_1 \perp T_2$) provided that $J^e(T_1,T_2)=J(T_1,T_2)=\{\mu_1\times\mu_2\}$. By a result of [13] if T_1 and T_2 are spectrally disjoint then $T_1 \perp T_2$. If $T_1=\dots=T_n=T$ then joinings are called <u>self-joinings</u> of T with the following notations $J_n^e(T), J_n(T)$ instead of $J^e(T,\dots,T)$, $J(T,\dots,T)$, respectively. Assume that

$T:(X,\mathcal{B},\mu)\circlearrowright$ is ergodic and let $S\in C(T)$. Then we can raise S to a self-joining of T by

(10) $\mu_S(A\times B)=\mu_{Id,S}(A\times B)=\mu(A\cap S^{-1}B)$.

More generally, if $S_1,\dots,S_n\in C(T)$, $2\leq n\leq+\infty$ then

(11) $\mu_{S_1,\dots,S_n}(A_1\times\dots\times A_n)=\mu(S_1^{-1}A_1\cap\dots\cap S_n^{-1}A_n)$

is a well-defined ergodic self-joining of T. Moreover, $T\times\dots\times T:(X\times\dots\times X,\mu_{S_1,\dots,S_n})\circlearrowright$ is isomorphic to $T:(X,\mu)\circlearrowright$. The joinings of the form (10) and (11) are called <u>graph joinings</u>. We say that a graph joining (11) is <u>invertible</u> as soon as $S_1,\dots,S_n$ are invertible. We can easily adapt the notion of a graph joining to the case of factor maps $f_i:(X_1,\mathcal{B}_1,\mu_1)\longrightarrow(X_i,\mathcal{B}_i,\mu_i)$, $i=1,\dots,n$, $f_iT_1=T_if_i$, where an application of (11) gives us an ergodic joining of $T_1,\dots,T_n$. If $\mathcal{E}_1,\mathcal{E}_2$ are two factors of $T:(X,\mathcal{B},\mu)\circlearrowright$ and if they are isomorphic, where the isomorphism map is $S:(\bar{X}_1,\mathcal{E}_1,\mu)\longrightarrow(\bar{X}_2,\mathcal{E}_2,\mu)$, then we can lift S to a self-joining λ of T by

(12) $\lambda(A\times B)=\int_{\bar{X}_1}E(A|\mathcal{E}_1)(\bar{x})E(B|\mathcal{E}_2)(S\bar{x})\,d\mu(\bar{x})$

($\bar{X}_i$ is the quotient corresponding to $\mathcal{E}_i$, $i=1,2$). Such a self-joining is called the <u>relatively independent extension of the isomorphism</u> S. Note that λ need not be ergodic. In particular, when $\mathcal{E}=\mathcal{E}_1=\mathcal{E}_2$, S=Id the corresponding λ is called the <u>relatively independent extension of the diagonal measure on</u> $\mathcal{E}\otimes\mathcal{E}$ and is denoted by $\mu\times_{\mathcal{E}}\mu$. This self-joining need not be ergodic: in fact it is ergodic iff $T:(X,\mathcal{B},\mu)\circlearrowright$ is the relatively weakly mixing extension of $\mathcal{E}$. More generally if λ is an n-self-joining of $T:(\bar{X},\mathcal{E},\mu)\circlearrowright$ then the measure $\hat{\lambda}$ given by

(13) $\hat{\lambda}(A_1\times\dots\times A_n)=\int_{\bar{X}\times\dots\times\bar{X}}\dots\int E(A_1|\mathcal{E})(\bar{x}_1)\dots E(A_n|\mathcal{E})(\bar{x}_n)\,d\lambda(\bar{x}_1,\dots,\bar{x}_n)$

or in case $n=+\infty$

(13a) $\hat{\lambda}(A_1\times\ldots\times A_k\times X\times X\times\ldots)=\int_{\tilde{X}^N} E(A_1|\mathcal{C})(\bar{x}_1)\ldots E(A_k|\mathcal{C})(\bar{x}_k)d\lambda(\bar{x}_1,\ldots,\bar{x}_k,\ldots)$

is called the <u>relatively independent extension of</u> λ and $\hat{\lambda}\in J_n(T)$.

Let us assume that $T:(X,\mathcal{B},\mu)\circlearrowleft$ is an ergodic automorphism with discrete spectrum, so T is an ergodic rotation on a compact monothetic group X with Haar measure μ. Let G be a compact abelian metric group with Haar measure ν. By a <u>cocycle</u> we mean a measurable map $\varphi:X\longrightarrow G$. Then, by Fubini theorem

(14) $T_\varphi:(X\times G,\tilde{\mu})\circlearrowleft$, $T_\varphi(x,g)=(Tx,\varphi(x)g)$

preserves the product measure $\tilde{\mu}=\mu\times\nu$. The automorphism given by (14) is called a G-<u>extension of</u> T. A cocycle $\varphi:X\longrightarrow G$ is called G-<u>coboundary</u> if there exists a measurable $f:X\longrightarrow G$ such that $\varphi(x)=f(Tx)/f(x)$. An S^1-coboundary cocycle will be called <u>coboundary</u>. Two G-cocycles are said to be <u>G-cohomologous</u> (<u>cohomologous</u>) if their quotient is G-coboundary (coboundary). In general a G-extension T_φ of T need not be ergodic. The following is classical.

(15) T_φ <u>is ergodic</u> iff <u>for each nontrivial character</u> $\chi\in\hat{G}$ <u>the</u> S^1<u>-cocycle</u> $\chi\varphi$ <u>is not coboundary</u>.

R e m a r k 0 . 3 . We will say that φ is <u>ergodic</u> whenever T_φ is ergodic, an ergodic φ is <u>weakly mixing</u> if $Sp(T_\varphi)=Sp(T)$ (i.e. if the maximal spectral type of T_φ in the orthocomplement of the eigenfunctions of T is continuous) and that φ is <u>rigid</u> whenever T_φ is rigid∎

Let $T_\varphi:(X\times G,\tilde{\mathcal{B}},\tilde{\mu})\circlearrowleft$ be an ergodic group extension of $T:(X,\mathcal{B},\mu)\circlearrowleft$. Then T is a factor of T_φ represented by the sub-σ-algebra $\{A\times G:\ A\in\mathcal{B}\}\subseteq\tilde{\mathcal{B}}$. By some abuse of notations we will use $\mathcal{B}$ to denote the factor. Then T is a canonical factor of T_{φ} as T itself is a canonical system.

Given a measurable function $f:X\longrightarrow G$, a continuous group epimorphism $v:G\circlearrowleft$ and $S\in C(T)$ we put

(16) $$S_{f,v}(x,g)=(Sx,f(x)v(g)),\ (x,g)\in X\times G.$$

Proposition 0.5 ([45]) _Assume that_ $\varphi_1,\varphi_2:X\longrightarrow G$ _are two ergodic cocycles. Let_ $\hat{S}$ _be a factor map from_ T_{φ_1} _to_ T_{φ_2} _satisfying_ $\hat{S}^{-1}\mathcal{B}=\mathcal{B}$. _Then_ $\hat{S}$ _must be of the form_ $\hat{S}=S_{f,v}$. _Moreover, the map_ $S_{f,v}$ _is invertible_ iff v _is invertible_ ■

From Proposition 0.5 we read immediately the form of the elements $\hat{S}\in C(T_{\varphi})$. To be precise, if $\hat{S}\in C(T_{\varphi})$ then $\hat{S}=S_{f,v}$ for appropriate S, f and v. Thus, $S_{f,v}$ belongs to $C(T_{\varphi})$ iff

(17) $$\varphi S/v\varphi=fT/f.$$

If this is the case we say that $S\in C(T)$ _can be lifted to the centralizer of_ T_{φ}.

Proposition 0.6. _Assume that_ $S_{f,v}$ _and_ $S_{f',v'}$ _are two liftings of_ $S\in C(T)$ _to the centralizer of_ T_{φ}. _Then_ $v=v'$ _and_ $f(x)=f'(x)g_0$ _for some_ $g_0\in G$.

Proof. From (17) we have

$$(f(Tx)f'(Tx)^{-1})(f(x)f'(x)^{-1})^{-1}=v'(\varphi(x))(v(\varphi(x)))^{-1}.$$

Let $\chi\in\hat{G}$ and $F(x)=\chi(f(x)f'(x)^{-1})$. Then, we get that $F:X\to S^1$ is measurable and that $F(Tx)/F(x)=(\overline{\chi\circ v}\cdot\chi\circ v')(\varphi(x))$. But φ is ergodic whence by (15) we have $\chi v=\chi v'$ for any $\chi\in\hat{G}$ and therefore $v=v'$. The second statement, now, easily follows from ergodicity of T ■

As an immediate consequence of Proposition 0.6 we obtain that if in Proposition 0.5 S=Id then $\hat{S}$ is invertible. In this case we say that T_{φ_1} and T_{φ_2} are <u>relatively isomorphic</u>.

Notice that the map

(18) $$g \longmapsto \sigma_g, \quad \sigma_g(x,h)=(x,hg)$$

is a continuous group embedding of G into $C(T_\varphi)$. The map $C(T_\varphi) \ni \hat{S} \xrightarrow{(*)} S \in C(T)$, where $\hat{S}$ is given by (16) is continuous in weak topology of the centralizers. Hence the T-invariant set $C_\varphi(T)$ of those S's in C(T) that can be lifted to the centralizer of T_φ is a continuous image of a Polish space and is a semigroup. Also, the map (*) is onto iff T has discrete spectrum ([20]). From these facts and by the obvious identification of X and C(T) given by (8) we deduce the following

Proposition 0.7. *The set $C_\varphi(T)$ is a measurable semigroup contained in X and one has*

$$\mu(C_\varphi(T))=0 \text{ or } C_\varphi(T)=X \blacksquare$$

We will also need the following.

Proposition 0.8. *Let $T_\varphi:(X\times G,\tilde{\mathcal{B}},\tilde{\mu})\circlearrowleft$ be a group extension. Let $\lambda \in J_2^e(T_\varphi)$ be such that $\lambda|\mathcal{B}\otimes\mathcal{B}=\mu_S$, $S\in C(T)$. Assume that S can be lifted to $C(T_\varphi)$. Then $\lambda=\tilde{\mu}_{\hat{S}}$ for some lifting $\hat{S}$ of S.*

Proof. We observe that λ is concentrated on the set $D=\{(x,g_1,Sx,g_2):\ x\in X,\ g_1,g_2\in G\}$. Now, S can be lifted, so the formula (17) holds. Denote $\hat{S}=S_{f,v}$ and observe that the support of $\tilde{\mu}_{\hat{S}}$ is the set $\{(x,g_1,Sx,f(x)v(g_1)):\ x\in X,\ g_1\in G\}=$ $\{((x,g_1),S_{f,v}(x,g_1)):\ (x,g_1)\in X\times G\}$. Consider the following measurable map $\xi: D \longrightarrow G$, $\xi(x,g_1,Sx,g_2)=f(x)v(g_1)/g_2$. Now, $\xi(T_\varphi\times T_\varphi((x,g_1),(Sx,g_2))=f(Tx)v(\varphi(x))v(g_1)/\varphi(Sx)g_2$ and in view of (17), $\xi(T_\varphi\times T_\varphi((x,g_1),(Sx,g_2))=\xi((x,g_1),(Sx,g_2))$.

In other words $f(x)v(g_1)/g_2=g_0'$ λ-a.e. . It implies that the support of λ is $\{(x,g_1,Sx,f(x)v(g_1)/g_0): x\in X, g_1\in G\}$ which is the support of $\tilde{\mu}_S^{\wedge}\delta_{g_0^{-1}}$ ∎

For some undefined terms concerning ergodic theory we refer the reader to [11], [14], [29], [52], [59] and [66].

Now, we recall some basic definitions from topological dynamics that we will need. Let $T:Y\circlearrowleft$ be a homeomorphism of a compact, metric space. We say that T is minimal if $\{T^n y: n\in Z\}$ is dense for every $y\in Y$. T is said to be uniquely ergodic whenever there is a unique (hence ergodic) T-invariant Borel probability measure. T is strictly ergodic if T is minimal and uniquely ergodic. A point $y\in Y$ is called almost periodic if for each nonempty open set U there is a number L such that for each n the set $\{T^n y, T^{n+1}y, \ldots, T^{n+L-1}y\}\cap U$ is nonempty. This is equivalent to saying that $T:\overline{O_T(y)}\circlearrowleft$ is minimal where $O_T(y)=\{T^n y: n\in Z\}$. A point $y\in Y$ is called strictly transitive if for each continuous function $f:Y\longrightarrow \mathbb{C}$

$$\lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1} f(T^n y)$$

exists and is uniform. This is equivalent to saying that $T:\overline{O_T(y)}\circlearrowleft$ is uniquely ergodic. A minimal homeomorphism T:Y is called topologically coalescent ([4]) if the set of those continuous maps $S:Y\circlearrowleft$ commuting with T (i.e. topological centralizer of T) forms a group. If $T_i:Y_i\circlearrowleft$, i=1,2, are homeomorphisms and $f:Y_1\longrightarrow Y_2$ is continuous and onto then T_2 is called a topological factor of T_1. Quite analogously to the measure-theoretic case we can define the notions of weak topological isomorphism and topological isomorphism.

In some examples we will make use of some basic facts on continued fraction expansion. Let a number $\alpha \in [0,1)$ have continued fraction expansion

$$\cfrac{1}{a_1+\cfrac{1}{a_2+\ddots}} = [0:a_1,a_2,\ldots],$$

where a_i are positive integers. The numbers a_i are called partial quotients of α. The rationals $p_k/q_k=[0:a_1,\ldots,a_k]$ are said to be convergents. The numbers q_k are called denominators of α. We say that α has bounded partial quotients if the set $\{a_i\}$ is finite. Let a,b be positive integers. The following is known as Legendre theorem ([27])

if a,b are coprime and $|\alpha - a/b| < 1/2b^2$ then a/b has to appear as a convergent of α.

For other definitions and results we refer to [27].

PART I

1. ERGODIC SELF-JOININGS FOR ABELIAN GROUP EXTENSIONS OF ROTATIONS. The set of ergodic measures for compact abelian group extensions has been described in [26]. In this section, we go-in some sense-further and study the ergodic self-joinings of abelian group extensions of discrete spectrum automorphisms. It turns out that every such a joining must be the relatively independent extension of an isomorphism between two natural factors.

We start with general results.

Lemma 1.1. _Let_ $T:(X,\mathcal{B},\mu)\circlearrowleft$ _be an ergodic automorphism and_ $\lambda\in J_2^e(T)$. _Then_ λ _is an invertible graph joining iff_

(19) $(\forall A\in\mathcal{B})\ (\exists\ B\in\mathcal{B})\quad \lambda(X\times A\,\triangle\, B\times X)=\lambda(A\times X\,\triangle\, X\times B)=0.$

Proof. Assume that $\lambda=\mu_S$, $S\in C(T)$, where S is invertible. Then

$\lambda(A\times X\,\triangle\, X\times B)=\lambda[(A\times B^c)\cup(A^c\times B)]=\mu(A\cap S^{-1}B^c)+\mu(A^c\cap S^{-1}B)$.

Therefore (19) holds for $B=SA$.

Now, assume that (19) is satisfied for λ. We will define a map $S^{-1}:\mathcal{B}\circlearrowleft$ by $S^{-1}A=B$ where

(20) $\quad\lambda(A\times X\,\triangle\, X\times B)=0.$

This definition is correct as there is a unique (up to μ) set B satisfying (20). The map S^{-1} is one-to-one and onto since $\lambda(B\times X\triangle X\times A)=\lambda(A\times X\triangle X\times B)$. Now, λ is a self-joining of T, hence $\lambda(A\times X\triangle X\times B)=\lambda((T\times T)(A\times X\triangle X\times B))=\lambda(TA\times X\triangle X\times TB)$ and therefore $S^{-1}(TA)=T(S^{-1}A)$. Moreover, (20) implies that $\lambda(A\times X)=\lambda(X\times B)$ so $\mu(S^{-1}A)=\mu(A)$. Notice that (20) implies also that $\lambda((A\times X)^c \triangle (X\times B^c)=0$ and therefore $S^{-1}(A^c)=(S^{-1}A)^c$. Moreover, the following inclusion

$$(\bigcup_{n\geq 1} A_n\times X)\triangle(X\times \bigcup_{n\geq 1} B_n)\subseteq \bigcup_{n\geq 1}(A_n\times X\triangle X\times B_n)$$

gives us $S^{-1}(\bigcup_{n\geq 1} A_n)=\bigcup_{n\geq 1} S^{-1}A_n$. Combining these properties of S^{-1} with the fact that the underlying space is Borel standard we find S to be a pointwise automorphism of (X,μ) (see e.g. [66], Chapter 2) and besides $S\in C(T)$. We have also $\lambda(A\times(S^{-1}A)^c\cup A^c\times S^{-1}A)=0$. Therefore

(21) $\lambda(A\times S^{-1}A\cup A^c\times(S^{-1}A)^c)=1$ for every $A\in\mathcal{B}$.

Now, take $B,C\in\mathcal{B}$. Using (21) we get that

$B\times C=B\times C\cap(SC\times C\cup(SC)^c\times C^c)=(B\cap SC)\times C$. Consequently

$\lambda(B\times C)\leq\lambda\,[(B\cap SC)\times X]=\mu(B\cap SC)=\mu_{S^{-1}}(B\times C)$ and we conclude that $\lambda=\mu_{S^{-1}}$ as the latter measure is ergodic∎

P r o p o s i t i o n 1.1. <u>For any ergodic automorphism</u> T <u>the following are equivalent.</u>

(i) T <u>has discrete spectrum.</u>

(ii) $J_2^e(T)=\{\mu_S:\ S\in C(T)\}$.

(iii) T <u>is a canonical system.</u>

P r o o f . (i)→(ii). Let $\lambda\in J_2^e(T)$. We will show that λ identifies the two marginal sub-σ-algebras $\tilde{\mathcal{B}}_1=\{A\times X:\ A\in\mathcal{B}\}$, $\tilde{\mathcal{B}}_2=\{X\times B:\ B\in\mathcal{B}\}$. To this end let us look at $L^2(X\times X,\mathcal{B}\otimes\mathcal{B},\lambda)$ and the corresponding marginal subspaces

$L^2(\tilde{\mathcal{B}}_1)=\{\tilde{f}:\ \tilde{f}(x,y)=f(x),\ f\in L^2(X,\mu)\}$,

$L^2(\tilde{\mathcal{B}}_2)=\{\tilde{\tilde{f}}:\ \tilde{\tilde{f}}(x,y)=f(y),\ f\in L^2(X,\mu)\}$.

Since $\lambda\in J_2(T)$, both $L^2(\tilde{\mathcal{B}}_1)$ and $L^2(\tilde{\mathcal{B}}_2)$ are identified with $L^2(X,\mu)$ in a natural way. Therefore, they are spanned by $\{\tilde{f}_\alpha:\ \alpha\in Sp(T)\}$, $\{\tilde{\tilde{f}}_\alpha:\ \alpha\in Sp(T)\}$, respectively, where

$f_\alpha T = \alpha f_\alpha$, $\alpha \in Sp(T)$. But λ is ergodic, so $\tilde{f}_\alpha = a_\alpha \tilde{\tilde{f}}_\alpha$ for some $a_\alpha \in \mathbb{C}$ and consequently $L^2(\tilde{\mathcal{B}}_1) = L^2(\tilde{\mathcal{B}}_2)$ (an equality of two subspaces of $L^2(X \times X, \lambda)$). Equivalently, $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$ are identified by λ . An application of Lemma 1.1 gives the result.

(ii)→(i). We can apply the implication (iii)→(i) of (2) to conclude that T is an ergodic group extension of the 1-point dynamical system. Hence it is an ergodic rotation on a compact group and therefore it has discrete spectrum.

(i)→(iii) by Proposition 0.3

(iii)→(ii). Assume that $T:(X, \mathcal{B}, \mu)\circlearrowleft$ is a canonical system. Let $\lambda \in J_2^e(T)$. Then, T being canonical, the self-joining must identify the two marginal sub-σ-algebras $\tilde{\mathcal{B}}_1$ and $\tilde{\mathcal{B}}_2$ since they realize T as a factor of $T \times T:(X \times X, \lambda)\circlearrowleft$. An application of Lemma 1.1 completes the proof ■

Remark 1.1 Assume that $T_i:(X_i, \mathcal{B}_i, \mu_i)\circlearrowleft$ are two ergodic automorphisms with discrete spectrum and $Sp(T_1)=Sp(T_2)$. When we extend the notion of a graph joining to the case of an isomorphism between T_1 and T_2 we see that Lemma 1.1 still works . Therefore the proof of (i)→(ii) in Proposition 1.1 shows that each ergodic joining of T_1 and T_2 is the graph of an isomorphism. Hence we obtain a new "joining" proof of the classical Halmos-von Neumann theorem ([15]) ■

Remark 1.2. Proposition 1.1 gives the positive answer to D. Newton question [45] of whether the class of canonical systems coincides with the class of ergodic automorphisms with discrete spectrum■

As a consequence of Proposition 1.1 we get

Corollary 1.1. *If $T:(X,\mathcal{B},\mu)\circlearrowleft$ is an ergodic automorphism with discrete spectrum then*

$$J_n^e(T)=\{\mu_{S_1,\ldots,S_n}:\ S_i\in C(T),\ i=1,\ldots,n\},\ n\geq 2 \blacksquare$$

Let $T:(X,\mathcal{B},\mu)\circlearrowleft$ be an ergodic automorphism with discrete spectrum. Assume that G is an abelian compact metric group with Haar measure ν. Suppose that $\varphi:X\longrightarrow G$ is an ergodic cocycle. If $H\subseteq G$ is a closed (compact) subgroup then we can consider the action of T_φ on $X\times G/H$. The factors of this form are called _natural factors_. By a slight discord with Remark 0.1 we denote the natural factor by $T_{\varphi,H}:(X\times G/H,\tilde{\mathcal{B}}_H,\tilde{\mu})\circlearrowleft$, where

(22) $\tilde{\mathcal{B}}_H=\{A\in\tilde{\mathcal{B}}:\ \sigma_h(A)=A \text{ for each } h\in H\}$

(see (18)). In fact the natural factors of T_φ are the only ones that contain the σ-algebra $\{A\times G:\ A\in\mathcal{B}\}$ ([20]). Our aim is a description of all ergodic self-joinings of T_φ. Without loss of generality we can assume that T is an ergodic rotation on a compact monothetic group X. Before proceeding to the point, we will first consider the case where $\psi:X\rightarrow G$ is a cocycle which is not necessarily ergodic.

Let $\pi:X\times G\longrightarrow X$, $\pi(x,g)=x$. Assume that λ is an ergodic T_ψ-invariant measure on $X\times G$. Then $\lambda\pi^{-1}$ is T-ergodic and therefore $\lambda\pi^{-1}=\mu$ since T is uniquely ergodic. The following well-known lemma is a consequence of the individual ergodic theorem.

Lemma 1.2. *There is a measurable T_ψ-invariant subset $Y\subseteq X\times G$, $\lambda(Y)=1$, such that for each $(x,g)\in Y$ and for each continuous function f on $X\times G$,* $\lim_{n\to\infty}(1/n)S_n(f)(x,g)=\int_X f\,d\lambda$ *where* $S_n(f)(x,g)=\sum_{k=0}^{n-1} f(T_\psi)^k(x,g)$ $\blacksquare$

Let us denote by H the stabilizer of λ in G, i.e. $H=\{g\in G:\ \lambda g=\lambda\}$, where $\lambda g(A\times B)=\lambda(A\times Bg^{-1})$ or $\int f(x,h)d(\lambda g)=\int f(x,hg)d\lambda$ for $f\in C(X\times G)$. Let us also denote $f\circ g(x,h)=f(x,hg)$.

Lemma 1.3. (i) H is a closed subgroup of G.
(ii) If $(x,g),(x,h)\in Y$ then $hH=gH$.

Proof. As (i) is obvious, we will prove (ii). Take $f\in C(X\times G)$. Then $(x,g)\in Y$ implies $\frac{1}{n}S_n(f)(x,g)\longrightarrow\int fd\lambda$. But $S_n(f)(x,g)=S_n(f)(x,hh^{-1}g)=S_n(f\circ h^{-1}g)(x,h)$ since the natural action of G on the second coordinate commutes with T_ψ. But from our assumption $(x,h)\in Y$, so $\frac{1}{n}S_n(f\circ h^{-1}g)(x,h)\longrightarrow\int f\circ h^{-1}gd\lambda=\int fd(\lambda h^{-1}g)$. Because f was arbitrary, $\lambda h^{-1}g=\lambda$ or which is the same $h^{-1}g\in H$ ■

Let us decompose λ over the factor $T:(X,\mathcal{B},\mu)\circlearrowleft$:

$$\lambda=\int_X\lambda_x d\mu(x).$$

Lemma 1.4. $\lambda_x=\delta_x\times\nu_H g$ μ-a.e., where ν_H is Haar measure on H, $g=g(x)$ and $(x,g)\in Y$.

Proof. Let A be a Borel subset of G and $h\in H$. Put

$$M=\{x\in X:\ \lambda_x(\{x\}\times Ah^{-1})<\lambda_x(\{x\}\times A)\}$$

and suppose that $\mu(M)>0$ (M is measurable as the function $x\longmapsto\lambda_x(\{x\}\times C)$ is measurable whenever $C\subseteq G$ is measurable). Then $\lambda(M\times A)=\lambda h(M\times A)=\lambda(M\times Ah^{-1})=\int_M\lambda_x(\{x\}\times Ah^{-1})d\mu(x)<\int_M\lambda_x(\{x\}\times A)d\mu(x)=\lambda(M\times A)$, a contradiction. Similarly, we show that $\mu\{x\in X:\ \lambda_x(\{x\}\times Ah^{-1})>\lambda_x(\{x\}\times A)\}=0$. As a conclusion we have that $\lambda_x h=\lambda_x$ μ-a.e. . Let $(x,g)\in Y$. From Lemma 1.3(ii) it follows that $Y\cap(\{x\}\times G)=\{x\}\times gH$. Hence $\lambda_x(\{x\}\times gH)=1$. This implies $\lambda_x g^{-1}(\{x\}\times H)=1$. But for $h\in H$, $(\lambda_x g^{-1})h=(\lambda_x h)g^{-1}=\lambda_x g^{-1}$. Thus $\lambda_x g^{-1}$ is invariant under all translations by elements of H and therefore $\lambda_x g^{-1} = \delta_x \times \nu_H$ ■

Remark 1.3. Lemma 1.4 implies that if we denote by $\tilde{\lambda} = \int_X \tilde{\lambda}_x d\mu(x)$ the image of λ on $(X \times G/H, \tilde{\mathfrak{B}}_H)$ then $\tilde{\lambda}_x$ is a Dirac measure. This allows us to define a measurable function $f: X \longrightarrow G/H$ by

(23) $\quad f(x) = gH$ if $\lambda_x = \delta_x \times \nu_H g$.

The function f is measurable since the map $x \overset{\Lambda_A}{\longmapsto} \tilde{\lambda}_x(A)$ is measurable for each $A \in \tilde{\mathfrak{B}}_H$ and then $f^{-1}(A) = \{x \in X: \ \tilde{\lambda}_x(A) = 1\} = \Lambda_A^{-1}(1)$. Moreover, the T_ψ-invariance of λ implies

(24) $\quad f(Tx) = \psi(x) f(x)$

(ψ is considered as a map from X into G/H) ■

Lemma 1.5. <u>The automorphism</u> $T_\psi: (X \times G, \lambda) \circlearrowleft$ <u>is isomorphic to</u> $T_\theta: (X \times H, \mu \times \nu_H) \circlearrowleft$ <u>where</u> $\theta: X \longrightarrow H$ <u>is a measurable map</u>.

Proof. Define $t: X \longrightarrow G$ by the formula $t(x) = U(f(x))$, where U is a measurable selector for the natural map $G \longrightarrow G/H$ (see [63], Chapter VIII p.5). Hence U satisfies $U(gH)H = gH$. Then $t(x) \in f(x)$ which means $t(x)H = f(x)$. In view of (24) we have $\psi(x)t(x)H = t(Tx)H$. Put

(25) $\quad \theta(x) = \psi(x) t(Tx)^{-1} t(x) \in H.$

Therefore from Lemma 1.4 it follows that the map $j: (X \times H, \mu \times \nu_H) \longrightarrow (X \times G, \lambda)$ acting by the formula $j(x,h) = (x, t(x)h)$ establishes an isomorphism of T_θ and T_ψ ■

Remark 1.4. One of the reasons why $T_\psi: (X \times G, \tilde{\mu}) \circlearrowleft$ is not ergodic is that the smallest closed subgroup H_ψ generated by $\{\psi(x): x \in X\}$ is not equal to G. In fact, if $H_\psi \neq G$ then $X \times gH_\psi$ is invariant for each $g \in G$, and we can take a measurable union of such sets with positive measure strictly smaller than 1. From the proof of Lemma 1.5 (see (25)) it follows that-up to cohomology of cocycles-this is the only reason to loose ergodicity. Let

$$\Gamma = \{\gamma\in\hat{G}:\ \text{there is a measurable } h:X\longrightarrow S^1 \text{ such that } h(Tx)h(x)^{-1}=\gamma(\Psi(x))\ \mu\text{-a.e.}\}.$$

Then Γ is a subgroup of $\hat{G}$ and we put

$$F=\operatorname{ann}\Gamma=\{g\in G:(\forall\gamma\in\Gamma)\quad \gamma(g)=1\}\ ■$$

Lemma 1.6. $F=H$.

Proof. Let $g_0\in H$ and $\gamma\in\Gamma$. Then $\gamma(\Psi(x))=h(Tx)h(x)^{-1}$ and let us define a function $w:X\times G\longrightarrow S^1$ by setting $w(x,g)=h(x)^{-1}\gamma(g)$. Then for μ-a.e. x and for <u>all</u> g we have $wT_\Psi(x,g)=w(x,g)$. The ergodicity of λ forces w to be constant λ-a.e., i.e. $h(x)^{-1}\gamma(g)=c\neq0$. Moreover, $c=\int w(x,g)d\lambda=\int h(x)^{-1}\gamma(g)d\lambda=\int h(x)^{-1}\gamma(g)d(\lambda g_0)=\int h(x)^{-1}\gamma(gg_0)d\lambda=\gamma(g_0)c$. Hence $\gamma(g_0)=1$ and therefore $g_0\in F$.

Now, let $g\in F$. If $g\notin H$ then there is a character γ such that $\gamma(g)\neq1$ and $\gamma(H)=1$. From (25) it follows that $\gamma(\Psi(x))=\gamma(\theta(x)t(x)^{-1}t(Tx))=\gamma(t(x))^{-1}\gamma(t(Tx))$ since $\theta(x)\in H$. This implies $\gamma\in\Gamma$ and consequently $\gamma(g)=1$ which is a contradiction ■

Remark 1.5. The results contained in Lemmas 1.3-1.6 can be deduced from [26]. We included these results for completeness as well as for their new and simple proofs ■

Now, we are in a position to pass to our main problem, namely, to describe all ergodic self-joinings of T_φ. We assume that

<u>T_φ is an ergodic G-extension of an ergodic rotation</u> $T:(X,\mathcal{B},\mu)\circlearrowleft$.

Let $\Pi: X\times G\times X\times G \longrightarrow X\times X$ be defined by $\Pi(x,g,y,h)=(x,y)$. Let $\bar{\lambda}\in J_2^e(T_\varphi)$. Then by Proposition 1.1 we have $\bar{\lambda}\Pi^{-1}=\mu_S$ for some $S\in C(T)$. Hence

Lemma 1.7. $\bar{\lambda}(\bigcup_{x\in X}\{x\}\times G\times\{Sx\}\times G)=1$ ■

We define a measure λ on $X\times G\times G$ as follows $\lambda(A\times B\times C)=\bar{\lambda}(A\times B\times SA\times C)$. Put $\alpha: \bigcup_{x\in X}\{x\}\times G\times\{Sx\}\times G \longrightarrow X\times G\times G$ by setting $\alpha(x,g,Sx,h)=(x,g,h)$. Then, we see that λ is just the image of $\bar{\lambda}$ via α. Also, $T_{\varphi\times\varphi S}\alpha=\alpha(T_\varphi\times T_\varphi)$. Therefore we have proved the following.

Lemma 1.8. <u>The function</u> α <u>is an isomorphism of</u> $T_\varphi\times T_\varphi:(X\times G\times X\times G,\bar{\lambda})\circlearrowleft$ <u>and</u> $T_{\varphi\times\varphi S}:(X\times G\times G,\lambda)\circlearrowleft$ ■

In what follows we will consider $T_{\varphi\times\varphi S}$ and the measure λ on $X\times G\times G$. Let $H\subseteq G\times G$ be defined as $H=\{(g_1,g_2)\in G\times G: \lambda(g_1,g_2)=\lambda\}$, and let $H_1=\{g\in G: (g,e)\in H\}$, $H_2=\{g\in G: (e,g)\in H\}$, where e is the unit element of G. Then obviously H_1,H_2 are closed subgroups of G. If we put

$$\Gamma=\{(\gamma_1,\gamma_2)\in\hat{G}\times\hat{G}: \text{ there is a measurable function } h:X\longrightarrow S^1 \text{ such that } \gamma_1(\varphi(x))\gamma_2(\varphi(Sx))=h(Tx)h(x)^{-1}\}$$

then from Lemma 1.6 it follows that $H=\operatorname{ann}\Gamma$ and therefore

(26) $\quad H_i=\operatorname{ann}\Gamma_i$

where $\Gamma_i=\Pi_i(\Gamma)$, $\Pi_i:\hat{G}\times\hat{G}\longrightarrow\hat{G}$, $\Pi_i(\gamma_1,\gamma_2)=\gamma_i$, $i=1,2$.

Lemma 1.9. Γ <u>is a "diagonal" subgroup of</u> $\hat{G}\times\hat{G}$, <u>i.e.</u>

$(\gamma_1,\gamma_2)\in\Gamma$, $(\gamma_1,\gamma_2')\in\Gamma$ <u>imply</u> $\gamma_2=\gamma_2'$,

$(\gamma_1,\gamma_2)\in\Gamma$, $(\gamma_1',\gamma_2)\in\Gamma$ <u>imply</u> $\gamma_1=\gamma_1'$.

Proof. This is an obvious consequence of ergodicity of T_φ and (15) ■

Let $\pi : X\times G\times X\times G \longrightarrow X\times X$ be defined by $\pi(x,g,y,h)=(x,y)$. Let $\bar{\lambda}\in J_2^e(T_\varphi)$. Then by Proposition 1.1 we have $\bar{\lambda}\pi^{-1}=\mu_S$ for some $S\in C(T)$. Hence

Lemma 1.7. $\bar{\lambda}\left(\bigcup_{x\in X}\{x\}\times G\times\{Sx\}\times G\right)=1$ ■

We define a measure λ on $X\times G\times G$ as follows $\lambda(A\times B\times C)=\bar{\lambda}(A\times B\times SA\times C)$. Put $\alpha : \bigcup_{x\in X}\{x\}\times G\times\{Sx\}\times G \longrightarrow X\times G\times G$ by setting $\alpha(x,g,Sx,h)=(x,g,h)$. Then, we see that λ is just the image of $\bar{\lambda}$ via α. Also, $T_{\varphi\times\varphi S}\alpha=\alpha(T_\varphi\times T_\varphi)$. Therefore we have proved the following.

Lemma 1.8. *The function* α *is an isomorphism of* $T_\varphi\times T_\varphi : (X\times G\times X\times G, \bar{\lambda})\circlearrowleft$ *and* $T_{\varphi\times\varphi S} : (X\times G\times G, \lambda)\circlearrowleft$ ■

In what follows we will consider $T_{\varphi\times\varphi S}$ and the measure λ on $X\times G\times G$. Let $H\subseteq G\times G$ be defined as $H=\{(g_1,g_2)\in G\times G:\ \lambda(g_1,g_2)=\lambda\}$, and let $H_1=\{g\in G:\ (g,e)\in H\}$, $H_2=\{g\in G:\ (e,g)\in H\}$, where e is the unit element of G. Then obviously H_1,H_2 are closed subgroups of G. If we put

$$\Gamma=\{(\gamma_1,\gamma_2)\in\hat{G}\times\hat{G}:\ \text{there is a measurable function } h:X\longrightarrow S^1 \text{ such that } \gamma_1(\varphi(x))\gamma_2(\varphi(Sx))=h(Tx)h(x)^{-1}\}$$

then from Lemma 1.6 it follows that $H=\operatorname{ann}\Gamma$ and therefore

(26) $\quad H_i=\operatorname{ann}\Gamma_i$

where $\Gamma_i=\pi_i(\Gamma)$, $\pi_i:\hat{G}\times\hat{G}\longrightarrow\hat{G}$, $\pi_i(\gamma_1,\gamma_2)=\gamma_i$, $i=1,2$.

Lemma 1.9. Γ *is a "diagonal" subgroup of* $\hat{G}\times\hat{G}$, *i.e.*

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$(\gamma_1,\gamma_2)\in\Gamma,\ (\gamma_1',\gamma_2)\in\Gamma$ *imply* $\gamma_1=\gamma_1'$.

Proof. This is an obvious consequence of ergodicity of T_φ and (15) ■

(31) $v(gH_1)=w(g^{-1}H_1)$.

Finally, let us define $S_{\bar{f},v}:X\times G/H_1 \longrightarrow X\times G/H_2$ by setting

(32) $S_{\bar{f},v}(x,gH_1)=(Sx,\bar{f}(x)v(gH_1))$.

L e m m a 1 . 1 2 . <u>The map $S_{\bar{f},v}$ establishes an isomorphism of the natural factors T_{φ,H_1} and T_{φ,H_2}</u> , $T_{\varphi,H_i}:(X\times G/H_i,\tilde{\mu})\circlearrowleft$, i=1,2.

P r o o f . It is sufficient to show that $S_{\bar{f},v}\,T_{\varphi,H_1}=T_{\varphi,H_2}S_{\bar{f},v}$. This is equivalent to proving the equality

(33) $\bar{f}(Tx)\bar{f}(x)^{-1}v(\varphi(x)H_1)\varphi(Sx)^{-1}H_2=H_2$ μ-a.e.

Using (24) and (30), the equality (33) can be reduced to showing that $p((\varphi(x),\varphi(Sx))H)v(\varphi(x)H_1)\varphi(Sx)^{-1}H_2=H_2$. Take $\gamma\in\Gamma_2$. Then by (28),(29) and (31) we have

$\gamma\,[p((\varphi(x),\varphi(Sx))H)v(\varphi(x)H_1)\varphi(Sx)^{-1}H_2]=$

$\gamma\cdot p((\varphi(x),\varphi(Sx))H)\cdot\gamma(v(\varphi(x)H_1))\cdot\gamma(\varphi(Sx)^{-1}H_2)=$

$\hat{p}(\gamma)((\varphi(x),\varphi(Sx))H)\cdot\gamma(v(\varphi(x)H_1))\cdot\gamma((\varphi(Sx)H_2)^{-1})=$

$\hat{w}(\gamma)(\varphi(x)H_1)\cdot\gamma(\varphi(Sx)H_2)\cdot\gamma(v(\varphi(x)H_1))\cdot\gamma(\varphi(Sx)H_2)^{-1}=1$ ∎

T h e o r e m 1 . 1 . <u>If $T_\varphi:(X\times G,\tilde{\mu})\circlearrowleft$ is an ergodic group extension of an automorphism with discrete spectrum and $\bar{\lambda}\in J_2^e(T_\varphi)$ then there exist closed subgroups</u> $H_1\subseteq G$, $H_2\subseteq G$ <u>and an isomorphism</u> $\bar{S}:(X\times G/H_1,\tilde{\mu})\longrightarrow(X\times G/H_2,\tilde{\mu})$ <u>of the natural factors</u> T_{φ,H_1} <u>and</u> T_{φ,H_2} <u>such that for all Borel sets</u> $A\subseteq X\times G$ <u>and</u> $B\subseteq X\times G$

$$\bar{\lambda}(A\times B)=\int_{X\times G/H_1}E(A|\tilde{\mathcal{B}}_{H_1})(x,gH_1)E(B\;|\tilde{\mathcal{B}}_{H_2})(\bar{S}(x,gH_1))\,d\tilde{\mu}(x,gH_1).$$

P r o o f . From Lemma 1.11 we obtain that

$(g_1,g_2)H=\bigcup_{g\in G} gH_1\times c(g_1,g_2)H_2\cdot v(gH_1)$ where $c(g_1,g_2)=g_2v(g_1^{-1}H_1)$. Moreover, from (30) it follows that if $f(x)=(g_1,g_2)H$, then $\bar{f}(x)=c(g_1,g_2)H_2$. From Lemma 1.4 and (23) we obtain that $\lambda=\int_X\delta_x\times\nu_{f(x)}d\mu(x)$ where $\nu_{f(x)}$ is the Haar measure ν_H

translated to $f(x)$. Therefore

$$\lambda = \int_X \delta_x \times \left(\int_{G/H_1} [\nu_{H_1} \circ gH_1 \times \nu_{H_2} \circ (\bar{f}(x)v(gH_1))] d\nu(gH_1) \right) d\mu(x) = \int_{X \times G/H_1} \delta_x \times \nu_{H_1} \circ gH_1 \times \nu_{H_2} \circ (\bar{f}(x)v(gH_1)) d\tilde{\mu}(x,gH_1).$$

From Lemma 1.8 it follows that

$$\bar{\lambda} = \int_{X \times G/H_1} \delta_x \times \nu_{H_1} \circ gH_1 \times \delta_{Sx} \times \nu_{H_2}(\bar{f}(x)v(gH_1)) d\tilde{\mu}(x,gH_1).$$

But $\tilde{\mu} = \int_{X \times G/H_1} \delta_x \times \nu_{H_1} \circ gH_1 d\tilde{\mu}(x,gH_1) = \int_{X \times G/H_1} \tilde{\mu}_{(x,gH_1)} d\tilde{\mu}(x,gH_1)$, so $\bar{\lambda} = \int_{X \times G/H_1} \tilde{\mu}_{(x,gH_1)} \times \tilde{\mu}_{\bar{S}(x,gH_1)} d\tilde{\mu}(x,gH_1)$. This latest equality means that $\bar{\lambda}$ is the relatively independent extension of the isomorphism $\bar{S}$ (see (9iii)) and the proof is complete ■

Although we have dealt with the case $\bar{\lambda} \in J_2^e(T_\varphi)$, all Lemmas 1.7-1.12 go through for $\bar{\lambda} \in J^e(T_{\varphi_1}, T_{\varphi_2})$ where $\varphi_i : X \longrightarrow G_i$ is an ergodic cocycle, $i=1,2$. Hence we have also proved the following.

Theorem 1.2. <u>Let</u> $T_{\varphi_i} : (X \times G_i, \tilde{\mu}_i) \circlearrowleft$ <u>be an ergodic group extension of</u> T, $i=1,2$. <u>If</u> $\bar{\lambda} \in J^e(T_{\varphi_1}, T_{\varphi_2})$ <u>then there exist two closed subgroups</u> $H_i \subseteq G_i$, $i=1,2$, <u>and an isomorphism</u> $\bar{S} : (X \times G_1/H_1, \tilde{\mu}_1) \longrightarrow (X \times G_2/H_2, \tilde{\mu}_2)$ <u>of the natural factors</u> T_{φ_1, H_1} <u>and</u> T_{φ_2, H_2} <u>such that for all Borel sets</u> $A \subseteq X \times G_1$, $B \subseteq X \times G_2$

$$\bar{\lambda}(A \times B) = \int_{X \times G_1/H_1} E(A|\tilde{\mathcal{B}}_{1,H_1})(x,g_1H_1) E(B|\tilde{\mathcal{B}}_{2,H_2})(\bar{S}(x,g_1H_1)) d\tilde{\mu}_1(x,g_1H_1) \blacksquare$$

Remark 1.6. A combination of Lemma 1.5 and Theorem 1.2 allows us to describe all ergodic joinings of ergodic group extensions $T_{\varphi_1}, \ldots, T_{\varphi_n}$, $\varphi_i : X \longrightarrow G_i$. Indeed, let $\lambda \in J^e(T_{\varphi_1}, \ldots, T_{\varphi_n})$. Then the measure λ_n given by

$$\lambda_n(A_1 \times \ldots \times A_{n-1}) = \lambda(A_1 \times \ldots \times A_{n-1} \times (X \times G_n))$$

is an ergodic joining of $T_{\varphi_1},\ldots,T_{\varphi_{n-1}}$. Applying Lemma 1.5 we know that $T_{\varphi_1}\times\ldots\times T_{\varphi_{n-1}}:((X\times G_1)\times\ldots\times(X\times G_{n-1}),\lambda_n)\circlearrowleft$ is isomorphic to some H-extension $T_{\psi_{n-1}}$ of T. Now, we apply Theorem 1.2 to $T_{\psi_{n-1}}$ and T_{φ_n} ∎

Remark 1.7. The result we have achieved by Theorem 1.1 can be generalized as follows. Let $T:(X,\mathcal{B},\mu)\circlearrowleft$ be an ergodic automorphism (not necessarily with discrete spectrum) If $\bar{\lambda}\in J_2^e(T_\varphi)$ where $\varphi:X\longrightarrow G$ is an ergodic cocycle and if $\bar{\lambda}$ projected on $X\times X$ is the graph joining of an $S\in C(T)$, then $\bar{\lambda}$ must satisfy the conclusion of Theorem 1.1∎

2. STRUCTURE OF FACTORS OF ABELIAN GROUP EXTENSIONS OF ROTATIONS. We will examine the structure of invariant sub-σ-algebras for an ergodic group extensions of rotations. We will prove that all invariant sub-σ-algebras are determined by compact subgroups in the centralizers of natural factors. Moreover, we show that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are two factors and $\tilde{S}$ establishes an isomorphism of them, then there exist two closed subgroups H_1, H_2 of G and an isomorphism $\hat{S}$ of the corresponding natural factors T_{φ,H_1} and T_{φ,H_2} such that $\hat{S}$ restricted to $\mathcal{E}_1$ is just $\tilde{S}$. From this, we conclude that, for an ergodic group extension of an automorphism with discrete spectrum, if all natural factors are coalescent then all factors are coalescent. Finally, we prove that any factor of abelian group extension of a rotation is an isometric extension of a rotation.

Let $T_\varphi : (X \times G, \tilde{\mathcal{B}}, \tilde{\mu})\circlearrowleft$ be an ergodic group extension of an automorphism with discrete spectrum $T:(X,\mathcal{B},\mu)\circlearrowleft$. Let $C_1(T_{\varphi,H})$ denote the group of all invertible elements of $C(T_{\varphi,H})$. Assume that $\mathcal{F} \subseteq C_1(T_{\varphi,H})$ is a subgroup. Then, this $\mathcal{F}$ determines a factor of $T_{\varphi,H}$ (and hence a factor of $T_{\varphi,\{e\}} = T_\varphi$) by

$$\mathcal{A}(\mathcal{F}) = \{A \in \tilde{\mathcal{B}} : \hat{S}A = A \text{ for each } \hat{S} \in \mathcal{F}\}.$$

The point is that when we pass through all compact subgroups of $C_1(T_{\varphi,H})$ for all possible closed subgroups $H \subseteq G$ we get all factors (Theorem 2.1).

Let $\mathcal{E} \subseteq \tilde{\mathcal{B}}$ be a T_φ -invariant sub-σ-algebra. Following (12), $\mathcal{E}$ gives rise to a self-joining of T_φ by

$$\tilde{\mu} \times_{\mathcal{E}} \tilde{\mu}(A \times B) = \int_{\bar{X}} E(A|\mathcal{E})(\bar{x}) E(B|\mathcal{E})(\bar{x}) \, d\bar{\mu}(\bar{x}),$$

where $\bar{X}$ is the quotient space corresponding to $\mathcal{E}$. Put

$\lambda=\tilde{\mu}\times_{\mathcal{E}}\tilde{\mu}$. Since λ is not necessarily ergodic, let

(34) $$\lambda=\int_{J_2^e(T_\varphi)} e\,d\gamma(e)$$

be its ergodic decomposition, γ being a probability measure on $J_2^e(T_\varphi)$. From the definition of λ it follows that

Lemma 2.1. <u>Let</u> A <u>be a Borel subset of</u> $X\times G$. <u>Then</u> $A\in\mathcal{E}$ iff $\lambda(A\times A^c\cup A^c\times A)=\lambda(A\times(X\times G)\triangle(X\times G)\times A)=0$ ∎

Let $E=\{e\in J_2^e(T_\varphi):\ e(A\times A^c\cup A^c\times A)=0$ for each $A\in\mathcal{E}\}$.

Lemma 2.2. $\gamma(E)=1$.

Proof. Let $\{A_n\}_{n\in N}$ be a dense family of sets in $\mathcal{E}$. Put $E_n=\{e\in J_2^e(T_\varphi):\ e(A_n\times A_n^c\cup A_n^c\times A_n)=0\}$. From lemma 2.1 and (34) it follows that $\gamma(E_n)=1$. Let $E_0=\bigcap_{n=1}^{\infty}E_n$. It is clear that $\gamma(E_0)=1$ and $E\subseteq E_0$. We will show that $E=E_0$. Given $\varepsilon>0$, $A\in\mathcal{E}$ we find A_n such that $\tilde{\mu}(A\triangle A_n)<\varepsilon/2$. Denoting $Y=X\times G$ we obtain that $e(A\times A^c\cup A^c\times A)=e(Y\times A\triangle A\times Y)\le e(Y\times A\triangle Y\times A_n)+e(Y\times A_n\triangle A_n\times Y)+e(A_n\times Y\triangle A\times Y)=2\tilde{\mu}(A\triangle A_n)<\varepsilon$ ∎

Let $e\in J_2^e(T_\varphi)$. By Theorem 1.1

$$e(A\times B)=\int_{X\times G/H_1}E(A|\tilde{\mathcal{B}}_{H_1})(x,gH_1)E(B|\tilde{\mathcal{B}}_{H_2})(S_{f,v}(x,gH_1))\,d\tilde{\mu}(x,gH_1),$$

where $S_{f,v}$ establishes an isomorphism of T_{φ,H_1} and T_{φ,H_2}.

Lemma 2.3. $e\in E$ iff $\mathcal{E}\subseteq\tilde{\mathcal{B}}_{H_1H_2}$ <u>and for each</u> $A\in\mathcal{E}$ <u>we have</u> $(S_{f,v})^{-1}(A)=A$.

Proof. We start with the following observation

(35) $$\tilde{\mathcal{B}}_{H_1}\cap\tilde{\mathcal{B}}_{H_2}=\tilde{\mathcal{B}}_{H_1H_2}$$

as (22) holds. Hence the sufficiency easily follows. Denote $\hat{S}=S_{f,v}$ and let $A\in\mathcal{E}$, $e\in E$. Then $e(A\times A^c)=0$ and $e(A^c\times A)=0$. The definition of e implies

(36) $$\int_{X\times G/H_1} E(A|\widetilde{\mathfrak{B}}_{H_1})E(A^c|\widetilde{\mathfrak{B}}_{H_2})\circ\hat{S}d\tilde{\mu}=0,$$

(37) $$\int_{X\times G/H_1} E(A^c|\widetilde{\mathfrak{B}}_{H_1})E(A|\widetilde{\mathfrak{B}}_{H_2})\circ\hat{S}d\tilde{\mu}=0.$$

Assume that $E(A|\widetilde{\mathfrak{B}}_{H_1})(x,gH_1)\neq 0,1$ for (x,gH_1) from a set of positive measure. Hence, by (36), $E(A^c|\widetilde{\mathfrak{B}}_{H_2})\circ\hat{S}(x,gH_1)=0$, which means that $E(A|\widetilde{\mathfrak{B}}_{H_2})\circ\hat{S}(x,gH_1)=1$. It follows that $\hat{S}(x,gH_1)\subseteq A$. But in view of (37) we have $E(A|\widetilde{\mathfrak{B}}_{H_2})\circ\hat{S}(x,gH_1)=0$ (since $E(A^c|\widetilde{\mathfrak{B}}_{H_1})(x,gH_1)\neq 0,1$), a contradiction. We conclude that $E(A|\widetilde{\mathfrak{B}}_{H_1})(x,gH_1)=0$ or 1 ($\tilde{\mu}$-a.a. (x,gH_1)) and therefore $A\in\widetilde{\mathfrak{B}}_{H_1}$. Suppose that $A\notin\widetilde{\mathfrak{B}}_{H_2}$. Then for a set of positive measure $E(A|\widetilde{\mathfrak{B}}_{H_2})\circ\hat{S}(x,gH_1)\neq 0,1$ and $E(A^c|\widetilde{\mathfrak{B}}_{H_2})\circ\hat{S}(x,gH_1)\neq 0,1$ since $\hat{S}$ is onto and measure-preserving. However, this implies (see (36), (37)) that $(x,gH_1)\subseteq A$ and $(x,gH_1)\subseteq A^c$ which is a conflict. Therefore from (35), $A\in\widetilde{\mathfrak{B}}_{H_1H_2}$. Moreover, $0=\int_{X\times G/H_1}\chi_A\cdot\chi_{A^c}\circ\hat{S}d\tilde{\mu}=\int_{X\times G/H_1}\chi_{A\cap\hat{S}^{-1}A}\,d\tilde{\mu}$ forces $\hat{S}^{-1}A=A$ to hold. This completes the proof■

Let H be the largest closed subgroup of G such that $\mathcal{E}\subseteq\widetilde{\mathfrak{B}}_H$. Such a group does exist as we can take H being the closure of the group generated by $\bigcup\{H_1:\ \mathcal{E}\subseteq\widetilde{\mathfrak{B}}_{H_1}\}$. Since for each $f\in L^2(X\times G,\tilde{\mu})$ the map $\tilde{f}:G\longrightarrow L^2(X\times G,\tilde{\mu})$ given by

$$\tilde{f}(g)(x,h)=f(x,gh)$$

is continuous, $\mathcal{E}\subseteq\widetilde{\mathfrak{B}}_H$. (Another proof of the existence of H easily follows from the fact that each invariant sub-σ-algebra that contains $\{A\times G:\ A\in\mathfrak{B}\}$ must be of the form $\widetilde{\mathfrak{B}}_J$ for a closed subgroup $J\subseteq G$, see [20], [65].) In other words, there exists the smallest natural factor $T_{\varphi,H}$ which is an extension of $\mathcal{E}$. We will only consider this natural factor. Denote

$$\mathcal{H}_{\varphi,H}(\mathcal{E})=\{\hat{S}\in C(T_{\varphi,H}):\ \hat{S}^{-1}A=A \text{ for each } A\in\mathcal{E}\}.$$

Lemma 2.4. $\mathcal{H}_{\varphi,H}(\mathcal{E}) \subseteq C_1(T_{\varphi,H})$.

Proof. If $\hat{S} \in \mathcal{H}_{\varphi,H}(\mathcal{E})$ is not invertible then $S^{-1}\tilde{\mathcal{B}}_H \subsetneq \tilde{\mathcal{B}}_H$. Then, $\hat{S}^{-1}$ carries the whole σ-algebra $\tilde{\mathcal{B}}_H$ to a smaller sub-σ-algebra which contains $\{A \times G: A \in \mathcal{B}\}$ (since the latter factor is a canonical system). Therefore $\hat{S}^{-1}\tilde{\mathcal{B}}_H$ is a natural factor. But $\mathcal{E} \subseteq \hat{S}^{-1}\tilde{\mathcal{B}}_H = \tilde{\mathcal{B}}_{H'}$, $H' \supsetneq H$ and we obtain a contradiction■

Let $\lambda_H = \tilde{\mu} \times_{\mathcal{E}} \tilde{\mu}$ ($\tilde{\mu}$ considered as the quotient measure on $\tilde{\mathcal{B}}_H$) and let $\lambda_H = \int_{J_2^e(T_{\varphi,H})} e\, d\gamma_H(e)$ be its ergodic decomposition. Denote

$$E_H = \{e \in J_2^e(T_{\varphi,H}) : e(A \times A^c \cup A^c \times A) = 0 \text{ for each } A \in \mathcal{E}\}.$$

Lemma 2.5. <u>For each</u> $e \in E_H$ <u>there is an</u> $\hat{S} \in C_1(T_{\varphi,H})$ <u>such that</u> $e = \tilde{\mu}_{\hat{S}}$.

Proof. It follows immediately from Lemma 2.3 that $e = \tilde{\mu}_{\hat{S}}$ and then from the argument we have used in the proof of Lemma 2.4 that $\hat{S}$ is invertible■

Remark 2.1. The arguments used in the following proof are due to Veech [65] (see also [20] for some details). We include the proof for the sake of completeness■

Theorem 2.1. <u>If</u> $\mathcal{E} \subseteq \tilde{\mathcal{B}}$ <u>is a</u> T_φ <u>-invariant sub-</u>σ<u>-algebra of an ergodic group extension</u> $T_\varphi : (X \times G, \tilde{\mu})\circlearrowleft$ <u>of an automorphism with discrete spectrum, then there exists a natural factor</u> $T_{\varphi,H}$ <u>of</u> T_φ <u>such that</u>

$$\mathcal{E} = \{A \in \tilde{\mathcal{B}}_H : \hat{S}(A) = A \text{ for each } \hat{S} \in \mathcal{H}_{\varphi,H}(\mathcal{E})\}.$$

<u>Moreover,</u> $\mathcal{H}_{\varphi,H}(\mathcal{E})$ <u>is a compact subgroup of</u> $C(T_{\varphi,H})$.

Proof. This natural factor $T_{\varphi,H}$ is taken as the smallest natural factor of T_φ which contains $\mathcal{E}$. Then from Lemma 2.5

it follows that

$$\lambda_H = \tilde{\mu} \times_{\mathcal{E}} \tilde{\mu} = \int_{C(T_{\varphi,H})} \tilde{\mu}_{\hat{S}} \, d\gamma_H(\hat{S})$$

for some probability measure γ_H on $C(T_{\varphi,H})$. From Lemma 2.4 it follows that $\mathcal{H}_{\varphi,H}(\mathcal{E})$ is a group. It is also clear that $\mathcal{H}_{\varphi,H}(\mathcal{E})$ is closed in weak topology. Hence, $\mathcal{H}_{\varphi,H}(\mathcal{E})$ is a separable, complete metric group. Moreover, if $A \in \mathcal{E}$ then by Lemma 2.1, $0 = \lambda_H(A \times (X \times G/H) \Delta (X \times G/H) \times A) = \int_{C(T_{\varphi,H})} \tilde{\mu}_{\hat{S}}(A^c \times A \cup A \times A^c) d\gamma_H(\hat{S}) = \int_{C(T_{\varphi,H})} \tilde{\mu}(A \Delta (\hat{S}^{-1}A)) d\gamma_H(\hat{S})$. This implies that $\gamma_H\{\hat{S} \in C(T_{\varphi,H}) : \tilde{\mu}(A \Delta \hat{S}^{-1}A) = 0\} = 1$. Hence by choosing a countable dense family $\{A_i\}$ we conclude that $\gamma_H\{\hat{S} : \tilde{\mu}(A \Delta S^{-1}A) = 0$ for each $A \in \mathcal{E}\} = 1$ which means that $\gamma_H(\mathcal{H}_{\varphi,H}(\mathcal{E})) = 1$. Therefore $\lambda_H = \int_{\mathcal{H}_{\varphi,H}(\mathcal{E})} \tilde{\mu}_{\hat{S}} d\gamma_H(\hat{S})$ for some probability Borel measure on $\mathcal{H}_{\varphi,H}(\mathcal{E})$. Let us suppose that $\hat{U} \in \mathcal{H}_{\varphi,H}(\mathcal{E})$. Then for all Borel sets $A, B \subseteq X \times G/H$ we have $\lambda_H(A \times \hat{U}^{-1}B) = \int_X E(A|\mathcal{E})E(\hat{U}^{-1}B|\mathcal{E})d\tilde{\mu} = \int_X E(A|\mathcal{E})E(B|\mathcal{E}) \circ \hat{U} d\tilde{\mu} = \int_X E(A|\mathcal{E})E(B|\mathcal{E})d\tilde{\mu}$ since $\hat{U}$ acts as the identity on $\mathcal{E}$. Hence $\lambda_H(A \times \hat{U}^{-1}A) = \lambda_H(A \times B)$. In other words, $(Id \times \hat{U})\lambda_H = \lambda_H$. Therefore

$$\int_{C(T_{\varphi,H})} (Id \times \hat{U})\tilde{\mu}_{\hat{S}} d\gamma_H(\hat{S}) = \int_{\mathcal{H}_{\varphi,H}(\mathcal{E})} \tilde{\mu}_{\hat{U}\hat{S}} d\gamma_H(\hat{S}) = \int_{\mathcal{H}_{\varphi,H}(\mathcal{E})} \tilde{\mu}_{\hat{S}} d\gamma_H(\hat{S}).$$

But the ergodic decomposition of λ_H is unique, so we are forced to conclude that γ_H is invariant under left multiplication by $\hat{U}$. Suppose for a moment that $(\mathcal{H}_{\varphi,H}(\mathcal{E}), d)$, where d is a metric giving weak topology, is not compact. Since $\mathcal{H}_{\varphi,H}(\mathcal{E})$ is separable and γ_H is left invariant, $\gamma_H(C) > 0$ for each open nonempty $C \subseteq \mathcal{H}_{\varphi,H}(\mathcal{E})$. By the completeness there must exist a sequence $\{S_n\} \subseteq \mathcal{H}_{\varphi,H}(\mathcal{E})$ from which we cannot select any Cauchy subsequence. Hence, there exists a ball $B(Id, \varepsilon_0)$, for some $\varepsilon_0 > 0$, such that $\hat{S}_n\hat{S}_m^{-1} \notin B(Id, \varepsilon_0)$ for each $n \neq m$. But $\mathcal{H}_{\varphi,H}(\mathcal{E})$ is a topological group, so there exists an open nonempty set $C \subseteq \mathcal{H}_{\varphi,H}(\mathcal{E})$ such that $CC^{-1} \subseteq B(Id, \varepsilon_0)$. But now $\hat{S}_nC \cap \hat{S}_mC = \emptyset$ for any pair $n \neq m$ and therefore γ_H cannot be finite measure. We have proved that $\mathcal{H}_{\varphi,H}(\mathcal{E})$ is compact. Now, if $A \in \mathcal{E}$, then for each $\hat{S} \in \mathcal{H}_{\varphi,H}(\mathcal{E})$ we have $\hat{S}A = A$. Assume that foe $A \in \tilde{\mathcal{B}}_H$ we have $\hat{S}A = A$ for each $\hat{S} \in \mathcal{H}_{\varphi,H}(\mathcal{E})$. Thus

$$\lambda_H(A \times (X \times G/H) \Delta (X \times G/H) \times A) = \int_{\mathcal{H}_{\varphi,H}(\mathcal{E})} \tilde{\mu}(A \Delta \hat{S}^{-1}A)\, d\gamma_H(\hat{S}) = 0$$

which combined with Lemma 2.1 yields $A \in \mathcal{E}$ ■

Remark 2.2. From Theorem 2.1 it follows that given factor $\mathcal{E}$ of an ergodic group extension $T_\varphi : (X \times G, \tilde{\mu})\circlearrowleft$ we can pass from $\mathcal{E}$ to $\tilde{\mathcal{B}}$ in two group extension steps ■

Remark 2.3. Although throughout we deal with a discrete spectrum rotation $T:(X,\mu)\circlearrowleft$, Theorem 2.1 is still valid for some weakly mixing automorphisms. Let us call a weakly mixing automorphism $T:(X,\mu)\circlearrowleft$ 2-<u>fold simple</u> ([20]) if the only ergodic self-joinings of T are $\mu \times \mu$ and graph joinings. Then Theorem 2.1 holds for all weakly mixing cocycles ■

Example 2.1. Let T be an irrational rotation by $\alpha \in [0,1)$. Let $\varphi(x) = x$, $x \in [0,1)$. Using the following classical result,

(38) <u>For</u> $m \in Z$, $m \neq 0$, $b \in [0,1)$ <u>the cocycle</u> $x \longmapsto mx + b$ <u>is not coboundary</u> <u>(with respect to</u> T) ([3]),

we can easily compute the centralizer of T_φ as well as of all its natural factors. Indeed, first notice that T_{φ, Z_m} is naturally isomorphic to $T_{m\varphi}$: see the proof of Proposition 4.2. Then, from (17) it follows that

$C(T_{m\varphi})=\{S_{f,v}: fT-f=(m\varphi)S-v(m\varphi)$, where $f:X\longrightarrow X$ is measurable and $v:X\circlearrowleft$ is a continuous group epimorphism$\}$.

Hence, we seek a measurable solution of

$$f(x+\alpha)-f(x)=m(x+\beta)-smx=m(1-s)x+m\beta,$$

where $Sx=x+\beta$, $v(x)=sx$, $x\in[0,1)$. Therefore, from (38), $s=1$ and using Lemma 1 [3] we obtain $m\beta=m'\alpha$ for an integer m'.

Thus the centralizer of T_φ does not contain compact subgroups whose projection on the first coordinate is different from $\{Id\}$. Consequently, from $C(T_\varphi)$ we can read merely all natural factors, while for instance the transformation on the two-dimensional torus $U(x,y)=(x+2\alpha, x+y)$ is a factor of T_φ via the map $(x,y)\longmapsto(2x,2y)$ and moreover U is not a natural factor of T_φ. However, this factor can be read from the centralizer of $T_{2\varphi}$ as the group $\{0,1/2\}$ can be lifted to $C(T_{2\varphi})$ ■

Remark 2.4. Since $s=1$ these circle extensions must be coalescent: actually this has been well-known. However, in Section 3 we will construct some Anzai skew products without coalescence property ■

It would be interesting to know whether for ergodic group extensions the following formula holds

(39) $\quad C((T_\varphi)^n)=C(T_\varphi),\ n\geq 2.$

This is not difficult to see that the following property of self-joinings of T_φ

(40) $\quad$ if $\lambda\in J_2^e(T_\varphi)$ then λ is totally ergodic

(i.e. $\lambda\in J_2^e((T_\varphi)^n)$ for each $n\neq 0$) forces (39) to be true. Indeed, let $\hat{S}\in C((T_\varphi)^n)$. Then take

$$\lambda=(1/n)(\tilde{\mu}_{\hat{S}}+\tilde{\mu}_{\hat{S}}\circ T_\varphi+\ldots+\tilde{\mu}_{\hat{S}}\circ(T_\varphi)^{n-1}).$$

It is not hard to see that $\lambda \in J_2(T_\varphi)$ is in fact ergodic. Then from (40) it follows that $\lambda \in J_2^e((T_\varphi)^n)$ and consequently $\lambda = \tilde{\mu}_{\hat{S}}$, i.e. $\hat{S} \in C(T_\varphi)$. Nevertheless, (40) does not hold in general for T_φ with total ergodicity property. For instance, for the example from Example 2.1 we have $C(T_\varphi) \neq C((T_\varphi)^2)$ as the rotation by 1/2 can be lifted to the centralizer of $(T_\varphi)^2$.

Now, we intend to describe a relationship between the isomorphism of factors $\mathcal{E}_1$, $\mathcal{E}_2$ of an ergodic G-extension and the corresponding subgroups $\mathcal{H}_{\varphi,H_i}(\mathcal{E}_i)$, i=1,2. Assume that U is an isomorphism of two T_φ-invariant sub-σ-algebras $\mathcal{E}_1$ and $\mathcal{E}_2$. Let $\tilde{\mathcal{B}}_{H_1}$, $\tilde{\mathcal{B}}_{H_2}$ be the smallest natural factors of T_φ containing $\mathcal{E}_1$, $\mathcal{E}_2$ respectively.

Theorem 2.2. _There exists an isomorphism_ $\hat{S}$, $\hat{S}:(X \times G/H_1, \tilde{\mu}) \longrightarrow (X \times G/H_2, \tilde{\mu})$, _of the natural factors_ T_{φ,H_1} _and_ T_{φ,H_2} _such that_ $\hat{S}$ _restricted to_ $\mathcal{E}_1$ _equals_ U.

Proof. The proof consists of two steps. First, we will establish the following property of ergodic joinings of $\mathcal{E}_1$, $\mathcal{E}_2$:

$$(41)\quad \begin{cases} \text{if } \nu \text{ is an arbitrary ergodic joining of } \mathcal{E}_1 \text{ and } \mathcal{E}_2 \\ \text{then } \nu \text{ is the projection of some ergodic joining of} \\ T_{\varphi,H_1} \text{ and } T_{\varphi,H_2}. \end{cases}$$

Indeed, put $\hat{\nu}$ to be the relatively independent extension of ν to a joining of T_{φ,H_1} and T_{φ,H_2}. This meaning that

$$\hat{\nu}(A \times B) = \int_{\overline{X}_1 \times \overline{X}_2} E(A|\mathcal{E}_1)(\overline{x})E(B|\mathcal{E}_2)(\overline{y})\,d\nu(\overline{x},\overline{y}),$$

where $\overline{X}_1, \overline{X}_2$ are the quotient spaces corresponding to $\mathcal{E}_1$, $\mathcal{E}_2$ respectively. Let

$$\hat{\nu} = \int_{J^e(T_{\varphi,H_1}, T_{\varphi,H_2})} \tau \, d\gamma(\tau)$$

be the ergodic decomposition of $\hat{\nu}$. Let $\pi_i : X \times G/H_i \longrightarrow \bar{X}_i$, $i=1,2$, be the corresponding factor map. Then

$$\nu = \hat{\nu}(\pi_1 \times \pi_2)^{-1} = \int_{J^e(T_{\varphi,H_1}, T_{\varphi,H_2})} \tau(\pi_1 \times \pi_2)^{-1} d\gamma(\tau)$$

and ergodicity of ν yields $\tau(\pi_1 \times \pi_2)^{-1} = \nu$ for γ-a.a τ. In particular, there exists an ergodic joining τ such that $\tau(\pi_1 \times \pi_2)^{-1} = \nu$ and the property (41) has been proved.

To finish the proof, let $\tilde{\mu}_U$ denote the graph joining on $\mathcal{C}_1 \otimes \mathcal{C}_2$ corresponding to U. In view of (41) there is a measure $\tau \in J^e(T_{\varphi,H_1}, T_{\varphi,H_2})$ such that

(42) $\tilde{\mu}_U = \tau(\pi_1 \times \pi_2)^{-1}$.

Take $A \in \mathcal{C}_1$. Then, by the definition of $\tilde{\mu}_U$,

(43) $\tilde{\mu}_U(A \times U(A^c)) = \tilde{\mu}(A \cap U^{-1}U(A^c)) = 0$.

On the other hand, using (42), we have

(44) $\tilde{\mu}_U(A \times U(A^c)) = \tau((\pi_1 \times \pi_2)^{-1}(A \times U(A^c))) = \tau(A \times U(A^c))$,

since $A \times U(A^c) \in \mathcal{C}_1 \otimes \mathcal{C}_2$. By Theorem 1.2 there are closed subgroups $\tilde{F}_i \subseteq G/H_i$, $i=1,2$, and an isomorphism $\hat{S}: X \times G/F_1 \longrightarrow X \times G/F_2$ (where F_i is the subgroup of G for which G/F_i is naturally isomorphic to $(G/H_i)/\tilde{F}_i$, $i=1,2$) of the natural factors T_{φ,F_1} anf T_{φ,F_2} satisfying $\tau(A \times B) = \int_{X \times G/F_1} E(A|\tilde{\mathcal{B}}_{F_1}) E(B|\tilde{\mathcal{B}}_{F_2}) \circ \hat{S} d\tilde{\mu}$.
Therefore by (43) and (44)

(45) $0 = \int_{X \times G/F_1} E(A|\tilde{\mathcal{B}}_{F_1}) E(U(A^c)|\tilde{\mathcal{B}}_{F_2}) \circ \hat{S} d\tilde{\mu} = \int_{X \times G/F_1} E(A|\tilde{\mathcal{B}}_{F_1}) E(\hat{S}^{-1}U(A^c)|\tilde{\mathcal{B}}_{F_2}) d\tilde{\mu}$.

Similarly

(46) $0 = \tilde{\mu}_U(A^c \times UA) = \int_{X \times G/F_1} E(A^c|\tilde{\mathcal{B}}_{F_1}) E(\hat{S}^{-1}U(A)|\tilde{\mathcal{B}}_{F_2}) d\tilde{\mu}$.

Let us suppose that $\tilde{F}_1$ is a nontrivial subgroup of G/H_1, equivalently that $F_1 \supsetneq H_1$. Then, for some set $A \in \mathcal{C}_1$ the function $E(A|\tilde{\mathcal{B}}_{F_1})$ is not a characteristic function (as there must exist an $A \in \mathcal{C}_1$ which does not belong to $\tilde{\mathcal{B}}_{F_1}$). In other words for this A on a set of positive measure $E(A|\tilde{\mathfrak{B}}_{F_1})(\bar{x})\neq 0,1$. By (45) and (46), for such an $\bar{x}$

$0=E(A|\tilde{\mathfrak{B}}_{F_1})(\bar{x})E(\hat{S}^{-1}U(A^c)|\mathfrak{B}_{F_2})(\bar{x})$ and $0=E(A^c|\tilde{\mathfrak{B}}_{F_1})(\bar{x})E(\hat{S}^{-1}U(A)|\tilde{\mathfrak{B}}_{F_2})(\bar{x})$.

Using the same arguments as in the proof of Lemma 2.3 we obtain a contradiction. Therefore $F_1=H_1$. Since U is an isomorphism, $F_2=H_2$. Thus $\hat{S}$ establishes an isomorphism of T_{φ,H_1} and T_{φ,H_2} and moreover (45) and (46) imply now that $\hat{S}|\mathcal{E}_1=U$ ∎

Corollary 2.1. *If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two isomorphic invariant sub-σ-algebras then there exists an isomorphism $\hat{S}$ of the smallest natural factors $\tilde{\mathfrak{B}}_{H_1}$, $\tilde{\mathfrak{B}}_{H_2}$ of T_φ containing $\mathcal{E}_1$, $\mathcal{E}_2$ respectively, such that* $\mathcal{H}_{\varphi,H_2}(\mathcal{E}_2)=\hat{S}\,\mathcal{H}_{\varphi,H_1}(\mathcal{E}_1)\hat{S}^{-1}$ ∎

Now, we will concern ourselves with the problem of whether a factor of an ergodic and coalescent group extension enjoys coalescence property. Before answering the question we will need a lemma.

Lemma 2.6. *Let $\mathcal{E}\subseteq\tilde{\mathfrak{B}}$ be a factor of $T_\varphi:(X\times G,\tilde{\mathfrak{B}},\tilde{\mu})\circlearrowleft$. Assume that $\tilde{\mathfrak{B}}_H$ is the smallest natural factor containing $\mathcal{E}$. Suppose that $S_{f,v}$ and $S_{f',v'}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$. Then $v=v'$ and $f=f'$.*

Proof. From Proposition 0.6 it follows that $v=v'$ and moreover that $f'(x)=f(x)g_0H$, $x\in X$, for some $g_0H\in G/H$. However

$$S_{f,v}\circ(S_{f',v})^{-1}=S_{f,v}\circ S^{-1}_{v^{-1}}[(f'S^{-1})^{-1}],v^{-1}=\sigma_{g_0^{-1}H}\in\mathcal{H}_{\varphi,H}(\mathcal{E}).$$

From the definition of H it follows that $\sigma_{g_0^{-1}H}$ must be the identity on $\tilde{\mathfrak{B}}_H$. Hence, g_0H is the unit element of G/H and the proof is complete ∎

The answer to our problem is given by the following.

Theorem 2.3. *Let $T_\varphi:(X\times G,\tilde{\mu})\circlearrowleft$ be an ergodic group extension of a rotation $T:(X,\mu)\circlearrowleft$. Assume that all natural factors of T_φ are coalescent. Then each factor of T_φ is coalescent.*

Proof. Assume that $U:(\tilde{X}_1,\mathcal{E}_1,\tilde{\mu})\longrightarrow(\tilde{X}_2,\mathcal{E}_2,\tilde{\mu})$ establishes an isomorphism of two factors $\mathcal{E}_1,\mathcal{E}_2$, where in addition $\mathcal{E}_1\subseteq\mathcal{E}_2$. Let $\tilde{\mathcal{B}}_{H_i}$ be the smallest natural factor containing $\mathcal{E}_i$, $i=1,2$. Then $\tilde{\mathcal{B}}_{H_1}\subseteq\tilde{\mathcal{B}}_{H_2}$. From Theorem 2.2 we can extend U to an isomorphism of T_{φ,H_1} and T_{φ,H_2}. But T_{φ,H_1} is a natural factor of T_{φ,H_2} and hence, by our standing assumption of coalescence of T_{φ,H_2}, we must conclude that $H_1=H_2$. Moreover, from Corollary 2.1 it follows that there exists an automorphism $\hat{S}\in C(T_{\varphi,H_2})$ such that $\mathcal{H}_{\varphi,H_2}(\mathcal{E}_2)=\hat{S}\,\mathcal{H}_{\varphi,H_2}(\mathcal{E}_1)\hat{S}^{-1}$. But $\mathcal{E}_1\subseteq\mathcal{E}_2$ implies $\mathcal{H}_{\varphi,H_2}(\mathcal{E}_1)\supseteq\mathcal{H}_{\varphi,H_2}(\mathcal{E}_2)$. Hence, an application of Lemma 2.5 gives us immediately $\mathcal{H}_{\varphi,H_2}(\mathcal{E}_1)=$ $\hat{S}\,\mathcal{H}_{\varphi,H_2}(\mathcal{E}_1)\hat{S}^{-1}=\mathcal{H}_{\varphi,H_2}(\mathcal{E}_2)$. We conclude that $\mathcal{E}_1=\mathcal{E}_2$ ∎

Remark 2.5. It follows from Theorem 2.3 that if an ergodic coalescent group extension has a factor which is not coalescent then there must exist a natural factor without coalescence property. We will see such a situation in Section 5 (Corollary 5.3) ∎

Now, we would like to describe the measure-theoretic structure of the factors of an ergodic abelian group extension of a rotation.

Let $T_\varphi:(X\times G,\tilde{\mu})\circlearrowleft$ be an ergodic group extension of a rotation. Denote by $\pi:C(T_\varphi)\longrightarrow C(T)$ the map given by

$$\pi(S_{f,v})=S \text{ where } S_{f,v}\in C(T_\varphi).$$

Suppose that $\mathcal{H}\subseteq C(T_\varphi)$ is a closed subgroup. Denote

$$\mathcal{H}/\sim = \{\tilde{\pi}^{-1}(S) : S \in \pi(\mathcal{H})\}, \quad \tilde{\pi}^{-1}(S) = \pi^{-1}(S) \cap \mathcal{H},$$

$$\tilde{d}(\tilde{\pi}^{-1}(S), \tilde{\pi}^{-1}(S')) = \inf_{S_{f,v} \in \mathcal{H},\, S'_{f',v'} \in \mathcal{H}} d(S_{f,v}, S'_{f',v'}),$$

where d is a metric whose topology is just the weak topology.

Lemma 2.7. <u>A closed subgroup</u> $\mathcal{H} \subseteq C(T_\varphi)$ <u>is compact iff</u> $\pi(\mathcal{H})$ <u>is compact and</u> $\tilde{\pi}^{-1} : \pi(\mathcal{H}) \longrightarrow \mathcal{H}/\sim$ <u>is continuous.</u>

Proof. Assume that $\mathcal{H}$ is compact. Since π is continuous, $\pi(\mathcal{H})$ is also compact. Suppose that $\tilde{\pi}^{-1} : \pi(\mathcal{H}) \longrightarrow \mathcal{H}/\sim$ is not continuous at some $S \in \pi(\mathcal{H})$. Hence there exists a sequence $\{S^{(n)}\} \subseteq C(T)$, $\lim_{n\to\infty} S^{(n)} = S \in \pi(\mathcal{H})$, such that $\tilde{\pi}^{-1}(S^{(n)}) \not\to \tilde{\pi}^{-1}(S)$. Therefore, there must exist an $\varepsilon_0 > 0$ and a subsequence $\{k_n\}$ such that if $S^{(k_n)}_{f_{k_n}, v_{k_n}}$, $S_{f,v} \in \mathcal{H}$ then $d(S^{(k_n)}_{f_{k_n}, v_{k_n}}, S_{f,v}) \geq \varepsilon_0$. But $S^{(k_n)} \in \pi(\mathcal{H})$, so we can find some corresponding f_{k_n}, v_{k_n} to get $S^{(k_n)}_{f_{k_n}, v_{k_n}} \in \mathcal{H}$. By compactness of $\mathcal{H}$ we can select, from $\{S^{(k_n)}_{f_{k_n}, v_{k_n}}\}$ a subsequence which converges to a lifting of S and we obtain a contradiction.

Conversely, suppose that $\{S^{(n)}_{f_n, v_n}\} \subseteq \mathcal{H}$. Then, from the sequence $\{S^{(n)}\}$ we can choose a converging subsequence, say, $S^{(n)} \longrightarrow S$, since $\pi(\mathcal{H})$ is compact. By continuity of $\tilde{\pi}^{-1}$ we can find f'_n, v'_n such that $S^{(n)}_{f'_n, v'_n}$ converges. Now, by Proposition 0.6, $v'_n = v_n$ and $f'_n(x) = f_n(x) h_n$ for some $h_n \in G$. From the sequence $\{h_n\} \subseteq G$ we select a converging subsequence, say, $h_n \longrightarrow h$. Then, as $C(T_\varphi)$ is a topological group, we get $\{\sigma_{h_n^{-1}} S^{(n)}_{f'_n, v'_n}\}$ converges. Since $\sigma_{h_n^{-1}} S^{(n)}_{f'_n, v'_n} = S^{(n)}_{f_n, v_n}$, the result follows■

Let $\mathcal{C} \subseteq \tilde{\mathcal{B}}$ be a factor of T_φ. Assume that $T_{\varphi,H}$ is the smallest natural factor containing $\mathcal{C}$. Denote $\pi_H : C(T_{\varphi,H}) \to C(T)$,

$\pi_H(S_{f,v})=S$. From Lemma 2.6 it follows that

$$\mathrm{card}(\mathcal{H}_{\varphi,H}(\mathcal{E})\cap \pi_H^{-1}(S))=1$$

for every $S\in \pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E}))$. Therefore, $\mathcal{H}_{\varphi,H}(\mathcal{E})$ is a choice of one lifting of S, for each $S\in \pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E}))$. In particular, $\mathcal{H}_{\varphi,H}(\mathcal{E})$ must be abelian.

Lemma 2.8. $\pi_H^{-1}(\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E})))$ <u>is a compact subgroup of</u> $C(T_{\varphi,H})$.

Proof. Denote $\bar{\mathcal{E}}=\{A\in\mathcal{B}:\ (\forall S\in \pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E})))\quad S^{-1}A=A\}$. By Theorem 2.1, $\bar{\mathcal{E}}\subseteq\mathcal{E}$. Notice that

$$\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E}))=\{S\in C(T):\ (\forall A\in\bar{\mathcal{E}})\quad S^{-1}A=A\}.$$

For the automorphism $T_{\varphi,H}$ let us consider the relatively independent extension of the diagonal measure on $\mathcal{E}\otimes\mathcal{E}$, whose ergodic decomposition is given by

$$\tilde{\mu}\times_{\mathcal{E}}\tilde{\mu}=\int_{J_2^e(T_{\varphi,H})}\gamma\, d\tau(\gamma).$$

Then, by Proposition 1.1, we have $\gamma|\mathcal{B}\otimes\mathcal{B}=\mu_{S_\gamma}$, for some $S_\gamma\in C(T)$. But for each $A\in\bar{\mathcal{E}}$, $\tilde{\mu}\times_{\mathcal{E}}\tilde{\mu}(A\times A^c\cup A^c\times A)=0$, whence $\gamma(A\times A^c\cup A^c\times A)=0$ and finally $\mu_{S_\gamma}(A\times X\triangle X\times A)=0$ (for a.a γ). In other words, $S_\gamma^{-1}(A)=A$. By a choice of a countable family $\{A_n\}$ generating $\bar{\mathcal{E}}$ we can conclude that for almost all S_γ we have $S_\gamma^{-1}A=A$ for each $A\in\bar{\mathcal{E}}$. Hence $S_\gamma\in\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E}))$. But S_γ can be lifted to $C(T_{\varphi,H})$, so by Proposition 0.8, we obtain that $\gamma=\tilde{\mu}_{(S_\gamma)_{f_\gamma,v_\gamma}}$. Now, in the ergodic decomposition of $\tilde{\mu}\times_{\mathcal{E}}\tilde{\mu}$ only graph measures occur. Therefore, in view of (2), $T_{\varphi,H}$ is a compact group (the group equal to $\pi_H^{-1}(\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E})))$) extension of $\bar{\mathcal{E}}$ as $\bar{\mathcal{E}}=\{A\in\tilde{\mathcal{B}}_H:(\forall\hat{S}\in\pi_H^{-1}(\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E}))))\ \hat{S}^{-1}A=A\}$ ∎

Let $S_{f_S,v_S}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$, where $S\in\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E}))$. Denote

$$\tilde{G}_0=\{gH:\ (\forall S\in\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E})))\ v_S(gH)=gH\}.$$

Thus $\widetilde{G}_0$ is a subgroup of G/H.

Lemma 2.9. $gH\in\widetilde{G}_0$ iff $\sigma_{gH}:\mathcal{E}\circlearrowleft$.

Proof. Suppose that $g_0H\in\widetilde{G}_0$, $A\in\mathcal{E}$. We wish to show that $\sigma_{g_0H}(A)\in\mathcal{E}$. All we have to prove is that

$(\forall\hat{S}\in\mathcal{H}_{\varphi,H}(\mathcal{E}))\qquad \hat{S}\,\sigma_{g_0H}(A)=\sigma_{g_0H}(A).$

Notice that A is a measurable union of the "points" of the form

$$(*)\qquad \bigcup_{\hat{S}\in\mathcal{H}_{\varphi,H}(\mathcal{E})}\{\hat{S}(x,gH)\}\quad,\ (\ (x,gH)\in X\times G/H\ ).$$

We also observe that

$$\sigma_{g_0H}\Big(\bigcup_{\hat{S}\in\mathcal{H}_{\varphi,H}(\mathcal{E})}\{\hat{S}(x,gH)\}\Big)=\bigcup_{\hat{S}\in\mathcal{H}_{\varphi,H}(\mathcal{E})}\{\hat{S}(x,g'H)\},$$

for g' satisfying $v_S(g'H)=v_S(gH)g_0H$, where $\hat{S}=S_{f_S,v_S}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$. But $g_0H=v_S(g_0H)$ and therefore $g'H=gg_0H$ does not depend on the choice of $S\in\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E}))$. Therefore $\sigma_{g_0H}(A)$ is also a measurable union of the "points" of the form (*) and consequently $\sigma_{g_0H}(A)\in\mathcal{E}$.

Now, assume that $\sigma_{g_0H}:\mathcal{E}\circlearrowleft$. Take any $\hat{S}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$, $\hat{S}=S_{f_S,v_S}$. Then, $\sigma_{g_0H}\hat{S}(\sigma_{g_0H})^{-1}\in\mathcal{H}_{\varphi,H}(\sigma_{g_0H}(\mathcal{E}))=\mathcal{H}_{\varphi,H}(\mathcal{E})$. Consequently $\sigma_{g_1H}\hat{S}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$, where $g_1H=g_0H\cdot v_S(g_0^{-1}H)$. But $\hat{S}^{-1}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$, so $\sigma_{g_1H}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$. From Lemma 2.6 applied to σ_{g_1H} and Id we obtain that $g_1H=H$, equivalently, that $g_0H=v_S(g_0H)$ ∎

Lemma 2.10. _If_ $\widetilde{S}\in C(T_{\varphi,H}|\mathcal{E})$ _and_ $\widetilde{S}|\overline{\mathcal{E}}=\mathrm{Id}$ _then there exists_ $g_SH\in\widetilde{G}_0$ _such that_ $\widetilde{S}=\sigma_{g_SH}|\mathcal{E}$.

Proof. Assume that $\widetilde{S}\in C(T_{\varphi,H}|\mathcal{E})$. Then, by Theorem 2.2, there exists an extension $\hat{S}$ of $\widetilde{S}$ such that $\hat{S}\in C(T_{\varphi,H})$. Since $\widetilde{S}|\overline{\mathcal{E}}=\mathrm{Id}$, $\hat{S}\in\pi_H^{-1}(\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E})))$. In particular, $\hat{S}$ is invertible. For the $\hat{S}$ there is a unique $g_SH\in G/H$ such that $\hat{S}\,\sigma_{g_S^{-1}H}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$. For each $A\in\mathcal{E}$ we have $\hat{S}\sigma_{g_S^{-1}H}(A)=A$, equivalently,

$\hat{S}^{-1}(A)=(\sigma_{g_S H})^{-1}(A)$. But for $A\in\mathcal{E}$, $\mathcal{E}\ni\tilde{S}^{-1}(A)=\hat{S}^{-1}(A)=(\sigma_{g_S H})^{-1}(A)$. In view of Lemma 2.9, $g_S H\in\tilde{G}_0$ ■

Let M be a compact metric space with a probability Borel measure ν_1. Let $U:(\bar{X},\nu)\circlearrowleft$ be an automorphism. Assume that $(\theta_{\bar{x}})_{\bar{x}\in\bar{X}}$ is a family of isometries of M, measurable as the map $\theta_{\cdot}(\cdot):\bar{X}\times M\to M$. We recall that then the automorphism $\tilde{U}:(\bar{X}\times M,\ \nu\times\nu_1)\circlearrowleft$ acting by the formula

$$\tilde{U}(x,m)=(Ux,\theta_x(m))$$

is called an <u>isometric extension</u> of U. We will need the following lemma ([20]).

Lemma 2.11. <u>Let</u> $T:(X,\mathcal{B},\mu)\circlearrowleft$ <u>be an ergodic automorphism with discrete spectrum</u>. <u>Let</u> $\varphi:X\to K$ <u>be an ergodic cocycle, where</u> K <u>is a compact metric</u> (<u>not necessarily abelian</u>) <u>group</u>. <u>Assume that</u> $\lambda\in J_2^e(T_\varphi)$ <u>satisfies</u> $\lambda|\mathcal{B}\otimes\mathcal{B}=\mu_{Id}$. <u>Then</u> $\lambda=\tilde{\mu}_{\sigma_k}$ <u>for some</u> $k\in K$.

Proof We can repeat the arguments from the proof of Proposition 0.8 since there exists a lifting $\hat{S}$ of S=Id of the form $\hat{S}=S_{f,v}$, where $f(x)=e$ ($x\in X$), $v(k)=k$ ($k\in K$) ■

Proposition 2.1. <u>Let</u> $\mathcal{E}$ <u>be a factor of an ergodic group extension</u> $T_\varphi:(X\times G,\tilde{\mu})\circlearrowleft$. <u>Let</u> $T_{\varphi,H}$ <u>be the smallest natural factor containing</u> $\mathcal{E}$. <u>Denote</u>

$$\bar{\mathcal{E}}=\{A\in\mathcal{B}:\ (\forall S\in\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E})))\ S^{-1}A=A\}.$$

<u>Then,</u> $\mathcal{E}$ <u>is an isometric extension of</u> $\bar{\mathcal{E}}$. <u>Moreover, the following statements are equivalent</u>.

(i) $\mathcal{E}$ <u>is a compact group extension of</u> $\bar{\mathcal{E}}$.
(ii) $\mathcal{E}$ <u>is a G/H-extension of</u> $\bar{\mathcal{E}}$.
(iii) $\tilde{G}_0=G/H$.
(iv) $\mathcal{H}_{\varphi,H}(\mathcal{E})$ <u>is a normal subgroup of</u> $\pi_H^{-1}(\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E})))$.

Proof. Let us denote by $\overline{X}$ the quotient space corresponding to $\overline{\mathcal{E}}$. Let $M' = \pi_H^{-1}(\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E})))$ and $M = M'/\mathcal{H}_{\varphi,H}(\mathcal{E})$. Then by Lemma 2.8, its proof and (2) we obtain that $T_{\varphi,H}$ can be represented as M'-extension of $\overline{T}:(\overline{X},\mu)\circlearrowleft$ (here by $\overline{T}$ we denote the action of T on the quotient space $\overline{X}$). Hence, there exists a cocycle $\Psi:\overline{X}\longrightarrow M'$ such that $T_{\varphi,H}$ is isomorphic to $\overline{T}_\Psi$. Now, $\mathcal{E}$ is a factor of $\overline{T}_\Psi$ containing $\overline{\mathcal{E}}$ and determined by $\mathcal{H}_{\varphi,H}(\mathcal{E})$, a subgroup of M', hence $\mathcal{E}$ is a "natural" factor of $\overline{T}_\Psi$ given by the formula

$$(**) \qquad (\overline{x},\hat{S}\,\mathcal{H}_{\varphi,H}(\mathcal{E}))\longmapsto(\overline{T}\overline{x},[\Psi(\overline{x})\hat{S}]\,\mathcal{H}_{\varphi,H}(\mathcal{E})),$$

where $(\overline{x},\hat{S}\,\mathcal{H}_{\varphi,H}(\mathcal{E}))\in\overline{X}\times M$. Therefore the first part of the proof is complete.

We proceed to prove the equivalence of (i)-(iv).

(ii)-(i) is obvious.

(i)-(iii) Suppose that $\widetilde{G}_0 \subsetneq G/H$. Hence, by Lemma 2.9, there exist $g_0H\in G/H$ and a set $A\in\mathcal{E}$ of positive measure such that $(\sigma_{g_0H})^{-1}(A)\notin\mathcal{E}$. Let $\lambda\in J_2^e(T_{\varphi,H})$ be the graph measure corresponding to $\sigma_{g_0H}\in C(T_{\varphi,H})$. Take the restriction $\overline{\lambda}$ of λ, where now $\overline{\lambda}\in J_2^e(T_{\varphi,H}|\mathcal{E})$. As λ projected on $\mathcal{B}\otimes\mathcal{B}$ is diagonal (which means supported on the graph of the identity), $\overline{\lambda}$ projected on $\overline{\mathcal{E}}\otimes\overline{\mathcal{E}}$ is diagonal as well. If $\mathcal{E}$ were a compact group extension of $\overline{\mathcal{E}}$, $\overline{\lambda}$ would have to be a graph measure (Lemma 2.11). Equivalently, $\overline{\lambda}$ would have to identify the two marginals. However, for each set $B\in\mathcal{E}$ we have

$$\overline{\lambda}(B\times(X\times G/H)\,\triangle\,(X\times G/H)\times A)=\lambda(B\times(X\times G/H)\,\triangle\,(X\times G/H)\times A)=\widetilde{\mu}(B\,\triangle\,(\sigma_{g_0H})^{-1}(A))>0,$$

a contradiction.

(iii)-(ii) Since $\widetilde{G}_0=G/H$ and Lemma 2.9 holds, $\{\sigma_{gH}:\ gH\in G/H\}=\{\sigma_{gH}:\ \sigma_{gH}:\mathcal{E}\circlearrowleft,\ gH\in G/H\}$. This and Lemma 2.10 imply that

$\bar{\mathcal{E}}=\{A\in\mathcal{E}:\ (\forall gH\in G/H)\ \ \sigma_{gH}(A)=A\}$. Therefore, in view of (2), $\mathcal{E}$ is F-extension of $\bar{\mathcal{E}}$, where $F=\{\sigma_{gH}:\ gH\in G/H\}$. It is clear that F is topologically isomorphic to G/H.

(iii)→(iv) Assume that $\hat{S}=S_{f,v}\in M'$ and $S_{f_S,v_S}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$. Then, from Proposition 0.6, it follows that $v=v_S$, while $\tilde{G}_0=G/H$ implies $v_S=id$ for an arbitrary $S\in\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E}))$. We intend to prove that for an arbitrary $\hat{S}'=S'_{f',id}\in M'$, $S'_{f',id}S_{f_S,id}=S_{f_S,id}S'_{f',id}$. To this end let $g'H$ be chosen in such a way that

$S'_{f',id}=S'_{f_{S'},id}\,\sigma_{g'H}$, where $S'_{f_{S'},id}\in M'$. Since $\mathcal{H}_{\varphi,H}(\mathcal{E})$ is abelian,

$S'_{f',id}S_{f_S,id}=(S'_{f_{S'},id}\,\sigma_{g'H})S_{f_S,id}=(S'_{f_{S'},id}S_{f_S,id})\,\sigma_{g'H}=$
$S_{f_S,id}(S'_{f_{S'},id}\,\sigma_{g'H})=S_{f_S,id}S'_{f',id}$.

(iv)→(i) This is an immediate consequence of the first part of the proof (see (**))■

Remark 2.6. As it follows from the proof of Proposition 2.1, a factor $\mathcal{E}$ is an isometric extension of $\bar{\mathcal{E}}$, where the family of isometries live on $M=\pi_H^{-1}(\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E})))/\mathcal{H}_{\varphi,H}(\mathcal{E})$ In fact, M is homeomorphic to G/H. Indeed, let us define

$$\Phi: M\longrightarrow G/H \text{ by setting}$$
$$\Phi(\hat{S}\,\mathcal{H}_{\varphi,H}(\mathcal{E}))=gH \text{ if } \hat{S}\,\sigma_{gH}\in\mathcal{H}_{\varphi,H}(\mathcal{E}).$$

The corectness of the definition is easy to check. Moreover, Φ is one-to-one and onto. We will show that Φ is continuous. Assume that $\hat{S}^{(n)}\mathcal{H}_{\varphi,H}(\mathcal{E})\longrightarrow\hat{S}\,\mathcal{H}_{\varphi,H}(\mathcal{E})$. Hence, $\hat{S}^{(n)}\hat{U}^{(n)}\xrightarrow{n}\hat{S}\hat{U}$ for some $\hat{U}^{(n)},\hat{U}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$. Suppose that $\hat{S}^{(n)}\sigma_{g_nH}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$ and $\hat{S}\,\sigma_{gH}\in\mathcal{H}_{\varphi,H}(\mathcal{E})$. We then have $S^{(n)}U^{(n)}\longrightarrow SU$. But $\mathcal{H}_{\varphi,H}(\mathcal{E})$ is compact and $S,S^{(n)},U,U^{(n)}\in\pi_H(\mathcal{H}_{\varphi,H}(\mathcal{E}))$. Therefore, in view of Lemma 2.7, $\tilde{\pi}_H^{-1}(S^{(n)}U^{(n)})\longrightarrow\tilde{\pi}_H^{-1}(SU)$. Thus, $\hat{S}^{(n)}\sigma_{g_nH}\hat{U}^{(n)}\longrightarrow\hat{S}\,\sigma_{gH}\hat{U}$. For each set $A\in\mathcal{E}$ we have

$A = \hat{S}^{(n)} \sigma_{g_n H}(A)$. Therefore $\hat{S}^{(n)}(A) = \sigma_{g_n^{-1}H}(A)$. But $\hat{S}^{(n)}\hat{U}^{(n)} \longrightarrow \hat{S}\hat{U}$, so for each $A \in \mathcal{E}$ we obtain that $\hat{S}^{(n)}(A) \longrightarrow \hat{S}(A)$. We conclude that $\sigma_{g_n^{-1}H}(A) \longrightarrow \sigma_{g^{-1}H}(A)$ for each $A \in \mathcal{E}$. Let $\tilde{g}H$ be any cluster point of $\{g_n^{-1}HgH\}$. Hence $\sigma_{g_{n_k}^{-1}HgH} \longrightarrow \sigma_{\tilde{g}H}$ on $\tilde{\mathcal{B}}_H$ and in particular on $\mathcal{E}$. We have proved that $\sigma_{\tilde{g}H}$ is the identity on $\mathcal{E}$ and thus by Lemma 2.6, it is the identity on $\tilde{\mathcal{B}}_H$ ∎

3. AN EXAMPLE OF NONCOALESCENT ANZAI SKEW PRODUCT. In this section by $T:(X,\mathcal{B},\mu)\circlearrowleft$ we mean an <u>irrational rotation</u>, so $X=R/Z=[0,1)$ with addition mod 1, μ is Lebesgue measure and $Tx=x+\alpha$, $x\in X$, for an irrational number $\alpha\in[0,1)$. If $\varphi:X\circlearrowleft$ is a cocycle then the corresponding extension $T_\varphi:(X\times X,\tilde{\mu})\circlearrowleft$ is called an <u>Anzai skew product</u> ([3]). We will concern ourselves with the conjecture of B. Kamiński and J.P. Thouvenot that all ergodic Anzai skew products are coalescent. (Note that this statement is true in topological setting, [53].) Here we disprove the conjecture by constructing an ergodic Anzai skew product which is isomorphic to a natural factor of itself, hence noncoalescent. We will also show that no ergodic Anzai skew product can be represented as an (essentially) infinite self-joining of another automorphism.

If $t\in\mathbb{R}$ then by $\|t\|$ we denote the distance of t from the set Z of all integers. This means that $\|t\|=\min_{n\in Z}|t-n|$. Given $t_1,t_2\in\mathbb{R}$ let $n_i\in Z$ be determined by $\|t_i\|=|t_i-n_i|$, $i=1,2$. Then, obviously, $|(t_1+t_2)-(n_1+n_2)|\le|t_1-n_1|+|t_2-n_2|$ and therefore $\|t_1+t_2\|\le\|t_1\|+\|t_2\|$ for each $t_1,t_2\in\mathbb{R}$. We will consider the corresponding "norm" $\|\cdot\|$ on the circle, hence $\|\cdot\|$ is given by $\|x\|=\min\{x,1-x\}$ if $x\in[0,1)$. As before we have $\|x_1+x_2\|\le\|x_1\|+\|x_2\|$ for each $x_1,x_2\in X$.

Definition 3.1. We say that $(\alpha,\beta)\in[0,1)\times[0,1)$ has the property (*) if for any integer $k\neq 0,\pm1$ there exists an **ergodic** (with respect to the rotation by α) cocycle $\varphi:[0,1)\circlearrowleft$ such that the equation

(47) $$\varphi(x+\beta)+k\varphi(x)=f(x+\alpha)-f(x)$$

has a measurable solution $f:X\circlearrowleft$.

Remark 3.1. If (47) holds then T_φ is not coalescent as $S_{f,-v}$, $Sx=x+\beta$, $vy=ky$, is not invertible and commutes with T_φ. Immediately from the definition of (*) it follows that α and β must be irrational. (If β were rational, which means $S^m=Id$, then $(S_{f,-v})^m$ would have to be invertible: see the remark after Proposition 0.6, so $S_{f,-v}$ would also be invertible) ■

Theorem 3.1. There are two irrationals α and β such that the property (*) holds.

Proof. The proof consists of several steps.

Step 1. Construction of α.

We require α to satisfy the following. There exists a sequence of rationals $\{p_i/q_i\}$ such that

(48) $$0<\alpha-p_i/q_i=o(1/q_i^3),$$

$q_i=2^{n_i}$, p_i is odd number. Besides, $n_{i+1}>n_i$, $i\geq 1$, and

(48a) $$n_{i+1}-n_i \xrightarrow[i]{} \infty .$$

We will put some other restrictions on the sequence $\{n_i\}$ later. Notice that α is irrational since by Legendre theorem p_n/q_n, $n\geq 1$, are convergents of α and there are infinitely many of them.

Step 2. Construction of β.

Let $\{s_i\}_{i\geq 1}$ be a sequence of natural numbers given by

(49) $$s_1=1$$
$$s_{i+1}=2q_i+s_i.$$

Then s_i is an odd number, $i\geq 1$. We require the sequence $\{n_i\}$ to be chosen in such a way that

(50) $$\sum_{i\geq 1} s_i/q_i<+\infty$$

(then $s_{i+1}<3q_i$ for i large enough). Denote $\alpha_i=p_i/q_i$ and put

$$\beta=\lim_{i\to\infty} s_i\alpha_i,$$

where the limit is considered in $(X, ||\cdot||)$. The definition of β is correct as $||s_{i+1}\alpha_{i+1} - s_i\alpha_i|| \le ||s_{i+1}\alpha_{i+1} - s_{i+1}\alpha_i|| + ||s_{i+1}\alpha_i - s_i\alpha_i|| = ||s_{i+1}(\alpha_{i+1} - \alpha_i)|| \le s_{i+1}||\alpha_{i+1} - \alpha_i|| <$ $3q_i||\alpha_{i+1} - \alpha_i||$ for i large enough.

Moreover, $||\beta - s_i\alpha_i|| \le \sum_{j\ge 1} ||s_{i+j}\alpha_{i+j} - s_{i+j-1}\alpha_{i+j-1}|| \le \sum_{j\ge 1} 3q_{i+j-1}||\alpha_{i+j} - \alpha_{i+j-1}|| \le 3\sum_{j\ge 1}[q_{i+j-1}||\alpha - \alpha_{i+j}|| + q_{i+j-1}||\alpha - \alpha_{i+j-1}||]$. Hence

$$(51) \qquad ||\beta - s_i\alpha_i|| = o(1/q_i^2).$$

Indeed, $q_i^2||\beta - s_i\alpha_i|| \le 3\sum_{j\ge 1}[q_i^2 q_{i+j}||\alpha - \alpha_{i+j}|| + q_i^2 q_{i+j-1}||\alpha - \alpha_{i+j-1}||] =$ $q_i^3||\alpha - \alpha_i|| + 6\sum_{j\ge 1} q_i^2 q_{i+j}||\alpha - \alpha_{i+j}|| \le q_i^3||\alpha - \alpha_i|| + 6q_i^2\sum_{j\ge 1} c_{i+j}/q_{i+j}^2$, where $|c_{i+j}| \le M$ for i large enough and an arbitrary $j \ge 1$. Therefore, for i large enough, since $n_{i+j} < n_{i+j+1}$, $j \ge 1$, $q_i^2||\beta - s_i\alpha_i|| \le$ $q_i^3||\alpha - \alpha_i|| + 6Mq_i^2\sum_{j\ge 1} 1/2^{2n_{i+j}} \le q_i^3||\alpha - \alpha_i|| + 8M2^{2n_i}(1/2^{2n_{i+1}}) =$ $q_i^3||\alpha - \alpha_i|| + 8M(2^{n_i - n_{i+1}})^2$. In view of (48) and (48a) the latest quantity goes to zero as i goes to infinity.

Step 3. Definition of T_i and S_i.

We define $Tx = x + \alpha$, $Sx = x + \beta$ and

$$T_i(x) = x + \alpha_i, \quad S_i(x) = x + s_i\alpha_i \quad \text{for } x \in [0,1).$$

At the i-th step we will consider a tower W_i given by the partition

$$W_i = \{[0,1/q_i), T_i[0,1/q_i) = [p_i/q_i, (p_i+1)/q_i), \ldots, T_i^{q_i-1}[0,1/q_i)\}.$$

This tower has q_i levels and the action of T on W_i can be viewed as the action of T_i on W_i with some error which is estimated by (48) (see Fig. 3.1).

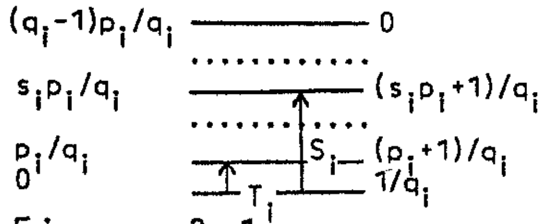


Figure 3.1.

The action of S_i on W_i is to send $[0,1/q_i)$ into $T_i^{s_i}[0,1/q_i)$. We will think about s_i as the "jump" of S_i. Hence, the jumps of S_i, $i \geq 1$, are summable by (50). Also, the action of S on W_i can be regarded as the action of S_i with an error estimated by (51).

Step 4. An infinite set of primes.

We fix an integer $k \neq 0, \pm 1$ and we intend to solve (47). Let $\mathcal{P}$ be an infinite set of primes such that for every $u \in \mathcal{P}$

(52) $u \geq 3$ and u does not divide k.

Let $\{u_i\}_{i \geq 1}$ be a sequence of the primes from $\mathcal{P}$ such that

(53) each $u \in \mathcal{P}$ appears infinitely many times in $\{u_i\}$.

Step 5. Construction of φ.

The definiton of φ is inductive. At the i-th step of the definition of φ we look at the i-th tower W_i. Then we take $T_i^{-1}[0,1/q_i), T_{i+1}^{-1}[0,1/q_{i+1}), T_{i+2}^{-1}[0,1/q_{i+2}), \ldots$ and we claim that φ has already been defined except for these intervals. More precisely, φ is defined on each interval $T_i^s[0,1/q_i)$, $s=0,\ldots,q_i-2$ and is constant equal to $a_s^{(i)} \in [0,1)$ on this interval except for possible smaller intervals of the form $T_{i+k}^{-1}[0,1/q_{i+k})$, $k \geq 1$, contained in $T_i^s[0,1/q_i)$ where φ is not defined yet (for not to stop the induction).

Later, we will compare the i-th step with the (i+1)-st one and we wish to neglect perturbations of the (i+1)-st step coming from the (i+s)-th steps, $s \geq 2$. This will be justified if the error we make is up to a set, say, A_i, satisfying $\sum_{i \geq 1} \mu(A_i) < +\infty$. This is made precise by the condition

$$\sum_{i \geq 1} (2^{n_{i+1}} \sum_{j \geq i+2} (\text{ the length of the interval in } W_j)) < +\infty ,$$

which means that besides (50) the sequence $\{n_i\}$ must satisfy

(54) $\sum_{i\geq 1} 1/2^{n_{i+1}-n_i} < +\infty$.

Now, fix $i \geq 1$. Without loss of generality we can assume that $s_{i+1} < 3q_i$ and that $0 < \alpha - p_j/q_j < 1/3q_j^2$ for all $j \geq i$. Let l_{i+1} be a natural number satisfying $p_{i+1}/q_{i+1} = p_i/q_i + l_{i+1}/q_{i+1}$. We will call $[0,1/q_i)$ the red interval, $T_i^{q_i-1}[0,1/q_i) = T_i^{-1}[0,1/q_i)$ the yellow interval and $T_{i+j}^{-1}[0,1/q_{i+j})$, $j \geq 1$, the green intervals (this does not exclude the possibility that some parts of the red and yellow intervals are in fact green).

In order to describe the induction step, we will look at the action of T_{i+1} and S_{i+1} on W_i (see Figure 3.2). We divide the red interval $[0,1/q_i)$ into the regular zone and the irregular zone. The regular zone Z_i consists of the red subintervals $[k/q_{i+1},(k+1)/q_{i+1})$ that after q_i-1 iterations of T_{i+1} hit the

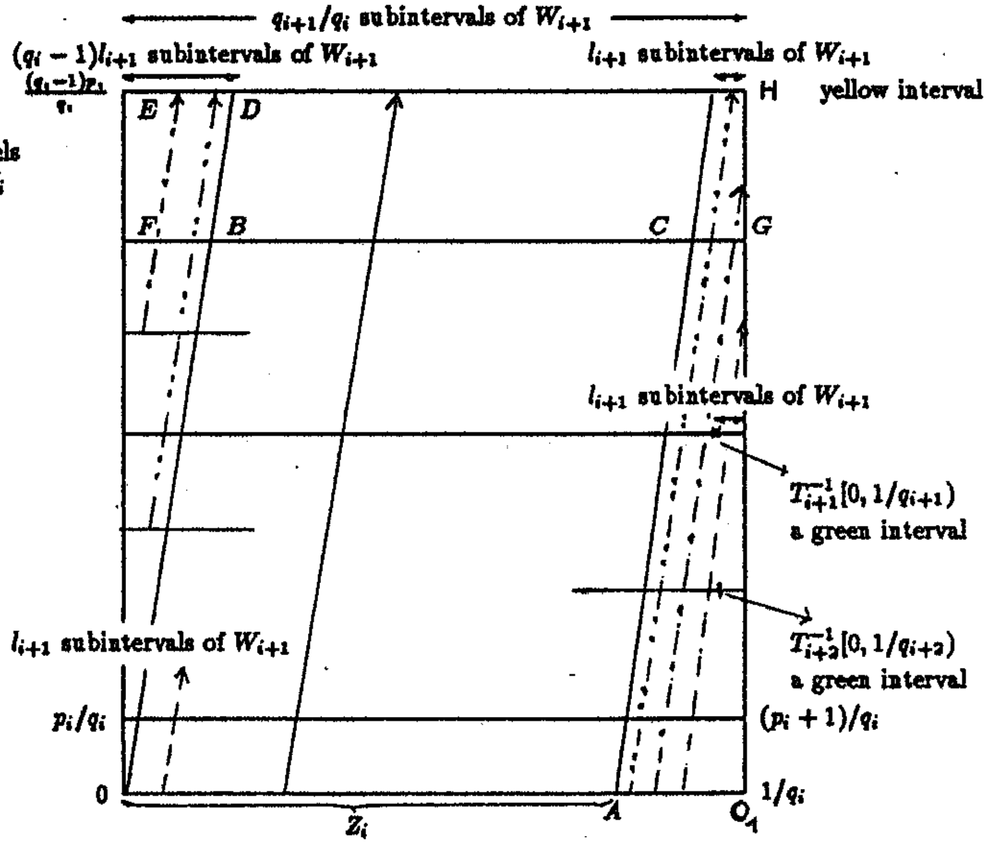


Figure 3.2.
→ regular trajectory of T_{i+1},
- - - → irregular trajectory of T_{i+1}, j=0,
-·-·→ irregular trajectory of T_{i+1}, j=1,
-··-··→ irregular trajectory of T_{i+1}, j=2.

yellow interval $T_i^{q_i-1}[0,1/q_i)$, the next iteration being red. Note that if we start with a red subinterval $[k/q_{i+1},(k+1)/q_{i+1})$ from the irregular zone then when looking at the trajectory $\{T_{i+1}^s[k/q_{i+1},(k+1)/q_{i+1})\}$ as long as we do not see another red subinterval, the number j of yellow subintervals contained in it is equal to 0,1 or 2 (see the three types of irregular trajectories in Fig. 3.2). The part of the tower W_i given by the area within OABC is called <u>regular area</u> and is denoted by $\mathcal{R}_i$. We have $\mathcal{R}_i = \bigcup_{r=0}^{q_i-s_i-1} T_{i+1}^r Z_i$. Then

(55) $$\sum_{i\geq 1} \mu(X \setminus \mathcal{R}_i) < +\infty.$$

Indeed, in view of (48), $q_i^2||\alpha - \alpha_i|| < \varepsilon_i/q_i$ and $q_{i+1}^2||\alpha - \alpha_{i+1}|| < \varepsilon_{i+1}/q_{i+1}$, where $\varepsilon_i \longrightarrow 0$. This gives us $q_i^2||\alpha - \alpha_{i+1}|| < \varepsilon_{i+1}/q_i$. Therefore, the area of ODE is bounded by $((q_i-1)l_{i+1}/q_{i+1})q_i \leq q_i^2 l_{i+1}/q_{i+1} = q_i^2||\alpha_i - \alpha_{i+1}|| < (\varepsilon_i + \varepsilon_{i+1})/q_i$ and moreover $\sum_{i\geq 1}(\varepsilon_i + \varepsilon_{i+1})/q_i < +\infty$. Similarly, the area $ACGO_1$ is bounded by δ_i such that $\sum \delta_i < +\infty$. The area of the rectangle EFGH is bounded by s_i/q_i which is summable by (50). Therefore (55) holds.

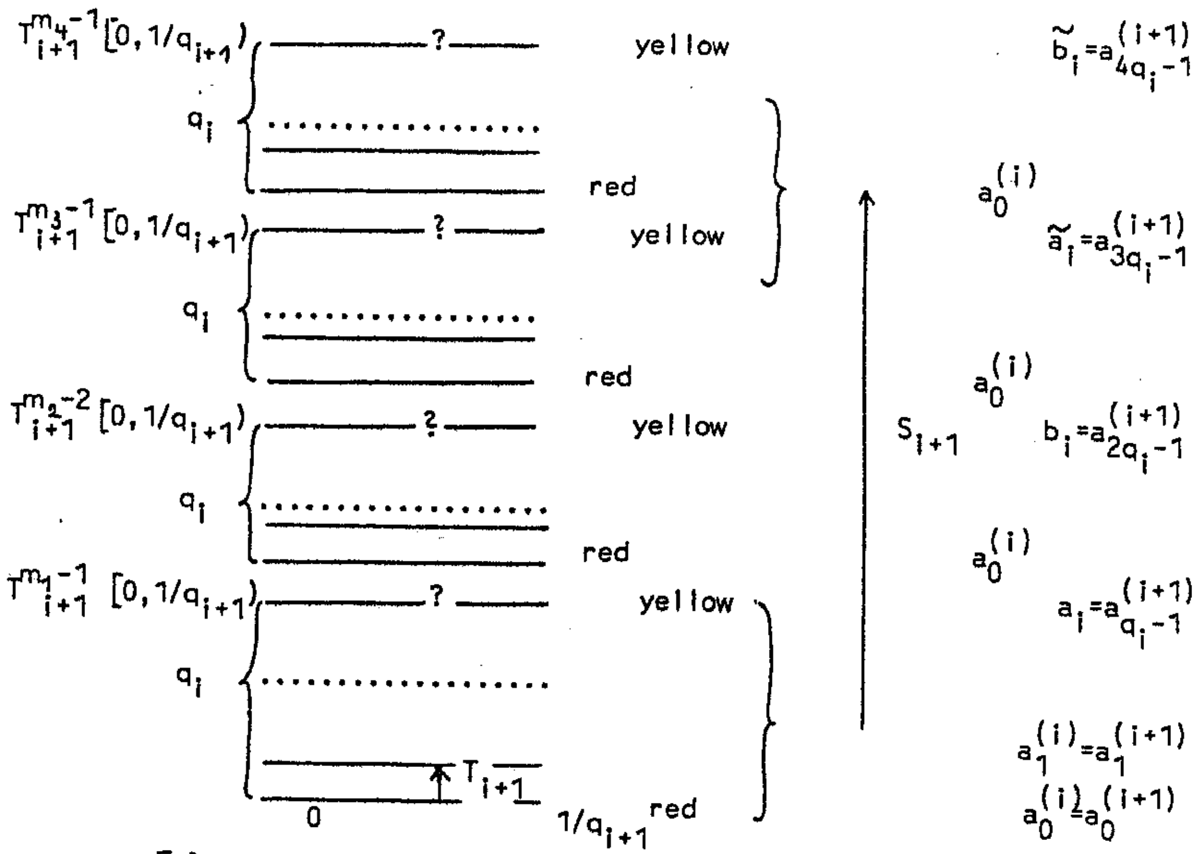
We proceed to a description of the induction step. We will look at the W_{i+1} and moreover the automorphism S_{i+1} will be involved in the procedure. We have to define φ on all yellow subintervals $[k/q_{i+1},(k+1)/q_{i+1})$ contained in $T_i^{q_i-1}[0,1/q_i)$ (except for the green part of them). First of all we take all irregular trajectories of T_{i+1} and for all yellow subintervals contained in those trajectories we define φ in an arbitrary way as a constant function on each yellow subinterval (for instance, we put zero). For those $x \in [0,1)$ where φ is defined at x and $S_{i+1}x$ we put

$$\Psi_{i+1}(x)=\varphi(S_{i+1}x)+k\varphi(x).$$

Let $m_1 < m_2 < \dots$ be the numbers of the consecutive red levels $[k/q_{i+1},(k+1)/q_{i+1})$ in W_{i+1}. We are going to define φ on the yellow subintervals in such a way that for some m_r's

(56) $\Psi_{i+1}(x)+\Psi_{i+1}(T_{i+1}x)+\dots+\Psi_{i+1}(T_{i+1}^{m_r-1}x)=0$

for all $x\in[0,1/q_{i+1})$. Moreover, the union of the corresponding m_r-th red levels of W_{i+1} almost covers the whole regular zone (see Fig. 3.3).



F i g u r e 3 . 3 .

We have to define φ on $T_{i+1}^{m_1-1}[0,1/q_{i+1})$, the first yellow interval of W_{i+1}. We let the constant value of φ on this interval (except for the green part, if any) be a_i, a u_i-th "root" of zero (see Step 4 for u_i). Then, in order to have (56) for r=1 we define φ also on $T_{i+1}^{m_3-1}[0,1/q_{i+1})$ as $\tilde{a}_i$ in such a way that (56) holds for r=1. We put $\varphi(\cdot)=b_i\neq a_i$, another

u_i-th "root" of zero, on $T_{i+1}^{m_2-1}[0,1/q_{i+1})$ and to get (56) for $r=2$ it is enough to know that

(57) $\Psi_{i+1}(x)+\Psi_{i+1}(T_{i+1}x)+\ldots+\Psi_{i+1}(T_{i+1}^{q_i-1}x)=0$

for all $x\in T_{i+1}^{q_i}[0,1/q_{i+1})$. Note that for such an x, $T_{i+1}^{q_i-1}x \in T_{i+1}^{m_2-1}[0,1/q_{i+1})$. Now, it suffices to define $\varphi(\cdot)=\tilde{b}_i$ on $T_{i+1}^{m_4-1}[0,1/q_{i+1})$ in such a way that (57) is satisfied.

Of course, $||a_i-b_i||=k(a_i,b_i)/u_i \geq 1/u_i$. We would like to argue that moreover

(58) $||\tilde{a}_i-\tilde{b}_i||=k(\tilde{a}_i,\tilde{b}_i)/u_i \geq 1/u_i$.

Toward this end we compute the values of φ on $T_{i+1}^{m_3-1}[0,1/q_{i+1})$ and $T_{i+1}^{m_4-1}[0,1/q_{i+1})$ we have already put. Immediately from Fig. 3.3 and the definition of the jump of S_{i+1} (Step 3) we obtain that for $x\in[0,1/q_{i+1})$, for an appropriate t, we have $\Psi_{i+1}(x)+\Psi_{i+1}(T_{i+1}x)+\ldots+\Psi_{i+1}(T_{i+1}^{m_1-1}x)=k(\varphi(x)+\varphi(T_{i+1}x)+\ldots+\varphi(T_{i+1}^{m_1-1}x))+\varphi(S_{i+1}x)+\varphi(S_{i+1}T_{i+1}x)+\ldots+\varphi(S_{i+1}T_{i+1}^{m_1-1}x)=k(a_0^{(i)}+a_1^{(i)}+\ldots+a_{q_i-2}^{(i)}+a_i)+(a_t^{(i)}+a_{t+1}^{(i)}+\ldots+a_{q_i-2}^{(i)}+\tilde{a}_i+a_0^{(i)}+\ldots+a_{t-1}^{(i)})=0$. But from (57), just by repeating the foregoing calculation (for $x\in T_{i+1}^{m_1}[0,1/q_{i+1})$) we get $k(a_0^{(i)}+a_1^{(i)}+\ldots a_{q_i-2}^{(i)}+b_i)+(a_0^{(i)}+a_1^{(i)}+\ldots+a_{q_i-2}^{(i)}+\tilde{b}_i)=0$. Hence, $||ka_i-kb_i||=||\tilde{a}_i-\tilde{b}_i||$ and in view of (52), the statement (58) is valid.

This procedure can be repeated as follows. Assume that (56) holds for some r and the red intervals $m_{r+1},\ldots,m_{r+4}$ are in Z_i. Assume, besides, that we have a_i,b_i, $a_i-b_i=k(a_i,b_i)/u_i \geq 1/u_i$, as the values of φ on $T_{i+1}^{m_{r+1}-1}[0,1/q_{i+1})$, $T_{i+1}^{m_{r+2}-1}[0,1/q_{i+1})$ respectively. Then, we define $\tilde{a}_i,\tilde{b}_i$ as above to conclude that φ has been defined on $T_{i+1}^{m_{r+3}-1}[0,1/q_{i+1})$ and on $T_{i+1}^{m_{r+4}-1}[0,1/q_{i+1})$ in such a way that (56) holds for m_{r+1},m_{r+2} with $||\tilde{a}_i-\tilde{b}_i||=k(\tilde{a}_i,\tilde{b}_i)/u_i \geq 1/u_i$. If r is so large that our procedure cannot be continued

(the red intervals $m_{r+1},\ldots,m_{r+4}$ are not all contained in Z_i) then we put whatever as the value of φ on the yellow interval immediately above the level m_r (unless φ is already defined there).

Now, we intend to describe the passage through an irregular trajectory (see Fig. 3.4). Suppose that φ has already been

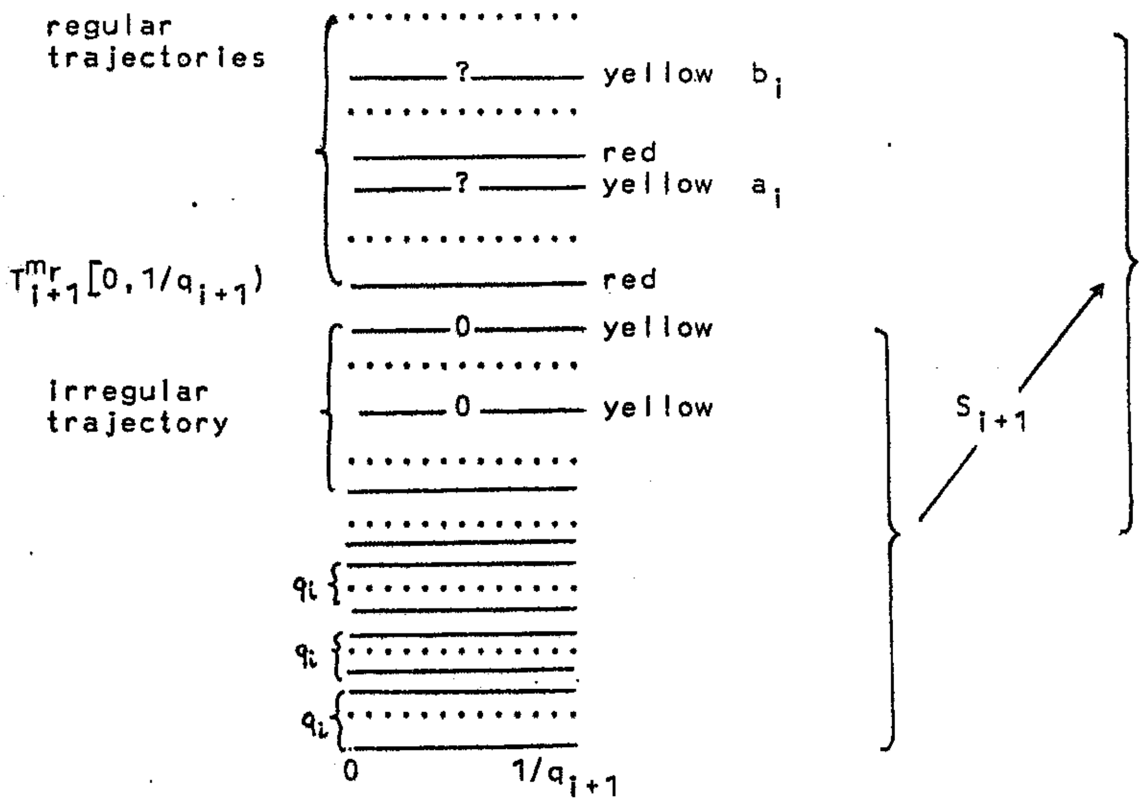


Figure 3.4.

defined on all yellow levels below the m_r-th (red) interval. We see that the image via S_{i+1} of the levels $0,1,\ldots,m_r-1$ of W_{i+1} contains two yellow levels (from some regular trajectories see Fig. 3.4 and 3.2) on which φ is not defined yet.We have some freedom in the choice of the values a_i,b_i on these two yellow intervals because if we want (56) to be true for m_r we have to take care only of a_i+b_i but not of a_i,b_i themselves. If a_i+b_i is to be equal to c ($\in[0,1)$) then we take

$$a_i=(1/u_i)+c/2,\quad b_i=(u_i-1)/u_i)+c/2$$

and we see that $||a_i-b_i||=(u_i-2)/u_i$, $k(a_i,b_i)=u_i-2 \geq 1$ and clearly (56) holds for m_r. Now, we can apply the procedure described before.

We continue defining φ until the level $q_{i+1}-1-s_{i+1}$. We put whatever on the remaining yellow levels above and we keep φ undefined on the green part of W_{i+1}.

As the measure of the green part tends to zero, φ is an (a.e.) well-defined measurable function from $[0,1)$ into itself.

Step 6. φ is ergodic.

Suppose φ is not ergodic. Then, by (15), there is an integer $t \neq 0$ and a measurable solution $f: X^2$ of

(59) $\quad t\varphi(x)=f(x+\alpha)-f(x)$.

Let $u \in \mathcal{P}$ be such that $u \nmid t$. In view of (53), the set M of all i's with $u_i=u$ is infinite. Fix an $\varepsilon > 0$. For some $i \in M$, there is an s, $0 \leq s \leq q_i-1$, such that

(60) $\begin{cases} \text{\underline{after discarding a set of measure} } \varepsilon/q_i \text{ \underline{from} } T^s[0,1/q_i) \\ \text{\underline{the remaining values of} } f \text{ \underline{are contained in a ball of radius} } \varepsilon. \end{cases}$

This is a consequence of the measurability of f. But, now, since (59) and (48) hold, the same is true for $T^{s+1}[0,1/q_i)$ with a slightly larger error, as φ and hence $t\varphi$ are "constant" on $T^s[0,1/q_i)$. By choosing $i \in M$ large enough, we can proceed up and down getting that (60) is satisfied on all the sets $T^s[0,1/q_i)$, $s=0,\dots,q_i-1$, with ε' equal, say, 2ε. But, we can reach most of $[0,1/q_i)$ through the "top" ($=T^{q_i-1}[0,1/q_i)$) just by looking at (59) for $x \in T^{q_i-1}[0,1/q_i)$.

Our definition of φ says that the top level of W_i has the property that when ignoring a small mass of $T_i^{q_i-1}[0,1/q_i)$ the remaining part splits into pairs A_j,B_j, $\mu(A_j)=\mu(B_j)$, $j=1,\dots,K$, of yellow subintervals in W_{i+1} such that if $x \in A_j$, $y \in B_j$ then $\|\varphi(x)-\varphi(y)\|=k(x,y)/u$, $k(x,y) \geq 1$. Therefore, since $u \nmid t$, $\|t\varphi(x)-t\varphi(y)\| \geq 1/u > 0$. As u is fixed, this, in view of (59), contradicts the property (60) of f on $[0,1/q_i)$.

Step 7. $\varphi S+k\varphi=fT-f$ can be solved.

We will prove this statement by exibiting a sequence $\{f_i\}$, $f_i:X\mathcal{R}$, of measurable functions such that

$$(61)\qquad \sum_{i\geq 1} \mu\{x\in X:\ f_i(x)\neq f_{i+1}(x)\} < +\infty,$$

$$(62)\qquad \mu\{x\in X:\ \varphi(S_i x)+k\varphi(x)=f_i(T_i x)-f_i(x)\} \xrightarrow[i]{} 1.$$

Then, from (61) we get $f_i \longrightarrow f$ a.e. and for f we will prove that (62) is still valid with f_i, S_i, T_i replaced by f,S,T, which means that $\mu\{x\in X:\ \varphi(Sx)+k\varphi(x)=f(Tx)-f(x)\}=1$.

At the i-th step we define f_i on W_i by first letting

$$(63)\qquad f_i=0 \text{ on } [0,1/q_i)$$

and then, using Ψ_i from Step 5, by

$$(64)\qquad f_i(T_i x)-f_i(x)=\varphi(S_i x)+k\varphi(x)=\Psi_i(x)\quad (x\in W_i\setminus T_i^{q_i-1}[0,1/q_i))$$

which allows us to extend f_i to the whole tower W_i. We appropriately restrict the regular zone Z_i to its subset $\tilde{Z}_i$ by discarding at most $s_{i+1}l_{i+1}$ subintervals of length $1/q_{i+1}$ and consider the corresponding subset $\tilde{\mathcal{R}}_i=\bigcup_{j=0}^{q_i-s_i-1} T_{i+1}^j\tilde{Z}_i$ (the area OA_2B_1B in Fig. 3.5). Then

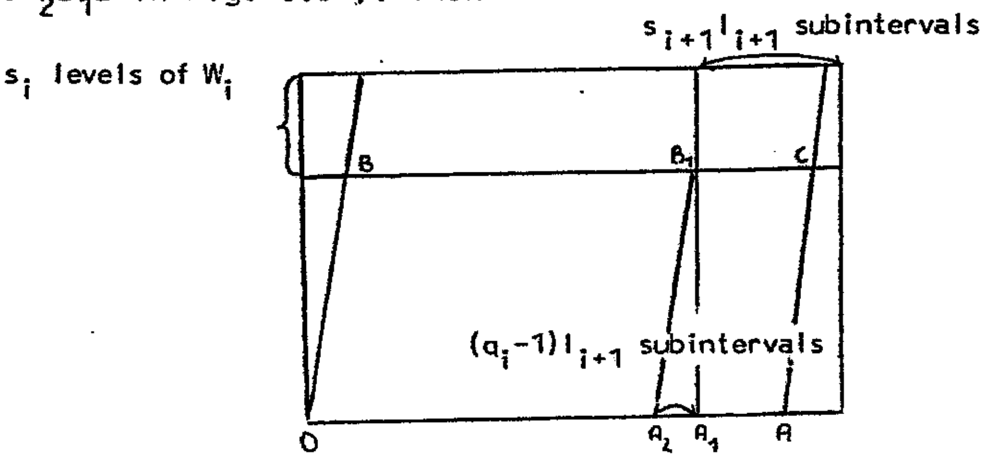


Figure 3.5.

(65) $\varphi(S_i x)=\varphi(S_{i+1}x)$

holds for $x\in\widetilde{\mathcal{R}}_i$ (except for the green part). As in (55) we still have

(66) $\sum_{i\geq 1}\mu(\widetilde{\mathcal{R}}_i^c)<+\infty.$

If we start with a red subinterval in $\widetilde{Z}_i$ whose number is m_r in W_{i+1} and (56) holds, then by (63) and (64) $f_{i+1}(T^{m_r}x)=\Psi_{i+1}(x)+\ldots+\Psi_{i+1}(T_{i+1}^{m_r-1}x)=0$, $x\in[0,1/q_{i+1})$ and then, by (64) and (65), $f_{i+1}=f_i$ on $\widetilde{\mathcal{R}}_i$ (except for the green part). Therefore, from (66), it follows that (61) is satisfied. Now, in view of (63) and (64), (62) holds. By (48), (51) and (54) we obtain that $\sum_{i\geq 1}\mu\{x:\ \varphi(Sx)\neq\varphi(S_i x)\}<+\infty$, $\sum_{i\geq 1}\mu\{x:\ f_i(T_i x)\neq f_i(Tx)\}<+\infty$, since φ and f_i are "constant" on almost all levels of W_i. Therefore, by (62), $\mu(Y_i)\longrightarrow 1$, where $Y_i=\{x:\ \varphi(Sx)+k\varphi(x)=f_i(Tx)-f_i(x)\}$. Consequently, $\mu(\limsup Y_i)=1$. But we have $\varphi(Sx)+k\varphi(x)=f(Tx)-f(x)$ for a.a. x in $\limsup Y_i$. The proof of Theorem 3.1 is complete■

Remark 3.2. Let (α,β) have the property (*). Let $\varphi:X\ \mathcal{R}$ be an ergodic continuous cocycle satisfying the assertion of Theorem 3.1 for some $k\neq 0,\pm 1$. (The existence of such a cocycle follows from [28].) Since φ is ergodic, the homeomorphism T_φ is minimal (see [9], [51]). Now, the (continuous) cocycle $\Psi=\varphi S+k\varphi$ is coboundary. However, for an arbitrary $t\neq 0$ we cannot solve the equation

$$t\varphi(x)=g(Tx)-g(x)$$

in continuous g, as otherwise the minimal homeomorphism $T_{t\varphi}$ would not be topologically coalescent contradicting a result of [53]. In particular, T_φ is a minimal homeomorphism which is not strictly ergodic (cf. [9], [52] p. 84)■

It should be noted that Anzai skew products are essentially different from infinite self-joinings.

Given an ergodic automorphism $\tau:(Y,\mathcal{C},\nu)\circlearrowleft$ and $\lambda\in J^e_\infty(\tau)$, we say that λ is <u>essentially infinite</u> if, for every $i\geq 1$,

$$\mathcal{C}_1\otimes\ldots\otimes\mathcal{C}_i\times X\times X\times\ldots \overset{(\lambda)}{\subsetneq} \mathcal{C}_1\otimes\ldots\otimes\mathcal{C}_i\otimes\mathcal{C}_{i+1}\times X\times X\times\ldots.$$

Let F be a compact metric abelian group. F is said to satisfy <u>descending chain condition</u> (d.c.c.) if for any sequence $\{F_i\}_{i\geq 1}$ of closed subgroups of F,

$$F_1\supseteq F_2\supseteq\ldots \text{ implies } F_{i_0}=F_{i_0+1}=\ldots \text{ for some } i_0.$$

Note that if F satisfies d.c.c. then (up to a topological group isomorphism) F is a closed subgroup of a finite-dimensional torus.

P r o p o s i t i o n 3 . 1 <u>Let</u> $T:(X,\mathcal{B},\mu)\circlearrowleft$ <u>be an ergodic rotation on a compact monothetic group</u>. <u>Let</u> G <u>be a compact metric and abelian group</u>. <u>Assume that</u> $\varphi:X\longrightarrow G$ <u>is an ergodic cocycle</u>. <u>Let</u> $\tau:(Y,\mathcal{C},\nu)\circlearrowleft$ <u>be an ergodic automorphism for which</u> T_φ <u>is isomorphic to</u> $\tau\times\tau\times\ldots:(Y\times Y\times\ldots,\mathcal{C}\otimes\mathcal{C}\otimes\ldots,\lambda)\circlearrowleft$ <u>for some</u> $\lambda\in J^e_\infty(\tau)$. <u>Then</u> λ <u>i s n o t essentially infinite as soon as</u> X <u>and</u> G <u>satisfy d.c.c.</u> . <u>In particular, no ergodic Anzai skew product is isomorphic to an essentially infinite self-joining</u>.

P r o o f . Suppose, on the contrary, that for some $\tau:(Y,\mathcal{C},\nu)\circlearrowleft$ and $\lambda\in J^e_\infty(\tau)$, essentially infinite, we have an isomorphism I of T_φ and $\tau\times\tau\times\ldots:(Y\times Y\times\ldots,\lambda)\circlearrowleft$. Denote

$$\mathcal{A}_i=I^{-1}\{Y\times\ldots\times Y\times\mathcal{C}_i\times Y\times\ldots\}$$

and let $\tilde{\mathcal{B}}_{H_i}$ be the smallest natural factor containing $\mathcal{A}_i$, $i\geq 1$. Assume that $\tilde{\mathcal{A}}_i$ is the smallest T_φ-invariant sub-σ-algebra generated by $\mathcal{A}_1,\ldots,\mathcal{A}_i$, $i\geq 1$. Define a sequence of closed subgroups of G by $F_1=H_1$, $F_{i+1}=H_{i+1}\cap F_i$, $i\geq 2$. Thus, $\tilde{\mathfrak{B}}_{F_i}$ is the smallest natural factor containing $\tilde{\mathfrak{A}}_i$, $i\geq 1$. However, $F_1\supseteq F_2\supseteq \dots$, so by d.c.c. of G we obtain that there exists i_0 such that $\tilde{\mathfrak{B}}_{F_{i_0}}=\tilde{\mathfrak{B}}_{F_{i_0+1}}=\dots$. Hence, necessarily, $F_{i_0}=\{e\}$ ($\tilde{\mathfrak{B}}_{F_{i_0}}=\tilde{\mathfrak{B}}$) as otherwise all the factors $\mathfrak{A}_i$, $i\geq 1$, are contained in $\tilde{\mathfrak{B}}_{F_{i_0}}$ and hence the smallest factor containing all the $\mathfrak{A}_i$'s is contained in $\tilde{\mathfrak{B}}_{F_{i_0}}$. We conclude that the smallest natural factor containing $\tilde{\mathfrak{A}}_{i_0+k}$, $k\geq 0$, is just $\tilde{\mathfrak{B}}$. From Theorem 2.2 it follows that $\tilde{\mathfrak{A}}_{i_0+k}$ is determined by $\mathcal{H}_{\varphi,\{e\}}(\tilde{\mathfrak{A}}_{i_0+k})=\mathcal{H}_{\varphi}(\tilde{\mathfrak{A}}_{i_0+k})$, $k\geq 0$. As λ is essentially infinite, $\tilde{\mathfrak{A}}_{i_0+k}\subsetneq\tilde{\mathfrak{A}}_{i_0+k+1}$, hence $\mathcal{H}_{\varphi}(\tilde{\mathfrak{A}}_{i_0+k})\supsetneq\mathcal{H}_{\varphi}(\tilde{\mathfrak{A}}_{i_0+k+1})$, $k\geq 0$. Since Lemma 2.6 holds, $\pi\mathcal{H}_{\varphi}(\tilde{\mathfrak{A}}_{i_0+k})\supsetneq\pi\mathcal{H}_{\varphi}(\tilde{\mathfrak{A}}_{i_0+k+1})$. However, the sequence $\{\pi\mathcal{H}_{\varphi}(\tilde{\mathfrak{A}}_{i_0+k})\}_{k\geq 0}$ is a decreasing sequence of closed subgroups of C(T). Therefore, C(T) cannot have d.c.c. . But, in view of (8), C(T) is topologically isomorphic to X and consequently X has no d.c.c., a contradiction■

We conclude this section with a few questions we have been unable to answer.

Question 3.1. Assume that φ is an ergodic coalescent Anzai cocycle. Is then $k\varphi$, $k\geq 2$, coalescent? (Compare with Corollary 5.3, Theorem 2.3 and Proposition 4.2)

Question 3.2. Do there exist two ergodic Anzai cocycles such that the corresponding Anzai skew products are weakly isomorphic but not isomorphic?

Question 3.3. What is the "size" of the set of all pairs (α,β) satisfying (*)? In particular, does the projection on the first coordinate give all irrationals?

4. COALESCENCE OF ANZAI SKEW PRODUCTS.

In this section we will deal with some well-known examples of Anzai skew products and will show that they are coalescent. Also, we exhibit a condition that guaratnees the coalescence property of the direct product of two disjoint Anzai skew products. In particular, this condition will be satisfied for our examples.

Let $X=[0,1)$ and T be the irrational rotation by α. By a quasi-coboundary cocycle $\varphi: X\circlearrowleft$ we mean a cocycle of the form

(67) $\varphi(x)=\lambda+f(x+\alpha)-f(x)$

for a measurable $f: X\circlearrowleft$ and some $\lambda\in[0,1)$. Given $n\in N$, $\varepsilon>0$ we define

$$\varphi^{(n)}(x)=\varphi(x)+\varphi(Tx)+\ldots+\varphi(T^{n-1}x)$$

and

$$a_n=\int \exp 2\pi i\, \varphi^{(n)}(x)\,d\mu(x).$$

Put

(68) $A_{\varepsilon}^{(n)}=\{x\in X:\ |\exp 2\pi i\, \varphi^{(n)}(x)-a_n|<\varepsilon\}$.

Lemma 4.1. $\lim_{\|n\alpha\|\to 0}\mu(A_{\varepsilon}^{(n)})=1$ *as soon as* φ *is quasi-coboundary*.

Proof. From (67) it follows that $\varphi^{(n)}(x)=n\lambda+f(T^n x)-f(x)$. Then, we have $\int_X|\exp 2\pi i f(T^n x)-\exp 2\pi i f(x)|^2 d\mu(x)$

$$=\int_X|\exp 2\pi i f(T^n x)|^2 d\mu(x)+\int_X|\exp 2\pi i f(x)|^2 d\mu(x)$$

$$-2\mathrm{Re}\int_X \exp 2\pi i f(T^n x)\overline{\exp 2\pi i f(x)}\,d\mu(x)=2\Big(1-\mathrm{Re}\int_X \exp 2\pi i f(T^n x)\overline{\exp 2\pi i f(x)}\,d\mu(x)\Big).$$

Moreover, the map $H_g: X\longrightarrow L^2(X,\mu)$ given by $H_g(x)(y)=g(x+y)$ is continuous whenever $g\in L^2(X,\mu)$. Therefore

$\lim_{\|n\alpha\|\to 0}\mathrm{Re}\int_X \exp 2\pi i f(T^n x)\overline{\exp 2\pi i f(x)}\,d\mu(x)=1$. But

$\mathrm{Re}\int_X \exp 2\pi i f(T^n x)\overline{\exp 2\pi i f(x)}\,d\mu(x)\le|a_n|\le 1$ which implies

$\lim_{||n\alpha||\to 0} |a_n|=1$. We conclude that $\exp 2\pi i\,\varphi^{(n)} - a_n \xrightarrow[||n\alpha||\to 0]{} 0$ in $L^2(X,\mu)$ and consequently in measure ∎

Remark 4.1. If φ is coboundary then from the proof of Lemma 4.1 we can deduce that $\lim_{||n\alpha||\to 0} a_n=1$ and hence $\mu(\tilde{A}_\varepsilon^{(n)}) \xrightarrow[||n\alpha||\to 0]{} 1$, where

(69) $$\tilde{A}_\varepsilon^{(n)}=\{x\in X:\ |\exp 2\pi i\,\varphi^{(n)}(x) - 1|<\varepsilon\}$$ ∎

We will consider three examples of Anzai skew cocycles $\varphi: X\to\mathbb{R}$ such that

(70) $(t\varphi)S + k\varphi$ <u>is not quasi-coboundary</u>

for each $S\in C(T)$, for each $k\in Z$, $k\neq 0,\pm 1$ and for each integer t, $|t|\neq|k|$. In particular, such a φ is ergodic since $t\varphi$ is not coboundary for $t\neq 0$ and also φ is coalescent since $\varphi S+k\varphi$ is not coboundary for each $S\in C(T)$ and $|k|\geq 2$.

Example 4.1. Assume that α admits an approximation by rationals p_i/q_i, $i\geq 0$, with speed $o(1/q_i^2)$. Let β be chosen in such a way that

(71) $$\min_j |\beta - j/q_n|\geq c/q_n$$

for a constant $c>0$ and infinitely many $n\geq 0$. Such a β can be constructed as the intersection of a descending sequence of closed intervals according to (71). By passing to a subsequence, we may assume that (71) holds for every n. We define $\varphi: X\to\mathbb{R}$ as $\varphi(x)=0$ if $x\in[0,\beta)$ and $\varphi(x)=\varrho$ if $x\in[\beta,1)$, where ϱ is irrational, $\varrho\in[0,1)$.

Denote $\Psi(x)=t\varphi(Sx)+k\varphi(x)$ and suppose that

(72) $$\Psi(x)=\lambda + f(x+\alpha) - f(x)$$

for a measurable $f: X\to\mathbb{R}$, $\lambda\in X$ and $S\in C(T)$ with $Sx=x+\gamma$. Then $\Psi^{(n)}(x)=t\varphi^{(n)}(Sx)+k\varphi^{(n)}(x)$. Also, we have $||q_n\alpha||\to 0$. We will look at the points of discontinuity of $\psi^{(q_n)}$. They are of the form $\{x_j: j=0,\dots,2q_n-1\}$, where $x_j=-i\alpha$ or $\beta-i\alpha$, $0\le i\le q_n-1$ and $\{y_l: l=0,\dots,2q_n-1\}$, where $y_l=-\gamma-m\alpha$ or $\beta-\gamma-m\alpha$, $0\le m\le q_n-1$. Because of the speed of approximation of α by $\{p_i/q_i\}$ and (71),

$$\min_{0\le i\ne j\le 2q_n-1} \|x_i - x_j\| \ge c'/q_n$$

for a constant $c\ge c'>0$ and n large enough. Therefore the same is true for $\{y_l\}$, that is

(73) $$\min_{0\le l\ne r\le 2q_n-1} \|y_l - y_r\| \ge c'/q_n.$$

For each x_i, $i=0,\dots,2q_n-1$, we take the interval $C_i=(x_i-c'/4q_n, x_i+c'/4q_n)$. These intervals are pairwise disjoint. Let us pick one i. If there are no other points of discontinuity of $\psi^{(q_n)}$ inside C_i then only one interval $(x_i-c'/4q_n, x_i)$, $(x_i, x_i+c'/4q_n)$ can be included in $A_\varepsilon^{(q_n)}$ (whenever ε is small enough) as the jump of $\psi^{(q_n)}$ at x_i is $k\rho$ or $-k\rho$ and $k\ne 0$. Inside C_i there can be at most one point y_r since (73) holds. Then from the three intervals only one can be included in $A_\varepsilon^{(q_n)}$ since the jump of $\psi^{(q_n)}$ at x_i is $k\rho$ or $-k\rho$ while at y_r it is equal to $t\rho$ or $-t\rho$ and $|t|\ne|k|$. If $y_r=x_i$ then still this is a point of discontinuity of $\psi^{(q_n)}$ as the jumps of $t\varphi$S and $k\varphi$ are different. As a conclusion we have that at least $2q_n(c'/q_n)$ $=c'/2$ of the mass of X is outside of $A_\varepsilon^{(q_n)}$ whenever ε is small enough and n is large enough. This contradiction to Lemma 4.1 proves our statement ■

Remark 4.2. A.E. Robinson in [56] has proved that for each α considered in Example 4.1 the set of those cocycles $\varphi: X\mathbb{R}$ such that T_φ has simple spectrum is residual in the topology given by the metric

$$d(\varphi,\varphi')=\int_X ||\varphi(x)-\varphi'(x)||d\mu(x).$$

Therefore, by Proposition 0.1, the set of those $\varphi:X\circlearrowleft$ that T_φ is coalescent is residual ■

We have been unable to answer the following question.

Question 4.1. Is the coalescence property a "typical" property for cocycles over each irrational rotation?

Example 4.2. Assume that α has bounded partial quotients. Let $\beta\in[0,1)$ and $\beta\notin Z\alpha$. Then, it is well-known (see [43],[64]) that the cocycle φ defined by $\varphi(x)=0$ if $x\in[0,\beta)$ and $\varphi(x)=\varrho$ if $x\in[\beta,1)$, where ϱ is irrational, is ergodic. Moreover, if we look at the proof of Theorem 2.4 [43], then we can see that for the points of discontinuity of $\varphi^{(q_n)}$, $\{x_i:\ i=0,\dots,2q_n-1\}$, we have

$$\min_{0\le i\ne j\le 2q_n-1} ||x_i-x_j||\ge c/q_n$$

for a constant $c>0$, where $\{q_n\}$ is the sequence of denominators of α ($||q_n\alpha||\to 0$). Therefore, we can repeat the arguments used in Example 4.1 to conclude that the corresponding Anzai skew product satisfies (70) ■

Example 4.3. For $x\in X$, $t\in R$ such that $x=t+Z=\bar{t}$ we set

$$||t||=||x|| \quad\text{and}\quad \langle t\rangle=t-[t].$$

Assume that $\varphi:X\circlearrowleft$ is continuous. Then, there exists a continuous map $a:[0,1]\longrightarrow R$ satisfying $a(0)=0$, $a(1)\in Z$ and

$$\varphi(\bar{t})=\varphi(\bar{0})+\overline{a(t)}.$$

The value $a(1)=d$ is called the <u>degree</u> $d(\varphi)$ of φ.

Lemma 4.2. ([9]) *If φ is uniformly Lipschitz continuous and $d(\varphi)\neq 0$ then the homeomorphism T_φ is uniquely ergodic (with the unique measure $\mu\times\mu$)* ■

Proposition 4.1. *If φ is uniformly Lipschitz continuous and $d(\varphi)\neq 0$ then T_φ is coalescent (actually (70) holds) and moreover if $S_{f,v}\in C(T_\varphi)$ then $v=id$.*

Proof. Assume that $(t\varphi)S+k\varphi=fT-f$. Then the degree of the cocycle $x\longmapsto t(\varphi(x+\beta))+k\varphi(x)$, where $Sx=x+\beta$, is equal to $(k+t)d(\varphi)$ and this cocycle is uniformly Lipschitz continuous. An easy application of Lemma 4.2 gives all the statements ■
■

Question 4.2. Assume that φ is uniformly Lipschitz continuous with $d(\varphi)=0$ and let φ be ergodic. Is then T_φ coalescent?

Proposition 4.2. *Let $\varphi:X\circlearrowleft$ be an ergodic cocycle given by Examples 4.1-3. Then all factors of T_φ are coalescent.*

Proof. In view of Theorem 2.3, it is enough to show that all natural factors of T_φ are coalescent. Let $Z_k=\{0,1/k,\dots,(k-1)/k\}$. We will prove that the natural factor T_{φ,Z_k} is isomorphic to $T_{k\varphi}$, $k\neq 0$. Indeed, the map $\Theta:X\times X/Z_k\longrightarrow X\times X$, given by $\Theta(x,y+Z_k)=(x,ky)$ is correctly defined, invertible and moreover $\Theta T_{\varphi,Z_k}=T_{k\varphi}\Theta$. Now, from (70) and Proposition 0.5 we can see that $T_{k\varphi}$, $k\neq 0$ is coalescent. Hence, all natural factors of T_φ are coalescent ■

Example 4.4. Below, we explore the results of [16] and [39] to prove that if $a(\cdot)$ is smooth enough, the cocycle φ has nonzero degree and α has bounded partial quotients then φ is cohomologous to an affine cocycle.

Let B be the set of irrational numbers α such that

$\sum_{i \geq 0} a_{i+1}/q_i < +\infty$, where $[0;a_1,a_2,\ldots]$ is the continuous fraction expansion of α and $\{q_i\}$ is the sequence of denominators of α . Note that all partial bounded quotients irrationals are contained in B and that the set B has full Lebesgue measure.

Proposition 4.3. *Let $\varphi:[0,1] \to R$ be a continuously differentiable map such that*

(i) $\int_0^1 \varphi(t)dt=0$,

(ii) φ' *is uniformly Lipschitz continuous*,

(iii) $\varphi'(0)=\varphi'(1)$,

(iv) $d=\varphi(1)-\varphi(0) \in Z\setminus\{0\}$.

Let $\alpha \in B$ *and* $Tx=x+\alpha$. *Put* $\varphi(\bar{t})=\overline{\varphi(t)}$, $t \in [0,1]$. *Then there exists* $c \in [0,1)$ *such that the circle cocycle* $x \longmapsto \varphi(x)-(dx+c)$ *is coboundary*.

Proof. Let $\varphi_1(x)=\varphi(x)-dx$, $x \in [0,1]$. Let $\Psi:R2$ be the continuous lifting of φ_1 (see Figure 4.1) given by

(74) $\Psi(t)=\varphi_1(t-[t])+[t](\varphi_1(1)-\varphi_1(0))=\varphi_1(t-[t])$.

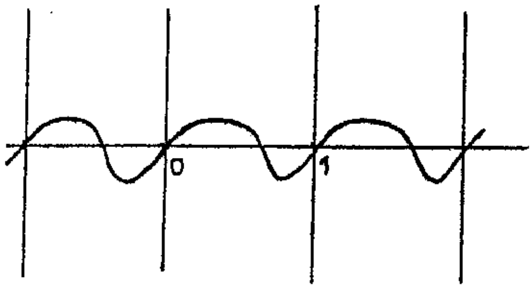


Figure 4.1.

Now, Ψ is periodic of period 1 and Ψ' is uniformly Lipschitz continuous. Moreover, $\Psi'(1)-\Psi'(0)=0$. Denote

(75) $c=\int_0^1 \Psi(x)dx$.

Let $\Psi^*(x)=\Psi(x)-c$. Then, from [16] the sequences

$$N \longmapsto \sum_{n<N} \Psi^*(\langle n\alpha+x\rangle)$$

are bounded for all $x \in [0,1]$. This implies (actually the boundness for an x is enough) that Ψ^* is a coboundary cocycle with a bounded solution, this meaning that there exists a (bounded) measurable, periodic of period 1 map $F:R\circlearrowleft$ such that

(76) $\Psi^*(x)=F(x+\alpha)-F(x)$ (see [39], [71]).

Hence, by (74) and (76), for $f:X\circlearrowleft$ induced by F we obtain that $\Psi_1(x)-c=f(x+\alpha)-f(x)$ (mod 1). From this it follows that $\Psi(x)-(dx+c)=f(x+\alpha)-f(x)$ for a measurable $f:X\circlearrowleft$ which completes the proof ■

■

Now, we pass to the problem of coalescence of the direct products of disjoint Anzai skew products.

As we have seen in Examples 4.1-3 the φ's considered there satisfy (70), hence are ergodic. Actually, all those cocycles are weakly mixing. Indeed, suppose that $F(T_\varphi(x,y))=\exp(2\pi ic)F(x,y)$ for some $c \in [0,1)$ and $\exp 2\pi ic \notin Sp(T)$. Since T_φ is ergodic, F is not a function of the first variable alone. Moreover,

(77) $L^2(X\times X,\mu\times\mu)=\bigoplus_{n=-\infty}^{+\infty}\exp(2\pi iny)L^2(X,\mu).$

Therefore, by ergodicity of T_φ, there exists $n\neq 0$ and $f_1\in L^2(X,\mu)$ such that $F(x,y)=f_1(x)\exp 2\pi iny$, whence $f_1(Tx)\exp 2\pi in\cdot(\varphi(x)+y)$ $=\exp(2\pi ic)f_1(x)\exp 2\pi iny$. Moreover, $f_2(Tx)/f_2(x)=\exp 2\pi in\varphi(x)\exp 2\pi i(-c)$, where $f_2(x)=(f_1(x))^{-1}$ (since by the ergodicity of T we can deduce that $|f_1|=\text{const.}$). Assuming that $|f_2(x)|=1$ we get a measurable $f:X\circlearrowleft$ such that $f(Tx)-f(x)=n\varphi(x)-c$ which means that $n\varphi$ is quasi-coboundary. This is a contradiction to (70).

Assume that we consider $T^{(i)}:X\circlearrowleft$, $T^{(i)}(x)=x+\alpha_i$, α_i is irrational, i=1,2. Assume also that $T^{(1)}\perp T^{(2)}$ or (which is the same for this particular case) that $Sp(T^{(1)})\cap Sp(T^{(2)})=\{1\}$. Then obviously $T^{(1)}\times T^{(2)}$ is ergodic and moreover

(78) $C(T^{(1)}\times T^{(2)})=C(T^{(1)})\times C(T^{(2)}).$

Lemma 4.3. *Let $\Psi: X\wp$ be a cocycle. Then there is a measurable $f: X\times X \longrightarrow X$ satisfying $\Psi(x)=f(T^{(1)}x, T^{(2)}y)-f(x,y)$ $\mu\times\mu$-a.e. iff there is an eigenvalue $\exp 2\pi ic \in Sp(T^{(2)})$, $c\in[0,1)$ such that $\Psi(x)=c+g(T^{(1)}x)-g(x)$ for a measurable $g: X\wp$.*

Proof. *Necessity.* Let $\tilde{f}_\chi(x)=\int_X \exp(2\pi i f(x,y))\,\chi(y)d\mu(y)$, where $\chi\in\hat{X}$. Then

(79) $\qquad \tilde{f}_\chi(T^{(1)}x)=\chi(\alpha_2)\exp(2\pi i\Psi(x))\tilde{f}_\chi(x)$

and we can assume that there exists a $\chi_0\in\hat{X}$ such that $\tilde{f}_{\chi_0}\neq 0$ as otherwise f would be a function of x alone (and Ψ would be of the form $\Psi(x)=f(T^{(1)}x)-f(x)$). Thus $|\tilde{f}_{\chi_0}(T^{(1)}x)|=|\tilde{f}_{\chi_0}(x)|$ and therefore by the ergodicity of $T^{(1)}$ we can assume that $|\tilde{f}_{\chi_0}|=1$. Hence $\tilde{f}_{\chi_0}(x)=\exp 2\pi i f_{\chi_0}(x)$ for a measurable $f_{\chi_0}: X\wp$. Then, (79) can be rewritten as $f_{\chi_0}(x+\alpha_1)=s\alpha_2+\Psi(x)+f_{\chi_0}(x)$, where $\exp 2\pi i s\alpha_2=\chi_0(\alpha_2)\in Sp(T^{(2)})$.

Sufficiency. If $\exp 2\pi ic\in Sp(T^{(2)})$ then there exists $\tilde{g}_1\in L^2(X,\mu)$, $|\tilde{g}_1|=1$ such that $\tilde{g}_1(T^{(2)}y)=\exp(2\pi ic)\tilde{g}_1(y)$. Hence there is a measurable $g_1: X\wp$ such that $g_1(T^{(2)}y)=c+g_1(y)$ and consequently $\Psi(x)=[\,g_1(T^{(2)}y)+g(T^{(1)}x)\,]-[\,g_1(y)+g(x)\,]$ ∎

Corollary 4.1. *Assume that $T^{(1)}_{\varphi_1}\times T^{(2)}$ is ergodic. Then $S_1\times S_2\in C(T^{(1)}\times T^{(2)})$ (see (78)) can be lifted to $C(T^{(1)}_{\varphi_1}\times T^{(2)})$ iff $\varphi_1(S_1x)-v\,\varphi_1(x)=c+f(T^{(1)}x)-f(x)$ for some $\exp 2\pi ic\in Sp(T^{(2)})$, a continuous group epimorphism $v: X\wp$ and a measurable $f: X\wp$.*

Proof. By (17), the automorphism $S_1\times S_2$ can be lifted to $C(T^{(1)}_{\varphi_1}\times T^{(2)})=C((T^{(1)}\times T^{(2)})_{\tilde{\varphi}_1})$, where $\tilde{\varphi}_1(x,y)=\varphi_1(x)$ iff $\tilde{\varphi}_1(S_1\times S_2)(x,y)-v(\tilde{\varphi}_1(x,y))=F(T^{(1)}x,T^{(2)}y)-F(x,y)$ for a continuous group epimorphism $v: X\wp$ and a measurable $F: X\times X\longrightarrow X$.

Hence, by the definition of $\tilde{\varphi}_1$, $\varphi_1(S_1x)-v\,\varphi_1(x)=F(T^{(1)}x,T^{(2)}y)-F(x,y)$. Now, an application of Lemma 4.3 gives the desired conclusion■

In [36], the authors raised the following question. Under which additional conditions to $U_1 \perp U_2$ the formula

$$C(U_1 \times U_2)=C(U_1)\times C(U_2)$$

holds. One might expect this to be true if U_1 and U_2 are spectrally disjoint. However, this fails to be true as the following example shows.

Example 4.5. Let $\varphi_1(x)=x$. As φ_1 satisfies (70) (see Proposition 4.1), φ_1 is weakly mixing. Therefore $T^{(1)}_{\varphi_1}$ and $T^{(2)}$ are spectrally disjoint. However

$$C(T^{(1)}_{\varphi_1})=\{(T^{(1)}_{\varphi_1})^n\circ \sigma_x:\ x\in[0,1),\ n\in \mathbb{Z}\},$$

since by Proposition 4.1 if $Sx=x+\beta$, can be lifted to $C(T^{(1)}_{\varphi_1})$ then $\varphi_1(x+\beta)-\varphi_1(x)=f(x+\alpha_1)-f(x)$ for a measurable $f:X\to\mathbb{Z}$. This means that $\beta=f(x+\alpha_1)-f(x)$ and therefore that $\beta=n\alpha_1$ for some $n\in\mathbb{Z}$. Moreover, $C(T^{(2)})=\{\sigma_y:\ y\in[0,1)\}$ (see (8)). Therefore only $\sigma_{(n\alpha_1,y)}\in C(T^{(1)}\times T^{(2)})$, $y\in[0,1)$, can be lifted to $C(T^{(1)}_{\varphi_1})\times C(T^{(2)})$ while by Corollary 4.1 it follows that $\sigma_{(n\alpha_2,y)}\in C(T^{(1)}\times T^{(2)})$ can be lifted to $C(T^{(1)}_{\varphi_1}\times T^{(2)})$ for each $n\in\mathbb{Z}$ and $y\in[0,1)$. Hence $C(T^{(1)}_{\varphi_1}\times T^{(2)})\neq C(T^{(1)}_{\varphi_1})\times C(T^{(2)})$■

Proposition 4.4. <u>Let φ_1 and φ_2 be two Anzai cocycles satisfying</u> (70). <u>Then $T^{(1)}_{\varphi_1}\perp T^{(2)}_{\varphi_2}$ and $T^{(1)}_{\varphi_1}\times T^{(2)}_{\varphi_2}$ is coalescent. Moreover, if $S_1\times S_2$ can be lifted to $C(T^{(1)}_{\varphi_1}\times T^{(2)}_{\varphi_2})$ then it can be lifted both to $C(T^{(1)}_{\varphi_1}\times T^{(2)})$ and to $C(T^{(1)}\times T^{(2)}_{\varphi_2})$.</u>

Proof. As $n\varphi_1, n\varphi_2$, $n\neq 0$ are not quasi-coboundary, φ_1 and φ_2 are weakly mixing and hence $T^{(1)}_{\varphi_1}\perp T^{(2)}_{\varphi_2}$ (see the proof of Thm. 1.4 in [10]).

Let us suppose that $S_1 \times S_2 \in C(T^{(1)} \times T^{(2)})$ can be lifted to $C(T^{(1)}_{\varphi_1} \times T^{(2)}_{\varphi_2}) = C((T^{(1)} \times T^{(2)})_{\varphi_1 \times \varphi_2})$. In view of (17) there are a continuous group epimorphism $w: X \times X\, ?$ and a measurable solution of the equation $(\varphi_1(S_1x), \varphi_2(S_2y)) - w(\varphi_1(x), \varphi_2(y)) = f(T^{(1)}x, T^{(2)}y) - f(x,y)$. Then by taking the two projections on the two coordinates we obtain that

(80) $$\varphi_1(S_1x) - [k_1\varphi_1(x) + l_1\varphi_2(y)] = f^{(1)}(T^{(1)}x, T^{(2)}y) - f^{(1)}(x,y),$$

(81) $$\varphi_2(S_2y) - [k_2\varphi_1(x) + l_2\varphi_2(y)] = f^{(2)}(T^{(1)}x, T^{(2)}y) - f^{(2)}(x,y).$$

Note that whenever $l_1 \neq 0$ then by (80) (see Proposition 0.5)

(82) $$T^{(1)}_{\varphi_1 S_1 - k_1\varphi_1} \times T^{(2)} \text{ is isomorphic to } T^{(1)} \times T^{(2)}_{l_1\varphi_2}$$

while by (81), when $k_2 \neq 0$,

(83) $$T^{(1)} \times T^{(2)}_{\varphi_2 S_2 - l_2\varphi_2} \text{ is isomorphic to } T^{(1)}_{k_2\varphi_1} \times T^{(2)}.$$

But $l_1\varphi_2$, $k_2\varphi_1$ are weakly mixing, thus $T^{(1)} \times T^{(2)}_{l_1\varphi_2}$ and $T^{(1)}_{k_2\varphi_1} \times T^{(2)}$ are ergodic. Therefore, all the product automorphisms in (82) and (83) are ergodic. Hence

$$Sp(T^{(1)}_{\varphi_1 S_1 - k_1\varphi_1} \times T^{(2)}) = Sp(T^{(1)} \times T^{(2)}) = Sp(T^{(1)} \times T^{(2)}_{\varphi_2 S_2 - l_2\varphi_2}).$$

This implies that $\varphi_1 S_1 - k_1\varphi_1$, $\varphi_2 S_2 - l_2\varphi_2$ are both weakly mixing. Suppose that $l_1 \neq 0$ (either l_1 or k_2 must be different from zero if w is not invertible). Then again by using the arguments from [10] we obtain that

(84) $$T^{(1)}_{\varphi_1 S_1 - k_1\varphi_1} \perp T^{(2)}_{l_1\varphi_2}$$

But $T^{(1)} \times T^{(2)}_{l_1\varphi_2}$ has $T^{(2)}_{l_1\varphi_2}$ as its factor and by (82) the automorphism $T^{(1)}_{\varphi_1 S_1 - k_1\varphi_1}$ has to appear as a factor of $T^{(1)} \times T^{(2)}_{l_1\varphi_2}$. Hence, by (17) the direct product $T^{(1)}_{\varphi_1 S_1 - k_1\varphi_1} \times T^{(2)}_{l_1\varphi_2}$

appears as a factor of $T^{(1)} \times T^{(2)}_{l_1\varphi_2}$ and therefore $T^{(1)} \times T^{(2)}_{l_1\varphi_2}$ is not coalescent. In view of Corollary 4.1 the equation $l_1\varphi_2(S_2y)-s(l_1\varphi_2)(y)=c+f(T^{(2)}y)-f(y)$ has a measurable solution $f:X\to\mathbb{R}$ for some $s\in\mathbb{Z}$, $|s|\geq 2$ and $c\in[0,1)$. Hence, for φ_2 the property (70) does not hold, a contradiction∎

Corollary 4.2. _If_ $T^{(1)}_{\varphi_1}$, $T^{(2)}_{\varphi_2}$ _are ergodic Anzai skew products coming from Examples_ 4.1-3 _and if_ $T^{(1)}_{\varphi_1} \perp T^{(2)}_{\varphi_2}$ _then_ $T^{(1)}_{\varphi_1} \times T^{(2)}_{\varphi_2}$ _is coalescent_∎

5. WEAKLY ISOMORPHIC AUTOMORPHISMS THAT ARE NOT ISOMORPHIC. In this section we show how to construct two ergodic group extensions of a rotation that are weakly isomorphic but not isomorphic. Also, we exhibit an ergodic coalescent group extension of a rotation with a natural noncoalescent factor. All these facts will be seen as properties of some ergodic infinite selfjoinigs of T_φ, where T is an ergodic rotation on a compact monothetic group X and $\varphi : X \longrightarrow Z_2$ is an ergodic cocycle with a special property.

Let X be a compact monothetic group with Haar measure μ. Assume that $T : X \circlearrowleft$ is an ergodic rotation, this meaning that $Tx = x + x_0$, where x_0 is a topological cyclic generator of X. Let $\varphi : X \longrightarrow Z_2$ be an ergodic cocycle and $S \in C(T)$.

Definition 5.1 We say that φ is S-<u>strongly ergodic</u> if for every $i_1 < \ldots < i_k$, $k \geq 2$ and for every $U \in C(T)$ the Z_2-cocycle $\varphi S^{i_1} + \varphi S^{i_2} + \ldots + \varphi S^{i_k} + \varphi U$ is ergodic.

Later, we will see that this condition is not vacuous. Notice that in particular $\varphi + \varphi S^r$, $r \geq 1$ is then ergodic. Indeed, $\varphi + \varphi S^r = \varphi + \varphi S^r + \varphi S^{r+1} + \varphi S^{r+1}$.

For φ having the S-strong ergodicity property we will consider infinite product group extensions of T of the form

(85) $\quad T_{\varphi S^{i_1} \times \varphi S^{i_2} \times \ldots} : (X \times Z_2 \times Z_2 \times \ldots, \mu \times \nu_2 \times \nu_2 \times \ldots) \circlearrowleft$,

$i_1 < i_2 < \ldots$ (ν_2 is Haar measure of Z_2). Let us denote for short $T_{i_1, i_2, \ldots} = T_{\varphi S^{i_1} \times \varphi S^{i_2} \times \ldots}$, $i_1 < i_2 < \ldots$. By (15) every such $T_{i_1, i_2, \ldots}$ is ergodic. Notice also that $T_{j_1, j_2, \ldots}$ is a factor of $T_{i_1, i_2, \ldots}$ whenever $\{j_1, j_2, \ldots\} \subseteq \{i_1, i_2, \ldots\}$. Indeed Let $i_{\sigma(k)} = j_k$ for $\sigma : N \circlearrowleft$. Then $\Theta : X \times Z_2 \times Z_2 \times \ldots \circlearrowleft$ defined by $\Theta(x, r_1, r_2 \ldots)$

$=(x, r_{\sigma(1)}, r_{\sigma(2)}, \ldots)$ is $\mu \times \nu_2 \times \nu_2 \times \ldots$-preserving map and moreover $T_{j_1, j_2, \ldots}\Theta = \Theta T_{i_1, i_2 \ldots}$. Furthermore, if there is a $c \in Z$ such that $-i_k + j_k = c$, $k \geq 1$, then $T_{i_1 i_2, \ldots}$ and $T_{j_1 j_2, \ldots}$ are isomorphic. Indeed, the map $\Theta(x, r_1, r_2, \ldots) = (S^c x, r_1, r_2 \ldots)$ establishes an isomorphism. The point is that only the maps of that form can establish an isomorphism of such automorphisms.

Theorem 5.1. _If_ $T_{i_1, i_2, \ldots}$ _and_ $T_{j_1, j_2, \ldots}$ _are isomorphic then there exists an integer_ c _such that_ $-i_k + j_k = c$, $k \geq 1$.

Proof. Assume that $T_{i_1 i_2, \ldots}$, $T_{j_1, j_2, \ldots}$ are isomorphic. These automorphisms are ergodic $Z_2 \times Z_2 \times \ldots$-extensions of T, so the isomorphism, by (16), must be of the form

$$U_{f,v}: (X \times Z_2 \times Z_2 \times \ldots, \mu \times \nu_2 \times \nu_2 \times \ldots) \circlearrowleft,$$

where $U \in C(T)$, $f: X \longrightarrow Z_2 \times Z_2 \times \ldots$ is measurable and v is a continuous group automorphism of $Z_2 \times Z_2 \times \ldots$. Let us notice that $f = (f_1, f_2, \ldots)$, where $f_i: X \longrightarrow Z_2$ is a cocycle. Also

(86) $\quad fT = (f_1 T, f_2 T, \ldots)$.

If v is a continuous group automorphism then $\pi_k v: Z_2 \times Z_2 \times \ldots \rightarrow Z_2$ ($\pi_k(r_1, r_2, \ldots) = r_k$) is a character of $Z_2 \times Z_2 \times \ldots$ and hence $\pi_k v(r_1, r_2, \ldots) = r_{j_1^{(k)}} + \ldots + r_{j_{s_k}^{(k)}}$, $k = 1, 2, \ldots$. In view of (16), $f(Tx) + f(x) = (\varphi S^{i_1} \times \varphi S^{i_2} \times \ldots)(Ux) + v(\varphi S^{j_1} \times \varphi S^{j_2} \times \ldots)(x)$. From this and (86) it follows that

$f_k(Tx) + f_k(x) = \varphi(S^{i_k} Ux) + \left[\varphi S^{j_1^{(k)}}(x) + \ldots + \varphi S^{j_{s_k}^{(k)}}(x)\right]$. Therefore, $\varphi U + \left[\varphi S^{j_1^{(k)} - i_k} + \ldots + \varphi S^{j_{s_k}^{(k)} - i_k}\right]$, $k \geq 1$, are coboundary, whence by S-strong ergodicity property $s_k = 1$. Thus $\varphi U + \varphi S^{j_1^{(k)} - i_k}$ are coboundary for each $k \geq 1$. Consequently for $k \neq l$ we have $\varphi S^{j_1^{(k)} - i_k} + \varphi S^{j_1^{(l)} - i_l}$ are coboundary. Equivalently

$\varphi+\varphi_S(j_1^{(k)}-i_k)-(j_1^{(l)}-i_l)$ are coboundary. Then, again by using S-strong ergodicity property of φ we have achieved that $(j_1^{(k)}-i_k)-(j_1^{(l)}-i_l)=0$, $k\neq l$. But $i_1<i_2<\dots$, whence $j_1^{(1)}<j_1^{(2)}<\dots$ Moreover, since v is an automorphism, $j_1^{(k)}=j_k$, so $j_k-i_k=c$, independently of k■

R e m a r k 5 . 1 . Actually by the results of Section 1, we have proved that if φ is S-strongly ergodic then for

$\lambda_{i_1,i_2,\dots}=\hat{\mu}_{S^{i_1},S^{i_2},\dots}$, $\lambda_{j_1,j_2,\dots}=\hat{\mu}_{S^{j_1},S^{j_2},\dots}$

(see (11) and (13a)) we have that $\lambda_{i_1,i_2,\dots}$, $\lambda_{j_1,j_2}\dots\in J_\infty^e(T_\varphi)$ and moreover that $\lambda_{i_1,i_2,\dots}$, $\lambda_{j_1,j_2,\dots}$ are isomorphic iff i_k-j_k is constant■

C o r o l l a r y 5 . 1 . $T_{0,1,2,3,4,\dots}$ _and_ $T_{0,2,3,4,\dots}$ _are weakly isomorphic but not isomorphic_. _Equivalently, the two ergodic infinite self-joinings_ $\lambda_{0,1,2,3,\dots}$ _and_ $\lambda_{0,2,3,\dots}$ _are weakly isomorphic but not isomorphic_■

Let $G=\dots\times Z_2\times Z_2\times\dots$.

C o r o l l a r y 5 . 2 . _There exists an ergodic_ G-_extension_ T_ψ _of an automrphism with discrete spectrum such that_ $C(T_\psi)$ _contains an automorphism_ $\hat{S}=S\times v$, _where_ v _is a Bernoulli automorphism_. _In particular, the entropy of_ $\hat{S}$ _is positive_.

P r o o f . Consider the "two-sided" version of the costruction above. To be precise, let

(87) $\psi(x)=(\dots,\varphi(S^{-1}x),\varphi(x),\varphi(Sx),\varphi(S^2x),\dots)$;

where $\varphi:X\longrightarrow Z_2$ is S-strongly ergodic. Then, still T_ψ is ergodic and moreover for the "two-sided" shift $\hat{S}\in C(T_\psi)$, where $\hat{S}(x,\dots,i_{-1},\overset{\downarrow}{i_0},i_1,\dots)=(Sx,\dots i_{-1},i_0,\overset{\downarrow}{i_1},i_2,\dots)$ we have $\hat{S}=S\times v$■

Remark 5.1a. We note that more general automorphisms of the form $\hat{S}=S_{f,v}$, where v is an ergodic group automorphism appeared in D. Lind [40]. Every such an automorphism is in fact of the form $S\times v$■

Corollary 5.3. <u>There exists an ergodic G-extension T_ψ of an automrphism with discrete spectrum such that T_ψ is coalescent but a natural factor of it is not.</u>

Proof. We prove this property for T_ψ defined by (87) for an S-strongly ergodic cocycle $\varphi: X \longrightarrow Z_2$. We claim that T_ψ is coalescent. Indeed, let $U \in C(T)$ can be lifted to $C(T_\psi)$. Then by (17), $fT+f=(\ldots\times\varphi S^{-1}\times\varphi\times\varphi S\times\ldots)U+v(\ldots\times\varphi S^{-1}\times\varphi\times\varphi S\times\ldots)$ holds for a measurable $f: X \longrightarrow G$ and a continuous group epimorphism $v: G \circlearrowleft$. Then, just by repeating the arguments from the proof of Theorem 5.1, we can see that $\varphi S^k U+\varphi S^{i_k}$ is coboundary for each $k \in Z$ and consequently $k-i_k=c$ for each $k \in Z$. But then $\{i_k : k \in Z\}=Z$ and hence v is merely a permutation of the coordinates, whence v is invertible. However $T_{0,1,2,3,\ldots}$ is a factor of T_ψ and the former automorphism is not coalescent■

Remark 5.2. Corollary 5.3 delivers the negative answer to D. Newton question [44] of whether the class of ergodic and coalescent automorphisms is closed under taking factors. We do not know if coalescence implies zero entropy. Note that the validity of Weak Pinsker Conjecture saying that each ergodic positive entropy automorphism is isomorphic to the direct product of a Bernoulli automorphism and an automorphism with the entropy smaller than ε would deliver the positive answer to our question. Indeed, it is then enough to show that no Bernoulli automorphism is coalescent. Let τ be Bernoulli and represent it as $\tau_1 \times \tau_1$ ([46]), where $\tau_1 : (X_1, \mathcal{B}_1, \mu_1) \circlearrowleft$ is also Bernoulli with entropy equal to one-half of the entropy of τ . Then take $f(x_1,x_2)=(x_2,x_1)$ and the corresponding factor $\mathcal{E}$ of $\tau_1 \times \tau_1$ of all f-invariant sets. We conclude that $\mathcal{E}$ is Bernoulli ([47]) with the same entropy as $\tau_1 \times \tau_1 = \tau$ (since $\tau_1 \times \tau_1$ is a 2-point extension of $\mathcal{E}$). Therefore τ is isomorphic to its proper factor and hence τ is not coalescent ∎

At present, we will pass to the problem of the existence of S-strongly ergodic cocycles.

Let $\{n_t\}_{t \geq 0}$ be a sequence of natural numbers satisfying: $n_t < n_{t+1}$, $n_t | n_{t+1}$. Denote $\lambda_0 = n_0$, $\lambda_t = n_t / n_{t-1}$, $t \geq 1$. Let X be the group of $\{n_t\}$-<u>adic numbers</u> which means that

$$X = \{ \bar{s} = (s_0, s_1, \ldots) : 0 \leq s_i \leq \lambda_i - 1, \ i \geq 0 \}$$

with addition "carry on the right". This is a compact monothetic group and $\hat{1} = (1,0,0,\ldots)$ is a topological cyclic generator. Hence, $T : X \circlearrowleft$ defined by $Tx = x + \hat{1}$, $x \in X$, is ergodic with respect to Haar measure μ on X. The automorphism $T : (X, \mathcal{B}, \mu) \circlearrowleft$ is said to be an $\{n_t\}$-<u>adic adding machine</u>. This automorphism is well-known to have pure point spectrum with $Sp(T) = G\{n_t : t \geq 0\}$, where the latter group is the subgroup of all roots of unity generated by $\exp 2\pi i / n_t$, $t \geq 0$. Denoting $D_0^t = \{ \bar{s} \in X : s_i = 0, \ i = 0, 1, \ldots, t \}$ we get a T-tower D^t of height n_t, where $D^t = (D_0^t, D_1^t, \ldots D_{n_t - 1}^t)$, $T^i D_0^t = D_i^t$ (mod n_t). Also, it is clear that D^t converges to $\mathcal{B}$ (see (6) for the definition). Actually, this latest condition is equivalent to saying that T has pure discrete spectrum with $Sp(T) = G\{n_t : t \geq 0\}$. Because of (8), $C(T) = \{ \sigma_{\bar{v}} : \bar{v} \in X \} \approx X$, where $\sigma_{\bar{v}}(x) = x + \bar{v}$. Let us consider the action of $S = \sigma_{\bar{v}}$ on D^t and D^{t+1}.

As T is ergodic, D^t is the only T-tower of height n_t filling up the whole space (up to a cyclic reordering of the levels, if necessary). Hence S must carry the levels of D^t into the levels. The action of S on D^t and D^{t+1} is represented in Figure 5.1.

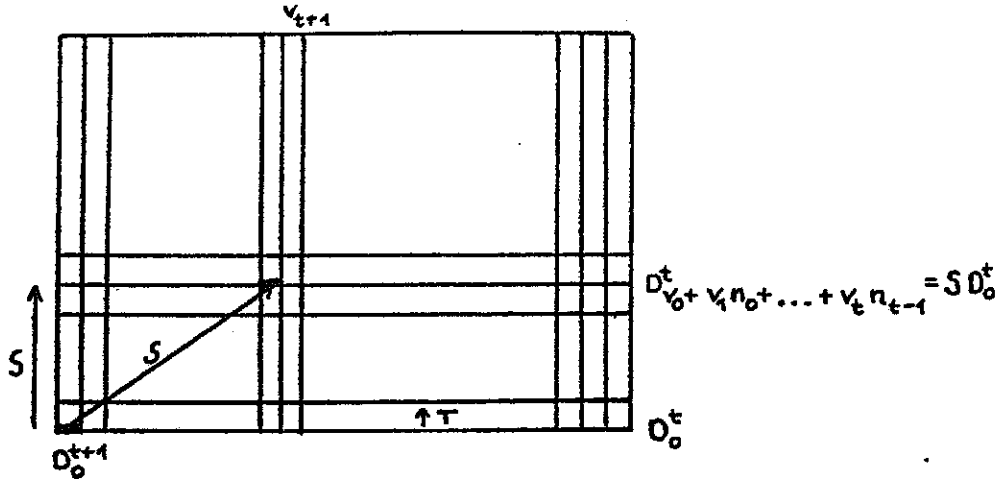


Figure 5.1.

Example 5.1. A rigid, S-strongly ergodic Z_2-cocycle.

Let $\lambda_t=2^{t+1}+1$, $t\geq 0$.

5.1.1. Definition of φ. The definition will be inductive. At the t-th step our cocycle will be defined on $D^t_0,\dots,D^t_{n_t-2}$ but not on $D^t_{n_t-1}$. Moreover

$$\varphi|D^t_i=\text{const}=a_i^{(t)}\in\{0,1\},\ i=0,\dots,n_t-2.$$

We now define the passage into the (t+1)-st step. First of all we define $a^{(t)}_{n_t-1}$ (=0 or 1) so that

(88) $$\sum_{i=0}^{n_t-1} a_i^{(t)}=1.$$

Then, we put

$$a^{(t+1)}_{sn_t-1}=\varphi|D^{t+1}_{sn_t-1}=\begin{cases} a^{(t)}_{n_t-1} & \text{if } s\neq\lambda_t,\lambda_{t+1} \\ 1-a^{(t)}_{n_t-1} & \text{if } s=\lambda_t \\ \text{undefined} & \text{if } s=\lambda_{t+1}\end{cases}$$

$s=1,2,\dots,\lambda_{t+1}$ (see Fig. 5.2).

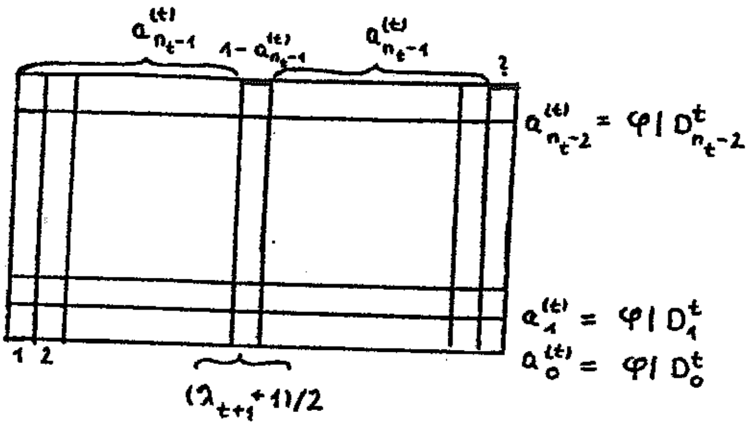


Figure 5.2.

Let us agree to call the set $D^{t+1}_{\lambda_t n_t-1}$ in Fig. 5.2 an <u>error</u> (as the value of φ on $D^t_{n_t-1}$ is nearly constant and equal to $a^{(t)}_{n_t-1}$ except the constant value of φ on $D^{t+1}_{\lambda_t n_t-1}$). This is a correct definition (a.e.) of a measurable function φ. Let us observe that our cocycle is "constant" on each level because for $i<n_t-1$ it is truly constant while for $i=n_t-1$ the (relative) measure of the error is less than $2/\lambda_{t+1} \to 0$. Hence $\varphi|D^t_{n_t-1}$ (and therefore $\varphi|D^{t+1}_{n_{t+1}-1}$) is almost constant. The number $a^{(t+1)}_{n_{t+1}-1}$ is defined in such a way that

$$(\lambda_{t+1}-2)\sum_{i=0}^{n_t-1} a_i^{(t)}+\Big[\sum_{i=0}^{n_t-2} a_i^{(t)}+(1-a^{(t)}_{n_t-1})\Big]+\Big[\sum_{i=0}^{n_t-2} a_i^{(t)}+a^{(t+1)}_{n_{t+1}-1}\Big]=1$$

as (88) must hold for t+1. Hence, by (88) applied to t we get $1=1+(1-a^{(t)}_{n_t-1})+a^{(t+1)}_{n_{t+1}-1}=a^{(t)}_{n_t-1}+a^{(t+1)}_{n_{t+1}-1}$. Therefore, $\varphi|D^{t+1}_{n_{t+1}-1}$ defines another "error" for the t-th step.

5.1.2. Definition of S. We take $S=\sigma_{\bar{v}}$, $\bar{v}=(v_i)_{i\geq 0}$, where $v_i=[\lambda_i/i]$, $i\geq 1$ and $v_0=0$. Then, given $k\geq 1$ there is i_0 such that

(89) $\quad S^k=(v_i^{(k)})_{i\geq 0}$, with $v_i^{(k)}=kv_i$ for $i\geq i_0$.

5.1.3. Keane's Criterion. We will need some criterion concerning ergodicity of Z_2-cocycles. Let $T:(X,\mathcal{B},\mu)\circlearrowleft$ be an $\{n_t\}$-adic adding machine. Suppose that $\varphi:X\to Z_2$ is a cocycle such that $\varphi|D_i^t=a_i^{(t)}\in\{0,1\}$, $i=0,1,\ldots,n_t-2$, $t\geq 0$ and $d_t=\max_{i=0,1}\{\mu\{x\in D^t_{n_t-1}: \varphi(x)=i\}\}>(1/2)\mu(D^t_{n_t-1})$. Let $d_t=\mu\{x\in D^t_{n_t-1}: \varphi(x)=i_t\}$. We say that $D^{t+1}_{kn_t-1}$ <u>defines an error</u> if $\varphi|D^{t+1}_{kn_t-1}\neq i_t$. Assume that $\lambda_t\longrightarrow\infty$ and let $m_1<m_2<\ldots<m_{r_t}$, $0<m_i\leq\lambda_{t+1}-1$ are the numbers such that the errors on $D^t_{n_t-1}$ occur at $D^{t+1}_{m_in_t-1}$. Assume that $r_t\geq 2$ and let $A_t=(m_2-m_1)+(m_4-m_3)+\ldots+(m_p-m_{p-1})$, where p is the largest even number not greater than r_t.

Proposition 5.1. (Keane's Criterion) φ *is ergodic iff* $\sum_{t\geq 0} A_t/\lambda_{t+1}=+\infty$ ∎

The proof follows from the observation that such a cocycle determines an almost periodic point $\omega\in\{0,1\}^Z$ and besides that φ is ergodic iff ω is strictly transitive (we will see all these facts in Section 8). Then, Lemma 3 [24] can be applied.

5.1.4. $\Psi=\varphi S^{i_1}+\ldots+\varphi S^{i_k}$ is ergodic, $0\leq i_1<\ldots<i_k$.

Let us look at the passage from the t-th step to the (t+1)-st one for Ψ, where t is large enough. We notice that $\varphi|D_i^t$ is constant except for k levels, say, $j_1,\ldots,j_k$. At each $D^t_{j_r}$ there are two errors. From (89), the "distance" (which means the number of columns of D^t) between errors in $D^t_{j_r}$ and $D^t_{j_{r'}}$, $r\neq r'$, is at least $[\lambda_{t+1}/(t+1)]$ and moreover the distance between these errors and the λ_t-th or λ_{t+1}-st column is at most $i_k[\lambda_{t+1}/(t+1)]$.

Now, we intend to define a new cocycle $\tilde\Psi$ satisfying the assumptions of Proposition 5.1 and making $\Psi+\tilde\Psi$ nonergodic (equivalently T_Ψ and $T_{\tilde\Psi}$ are relatively isomorphic). The definition of $\tilde\Psi$ is inductive. At each step t, $\tilde\Psi|D_i^t=c_i^{(t)}$, $i=0,\ldots,n_t-2$ and $\tilde\Psi$ is not defined on $D^t_{n_t-1}$. First, we define

$c^{(t)}_{n_t-1}$ so that

(90) $$\sum_{i=0}^{n_t-1} c_i^{(t)} = \sum_{i=0}^{n_t-1} b_i^{(t)},$$

where $b_i^{(t)}$ are the prevailing constant values of Ψ on D_i^t. Then, we put

$$c^{(t+1)}_{sn_t-1} = \begin{cases} c^{(t)}_{n_t-1} & \text{if } s<\lambda_{t+1} \text{ and there is no error in the } s\text{-th colum of } D^t \\ 1-c^{(t)}_{n_t-1} & \text{if } s<\lambda_{t+1} \text{ and there is an error in the } s\text{-th colum of } D^t \\ \text{undefined} & \text{if } s=\lambda_{t+1}. \end{cases}$$

We prove that Ψ and $\tilde{\Psi}$ are Z_2-cohomologous. Consider the passage from the t-th step to (t+1)-st step for $\Psi+\tilde{\Psi}$. We observe that $\Psi+\tilde{\Psi}$ is almost constant on each level D_i^t, $(\Psi+\tilde{\Psi})|D_i^t=d_i^{(t)}$, $i=0,\dots,n_t-1$. Since (90) holds,

(91) $$\sum_{i=0}^{n_t-1} d_i^{(t)}=0.$$

Let u_t denote the number of columns with errors. Then, $u_t=2k$ and moreover in every such a column there are two errors. Consider $T_{\Psi+\tilde{\Psi}}:(X\times Z_2,\tilde{\mu})\circlearrowleft$ and the following sequence of sets

$$C_t=\bigcup_{i=0}^{n_t-1} D_i^t\times(d_0^{(t)}+\dots+d_{i-1}^{(t)}).$$

Since (91) holds and in each column there is an even number of errors,

(i) $\mu(C_t)=1/2$,

(ii) $\mu(T_{\Psi+\tilde{\Psi}}C_t\Delta C_t)\longrightarrow 0$,

(iii) $\sum_{t\geq 0}\mu(C_{t+1}\Delta C_t)<+\infty$.

Hence, $\{C_t\}$ is a Cauchy sequence and $C=\lim C_t$ is $T_{\Psi+\tilde{\Psi}}$ invariant, whence $T_{\Psi+\tilde{\Psi}}$ cannot be ergodic.

Now, we will prove that $\tilde{\Psi}$ is ergodic. As $D^t_{n_t-1}=\bigcup_{s=1}^{\lambda_{t+1}} D^{t+1}_{sn_t-1}$, it is divided into λ_{t+1} pieces. We group them into t consecutive groups $\Lambda_1, \Lambda_2, \ldots, \Lambda_t$ of $[\lambda_{t+1}/t]$ pieces (the last group need not have $[\lambda_{t+1}/t]$ pieces). We observe that there is no possibility for two different errors (for $\tilde{\Psi}$) to be in the same group Λ_t, since $i_1 < \ldots < i_k$ and (89) holds. This means that $A_t \geq (1/2)v_{t+1}$. Since the series $\sum_{t \geq 0} v_{t+1}/2\lambda_{t+1}$ is divergent, an application of Proposition 5.1 completes the proof of ergodicity of Ψ.

5.1.5. Cocycles $\varphi S^{i_1} + \ldots + \varphi S^{i_k} + \varphi U$, $k \geq 2$ are ergodic.

It is now enough to apply the foregoing arguments as it is not possible to destroy the divergence of $\sum_{t \geq 0} A_t/\lambda_{t+1}$ using only two more errors (φU adds merely two new errors to the $2k \geq 4$ already existing ones).

5.1.6. φ is rigid. It is not hard to see that $(T_\varphi)^{2n_t}$ tends weakly to the identity automorphism■

Remark 5.3. Compact rank need not imply coalescence. We are in a position to answer J.P. Thouvenot question of whether each compact rank automorphism is coalescent (by compact rank automorphism we mean any $\bar{d}$-limit ([59]) of ergodic finite rank automorphisms). We recall that by Proposition 0.2 and 0.1 all ergodic finite rank automorphisms are coalescent. We observe that φ constructed in Ex. 5.1 enjoys the following T_φ has rank 1 (by (88)), $T_{\varphi \times \varphi S}$ has rank at most $2^2 \cdot 2$, $T_{\varphi \times \varphi S \times \varphi S^2}$ has rank at most $2^3 \cdot 3$, ... Therefore, $T_{\varphi \times \varphi S \times \varphi S^2 \times \ldots}$ has compact rank property as it is the inverse limit $T_{\varphi \times \varphi S \times \ldots \times \varphi S^k}$, $k \geq 1$. By that, we have constructed a compact rank automorphism that is not coalescent (see Corollary 5.1)■

Example 5.2. A substitution-like cocycle φ is S-strongly ergodic for each "typical" $S \in C(T)$.

We will consider some ergodic cocycles over an adding machine given by $\lambda_t = \lambda \geq 2$, $t \geq 0$. A cocycle $\varphi : X \longrightarrow Z_2$ is said to have an α-<u>regular hole with a constant</u> β ($0 < \alpha, \beta < 1$) with respect to $\{D^t\}$ if for each t there exists v_t, $0 \leq v_t \leq n_t - 1$ such that

$$(92) \left\{ \begin{array}{ll} \text{(i)} & \min\left\{\mu(\{\varphi=0\} \cap D^t_{v_t}),\ \mu(\{\varphi=1\} \cap D^t_{v_t})\right\} \geq \alpha\, \mu(D^t_{v_t}), \\ \text{(ii)} & \varphi \text{ is constant on each } T^j D^t_{v_t},\ 0 < |j - v_t| \leq \beta\, n_t \end{array} \right.$$

(here $|\cdot|$ is the norm in the group Z_{n_t}). A cocycle for which there are some α, β satisfying (i) and (ii) will be called <u>regular</u> (with respect to $\{D^t\}$).

A motivation to consider such cocycles lies in the following.

Proposition 5.2. <u>If φ is regular then it is ergodic.</u>

Proof. Suppose φ has an α-regular hole with a constant β and is not ergodic. Then by (15), there is a measurable $g : X \to Z_2$ such that

(93) $\quad gT + g = \varphi$.

As D^t converges to $\mathfrak{B}$ and g is measurable, we can conclude that for each $\varepsilon > 0$ and t large enough card $\Lambda > (1-\varepsilon) n_t$, where $\Lambda = \{ i :$ g is constant on D^t_i but for a set of measure less than $\varepsilon \mu(D^t_i)\}$. Hence, for ε small enough in consecutive βn_t levels we can find a level from Λ. We choose $k_1, k_2 \in \Lambda$ such that

$$v_t - \beta n_t \leq k_1 < v_t < k_2 \leq v_t + \beta n_t \quad (\text{mod } n_t).$$

Because of (93) and (92ii) we get that all the levels above $D^t_{k_1}$ including $D^t_{v_t}$ are in Λ, while certainly $D^t_{v_t+1}$ is not in Λ. However, by starting from $D^t_{k_2}$ and using (93) to go down we get that $D^t_{v_t+1}$ is in Λ which is a contradiction■

Let $X(\lambda)$ denote the group of λ-adic numbers (strictly speaking $\{\lambda^t\}$-adic numbers). For a fixed $r \geq 1$ we denote $\lambda' = \lambda^r$.

The mapping $\phi_r : X(\lambda) \longrightarrow X(\lambda')$ defined by

$\phi_r(s_0,s_1,\ldots)=(s_0+s_1\lambda+\ldots+s_{r-1}\lambda^{r-1},\ s_r+s_{r+1}\lambda+\ldots s_{2r-1}\lambda^{r-1},\ldots)$

is a group isomorphism. It follows that ϕ_r also establishes an isomorphism between the automorphisms $T=\sigma_{\bar{1}}$ of $X(\lambda)$ and $T'=\sigma_{\bar{1}}$ of $X(\lambda')$. By this isomorphism the sequence of T'-towers $\{D'^t\}$ is sent back to a subsequence of $\{D^t\}$. More precisely, $\phi_r^{-1}(D'^t)=D^{t'}$ where $t'=(t+1)r-1$. If $S=\sigma_{\bar{s}}\in C(T)$ then the corresponding $S'=\phi_r S\phi_r^{-1}\in C(T')$ is equal to $\sigma_{\bar{s}'}$, where $s'=\phi_r(s)$.

Given a cocycle $\varphi : X(\lambda)\longrightarrow Z_2$ we put $\varphi'=\varphi\phi_r^{-1} : X(\lambda')\longrightarrow Z_2$. We point out an uncountable subset $\mathcal{S}\subseteq C(T)$ as follows. The set $\mathcal{S}$ consists of those $\sigma_{\bar{s}}$ such that for every $r\geq 1$ the element $\phi_r(\bar{s})=(s'_0,s'_1,\ldots)$ satisfies $s'_p\neq 0$, $s'_{p+1}=0$ for infinitely many p. This is equivalent to saying that $\bar{s}=(s_0,s_1,\ldots)$ is not eventually periodic and contains arbitrarily long blocks of zeros.

Proposition 5.3. _If $\varphi : X(\lambda)\longrightarrow Z_2$ is regular with respect to $\{D^t\}$ then for every $S\in\mathcal{S}$ and $j_1<\ldots<j_k$, $k\geq 1$, the Z_2-cocycle $\varphi S^{j_1}+\ldots+\varphi S^{j_k}$ is regular with respect to a subsequence of $\{D^t\}$._

Proof. The cocycle φ has an α-regular hole with a constant β for the sequence $\{D^t\}$. We can assume that $0\leq j_1<\ldots<j_k$, $k\geq 2$. It is enough to prove that $\varphi S^{j_1}+\ldots+\varphi S^{j_k}$ has an α-regular hole with a constant γ for a subsequence of $\{D^t\}$. Take $S=\sigma_{\bar{s}}\in\mathcal{S}$. Let us select $r\geq 1$ so that $\lambda'=\lambda^r>2j_k/\beta$ and consider

$$\mathcal{P}_r=\{p:\ s'_p\neq 0 \text{ and } s'_{p+1}=0\}$$

which is an infinite set as $S\in\mathcal{S}$. We will prove that $\varphi'S'^{j_1}+\ldots+\varphi'S'^{j_k}$ has an α-regular hole with a constant γ for the sequence of T'-towers $\{D'^{p+1}\}_{p\in\mathcal{P}_r}$. Denote

$u_p = s'_0 + \ldots + s'_p n'_{p-1}$, $u_{p+1} = s'_0 + \ldots + s'_{p+1} n'_p$. Now, fix $p \in \mathcal{P}_r$. Since $s'_{p+1} = 0$, we have $n'_{p-1} \leq u_p = u_{p+1} \leq n'_p$, $S' D'^{\,p}_0 = D'^{\,p}_{u_p}$ and $S' D'^{\,p+1}_0 = D'^{\,p+1}_{u_p}$ (see Fig. 5.3)

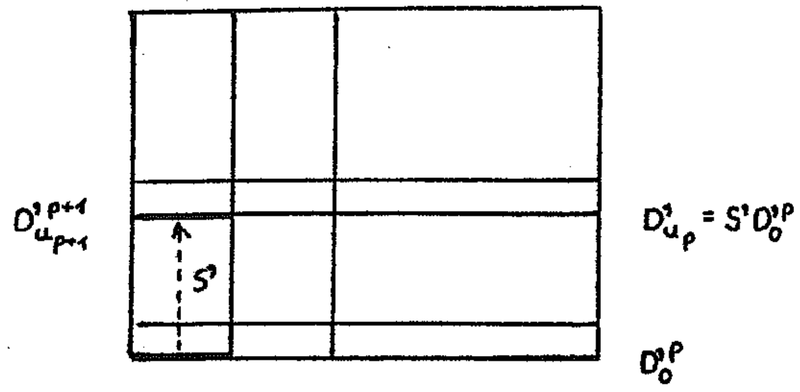


Figure 5.3.

Since φ has an α-regular hole with a constant β in the tower $D^{(p+1)'}$ at level $v_{(p+1)'}$, φ' has such a hole in the tower $D'^{\,p+1}$ at level $v'_{p+1} = v_{(p+1)'}$. It is clear that $\varphi' S'^{\,j_m}$ has also such a hole in the same tower at level $v'_{p+1} - j_m u_{p+1}$ (mod n'_{p+1}) (see Figure 5.4)

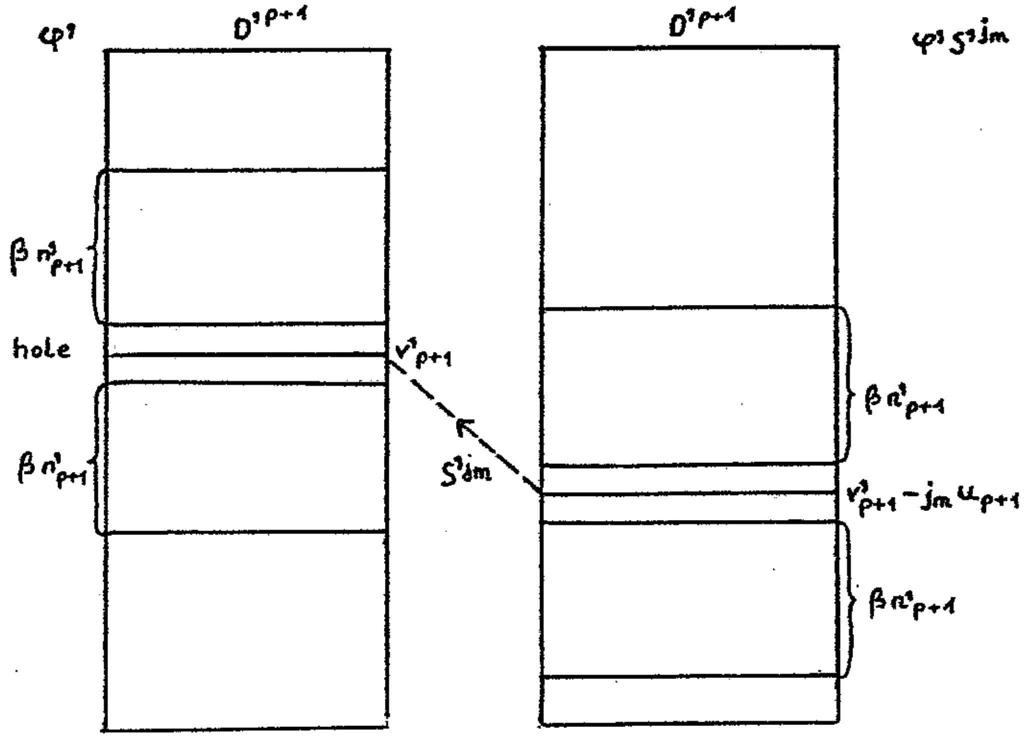


Figure 5.4.

Since $j_k < \lambda' \beta/2$, it follows that $\varphi' S'^{j_1} + \ldots + \varphi' S'^{j_k}$ has k α-regular holes in this tower with a sufficiently small constant γ. Since $u_{p+1} \geq n'_{p-1} = n'_{p+1}/\lambda'^2$ and $(j_k - j_1)u_{p+1} \leq j_k n'_p < \beta n'_{p+1}/2$, we may choose $\gamma = \min\{1/\lambda'^2, \beta/2\}$ ∎

Looking at the proof of Proposition 5.3 we notice that though $\varphi S^{j_1} + \ldots + \varphi S^{j_k}$ is ergodic, $\varphi S^{j_1} + \ldots + \varphi S^{j_k} + \varphi U$ need not be ergodic as the $U \in C(T)$ can shift an "unknown" part of the tower to the levels within $\beta n'_{p+1}$ of $v'_{p+1} - j_1 u_{p+1}$. We will overcome this difficulty by considering some nice cocycles.

We say that a cocycle $\varphi : X(\lambda) \longrightarrow Z_2$ is a <u>substitution-like</u> cocycle if there exist numbers α, β ($0 < \alpha, \beta < 1$) such that φ is constant on each level of D^t ($t \geq 1$) which is not an α-regular hole with constant β.

If this is the case then all the "holes" of φ are isolated and moreover the number of them is bounded. The class of substitution-like cocycles includes the following well-known automorphisms: continuous substitutions on two symbols [7], generalized Morse sequences in the sense of [6], Mathew-Nadkarni examples [41]. We will look at these examples more carefully in Part II.

Proposition 5.4. <u>Let</u> $\varphi : X(\lambda) \longrightarrow Z_2$ <u>be a substitution-like cocycle</u>. <u>Then for each</u> $S \in \mathcal{S}$ <u>the cocycle</u> φ <u>is S-strongly ergodic</u>.

Proof. Apply the arguments from the proof of Proposition 5.3 combined with the definition of a substitution-like cocycle ∎

The set $\mathcal{S}$ turns out to be a "large" set.

Proposition 5.5. <u>The Haar measure of the set</u> $\{\bar{s} \in X(\lambda): \sigma_{\bar{s}} = S \in \mathcal{S}\}$ <u>is equal to one</u>.

Proof. We will consider $X(\lambda)$ as $W=\{0,1,\ldots,\lambda-1\}^N$ with the one-sided shift $\tau(x_0,x_1,\ldots)=(x_1,x_2,\ldots)$. The Haar measure on $X(\lambda)$ is just the Bernoulli measure $(1/\lambda,\ldots,1/\lambda)$. It is well-known that this measure is concentrated on the set of generic points ([11], Prop. 3.7). Now, if $\bar{s}=(s_0,s_1,\ldots)\in W\setminus\mathcal{S}$ then there is a number $M>0$ such that the M-block $(0,\ldots,0)$ cannot be seen on $\bar{s}$ as a sequence $(s_i,\ldots,s_{i+M-1})$ for any i. Hence such an $\bar{s}$ is not generic ∎

∎

We have been unable to answer the following.

Question 5.1. Assume that $T:(X,\mathcal{B},\mu)\circlearrowleft$ is an ergodic rotation on a compact monothetic group. Let $\varphi:X\longrightarrow Z_2$ be an ergodic cocycle. Does there exist an $S\in C(T)$ such that φ is S-strongly ergodic?

The example below delivers the affirmative answer in the case of adding machines and some special regular cocycles.

Example 5.3. Let T be an $\{n_t\}$-adic adding machine and let $0<\alpha<1$. Assume that $\varphi:X\longrightarrow Z_2$ has an α-regular hole at $D^t_{n_t-1}$ for all $t\geq 0$ and, besides, that this is the only hole of φ in D^t. We will prove that <u>there exists an</u> $S\in C(T)$ such that φ is S-strongly ergodic.

Let $\{a_k\}$ be a sequence of natural numbers such that

(94) each natural number occurs infinitely often in $\{a_k\}$.

Let $\{\varepsilon_k\}$ be a sequence of positive numbers such that $1>\varepsilon_k/(1/a_k)\searrow 0$. We will construct $S\in C(T)$ inductively. At step k we will define the action of S on a tower D^{t_k} so that $SD_0^{t_k}=D_{i_k}^{t_k}$, where

(95) $$i_k \in (((1/a_k)-\varepsilon_k)n_{t_k},((1/a_k)+\varepsilon_k)n_{t_k}).$$

Now, we describe the induction step (see Fig. 5.5)

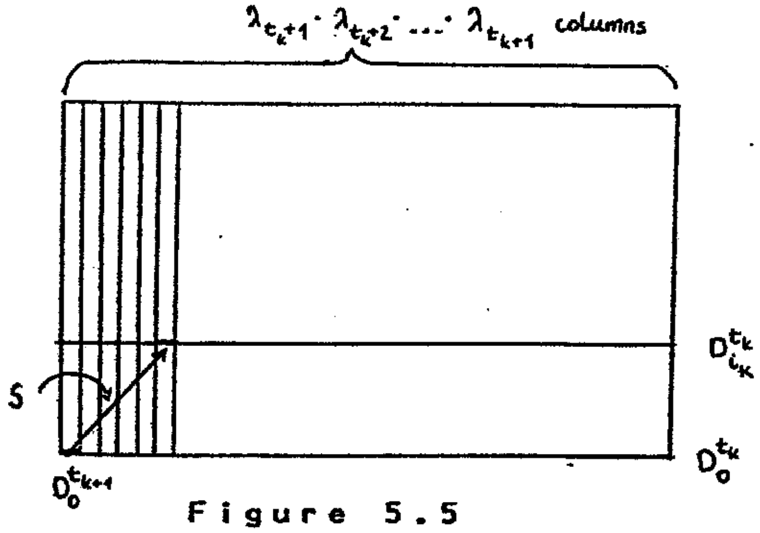


Figure 5.5

We choose the number t_{k+1} (depending on a_{k+1}, ε_{k+1}) in such a way that there exists a number $i_{k+1}=jn_{t_k}+i_k$ satisfying (95) for k+1. As there is no obstacle to do this we have defined an $S\in C(T)$. The proof that φ is S-strongly ergodic is much the same as that of Proposition 5.3. For the cocycle $\psi=\varphi S^{j_1}+\ldots+\varphi S^{j_s}$ $0\le j_1<\ldots<j_s$ we consider the infinite set $\mathcal{A}=\{k:\ a_k=2j_s\}$. By (95), the behavior of ψ on the towers D^{t_k} for $k\in\mathcal{A}$ is essentially the same as that of the cocycles of the form $\sum_p \varphi S^{j_p}$ for a substitution-like φ and S in $\mathcal{S}$. Therefore, the arguments from the proof of Proposition 5.3 work and φ is S-strongly ergodic■

6. ON THE WEAK ISOMORPHISM OF STRICTLY ERGODIC HOMEOMORPHISMS. Let $X=[0,1)$ with addition modulo 1 and Lebesgue measure μ. Let $T:(X,\mathcal{B},\mu)\circlearrowleft$ be an irrational rotation: $Tx=x+\alpha$. Assume that $S\in C(T)$. In this section we will mainly deal with automorphisms of the form

(96) $\qquad T_{\varphi\times\varphi S\times\varphi S^2\times\ldots}:(X\times(X\times X\times\ldots),\mu\times(\mu\times\mu\times\ldots))\circlearrowleft,$

where $\varphi:X\circlearrowleft$ is an ergodic cocycle.

In 1987 J.-P. Thouvenot raised the following question.

(97) $\begin{cases}\text{If } \varphi:X\circlearrowleft \text{ is ergodic and not cohomologous to a constant then does}\\ \text{there exist an } S\in C(T) \text{ such that the automorphism (96) is ergodic?}\end{cases}$

As we have already seen, the ergodicity of the automorphism (96) is equivalent to saying that the relatively independent extension of $\mu_{Id,S,S^2,\ldots}\in J^e_\infty(T)$ is an ergodic infinite self-joining of T_φ. Moreover, $T_\varphi\times T_\varphi\times\ldots:((X\times X)\times(X\times X)\times\ldots,\hat{\mu}_{Id,S,S^2,\ldots})\circlearrowleft$ is not coalescent since the formula $\hat{S}(x,y_1,y_2,\ldots)=(Sx,y_2,y_3,\ldots)$ defines a noninvertible element of $C(T_{\varphi\times\varphi S\times\varphi S^2\times\ldots})$.

Instead of (97) we will consider a more general question. Assume that $\tau:(Y,\mathcal{C},\nu)\circlearrowleft$ is an ergodic automorphism with partly continuous spectrum.

(98) $\begin{cases}\text{Does there exist a } \lambda\in J^e_\infty(\tau) \text{ such that } \tau\times\tau\times\ldots:(Y\times Y\times\ldots,\lambda)\circlearrowleft\\ \text{is not coalescent?}\end{cases}$

We will work in two directions. First, we will show that there exists an ergodic Anzai skew product T_φ, where φ is not cohomologous to a constant, such that for the T_φ the answer to (98) (and hence to (97)) is negative. On the other hand we will construct some special φ giving the positive answer to (97). This will lead to a construction of strictly ergodic homeomorphisms that are topological factor of each other but that are not measure-theoretically (m.t.) isomorphic.

We start with the following.

Theorem 6.1. <u>Let</u> $\varphi : X \circlearrowleft$ <u>be a weakly mixing cocycle. Assume that the cocycle</u> $x \longmapsto \varphi(x+\beta) - \varphi(x)$, $x \in X$, <u>is quasi-coboundary for each</u> $\beta \in [0,1)$. <u>Then, each</u> $\lambda \in J^e_\infty(T_\varphi)$ <u>is coalescent</u>.

Proof. Assume that $\lambda \in J^e_\infty(T_\varphi)$ and let $\mu_{S_1,S_2,\ldots}$, $S_i x = x+\beta_i$, be the corresponding projection of λ on $J^e_\infty(T)$. According to our assumption, for each $S \in C(T)$, $Sx = x+\beta$, we have $\varphi S - \varphi = c_\beta + \mathrm{cob}(T)$, where we will use cob(T) to denote any coboundary cocycle. Note that the constant c_β is well-defined up to $Z\alpha$. Denote
$\psi = \varphi S_1 \times \varphi S_2 \times \ldots$.

As in Lemma 1.8, instead of considering $T_\varphi \times T_\varphi \times \ldots : ((X \times X) \times \ldots, \lambda) \circlearrowleft$ we can study ergodic components of $T_\psi : (X \times (X \times X \times \ldots), \mu \times (\mu \times \mu \times \ldots)) \circlearrowleft$. In view of Lemma 1.5 (and its proof) , each ergodic component of T_ψ is isomorphic to the automorphism $T_{\psi_1} : (X \times G, \mu \times \nu_G)$, where $\psi_1 : X \longrightarrow G$ and

(99) $\psi_1 = \psi + fT - f$

for a measurable $f : X \longrightarrow X^N$ and $G = \mathrm{Ann}\widetilde{H}$, where

$$\widetilde{H} = \{\chi \in \hat{X}^N : \ \chi\psi = \mathrm{cob}(T)\}.$$

If $\chi \in \hat{X}^N$ then $\chi = \chi^{n_1,\ldots,n_k}_{i_1,\ldots,i_k}$, where

$$\chi^{n_1,\ldots,n_k}_{i_1,\ldots,i_k}(y_1, y_2, \ldots) = n_1 y_{i_1} + \ldots + n_k y_{i_k}$$

for a choice of integers $n_1, \ldots, n_k$ and coordinates $i_1, \ldots, i_k$.

Assume, now, that

(100) $\psi_1 S - v\psi_1 = gT - g$

for a measurable $g : X \to G$ and a continuous group epimorphism $v : G \circlearrowleft$. All we need to show is that v is invertible (Proposition 0.5).

Now, $\pi_k(y_1,y_2,\ldots)=y_k$ is a character of G, whence so is $\pi_k v:G\longrightarrow X$. Therefore, there exists an extension $\widetilde{\pi_k v}$ to a character of X^N. Hence

(101) $\widetilde{\pi_k v}(y_1,y_2,\ldots)=n_1^{(k)}y_{i_1^{(k)}}+\ldots n_{s_k}^{(k)}y_{i_{s_k}^{(k)}}$.

From (100) it follows that $\pi_k(gT)-\pi_k(g)=\pi_k(\Psi_1 S)-\pi_k(v\Psi_1)$. In view of (99) and the following equalities $\pi_k(gT)=(\pi_k g)T$, $\pi_k(\Psi_1 S)=\pi_k(\Psi S)+(\pi_k fS)T-\pi_k fS$, $\pi_k(v\Psi_1)=(\widetilde{\pi_k}v)\Psi+(\widetilde{\pi_k}vf)T-\widetilde{\pi_k}vf$ we obtain that

(102) $\varphi S_k S-(\widetilde{\pi_k}v)\Psi=g_kT-g_k$

for the measurable $g_k:X\to X$ given by $\pi_k g-\pi_k fS-\widetilde{\pi_k}vf$. Furthermore

(103) $\chi^{n_1,\ldots,n_k}_{i_1,\ldots,i_k}\in\tilde{H}$ iff $\begin{cases} n_1+\ldots+n_k=0 \text{ and} \\ n_1c_{\beta_{i_1}}+\ldots+n_kc_{\beta_{i_k}}\in Z\alpha. \end{cases}$

Indeed, $\chi^{n_1,\ldots,n_k}_{i_1,\ldots,i_k}(\Psi(x))=n_1\varphi(S_{i_1})+\ldots+n_k\varphi(S_{i_k}x)=$ $(n_1+\ldots+n_k)\varphi(x)+n_1c_{\beta_{i_1}}+\ldots+n_kc_{\beta_{i_k}}+\mathrm{cob}(Tx)$, since $n_r\varphi(S_{i_r}x)=n_r\varphi(x)+n_rc_{\beta_{i_r}}+\mathrm{cob}(Tx)$, $r=1,\ldots,k$. As φ is assumed to be weakly mixing, (103) is valid.

In view of (101) and (102) we have

$$g_k(x+\alpha)-g_k(x)=\varphi(x+\beta_k+\beta)-\left[n_1^{(k)}\varphi(x+\beta_{i_1^{(k)}})+\ldots+n_{s_k}^{(k)}\varphi(x+\beta_{i_{s_k}^{(k)}})\right],$$

$k=1,2,\ldots$. Therefore, by the argument we have just used to prove (103) we obtain that

(104a) $1-\left[n_1^{(k)}+\ldots n_{s_k}^{(k)}\right]=0$,

(104b) $c_{\beta+\beta_k}-M_k\in Z\alpha$,

where $M_k=n_1^{(k)}c_{\beta_{i_1^{(k)}}}+\ldots+n_{s_k}^{(k)}c_{\beta_{i_{s_k}^{(k)}}}$. Then, by taking the difference for k and l in (104b) we have $(c_{\beta_k+\beta}-M_k)-(c_{\beta_l+\beta}-M_l)\in Z\alpha$. But

(104c) $c_{\beta_k+\beta}-c_{\beta_l+\beta}=c_{\beta_k}-c_{\beta_l}+s\alpha$, for some $s\in Z$.

Indeed, $\varphi(x+\beta_r+\beta)-\varphi(x)=c_{\beta_r+\beta}+\mathrm{cob}(Tx)$, $r=k,l$. Thus

$\varphi(x+\beta_k+\beta)-\varphi(x+\beta_l+\beta)=c_{\beta_k+\beta}-c_{\beta_l+\beta}+\mathrm{cob}(Tx)$ and then
$\varphi(x+\beta_k-\beta_l)-\varphi(x)=c_{\beta_k+\beta}-c_{\beta_l+\beta}+\mathrm{cob}(Tx)$. Also
$\varphi(x+\beta_k)-\varphi(x+\beta_l)=c_{\beta_k}-c_{\beta_l}+\mathrm{cob}(Tx)$, which gives us
$\varphi(x+\beta_k-\beta_l)-\varphi(x)=c_{\beta_k}-c_{\beta_l}+\mathrm{cob}(Tx)$. Hence $c_{\beta_k+\beta}-c_{\beta_l+\beta}+Z\alpha=$
$c_{\beta_k-\beta_l}+Z\alpha=c_{\beta_k}-c_{\beta_l}+Z\alpha$ and, in particular, (104c) follows.
Therefore, by (104c), we get $(c_{\beta_k}-M_k)-(c_{\beta_l}-M_l)\in Z\alpha$. This combined with (103) and (104a) gives us

(105) $$\widetilde{\chi}=\chi^{1,-n_1^{(k)},\dots,-n_{s_k}^{(k)}}_{k,\ i_1^{(k)},\dots,\ i_{s_k}^{(k)}}-\chi^{1,-n_1^{(l)},\dots,-n_{s_l}^{(l)}}_{l,\ i_1^{(l)},\dots,\ i_{s_l}^{(l)}}\in\widetilde{H}.$$

Let $y\in G$ and $v(y)=0$. Since $G=\mathrm{Ann}\widetilde{H}$, for any k,l we have $0=\widetilde{\chi}(y)=[y_k-\pi_k v(y)]-[y_l-\pi_l v(y)]=y_k-y_l$, whence $y=(y_1,y_1,\dots)$ and by (104a), $0=\pi_k v(y)=(n_1^{(k)}+\dots+n_{s_k}^{(k)})y_1=y_1$ implying $y=0$. This proves that v is invertible■

R e m a r k 6 . 1 . The assumptions of Theorem 6.1 are satisfied for all affine cocycles, $\varphi(x)=nx+\lambda$, ($n\in Z$, $\lambda\in X$). We do not know whether these are the only such cocycles (up to cohomology)■

Now, we will pass to a construction of an ergodic φ yielding the affirmative answer to (97). Actually, we replace ergodicity of (96) by some stronger condition and seek $S\in C(T)$ such that

(106) $$\begin{cases} m_1\varphi S^{i_1}+\dots+m_k\varphi S^{i_k}+m\varphi U\ (:X^2) \text{ is ergodic whenever } i_1<\dots<i_k \\ (k\geq 2),\ m_1,\dots,m_k \text{ are nonzero integers, } m\in Z, \text{ and } U \text{ is an arbitrary} \\ \text{element of } C(T). \end{cases}$$

Once (106) holds, the automorphism (96) is ergodic (see (15)).

T h e o r e m 6 . 2 . <u>If</u> α <u>is an irrational number that admits an approximation by rationals</u> $\{p_n/q_n\}$ <u>with speed</u> $o(1/q_n^2)$ <u>then there exist an ergodic</u> φ <u>and an irrational number</u> γ <u>such that (106) is fullfilled for</u> $Sx=x+\gamma$.

P r o o f . For a number $\delta \in [0,1)$ we denote

$$||\delta||_n = \min_{0 \le r \le q_n - 1} |\delta - r/q_n|.$$

Then

(107) $\quad ||\delta||_n = ||1-\delta||_n$

and if $||\delta||_n < 1/2kq_n$ then

(108) $\quad ||s\delta||_n = s||\delta||_n, \; s=1,2,\ldots,k-1.$

From our assumption

(109) $\quad ||\alpha||_n = o(1/q_n^2).$

Then, by passing to a subsequence of $\{p_n/q_n\}$, if necessary, we can find another irrational $\beta \in [0,1)$ such that

(110) $\quad ||\beta||_n \ge c/q_n$ for some $0 < c < 1/2$.

Assume that $\{a_n\}$ is a sequence of natural numbers such that each natural number appears infinitely often in $\{a_n\}$. Hence, by passing to a subsequence of $\{p_n/q_n\}$, we can find an irrational number γ such that

(111) $\quad \big| \; ||\beta||_n - a_n||\gamma||_n \; \big| = o(1/q_n^2).$

The construction of γ is essentially the same as that of β beacuse $||\beta||_n$ and a_n are determined prior to the construction.

Now, define $S \in C(T)$ by $Sx = x+\gamma$ and let

(112) $\quad \varphi(x) = \rho_1$ for $x \in [0,\beta)$ and $\varphi(x) = \rho_2$ for $x \in [\beta,1)$,

where

(113) $\quad \rho_1 - \rho_2$ is irrational.

First, we will prove the following.

(113) For every $m_1,\ldots,m_k \in Z$, $m_i \ne 0$ and $0 < i_1 < \ldots < i_k$ the cocycle $m_1\varphi S^{i_1} + \ldots + m_k\varphi S^{i_k}$ is not coboundary.

Let us denote $\Psi(x) = m_1\varphi(S^{i_1}x) + \ldots + m_k\varphi(S^{i_k}x) =$

$$m_1\varphi(x+i_1\gamma) + \ldots + m_k\varphi(x+i_k\gamma).$$

Suppose Ψ is coboundary. We will consider only those n for which

(115) $\quad a_n=3i_k$

(thereare infinitely many such n's).

By (111) we have $\|\beta\|_n/(3i_k)=\|\gamma\|_n+o(1/q_n^2)$, hence by (110) for n large enough

(116) $(7/(24i_k))\|\beta\|_n \le \|\gamma\|_n \le (9/(24i_k))\|\beta\|_n < 1/(2(i_k+1)q_n)$.

Also, for each $\delta\in[0,1)$ and $j=0,\dots,q_n-1$

(117) $\quad \big|\ \|\delta-j\alpha\|_n-\|\delta\|_n\ \big|<\varepsilon_n/q_n$, where $\varepsilon_n\to 0$.

We will study the discontinuity points of

$$\psi^{(q_n)}(x)=\psi(x)+\psi(Tx)+\dots+\psi(T^{q_n-1}x).$$

They are of the form: $-i_s\gamma-j\alpha$, $\beta-i_s\gamma-j\alpha$, where $s=1,\dots,k$ and $j=0,\dots,q_n-1$. Let us order these points in an arbitrary way, say, $\{y_r\}_{r=0}^{2kq_n-1}$. Then

(118) $\quad \|y_j-y_{j'}\|\ge c_1/q_n$ for every $j\ne j'$

for a constant $c_1>0$ and n large enough. Indeed, from (107), (108) and (116) we obtain that $\|-i_s\gamma\|_n=i_s\|\gamma\|_n$. Moreover, it follows from (117) that $\big|\ \|-i_s\gamma-j\alpha\|_n-\|-i_s\gamma\|_n\ \big|<\varepsilon_n/q_n$ and $\big|\ \|\beta-i_s\gamma-j\alpha\|_n-\|\beta-i_s\gamma\|_n\ \big|<\varepsilon_n/q_n$. Therefore, the distance between the points of the form $-i_s\gamma-j\alpha$, $-i_{s'}\gamma-j'\alpha$ is at least $\|\gamma\|_n-|\varepsilon_n|/q_n>\|\gamma\|_n/2$ for n large enough. The same is true for the points of the form $\beta-i_s\gamma-j\alpha$, $\beta-i_{s'}\gamma-j'\alpha$. Now, the distance between $-i_s\gamma-j\alpha$ and $\beta-i_{s'}\gamma-j'\alpha$ is at least $(6/24)\|\beta\|_n\ge(1/8)c/q_n$ because by (116) $\|\beta-i_{s'}\gamma\|_n\ge\|\beta\|_n-\|i_{s'}\gamma\|_n\ge\|\beta\|_n-i_{s'}(9/(24i_k))\|\beta\|_n\ge 5\|\beta\|_n/8$. Therefore (118) holds. At each point y_j the cocycle $\psi^{(q_n)}$ has a jump equal to $m_i(\rho_1-\rho_2)$ or $-m_i(\rho_1-\rho_2)$. Thus, if $\varepsilon>0$ is small enough and we consider the intervals $(y_j-c_1/(3q_n),y_j)$, $(y_j,y_j+c_1/(3q_n))$ only one can belong to $\tilde{A}_\varepsilon^{(q_n)}$ (for the definition of $\tilde{A}_\varepsilon^{(q_n)}$ for ψ see (69)). Hence, $\mu(\tilde{A}_\varepsilon^{(q_n)})\not\to 1$ and in view of Remark 4.1 we get a contradiction. Therefore (114) has been proved.

Now, consider

(119) $\Psi_1 = m_1 \varphi S^{i_1} + \ldots + m_k \varphi S^{i_k} + m \varphi U,$

where $k \geq 2$. Then the jumps of $\Psi_1^{(q_n)}$ occur at $\{y_r\}_{r=0}^{2kq_n-1}$ $\{z_j\}_{j=0}^{2q_n-1}$ (with the possibility $y_r = z_j$ for some j,r). In at least $2(k-1)q_n \geq 2q_n$ intervals $(y_r - c_1/(3q_n), y_r + c_1/(3q_n))$ there are no z_j. Hence, the above argument shows that Ψ_1 is not coboundary and the proof of Theorem 6.2 is complete■

R e m a r k 6 . 2 . In the same manner we prove that the cocycle $m_1 \varphi S^{i_1} + \ldots + m_k \varphi S^{i_k} + n_1 \varphi U_1 + \ldots + n_{k-1} \varphi U_{k-1}$ is not coboundary for arbitrary $U_1, \ldots, U_{k-1} \in C(T)$ and $m_i, n_j \in Z \setminus \{0\}$■

It is clear that the property (106) is stable under the passage to cohomologous cocycles. Therefore, if $\psi(x) = \varphi(x) + \mathrm{cob}(Tx)$ is continuous then $\overline{T}_{i_1,i_2,\ldots} = T_{\psi S^{i_1} \times \psi S^{i_2} \times \ldots}$ is a strictly ergodic model of $T_{i_1,i_2,\ldots} = T_{\varphi S^{i_1} \times \varphi S^{i_2} \times \ldots}$ if $\overline{T}_{i_1,i_2,\ldots}$ or equivalently $T_{i_1,i_2,\ldots}$ is ergodic ([9] p. 66). The existence of such ψ follows from [28], [58].

T h e o r e m 6 . 3 . <u>Assume that</u> φ <u>and</u> S <u>satisfy the conclusion of Theorem</u> 6.2 <u>and let</u> $j_1 < j_2 < \ldots$, $i_1 < i_2 < \ldots$. <u>Then the following are equivalent</u>.

(i) $\overline{T}_{i_1,i_2,\ldots}$ <u>and</u> $\overline{T}_{j_1 j_2,\ldots}$ <u>are topologically isomorphic</u>.

(ii) $T_{i_1,i_2,\ldots}$ <u>and</u> $T_{j_1,j_2,\ldots}$ <u>are m.t. isomorphic</u>.

(iii) $i_k - j_k = c$, $k = 1, 2, \ldots$.

P r r o f . The proof is based on the same calculations as those in the proof of Theorem 5.1, so we omit the details■

Corollary 6.1. <u>Strictly ergodic homeomorphisms $\bar{T}_{0,1,2,3,\dots}$ and $\bar{T}_{0,2,3,4,\dots}$ are topological factors of each other but they are not m.t. isomorphic. In particular, they are not topologically isomorphic</u>■

Corollary 6.2. <u>Strictly ergodic homeomorphism $\bar{T}_{\dots,-1,0,1,2,\dots}$ is topologically coalescent and it has a topologically noncoalescent factor</u> $\bar{T}_{0,1,2,\dots}$ ■

Remark 6.3. Note that the three known constructions of weakly isomorphic automorphisms that are not isomorphic do not give examples as in Corollary 6.1 (see [57],[62] and Theorem 5.1). Assume , now, that

$$\begin{array}{ccccc} (X,\mu) & \xrightarrow{f} & (Y,\nu) & \xrightarrow{g} & (X,\mu) \\ T\circlearrowleft & & \tau\circlearrowleft & & T\circlearrowleft \end{array}$$

is a m.t. diagram of weak isomorphism. Then, it follows from [67] that this diagram has a strictly ergodic model

$$\begin{array}{ccccc} (X_1,\mu_1) & \xrightarrow{f_1} & (Y_1,\nu_1) & \xrightarrow{g_1} & (X_2,\mu_2) \\ T_1\circlearrowleft & & \tau_1\circlearrowleft & & T_2\circlearrowleft \end{array} \qquad (120)$$

where possibly $T_1 \neq T_2$. (It is now understood that T_1, T_2, τ_1 are strictly ergodic homeomorphisms and f_1, g_1 are continuous). Theorem 6.3 delivers concrete examples of strictly ergodic models for weak isomorphisms diagrams. We do not know whether for each m.t. diagram of weak isomorphism there exists its strictly ergodic model (in the sense that $T_1=T_2$ in (120)) such that T_1 and τ_1 are not m.t. isomorphic.

Remark 6.4. The result of Corollary 6.1 strenghtens the result of W. Parry and P. Walters [53] saying that there is a minimal noncoalescent homeomorphism of the infinite dimensional torus. Another result of [53] shows that for finite dimensional tori extensions of rotations we obtain topologically coalescent homeomorphisms. In particular, the result of Corollary 6.1 is no longer true for strictly ergodic cocycles taking values in the circle.

Corollary 6.2 shows that the class of strictly ergodic topologically coalescent homeomorphisms is not closed under taking topological factors. It has been communicated to the author by J. Auslander that there exist minimal homeomorphisms having both positive entropy and topological coalescence property ■

Remark 6.5. Let us observe that if $\tau:(Y,\mathcal{C},\nu)\circlearrowleft$ is weakly mixing then the answer to (98) is affirmative. Indeed, if this is the case then $\lambda=\nu\times\nu\times\ldots\in J^e_\infty(\tau)$ and the one-sided shift $\sigma(x_1,x_2,x_3,\ldots)=(x_2,x_3,\ldots)$ commutes with $\tau\times\tau\times\ldots$ and is not invertible. More generally, suppose that $\tau:(Y,\mathcal{C},\nu)\circlearrowleft$ is ergodic and that there is a proper factor $\mathcal{C}_1\subseteq\mathcal{C}$ such that τ is relatively weakly mixing with respect to $\mathcal{C}_1$. Let $\bar{Y}$ be the quotient space corresponding to $\mathcal{C}_1$ and

$$\nu=\int_{\bar{Y}}\nu_{\bar{y}}\,d\nu(\bar{y})$$

be the decomposition of ν over the factor $\mathcal{C}_1$. Then the measure

$$\lambda=\int_{\bar{Y}}(\nu_{\bar{y}}\times\nu_{\bar{y}}\times\ldots)\,d\nu(\bar{y})$$

is in $J^e_\infty(\tau)$ ([11] p. 110-115). Moreover, the measure λ is concentrated on $W=Y\times_{\mathcal{C}_1}Y\times_{\mathcal{C}_1}\ldots=\{(y_1,y_2,\ldots)\in Y\times Y\times\ldots:\pi(y_1)=\pi(y_2)=\ldots\}$, where $\pi:(Y,\mathcal{C},\nu)\to(\bar{Y},\mathcal{C}_1,\nu)$ establishes the homomorphism of τ on $\mathcal{C}$ and τ on $\mathcal{C}_1$. This allows us to define $\sigma:W\circlearrowleft$ as $\sigma(y_1,y_2,\ldots)=(y_2,y_3,\ldots)$. Actually σ preserves λ since $\sigma^{-1}(A_1\times A_2\times\ldots\times A_k\times Y\times Y\times\ldots)=(Y\times A_1\times A_2\times\ldots\times A_k\times Y\times Y\times\ldots)$. Moreover,

σ commutes with $\tau\times\tau\times\ldots$ and σ is not invertible since π is not trivial. Therefore if τ is ergodic and the chain from Structure Theorem ends with a relatively weakly mixing extension then, for such a τ, the answer to (98) is positive. In particular, if τ is ergodic and has positive entropy then there exists $\lambda \in J^{e}_{\infty}(\tau)$ which is not coalescent (but still it is open whether τ itself is coalescent).

If τ is ergodic and in Structure Theorem only compact extensions occur then the answer to (98) is in general unknown. One could hope that the negative answer to this question gives another characterization of the discrete spectrum class but this is not the case as Theorem 6.1 says. It is well-known that the Anzai skew products given by affine cocycles have quasi-discrete spectrum ([1]). Actually, one can show that for each automorphism with quasi-discrete spectrum the answer to (98) is negative (this will be published elsewhere). We do not know whether the negative answer for totally ergodic τ enjoying partly continuous spectrum yields quasi-discrete spectrum property ■

Another property of the infinite product measure is that this self-joining has infinite spectral multiplicity. We have been unable to answer the following question.

Question 6.1. Let τ be ergodic with partly continuous spectrum. Does there exist an ergodic infinite self-joining λ of τ possessing infinite spectral multiplicity?

Note that if for τ the answer to (98) is positive then automatically it is the same for Question 6.1 (see Proposition 0.1). Note that T_{φ} given by affine cocycle cannot be a counterexample as T_{φ} has infinite spectral multiplicity itself ([1]).

PART II

7. EVERY ERGODIC Z_2-COCYCLE IS A FACTOR OF AN ERGODIC JOINING OF TWO Z_2-COCYCLES WITH SIMPLE SINGULAR SPECTRA. In this section we will work on Z_2-extensions over an $\{n_t\}$-adic adding machine. We show that if a Z_2-cocycle admits an approximation with speed high enough then it is cohomologous to a Morse cocycle. Then we will conclude that every ergodic Z_2-extension of an adding machine is a factor of an ergodic joining of two Morse cocycles.

Let $T:(X,\mathcal{B},\mu)\circlearrowleft$ be an $\{n_t\}$-adic adding machine with the corresponding sequence of towers $D^t=(D_0^t,\dots D_{n_t-1}^t)$, $t\geq 0$ (see Section 5). Let $\mathcal{K}$ denote the class of all Z_2-cocycles on $(X,\mathcal{B},\mu)$: this meaning that

$$\varphi\in\mathcal{K} \quad \text{whenever} \quad \varphi: X\to Z_2 \text{ is measurable.}$$

We endow $\mathcal{K}$ with the natural topology given by the metric

$$\rho(\varphi,\varphi')=\mu(\varphi^{-1}(0)\vartriangle\varphi'^{-1}(0))+\mu(\varphi^{-1}(1)\vartriangle\varphi'^{-1}(1)).$$

With this metric, $\mathcal{K}$ becomes a Polish topological group (with pointwise addition mod 2).

Definition 7.1. ([23]) We say that φ <u>is oddly approximated with speed</u> $o(r(n))$ if for a subsequence $\{n_{t_k}\}$ there exists for each k a union F_k of an odd number of levels of D^t, such that

$$\mu(\varphi^{-1}(1)\vartriangle F_k)=o(r(n_{t_k})). \tag{121}$$

The odd approximation of φ with speed $o(1/n)$ guarantees the ergodicity of T_φ ([23]).

A cocycle $\varphi\in\mathcal{K}$ is called a <u>Morse cocycle</u> if there is a subsequence $\{n_{t_k}\}$ such that $\varphi|D_i^{t_k}=a_i^{(t_k)}\in\{0,1\}$, $i=0,\dots,n_{t_k}-2$, which means that φ is constant on each level of D^{t_k} except for the top.

Theorem 7.1. <u>Let $\varphi \in \mathcal{K}$ be ergodic and admit an odd approximation with speed</u> $o(1/n^{1+\varepsilon})$, $\varepsilon > 0$. <u>Then there exists a Morse cocycle Ψ such that φ and Ψ are cohomologous.</u>

Proof. From our assumption it follows that there exists a subsequence $\{n_{t_k}\}$ of $\{n_t\}$ satisfying (121). For simplicity, we assume that this subsequence is equal to $\{n_t\}$, whence

(122) $$n_t^{1+\varepsilon}\,\mu(\varphi^{-1}(1) \triangle F_t) \longrightarrow 0,$$

where F_t is the union of an odd number of levels of D^t. Now, the function φ restricted to each level D_i^t, $i=0,\dots,n_t-1$ is "almost" constant. This is made precise by the following.

$\varphi|D_i^t = a_i^{(t)}$ except for a part V_i^t of D_i^t, where $\mu(\bigcup_{i=0}^{n_t-1} V_i^t) < \varepsilon_t/n_t^{1+\varepsilon}$ with $\varepsilon_t \to 0$.

Since the approximation is odd,

(123) $$\sum_{i=0}^{n_t-1} a_i^{(t)} = 1.$$

In order to obtain D^{t+1} we divide the tower D^t into $\lambda_{t+1} = n_{t+1}/n_t$ columns of the same measure $1/\lambda_{t+1}$. If $i=j \pmod{n_t}$ and $a_i^{(t)} \neq a_j^{(t+1)}$ then we say that <u>there is an error at</u> D_j^{t+1}. Denote by m_t the total number of all columns with errors. As the measure of each column is equal to $1/\lambda_{t+1}$, the measure CE^{t+1} of all columns with errors equals

(124) $$CE^{t+1} = m_t/\lambda_{t+1}.$$

Moreover, the measure of each error (i.e. $\mu(D_j^{t+1})$) is equal to $1/n_{t+1}$, so by (122)

(125) $$n_t^{1+\varepsilon}(k_t/n_{t+1}) \longrightarrow 0,$$

where k_t is the number of all errors. But $m_t \le k_t$ and hence a combination of (124) and (125) gives us $n_t^{\varepsilon} CE^{t+1} \longrightarrow 0$, whence

(126) $$\sum_{t\ge 0} CE^{t+1} < +\infty.$$

Now, we pass to a construction of a Morse cocycle Ψ. At each stage t our cocycle will be constant on each level D_i^t, $i=0,\dots,n_t-2$ and will not be defined on $D_{n_t-1}^t$. We define Ψ on D_i^0, $i=0,\dots,n_0-2$ in an arbitrary way. Assume inductively that $\Psi|D_i^t=b_i^{(t)}$ $(\in\{0,1\})$, $i=0,\dots,n_t-2$. First, we define $b_{n_t-1}^{(t)}$ so that

(127) $$\sum_{i=0}^{n_t-1} b_i^{(t)}=1.$$

To define Ψ on D^{t+1} we do not change its values on the levels $D_{jn_t+i}^{t+1}$, $j=0,\dots,\lambda_{t+1}-1$, $i=0,\dots,n_t-2$, so we put $\Psi|D_{jn_t+i}^{t+1}=b_{jn_t+i}^{(t+1)}$ $=b_i^{(t)}$. Now, we should define Ψ on the levels $D_{jn_t-1}^{t+1}$, $j=1,\dots,\lambda_{t+1}-1$. If the number of errors in the j-th column is even (in particular if no errors occur) then we put $\Psi|D_{jn_t-1}^{t+1}=b_{jn_t-1}^{(t+1)}$ $=b_{n_t-1}^{(t)}$; put $1+b_{n_t-1}^{(t)}$ otherwise. Of course, this procedure leads to an (a.e.) correct definition of a Morse cocycle. Moreover, the cocycle Ψ admits an odd approximation with speed $o(1/n^{1+\varepsilon})$ (by (127) and (125)).

It remains to prove that the cocycle $\varphi+\Psi$ is coboundary or which is the same that $T_{\varphi+\Psi}$ is not ergodic. We will define a sequence of measurable sets $A_t\subseteq X\times Z_2$ such that

(128) $$\tilde{\mu}(A_t)=1/2,$$

(129) $$\sum_{t\geq 0}\tilde{\mu}(A_t\triangle A_{t+1})<+\infty,$$

(130) $$\tilde{\mu}(T_{\varphi+\psi}A_t\triangle A_t)\longrightarrow 0.$$

For every $t\geq 0$ we look at the partition $D^t\times\{0,1\}$ of $X\times Z_2$. Put $c_i^{(t)}=a_i^{(t)}+b_i^{(t)}$, $i=0,\dots,n_t-1$, $t\geq 0$. Then, by (127) and (123),

(131) $$\sum_{i=0}^{n_t-1} c_i^{(t)}=0.$$

The set A_t is defined as $A_t=\bigcup_{i=0}^{n_t-1}(D_i^t\times d_i^{(t)})$, where $d_0^{(t)}=0$ and

$d_i^{(t)} = c_0^{(t)} + \ldots + c_{i-1}^{(t)}$, $i=1,\ldots,n_t-1$. From this definition it follows that (128) holds. We now prove that (129) is valid as well. A key observation is that for the cocycle $\varphi+\psi$ the number of errors in each column is even (because ψ is so defined). We have

$$A_{t+1} = \bigcup_{p=0}^{n_{t+1}-1} (D_p^{t+1} \times d_p^{(t+1)}) = \bigcup_{j=0}^{\lambda_{t+1}-1} \bigcup_{i=0}^{n_t-1} (D_{jn_t+i}^{t+1} \times d_{jn_t+i}^{(t+1)}).$$

We will show that

(132) $\qquad d_{jn_t}^{(t+1)} = 0, \; j=0,\ldots,\lambda_{t+1}-1.$

Indeed, $d_0^{(t+1)}=0$ and assume that $d_{jn_t}^{(t+1)}=0$ for $j<\lambda_{t+1}-1$. Since $d_{(j+1)n_t}^{(t+1)} = d_{jn_t}^{(t+1)} + c_0^{(t+1)} + \ldots + c_{n_t-1}^{(t+1)}$ and $\sum_{i=0}^{n_t-1} c_i^{(t+1)} = \sum_{i=0}^{n_t-1} c_i^{(t)}$ (as in the $(j+1)$-st column of D^t there is an even number of errors), we conclude that (132) holds. Hence, $\tilde{\mu}(A_{t+1} \triangle A_t) \le m_t/\lambda_{t+1}$ and then (126) and (124) yield the validity of (129). Now, we estimate $\tilde{\mu}(T_{\varphi+\psi} A_t \triangle A_t)$. Let us observe that

$(T_{\varphi+\psi})^s(x,i) = (T^s x, (\varphi+\psi)(x) + (\varphi+\psi)(Tx) + \ldots + (\varphi+\psi)(T^{s-1}x) + i)$

In other words, if $\varphi+\psi$ were constant on all levels of D^t (and equal to $c_i^{(t)}$ on D_i^t) then we would get

$(T_{\varphi+\psi})^s (D_0^t \times 0) = D_s^t \times (c_0^{(t)} + \ldots + c_{s-1}^{(t)}) = D_s^t \times d_s^{(t)}$,

$s=0,\ldots,n_t-1$ and then by (131) that

$(T_{\varphi+\psi})^{n_t} (D_0^t \times 0) = D_0^t \times 0$. So, this would mean that $T_{\varphi+\psi}(A_t) = A_t$. However, $\varphi+\psi$ need not be constant on the levels of D^t. But by the arguments above it follows that $\mu(T_{\varphi+\psi} A_t \triangle A_t) = \varepsilon_t / n_t^{\varepsilon}$, where $\varepsilon_t \to 0$. This implies (130).

The set A given as the limit of the sequence $\{A_t\}$ gives us a $T_{\varphi+\psi}$-invariant set with measure 1/2, whence $T_{\varphi+\psi}$ is not ergodic ∎

Theorem 7.2. *Let $\varphi \in \mathcal{K}$ be ergodic. Then there exist two Morse cocycles φ_1 and φ_2 such that φ is cohomologous to $\varphi_1 + \varphi_2$. In particular, T_φ is a factor of an ergodic joining of T_{φ_1} and T_{φ_2}.*

Proof. Let $\varepsilon > 0$ and $\mathcal{K}^\varepsilon \subseteq \mathcal{K}$ be the set of those cocycles φ that admit an odd approximation with speed $o(1/n^{1+\varepsilon})$. From [23] it follows that $\mathcal{K}^\varepsilon$ is residual in $\mathcal{K}$. As $\mathcal{K}$ is a topological group, $\varphi + \mathcal{K}^\varepsilon$ is residual. Since $\mathcal{K}$ is a Polish space, there is $\psi \in \mathcal{K}^\varepsilon \cap (\varphi + \mathcal{K}^\varepsilon)$. Consequently, by Theorem 7.1, $\psi = \varphi_1 + f_1 T + f_1$, where φ_1 is a Morse cocycle. Also $\psi = \varphi + \psi_2$, where $\psi_2 \in \mathcal{K}^\varepsilon$ and $\psi_2 = \varphi_2 + f_2 T + f_2$ for a Morse cocycle φ_2. Therefore $\varphi = \varphi_1 + \varphi_2 + fT + f$ for $f \in \mathcal{K}$. So T_φ is relatively isomorphic to $T_{\varphi_1 + \varphi_2}$. The latter automorphism is a factor of $T_{\varphi_1 \times \varphi_2} : (X \times Z_2 \times Z_2, \mu \times \nu_2 \times \nu_2)\circlearrowleft$. Now $T_{\varphi_1 \times \varphi_2}$ is ergodic as otherwise we could solve either $\varphi_1 + \varphi_2 = gT + g$ or $\varphi_1 = gT + g$ or else $\varphi_2 = gT + g$ (by (15)) which leads to a contradiction in any case. Finally, $T_{\varphi_1 \times \varphi_2}$ is isomorphic to $T_{\varphi_1} \times T_{\varphi_2} : ((X \times Z_2) \times (X \times Z_2), \lambda)\circlearrowleft$, where λ is the relatively independent extension of the diagonal measure on $X \times X$ ■

Corollary 7.1 *Let $\varphi \in \mathcal{K}$ be ergodic. Then there are two $\varphi_1, \varphi_2 \in \mathcal{K}$ where $T_{\varphi_1}, T_{\varphi_2}$ have simple singular spectra and such that T_φ is spectrally isomorphic to $T_{\varphi_1} \times T_{\varphi_2}$ restricted to a subspace of $L^2((X \times Z_2)^2, \lambda)$, where $\lambda \in J^e(T_{\varphi_1}, T_{\varphi_2})$.*

Proof. The cocycles φ_1, φ_2 are as in Theorem 7.2. We will see in Section 8 that each ergodic Z_2-extension given by a Morse cocycle is isomorphic to a dynamical system generated by a Morse sequence. As it follows from [25], [30], the latter dynamical systems have simple and singular spectra ■

Remark 7.1 It follows from [52] pp. 85-87 that there is an adding machine and its ergodic Z_2-extension with countable Lebesgue spectrum in the orthocomplement of the space of eigenfunctions. From Corollary 7.1 it follows that this is still a property of the sum of two cocycles with simple singular spectra. Equivalently, this is a property of a joining of two automorphisms with simple singular spectra.

We have been unable to answer the following question. Let $\tau:(Y, \mathcal{E}, \nu)^2$ be an ergodic zero entropy automorphism. Do there exist T_i, $i=1,2$ with simple and singular spectra such that τ is spectrally isomorphic to a factor of an ergodic joining of T_1 and T_2?■

8. TOEPLITZ Z_2-COCYCLES AND SHIFT DYNAMICAL SYSTEMS. In this section we introduce the notion of a Toeplitz Z_2-extension. This will be useful to see a passage from the language of cocycles into some objects considered in combinatorial ergodic theory. The objects we have in mind are Toeplitz sequences [19], [68], substitution minimal sets [7], [18], [36], [50], Morse sequences [24], Morse sequences in the sense of [6]. Then, we briefly describe spectral theory of Toeplitz Z_2-extensions.

Let us assume that $T:(X,\mathcal{B},\mu)\circlearrowleft$ is an ergodic $\{n_t\}$-adic adding machine. Let $\{D^t\}$ be the corresponding sequence of towers. We will introduce some special measurable φ's in $\mathcal{K}$. We will assume that $\varphi^{-1}(i)$ contains some levels of D^t, $t \geq 0$. More precisely, there exists a partition of the set $\{0,\dots,n_t-1\}$ into A^t, B^t, C^t (with C^t relatively small) such that

$$(134)\quad \begin{cases} i \in A^t & \text{if} \quad \varphi|D_i^t = 0 \\ i \in B^t & \text{if} \quad \varphi|D_i^t = 1, \\ i \in C^t & \text{if} \quad \varphi|D_i^t \text{ is not constant.} \end{cases}$$

If $\lambda_{t+1}=n_{t+1}/n_t$ then for every $i=0,\dots,\lambda_{t+1}-1$

$$(135)\quad \begin{cases} A^t+in_t \subseteq A^{t+1} \\ B^t+in_t \subseteq B^{t+1} \end{cases}$$

Moreover, given A^t, B^t, C^t we can define a finite sequence $\varphi^{(t)}=(\varphi_0^{(t)},\dots,\varphi_{n_t-1}^{(t)})$ of the symbols $0,1,\infty$, where

$$\varphi_i^{(t)} = \begin{cases} 0 & \text{if } i \in A^t \\ 1 & \text{if } i \in B^t \\ \infty & \text{if } i \in C^t. \end{cases}$$

The " ∞ " will be referred to as "hole". In view of (135), $\varphi^{(t+1)}$ arises from $\varphi^{(t)}$ by repeating $\varphi^{(t)}$ λ_{t+1} times and then by filling up some "holes". Let φ be the limit of $\varphi^{(t)}$.

if φ has no holes then we call it a <u>Toeplitz sequence</u>. The finite sequence $\varphi^{(t)}$ will be called the t-<u>skeleton of</u> φ. We will assume that the resulting sequence is <u>regular</u>, meaning that

$$(1/n_t) \text{ card } C^t \longrightarrow 0.$$

By abuse of notation we use φ to denote both the Toeplitz sequence and the cocycle such that $\varphi(x)=0$ (resp. $\varphi(x)=1$) if $x \in D_j^t$ with $j \in A^t$ (resp. $j \in B^t$). There exists an extension of $\varphi \in \{0,1\}^N$ to a sequence in $\{0,1\}^Z$ also denoted by φ such that the bilateral shift τ is strictly ergodic on $Y=\overline{O_\tau(\varphi)} \subseteq \{0,1\}^Z$ ([19], [68]). The unique (ergodic) τ-invariant probability measure on Y will be denoted by ν. Then there is a sequence $E^t=(E_0^t,\dots,E_{n_t-1}^t)$, $t \geq 0$, where E_i^t are simultaneously open and closed in Y and $\tau^i E_0^t = E_i^t$ (mod n_t). In fact, E_i^t consists of those $y \in Y$ that have the same t-skeleton as $\tau^i\varphi$ (see [68]). It is also well-known that $\tau:(Y,\nu)\circlearrowleft$ has discrete spectrum with $Sp(\tau)=G\{n_t:\ t \geq 0\}$. Consider the continuous function $\varphi: Y \longrightarrow Z_2$ given by

(136) $\varphi(y)=y[0]$.

By taking into consideration E^t and the t-skeleton $\varphi^{(t)}$ of φ we observe that

$\varphi|E_i^t = \varphi[i]$ whenever $\varphi[i]$ is not a "hole".

The extension τ_φ, where φ is given by (136) is called a <u>Toeplitz</u> Z_2-<u>extension</u>.

Evidently, we can consider regular Toeplitz sequences as measurable maps $\varphi: X \to Z_2$. This is our first point of view. To present our second (combinatorial) approach we have to introduce some operations on 0-1 sequences and blocks.

Let $B=(b_0,\dots,b_{r-1})$ be a $\{0,1\}$-block. The <u>length</u> r of B will be denoted by $|B|$. We say that the <u>parity of</u> B <u>is even</u> (odd) if B contains an even (odd) number of 1's. By $\hat{B}$ we mean the block $\hat{B}=(\hat{b}_0,\dots,\hat{b}_{r-2})$, where $\hat{b}_i=b_i+b_{i+1}$. The inverse operation yields two blocks C and $\tilde{C}$ (where $\tilde{C}$ means interchanging of all 0's and 1's in C). $\tilde{C}$ will be said to be the <u>mirror image</u> of C. These notations are easily adapted to the case of infinite one-sided or two-sided sequences. To avoid some difficulties, from now on, we will assume that if φ is a regular (two-sided) Toeplitz sequence then it enjoys the following property

(137) A block B appears on η iff $\tilde{B}$ does,

where η is a two-sided sequence satisfying $\hat{\eta}=\varphi$. In this case, $\overline{O_\tau(\eta)}=\overline{O_\tau(\tilde{\eta})}=\check{Y}$ and therefore it is harmless to denote $\check{Y}$ by $\overline{O_\tau(\check{\varphi})}$. Notice also that $\overline{O_\tau(\check{\varphi})}$ is mirror-invariant (here and before $\check{\varphi}=\{\eta,\tilde{\eta}\}$, where $\hat{\eta}=\varphi$).

Now, we will describe some connections between $\tau:\overline{O_\tau(\check{\varphi})}\circlearrowleft$ and $\tau_\varphi:\overline{O_\tau(\varphi)}\times Z_2\circlearrowleft$. Let $\varphi\in\{0,1\}^Z$ be a regular Toeplitz sequence. We recall that τ_φ is ergodic with respect to $\tilde{\nu}$ iff τ_φ is uniquely ergodic ([11]p. 66).

Proposition 8.1 <u>Let φ be a regular Toeplitz sequence. The topological dynamical systems $\tau_\varphi:\overline{O_\tau(\varphi)}\times Z_2\circlearrowleft$ and $\tau:\overline{O_\tau(\check{\varphi})}\circlearrowleft$ are topologically isomorphic. In particular,</u>
(i) <u>$\check{\varphi}$ is strictly transitive</u> iff <u>τ_φ is (uniquely) ergodic</u>.
(ii) <u>If τ_φ is ergodic then $\tau:\overline{O_\tau(\check{\varphi})}\circlearrowleft$ is uniquely ergodic and they are m.t. isomorphic</u>.

Proof. Immediately from (137) it follows that

(138) $\overline{O_\tau(\check{\varphi})}=\{\eta\in\{0,1\}^Z:\ \hat{\eta}\in\overline{O_\tau(\varphi)}\}$.

Define $F:\overline{O_\tau(\check{\varphi})}\longrightarrow\overline{O_\tau(\varphi)}\times Z_2$ by $F(\eta)=(\hat{\eta},\eta[0])$. Now we have $\tau_\varphi F(\eta)=(\tau\hat{\eta},\hat{\eta}[0]+\eta[0])=(\widehat{\tau\eta},\eta[1])=(\widehat{\tau\eta},\tau\eta[0])=F\tau(\eta)$. Moreover, using (138), we easily get that F is a homeomorphism.

Therefore, τ_φ is uniquely ergodic iff $\tau:\overline{O_\tau(\check{\varphi})}\circlearrowleft$ is uniquely ergodic iff $\check{\varphi}$ is strictly transitive. Also, in the case of unique ergodicity of τ_φ, F has to preserve the corresponding unique invariant measure, hence, it establishes the desired m.t. isomorphism∎

As the sum of finitely many Morse cocycles is a Toeplitz cocycle, from Theorem 7.2, we get the following.

C o r o l l a r y 8 . 1 . <u>If $\varphi\in\mathcal{K}$ is ergodic then φ is cohomologous to a Toeplitz cocycle</u>∎

Let φ be a regular Toeplitz sequence such that $\check{\varphi}$ is strictly transitive. We will pass to a brief description of spectral theory of $T_\varphi:(X\times Z_2,\tilde{\mu})\circlearrowleft$ (or of $\tau_\varphi:(\overline{O_\tau(\varphi)}\times Z_2, \tilde{\nu})\circlearrowleft$ or else of $\tau:\overline{O_\tau(\check{\varphi})}\circlearrowleft$). We have

$$L^2(X\times Z_2,\tilde{\mu})=\mathcal{D}\oplus\mathcal{L},$$

where $\mathcal{D}=\{f\in L^2(X\times Z_2,\tilde{\mu}):\ f\sigma=f\}$, $\mathcal{L}=\{f\in L^2(X\times Z_2,\tilde{\mu}):\ f\sigma=-f\}$ with $\sigma:(X\times Z_2,\tilde{\mu})\circlearrowleft$ defined by $\sigma(x,i)=(x,i+1)$. Then $U_{T_\varphi}|\mathcal{D}$ is spectrally isomorphic to $U_T:L^2(X,\mu)\circlearrowleft$. It is also clear that if there is an eigenfunction in $\mathcal{L}$ then T_φ has pure point spectrum. Let us define $V^\varphi:L^2(X,\mu)\circlearrowleft$ by letting

$$V^\varphi(f)(x)=(-1)^{\varphi(x)}f(Tx).$$

Then, V^φ is unitary and moreover $U_{T_\varphi}|\mathcal{L}$ is spectrally isomorphic to V^φ ([17]). It turns out that when dealing with spectral properties of T_φ the base role is played by the sequence $(\ (V^\varphi)^n(1),1\ )=\int_X (V^\varphi)^n(1)d\mu$. It is clear that

$$\int_X (V^\varphi)^n(1)d\mu=1-2\mu\{x:\ \varphi(x)+\ldots+\varphi(T^{n-1}x)=1\}.$$

Let $\mathcal{A}_n$ be the union of cylinder sets determined by blocks of length n with odd parity. Denote

(139) $\quad a_n^{\varphi} = 1 - 2\nu(\mathcal{A}_n)$,

$n=1,2,\ldots$.

Proposition 8.2. *Let φ be a regular Toeplitz sequence. Then* $((V^{\varphi})^k(1),1) = a_k^{\varphi}$, $k=1,2,\ldots$.

Proof. From the regularity of φ the average frequency of each block B exists in φ and equals the measure ν of the corresponding cylinder set. Hence $\nu(\mathcal{A}_n)=\mu\{x: \varphi(x)+\ldots+\varphi(T^{n-1}x)=1\}$ and the assertion follows ■

Assume that $\check{\varphi}$ is strictly transitive and denote by $\mu_{\check{\varphi}}$ the measure given by the average frequencies of blocks on $\check{\varphi}$. Let

$$ij^{(n)} = \{y \in \overline{O_{\tau}(\check{\varphi})}: y[0]=i,\ y[n-1]=j\}.$$

Proposition 8.3. $a_k^{\varphi} = 1 - 2\mu_{\check{\varphi}}(01^{(k)} \cup 10^{(k)})$.

Proof. Let us recall that $\varphi[i]=\check{\varphi}[i]+\check{\varphi}[i+1]$. Therefore, if $(\varphi[i],\varphi[i+1],\ldots,\varphi[j])$ contains an odd (even) number of 1's then $(\check{\varphi}[i],\check{\varphi}[i+1],\ldots,\check{\varphi}[j],\check{\varphi}[j+1])$ contains an odd (even) number of the blocks of the form 01, 10. Notice that the k-blocks corresponding to $01^{(k)} \cup 10^{(k)}$ are exactly those containing an odd number of the 2-blocks 01,10 ■

The following result is well-known.

Propositon 8.4. *The maximal spectral type of V^{φ} is equal to* $\sigma_m = \sum_{k\geq 1} 2^{-k}(\sigma * \delta_{\alpha_k})$, *where σ is the spectral type of 1, i.e. $\sigma=\sigma_1$, and $\{\alpha_k\}$ is the set of eigenvalues of T* ■

Suppose that T_{φ} is ergodic and take $f(x,i)=(-1)^i$. Then by using Birkhoff theorem we obtain that

$$(1/N)\sum_{k=0}^{N-1}\left[f((T_{\varphi})^k(x,i))\right] \xrightarrow[N]{} 0.$$

By Lebesgue integral theorem

$$(1/N)\sum_{k=0}^{N-1} \int_X \left[(-1)^{\varphi(x)+\dots+\varphi(T^{n-1}x)}\right] d\mu(x) \longrightarrow 0.$$

Hence, we have

$$(1/N)\sum_{k=0}^{N-1} \left((V^{\varphi})^k(1), 1 \right) \longrightarrow 0.$$

as a necessary condition of ergodicity of T_φ.

C o r o l l a r y 8 . 2 . T_φ <u>is ergodic</u> iff $(1/N)\sum_{k=0}^{N-1} \alpha^k\left((V^{\varphi})^k(1), 1 \right) \longrightarrow 0$ <u>for all</u> $\alpha \in Sp(T)$.

P r o o f . The latter condition gives us $(1/N)\sum_{s=0}^{N-1} (U_T^s F, F) \longrightarrow 0$, where $F(x,r)=(-1)^r g(x)$ and g is an eigenfunction corresponding to $\alpha \in Sp(T)$. Since the F's form an orthonormal base on $\mathcal{L}$, the result follows■

C o r o l l a r y 8 . 3 . T_φ <u>is weakly mixing on</u> $\mathcal{L}$ iff

(140) $(1/N)\sum_{k=0}^{N-1} \left| \left((V^{\varphi})^k(1), 1 \right) \right| \longrightarrow 0$. <u>Equivalently, there exists a set</u> $M \subseteq \mathbb{N}$ <u>of density zero such that</u> $\left((V^{\varphi})^k(1), 1 \right) \underset{k \notin M}{\longrightarrow} 0$.

P r o o f . Let us notice that σ_1 is continuous iff (140) is valid and moreover that σ_1 is continuous iff σ_m is continuous by Proposition 8.4■

C o r o l l a r y 8 . 4 . T_φ <u>is mixing on</u> $\mathcal{L}$ iff $\left((V^{\varphi})^k(1), 1 \right) \to 0$

P r o o f . Let F be defined as in the proof of Corollary 8.2. Hence, $\left((V^{\varphi})^k F, F \right) = \alpha^k \left((V^{\varphi})^k(1), 1 \right)$ and the sufficiency easily follows. Conversely, as a consequence of Proposition 8.4 we have that $\int_{S^1} z^n d\sigma_m(z) \underset{n}{\longrightarrow} 0$. Therefore, from the generalized Lebesgue-Riemann lemma $\int_{S^1} z^n d\rho(z) \underset{n}{\longrightarrow} 0$ whenever $\rho \ll \sigma_m$, so $\left((V^{\varphi})^k(1), 1 \right) \longrightarrow 0$■

Corollary 8.5. Let λ be Lebesgue measure on S^1.

(a) $\sigma_m \equiv \lambda$ iff $\sigma_1 << \lambda$.

(b) If $\sigma_m << \lambda$ then $\sigma_m \equiv \lambda$.

(c) If $\sum_{n \geq 1} |\sigma_1(n)|^2 < +\infty$ then $\sigma_m \equiv \lambda$.

(d) σ_m is singular iff σ_1 is singular.

Proof. For the proof of (a) we notice that the set Sp(T) is dense in S^1. The remaining statements are then clear ∎

Proposition 8.5. (An estimation of spectral multiplicity) Let φ be a regular Toeplitz sequence such that τ_φ is ergodic and

$$(\exists\ 0 < C < 1)(\forall\ t \geq 1)(\exists\ i < j) \text{ the block } (\varphi^{(t)}[i], \varphi^{(t)}[i+1], \ldots, \varphi^{(t)}[j]) \text{ does not contain any "hole" and } |j-i| \geq Cn_t. \tag{141}$$

Then, the maximal spectral multiplicity of V^φ is at most 1/C.

Proof. We will use Spectral Multiplicity Theorem. Given K and $f_1, \ldots, f_K$ of norm 1 let us fix $\varepsilon > 0$. Since $\{D^t\}$ converges to $\mathcal{B}$, we can find $f_1^{t_0}, \ldots, f_K^{t_0}$ which are constant on all levels of D^{t_0} and such that

$$||f_i - f_i^{t_0}|| < \varepsilon . \tag{142}$$

By (141), for the tower D^{t_0} we can find $i_0 < j_0$ such that $\varphi|D_s^{t_0} = b_s^{(t_0)} \in \{0,1\}$, $s = i_0, i_0+1, \ldots, j_0$ and $|j_0 - i_0| \geq Cn_{t_0}$. Let us take $g = \chi_{D^{t_0}_{j_0}}$. Then, the function $g_i = f_i^{t_0} \cdot \chi_{\bigcup_{k=i_0}^{j_0} D_k^{t_0}}$

belongs to Z(g), i=1,...,K since $(V^\varphi)^s(g)(x) = (-1)^{\varphi(x) + \ldots + \varphi(T^{s-1}x)} g(T^s x)$. Hence $(V^\varphi)^s(g)$ is constant on $D^{t_0}_{j_0 - s}$ for $s = 0, 1, \ldots, j_0 - i_0$. Moreover,

$$||f_i^{t_0} - g_i||^2 = \int_{(\bigcup_{k=i_0}^{j_0} D_k^{t_0})^c} |f_i^{t_0}|^2 d\mu \leq (1 - \mu(\bigcup_{k=i_0}^{j_0} D_k^{t_0}))||f_i||^2 = 1-C.$$

From this and (142) it follows that

$||f_i - g_i|| \leq ||f_i - f_i^{t_0}|| + ||f_i^{t_0} - g_i|| \leq \varepsilon + \sqrt{1-C}$. Therefore

$K-1 \leq \sum_{i=1}^{K} ||f_i - g_i||^2 \leq K(\varepsilon^2 + 2\varepsilon\sqrt{1-C} + (1-C))$ which means that

$K(C - \varepsilon^2 - 2\varepsilon\sqrt{1-C}) \leq 1$. As ε was arbitrary, $K \leq 1/C$ ∎

9. A CLASS OF TOEPLITZ Z_2-EXTENSIONS WITH NONSINGULAR SPECTRA OF FINITE MULTIPLICITY. In this section, we will show how to construct, for a given even number m, a regular Toeplitz sequence giving rise to a Toeplitz Z_2-extension with a Lebesgue component of uniform multiplicity m.

We will consider a Toeplitz sequence $\varphi \in \{0,1\}^N$ as a limit of some periodic sequences $Per_r \varphi \in \{0,1,\infty\}^N$,

$$Per_r \varphi = b_1^{(r)} \dots b_{k_r}^{(r)} | b_1^{(r)} \dots b_{k_r}^{(r)} | \dots .$$

To obtain φ from the sequence $\{Per_r \varphi\}_{r>0}$, we proceed as follows. We write down the periodic sequence $Per_1 \varphi$. The places where $Per_1 \varphi$ is either 0 or 1 are also the places where φ is defined and we will not change the values. Then, we take $Per_2 \varphi$ and write it down into the "holes" of $Per_1 \varphi$ from the left to the right. The places, where the new sequence is equal to 0 or 1 are those where φ is already defined. Then, we take $Per_3 \varphi$ and write it down into the "holes" of the sequence we have from the left to the right. This is continued to obtain φ. Under some assumptions, this procedure leads to a well-defined nonperiodic and regular Toeplitz sequence. The periodic part of $Per_r \varphi$ will be called r-<u>base-skeleton</u> of φ and denoted by $per_r \varphi$. Hence

(143) $per_r \varphi = b_1^{(r)} \dots b_{k_r}^{(r)}$ and $Per_r \varphi = per_r \varphi | per_r \varphi | \dots .$

We recall that the r-skeleton $\varphi^{(r)}$ of φ is just the periodic part of the sequence we have described above after the first r steps, $r \geq 1$.

Let $m=2^k$ for some $k \geq 1$. Consider the family of regular Toeplitz sequences, where

(144) $\quad \mathrm{per}_r \varphi = b_1^{(r)} \infty b_2^{(r)} \infty \ldots \infty b_{2^{k-1}}^{(r)} \infty \tilde{b}_1^{(r)} \infty \; \tilde{b}_2^{(r)} \infty \; \ldots \infty \tilde{b}_{2^{k-1}}^{(r)} \infty,$

where $r=1,2,\ldots,\; i=1,\ldots,2^{k-1}$ and $b_i^{(r)} \in \{0,1\}$ with $\tilde{b}_i^{(r)} = 1-b_i^{(r)}$. By the procedure we have just described, the corresponding Toeplitz sequence arises as follows.

$$\begin{array}{l} b_1^{(1)} \infty\, b_2^{(1)} \infty\, b_3^{(1)} \infty \ldots b_{2^{k-1}}^{(1)} \ldots \infty \tilde{b}_{2^{k-1}}^{(1)} \infty \,|\, b_1^{(1)} \infty\, b_2^{(1)} \infty\, b_3^{(1)} \infty \ldots \quad b_{2^{k-1}}^{(1)} \ldots \infty \tilde{b}_{2^{k-1}}^{(1)} \infty \,|\, \ldots \\ \quad b_1^{(2)} \;\; \infty \;\; b_2^{(2)} \quad \ldots \quad b_{2^{k-1}}^{(2)} \; \infty \quad \tilde{b}_1^{(2)} \; \infty \; \tilde{b}_2^{(2)} \quad \ldots \quad \tilde{b}_{2^{k-1}}^{(2)} \; \infty \,|\, \ldots \\ \qquad\quad b_1^{(3)} \qquad \ldots \\ \qquad\qquad\quad \ldots \end{array}$$

Since on each r-base-skeleton of the sequence φ the frequencies of 0's and 1's are the same,

(145) $\quad a_1^{\varphi} = 0.$

Moreover, for each $r \geq 1$

(146) $\quad \varphi^{(r)} = A_1^{(r)} \infty \; A_2^{(r)} \infty \; \ldots \infty A_m^{(r)} \infty \;,$

where $|A_i^{(r)}| = 2^r - 1$. We wish to show that

(147) $\quad a_n^{\varphi} = 0$ for all $n \geq 1$.

Let $n \geq 2$ and select $r \geq 1$ such that

(148) $\quad 2^r \leq n < 2^{r+1}.$

Consider the sequence $\varphi^{(r)} \varphi^{(r)} \ldots$. We write

$$\varphi^{(r)} \varphi^{(r)} \ldots = \ell_1 \ell_2 \ldots$$

and we label the holes subsequently $1,\ldots,2^k$ in each ℓ_i $(= \varphi^{(r)})$. Now, we fix our attention on $r+1, r+2, \ldots, r+k$-base-skeletons of φ. We obtain that

$(r+j)$-<u>base-skeleton fills up</u> 2^{k-j} <u>"holes" labeled</u> $2^{j-1} + 2^j s$ <u>in each</u> ℓ_i $(0 \leq s \leq 2^{k-j})$

for each $j = 1, \ldots, k$.

Note that for each $1 \leq j \leq k$, the $(r+j)$-base-skeleton $\mathrm{per}_{r+j}\varphi$ can be written as

$$(e_1^{(1)}\infty \dots \infty e_{2^{k-j}}^{(1)})\infty(e_1^{(2)}\infty \dots \infty e_{2^{k-j}}^{(2)})\infty \dots \infty(e_1^{(2^{j-1})}\infty \dots \infty e_{2^{k-j}}^{(2^{j-1})})\infty$$

$$(\tilde{e}_1^{(1)}\infty \dots \infty \tilde{e}_{2^{k-j}}^{(1)})\infty(\tilde{e}_1^{(2)}\infty \dots \infty \tilde{e}_{2^{k-j}}^{(2)})\infty \dots \infty(\tilde{e}_1^{(2^{j-1})}\infty \dots \infty e_{2^{k-j}}^{(2^{j-1})})\infty .$$

Fix $i < 2^k$. Then there is a unique $1 \leq j \leq k$ that the i-th "hole" is filled in by $(r+j)$-base-skeleton of φ . We will consider the consecutive $(2^{r+1}-1)$-blocks $P_1, P_2, \dots$ of the form $A_i^{(r)}\infty A_{i+1}^{(r)}$ in $\mathrm{Per}_r\varphi$. Notice that if the hole in P_k is filled in by $\alpha \in \{0,1\}$ then the hole in $P_{k+2^{j-1}}$ is filled in by $1-\alpha$. Hence, an n-subblock B in P_s and the equally positioned B' in $P_{s+2^{j-1}}$ (see Fig. 9.1) are equal except for the hole where they must differ because of the form of $\mathrm{per}_{r+j}\varphi$.

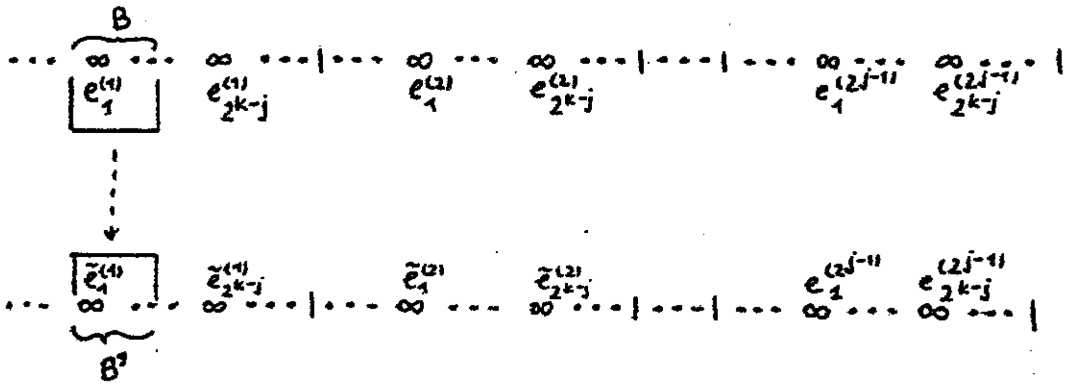


Figure 9.1.

Hence the frequencies of the odd parity n-blocks and the even parity n-blocks contained in the P_s's are equal. Now, to conclude the same for the n-subblocks contained in $A_{2^k}^{(r)}\infty A_1^{(r)}$ it is enough to notice that the Toeplitz sequence arising as the limit of $\mathrm{per}_{r+k+j}\varphi$, $j \geq 1$, has the same frequencies of 0's and 1's.

Now, let us consider the (more complicated) case where B contains two holes. Then one of them must be filled in by $(r+1)$-base-skeleton and there is a unique $2 \leq j \leq k$ such that the second one is filled in by $(r+j)$-base-skeleton (we exclude for a moment the case where one of the holes has label 2^k). Again we pair n-blocks with different parities. This can be done as in Figure 9.2.

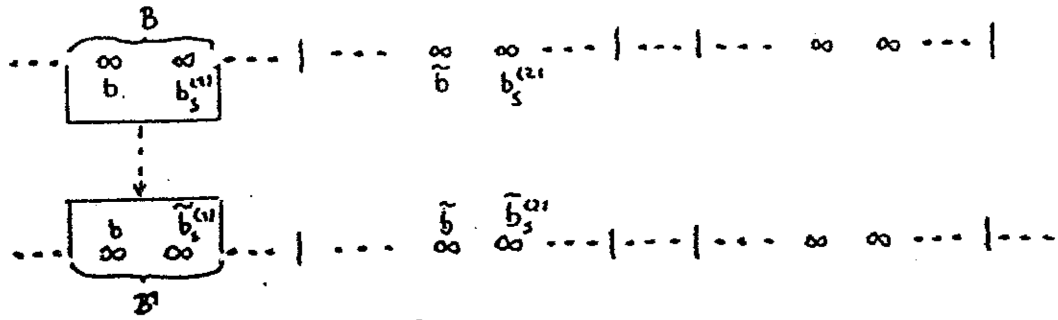


Figure 9.2.

In order to complete the proof of (147), it remains to consider the case where B is an n-subblock of $A_{2^k}^{(r)} \infty A_1^{(r)} \infty A_2^{(r)}$ or $A_{2^k-1}^{(r)} \infty A_{2^k}^{(r)} \infty A_1^{(r)}$. Here, the pairing process can be done as in Figure 9.3.

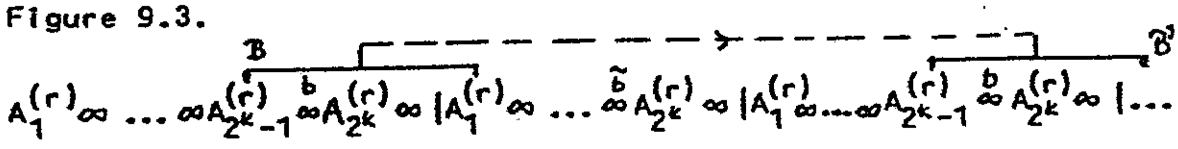


Figure 9.3.

The $(r+k+1)$-base-skeleton filles up precisely those holes following the b. The Toeplitz sequence arising as the limit of $\mathrm{per}_{r+k+j}\varphi$, $j \geq 2$, filles in the remaining holes. In both cases the frequencies of 0's and 1's are the same. Therefore, the pairing process as described in Fig. 9.3 leads to equal frequencies of the odd and the even parity n-subblocks contained in $A_{2^k-1}^{(r)} \infty A_{2^k}^{(r)} \infty A_1^{(r)}$. The same can be applied to $A_{2^k}^{(r)} \infty A_1^{(r)} \infty A_2^{(r)}$. The proof of (147) is complete.

All we have shown so far is that

(149) $a_n=((V^{\varphi})^n(1),1)=0$ for all $n\geq 1$

and therefore for all $n\leq -1$. Thus by Corollary 8.5 and Proposition 8.5, V^{φ} has Lebesgue spectrum with multiplicity at most 2^k. It remains to prove that the multiplicity is at least 2^k. Toward this end, let us take $0\leq i\leq 2^k-1$ and consider all blocks (of a fixed length) that start at the places of the form

(150) $i+s\cdot 2^k$, $s=0,1,\ldots$.

It turns out that for each n the frequencies of all blocks of length n with different parities starting at the places (150) are equal. Indeed, for $n\geq 2$ this is the reasoning above. If n=1 and i is even the result is obvious, while if i is odd we apply the arguments we have used in the case B contains one hole. But

$$((V^{\varphi})^n(1),\chi_{D_i^k})=1-2\nu(\tilde{\mathcal{A}}_n),$$

where $\tilde{\mathcal{A}}_n$ is the union of cylinder sets determined by blocks of length n, starting at the places (150) and having odd parity. This yields

(151) $((V^{\varphi})^n(1),\chi_{D_i^k})=0$

for $n\geq 1$ and $i=0,\ldots,2^k-1$. But for $n\geq 1$

$$((V^{\varphi})^n\chi_{D_i^k},\chi_{D_i^k})=\begin{cases}((V^{\varphi})^n(1),\chi_{D_i^k}) & \text{if } n=i \bmod 2^k\\ 0 & \text{otherwise}\end{cases}$$

$i=0,\ldots,2^k-1$. Therefore, by (151), the spectral type of $f_i=\chi_{D_i^k}$ is Lebesgue. Moreover,

$$((V^{\varphi})^n \chi_{D_i^k}, \chi_{D_j^k}) = \begin{cases} ((V^{\varphi})^n 1, \chi_{D_j^k}) & \text{if } j+n \ (\mathrm{mod}\ 2^k) = i \\ 0 & \text{otherwise.} \end{cases}$$

Hence the cyclic spaces $Z(f_i)$, $i=0,\dots,2^k-1$ are mutually orthogonal and therefore spectral multiplicity is at least 2^k.

Theorem 9.1. For each even number m there exists an ergodic Toeplitz Z_2-extension T_φ such that V^φ has Lebesgue spectrum with uniform multiplicity m.

Proof. Let $m=2^k s$, $k \geq 1$ and s be odd. Then, for 2^k we have just constructed a Toeplitz Z_2-extension T_φ such that V^φ has Lebesgue spectrum with uniform multiplicity 2^k. But

$$Sp(T_\varphi)=Sp(T)=G\{2^t:\ t\geq 0\}.$$

Therefore $(T_\varphi)^s$ is ergodic and $(T_\varphi)^s=(T^s)_{\varphi+\varphi T+\dots+\varphi T^{s-1}}$ is still a Toeplitz Z_2-extension. Moreover, $(T_\varphi)^s$ has Lebesgue spectrum with uniform multiplicity $s\cdot 2^k$ on $\mathcal{L}$. The result follows. ∎

Immediately from Theorem 7.2 we obtain the following.

Corollary 9.1. For each even number m there exist two Morse cocycles φ_1 and φ_2 such that $V^{\varphi_1+\varphi_2}$ has Lebesgue spectrum with uniform multiplicity m ∎

In [6] the authors proposed to study the automorphisms arising from some automatic sequences.

Let $u=u_{l-1}\dots u_0 \in \{0,1,\infty\}^l$, $u_{l-1}\neq\infty$. Define an infinite sequence $x_u \in \{0,1\}^N$ by putting

$$x_u[n]=\mathrm{card}\{l \geq 0: (\forall j=0,\dots,l-1)\ u_j\neq\infty \Rightarrow u_j=e_{j+l}(n)\} \mod 2,$$

where

$$n=\sum_{i=0}^{\infty} e_i(n)2^n \quad (e_i(n)=0 \text{ for } i > \log n/\log 2).$$

This means that $x_u[n]$ is the number of appearences of u in n written in binary expansion. In [6] there is a proof of the fact that if u is not trivial (i.e. if u contains at least one 1) then u is recognizable by a 2-substitution. To be more precise, let u be nontrivial and let

$$E(u)=\{v=(v_0,\dots,v_{l-1})\in\{0,1\}^l:(\forall\ i=0,\dots,l-1)\ u_i\neq\infty \Rightarrow u_i=v_i\}.$$

Put

$$Sp(u)=\left\{\sum_{i=0}^{l-1} v_i 2^i:\ v\in E(u)\right\},\qquad k=2^{|u|-1}.$$

Then x_u is a fixed point of the (k,2k)-substitution given by

(152) $\sigma(y_0,\dots,y_{k-1})=\theta^{(\alpha_0)}y_0\theta^{(\alpha_1)}y_1\dots\theta^{(\alpha_k)}y_{k-1}$

$y_i\in\{0,1\}$ and

$\alpha_j=0$ if $2j\notin Sp(u)$ and $2j+1\notin Sp(u)$

$\alpha_j=1$ if $2j\in Sp(u)$ and $2j+1\notin Sp(u)$

$\alpha_j=2$ if $2j\notin Sp(u)$ and $2j+1\in Sp(u)$

$\alpha_j=3$ if $2j\in Sp(u)$ and $2j+1\in Sp(u)$,

where

$\theta^{(0)}(0)=00,\theta^{(1)}(0)=01,\theta^{(2)}(0)=10,\theta^{(3)}(0)=11,\qquad \theta^{(i)}(1)=\widetilde{\theta^{(i)}}(0)$, $i=0,1,2,3.$

P r o p o s i t i o n 9.1. <u>For each substitution</u> (152) <u>satisfying</u>

(153) $\sigma(\underbrace{0\dots0}_{2^k})=\underbrace{0\dots0}_{2^k}(i_1,1+i_1)(i_2,1+i_2)\dots(i_{2^k-1},1+i_{2^k-1})$

<u>the dynamical system arising from a fixed point of</u> σ <u>has Lebesgue spectrum with multiplicity</u> 2^k <u>in the orthocomplement of the space of eigenfunctions. In particular,</u> x_{11} <u>and</u> x_{10} <u>induce Lebesgue spectrum with multiplicity</u> 2 <u>and in general</u> $x_{1\underbrace{\infty\dots\infty}_{u}0}$, $x_{1\underbrace{\infty\dots\infty}_{u}1}$ <u>induce Lebesgue spectrum with multiplicity</u> $2^{|u|-1}$.

Proof. We will prove Proposition 9.1 by showing that a fixed point x of σ given by (153) has the property that $\hat{x}$ is a regular Toeplitz sequence with the structure of the base-skeletons as in (144).

Let K be a natural number, $K \geq 1$. We will consider blocks whose lengths are multiples of K. Let $B \in \{0,1\}^{rK}$ and $C \in \{0,1\}^{sK}$. Then, we can write $B=B_1 \dots B_K$, where $|B_i|=r$, $i=1,\dots,K$ and

$C=(c_0,c_1,\dots,c_{K-1})(c_K,\dots,c_{2K-1})\dots(c_{(s-1)K},\dots,c_{sK-1})$.

Define the K-<u>product</u> of B and C by putting

$$(154)\quad B \times_K C = B_1^{c_0} B_2^{c_1} \dots B_K^{c_{K-1}} B_1^{c_K} \dots B_K^{c_{2K-1}} \dots B_1^{c_{(s-1)K}} \dots B_K^{c_{sK-1}},$$

where $B_i^0=B_i$, $B_i^1=\tilde{B}_i$, $i=1,\dots,K$.

Hence $|B \times_K C|=Krs$. This is also worthwhile to note that the K-product is associative. Assume now that K is an even number. Let σ be a $(K,2K)$-substitution (152). Then σ is determined by its value on $(\underbrace{0,\dots,0}_{K})$. Assume in addition that $\sigma(0,\dots,0)=0\dots 0 b_{k+1}\dots b_{2k+1}$ with

$\sigma(0,\dots 0)=\theta^{(\alpha_0)}(0)\dots\theta^{(\alpha_K)}(0)=B$ and $B=B_1B_2\dots B_K$, where $|B_i|=2$.

Now, $|B|=2K$ and moreover $x=B\times_K B \times_K B \times_K \dots$ is a fixed point of σ.

We intend to prove that $\hat{x}$ is a regular Toeplitz sequence by calculating its base-skeletons. We have

$$\underbrace{B \times_K B \times_K \dots \times_K B}_{v-1} \times_K B = C_1^{b_0} \dots C_K^{b_{K-1}} C_1^{b_K} \dots C_K^{b_{2K-1}},$$

where

$$\underbrace{B \times_K B \times_K \dots \times_K B}_{v-1} = C_1 \dots C_K, \quad |C_i|=|C_j|.$$

But K is an even number, so

$$(155)\quad \underbrace{B \times_K B \times_K \dots \times_K B}_{v-1} \times_K B = (C_1^{b_0}C_2^{b_1})\dots(C_{K-1}^{b_{K-2}}C_K^{b_{K-1}})(C_1^{b_K}C_2^{b_{K+1}})\dots(C_{K-1}^{b_{2K-2}}C_K^{b_{2K-1}})$$

Let $K=2^k$, $k \geq 1$ and σ be given by (153). We have

(156) $x=(B\varkappa_K\ldots\varkappa_K B)\varkappa_K x$.

But x is a juxtaposition of $\alpha_1=(\underbrace{0,\ldots,0}_{2^k})$, $\alpha_2=(\underbrace{1,\ldots,1}_{2^k})$

$\beta_1=((i_1,i_1+1),(i_2,i_2+1),\ldots,(i_{2^{k-1}},i_{2^{k-1}}+1))$,

$\beta_2=((i_1+1,i_1)(i_2+1,i_2),\ldots,(i_{2^{k-1}}+1,i_{2^{k-1}}))$,

where this juxtaposition is of the form

$$x=\alpha_{t_1}\beta_{t_1}\alpha_{t_2}\beta_{t_2}\alpha_{t_3}\beta_{t_3}\ldots,$$

$t_w\in\{1,2\}$, $w=1,2,\ldots$. Hence, by (155) and (156) we obtain that

$$x=(C_1^{\alpha_{t_1}}[0]C_2^{\alpha_{t_1}}[1])\ldots(C_{K-1}^{\alpha_{t_1}}[K-2]C_K^{\alpha_{t_1}}[K-1])$$
$$(C_1^{\beta_{t_1}}[0]C_2^{\beta_{t_1}}[1])\ldots(C_{K-1}^{\beta_{t_1}}[K-2]C_K^{\beta_{t_1}}[K-1])$$
$$(C_1^{\alpha_{t_2}}[0]C_2^{\alpha_{t_2}}[1])\ldots(C_{K-1}^{\alpha_{t_2}}[K-2]C_K^{\alpha_{t_2}}[K-1])\ldots .$$

This proves immediately that $\hat{x}$ is a Toeplitz sequence with the v-base-skeleton equal to

$$\mathrm{per}_v(\hat{x})=(C_1[|C_1|-1]+C_2[0]+\alpha_{t_1}[0]+\alpha_{t_1}[1])\infty$$
$$(C_3[|C_3|-1]+C_4[0]+\alpha_{t_1}[2]+\alpha_{t_1}[3])\infty$$
$$\ldots$$
$$(C_{K-1}[|C_{K-1}|-1]+C_K[0]+\alpha_{t_1}[K-2]+\alpha_{t_1}[K-1])\infty$$
$$(C_1[|C_1|-1]+C_2[0]+\beta_{t_1}[0]+\beta_{t_1}[1])\infty$$
$$\ldots$$
$$(C_{K-1}[|C_{K-1}|-1]+C_K[0]+\beta_{t_1}[K-2]+\beta_{t_1}[K-1])\infty ,$$

as $\alpha_1[i]+\alpha_1[i+1]=\alpha_2[i]+\alpha_2[i+1]$, $\beta_1[i]+\beta_1[i+1]=\beta_2[i]+\beta_2[i+1]$. Therefore $\mathrm{per}_v(\hat{x})$, $v\geq 1$, is of the form (144) and the first part of Proposition 9.1 has been proved.

To complete the proof we notice that

$$Sp(1\underbrace{\infty\ldots\infty}_{u-2}1)\subseteq\{2^{u-1}+1,2^{u-1}+2+1,2^{u-1}+2+2+1,\ldots\},$$

$$Sp(1\underbrace{\infty\ldots\infty}_{u-2}0)\subseteq\{2^{u-1},2^{u-1}+2,2^{u-1}+2+2,\ldots\}.$$

Therefore σ corresponding to $1\infty\ldots\infty 1$ and $1\infty\ldots\infty 0$ is of the form (153)∎

R e m a r k 9.1 In a conversation M. Mendès-France has communicated to me that he knows another (arithmetical) proof of the fact that $x_{1\infty\ldots\infty 1}$, $x_{1\infty\ldots\infty 0}$ induce Lebesgue measure, though without calculating the multiplicity function∎

R e m a r k 9.2. Consider the case of Lebesgue multiplicity equal to two. Actually, we have constructed uncountably many Toeplitz sequences with this property since (see (144))

$$\mathrm{per}_t \varphi \in \{0\infty 1\infty,\ 1\infty 0\infty\}.$$

In [32] uncountably many non m.t. isomorphic automorphisms with this spectral property have been constructed.

Independently of the author, O.N. Agejev [2] obtained a similar result in a class of Z_2-extensions of a rank 1 automorphism∎

E x a m p l e 9.1. It follows from Theorem 7.2 that all examples with a nonsingular part in the spectrum must come from invariant subspaces of some ergodic joinings between Morse cocycles. Sometimes, it is possible to recognize these Morse cocycles. Below, we consider two Morse cocycles φ_1, φ_2 (hence V^{φ_1}, V^{φ_2} with simple and singular spectra) such that $V^{\varphi_1+\varphi_2}$ has Lebesgue spectrum of multiplicity 2. To do it, first, we briefly recall the theory of Morse sequences.

Let $b^i \in \{0,1\}^{\lambda_i}$, $\lambda_i \geq 2$, $b^i[0]=0$. Then

(157) $x=b^0\times b^1\times b^2\times\ldots$,

where "$\times$" denotes 1-product, is called a <u>Morse sequence</u> if infinitely many of the b^i's are different from $0\ldots0$, infinitely many of them is different from 01...010 and moreover

$$\sum_{i\geq 0} \min\left((1/\lambda_i) fr(0,b^i), (1/\lambda_i) fr(1,b^i)\right) = +\infty.$$

If we denote $n_t = \lambda_0 \cdot \ldots \cdot \lambda_t$ then the strictly ergodic dynamical system arisen from x by taking its closure via the shift in $\{0,1\}^Z$ (see [24]) is isomorphic to a Z_2-extension T_φ of the $\{n_t\}$-adic adding machine T, where

$$\varphi|D_i^{n_t} = \hat{c}_t[i], \quad i=0,\ldots,n_t-2.$$

Here $c_t = b^0 \times b^1 \times \ldots \times b^t$, $t \geq 0$. In particular, the meaning of this is that φ is a Morse cocycle.

If $B \in \{0,1\}^n$, $C \in \{0,1\}^m$ then we define their $*$-<u>product</u> by

$$B*C = \begin{cases} Bc_0Bc_1\ldots Bc_{m-1}B & \text{if } B \text{ contains an even number of } 1\text{'s} \\ B\tilde{c}_0B\tilde{c}_1\ldots B\tilde{c}_{m-1}B & \text{otherwise.} \end{cases}$$

Thus, if $d^0, d^1, d^2, \ldots$ is a sequence of blocks with lengths at least 1 then the formula

$$z = d^0 * d^1 * d^2 * \ldots$$

defines a Toeplitz sequence. The following formula

$$(B \times C)^\wedge = \hat{B} * \hat{C}$$

holds. Therefore if $x = b^0 \times b^1 \times \ldots$ is a Morse sequence then the corresponding Toeplitz sequence is given by $\hat{x} = \hat{b}^0 * \hat{b}^1 * \hat{b}^2 * \ldots$. and t-skeleton of x is equal to

$$\hat{x}^{(t)} = \hat{c}_t \infty = \hat{b}^0 * \hat{b}^1 * \ldots * \hat{b}^t.$$

Now, let us take

$x = b^0 \times b^1 \times b^2 \times b^3 \times \ldots$, $b^0 = 01$,

$y = \beta^0 \times \beta^1 \times \beta^2 \times \beta^3 \times \ldots$,

where $b^i, \beta^j \in \{0001, 0100\}$, $i \geq 1$, $j \geq 0$.

By the above we obtain that

$per_t(\hat{x}) = \hat{b}^t \infty$, where $\hat{b}^t \in \{001, 110\}$, $t \geq 1$, $per_0(\hat{x}) = 1\infty$,

$per_t(\hat{y}) = \hat{\beta}^t \infty$, where $\beta^t \in \{001, 110\}$, $t \geq 0$.

We intend to calculate $per_t(\hat{x}+\hat{y})$ (see Fig. 9.4)

x: 1∞1∞1∞1∞1∞1∞1∞1∞1∞1∞1∞1∞1∞1∞1∞1∞1∞ ...

{0 ∞ 1 ∞ 0 ∞ 1 ∞ 0 ∞ 1 ∞ 0 ∞ 1 ∞ ...
{ 0 ∞ 0 ∞ 0 ∞ 0 ∞ ...

{1 ∞ 0 ∞ ...
{ 1 ...

...

y: {0∞1∞0∞1∞0∞1∞0∞1∞0∞1∞0∞1∞0∞1∞0∞1∞0∞1∞ ...
{ 0 ∞ 0 ∞ 0 ∞ 0 ∞ 0 ∞ 0 ∞ 0 ∞ 0 ∞ 0 ∞ ...
1 ∞ 0 ∞ 1 ∞ 0 ∞ 1 ...
1 ∞ 1 ∞ ...

Figure 9.4. x=01×0001×0100×...
y=0001×0100×... .

We "decompose" each t-skeleton into two periodic sequences: for instance

001∞001∞001∞... = {0∞1∞0∞1∞0∞1∞0∞1∞...
{ 0 ∞ 0 ∞ 0 ∞ 0 ∞...

From this we obtain that $per_t(\hat{x}+\hat{y})=0\infty 1\infty$ or $1\infty 0\infty$, hence of the form (144) ■

BIBLIOGRAPHICAL REMARKS AND COMMENTS.

Section 1. The results of this section have been published in [37]. The proofs of Lemma 1.1 and Proposition 1.1 are based on ideas from [33]. The important Proposition 0.5 (proved in [45]) follows directly from Theorem 1.2.

Section 2. Theorem 2.1, 2.2 and Corollary 2.1 come from [37].
Added in April 1990: Most of the results of Section 1 and 2 have been recently extended to the nonabelian case by M.K. Mentzen.

Section 3. The example of a noncoalescent ergodic Anzai skew product comes from [38]. The notion of essentially infinite self-joining has appeared in [73] and Proposition 3.1 has been proved there.
Added in April 1990: The affirmative answer to Question 3.2 has been found by J. Kwiatkowski, P. Liardet and the author.

Section 4. The results of this section are contained in [38] (except for Proposition 4.2 and 4.3).
Added in April 1990: Recently, in [72], the authors showed that Lemma 4.2 and Proposition 4.1 are still valid when replace the uniform Lipschitz continuity condition by the absolute continuity of cocycles. Also, the coalescence property of ergodic step cocycles over locally bounded partial quotients irrational rotation has been proved there.

S e c t i o n 5 . The concept of the S-strongly ergodic Z_2-cocycle has appeared in [34]: Theorem 5.1, Corollary 5.1,5.2, 5.3, Remark 5.3 and Example 5.1 come from [34]. Example 5.2 has been published in [12].

S e c t i o n 6 . Theorem 6.1 has been shown in [73]. The remaining results have been published in [35].

S e c t i o n 7 . Theorem 7.1 comes from [8].

S e c t i o n 8 . The notion of the Toeplitz Z_2-extension has been introduced in [32]. All the results of this section (except for Proposition 8.5) are contained in [32]. A simplification concerning the proof of Proposition 8.1 has been communicated to the author by T. Rojek.

S e c t i o n 9 . The construction of Toeplitz Z_2-extensions with a given even Lebesgue multiplicity comes from [32].

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LIST OF SYMBOLS

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