SOME ERGODIC PROPERTIES OF MORSE SEQUENCES

Mariusz Lemańczyk
Toruń, Poland

Let $B=(b_0,\dots,b_{n-1})$, $C=(c_0,\dots,c_{m-1})$, $b_i,c_j\in\{0,1\}$, be two blocks with length $|B|=n$, $|C|=m$. Then by $B\times C$ we mean the following juxtaposition of blocks

/1/ $$B\times C = B^{c_0}B^{c_1}\dots B^{c_{m-1}}$$

where $B^0=B$, $B^1=\tilde{B}=(1-b_0,\dots,1-b_{n-1})$. B is called the mirror image of B. Denote

$$fr(B,C) = card\{i:\ B=C[i,i+|B|-1]\}.$$

Now, let $b^0,b^1,b^2,\dots$ be a sequence of blocks starting with 0 and with the length at least 2, i.e. $\lambda_i=|b^i|\geq 2$. Due to the associativity property of the product in /1/ we can define

/2/ $$x = b^0\times b^1\times b^2\times\dots\ .$$

x is said to be a Morse sequence if

/i/ infinitely many of the b^i's are different from 0...0,
/ii/ infinitely many of the b^i's are different from 01...010,
/iii/ $$\sum_{r\geq 0}\min\left\{\frac{1}{\lambda_r}fr(0,b^r)\ ,\ \frac{1}{\lambda_r}fr(1,b^r)\right\}=\infty.$$

These conditions garantee x to have almost periodicity and strict transitivity properties [8].

Let T be the shift in the space $\{0,1\}^Z$. Then we obtain $(X(x),T,\mu_x)$ a strictly ergodic dynamical system, called a Morse shift. Here $X(x)=\overline{\{T^ix:\ i\in Z\}}$ and μ_x is given by

/3/ $$\mu_x(B) = \lim\frac{1}{n}fr(B,x[0,n-1]).$$

x is said to be regular provided there exists $\varsigma>0$ that

$$\varsigma>\mu_{x_t}(00),\quad \mu_{x_t}(01)>\frac{1}{2}-\varsigma$$

where $x_t=b^t\times b^{t+1}\times\dots$ is also a Morse sequence.

In this note we intend to answer why this class of shifts is an important class of automorphisms in ergodic theory.

I. Examples and counterexamples.

I.1. Spectral multiplicity and finite rank.

It was shown in [1] that rank 1 implies simple spectrum. The problem arises whether the reverse implication is true. In 1977 del Junco [4] proved that the dynamical system induced by $x = 01 \times 01 \times \dots$ has simple spectrum and rank 2. In 1980 Kwiatkowski [10] proved that any Morse sequence has simple spectrum. Now, using different methods from those presented in [4] we get

Theorem 1 /[11]/. If $x = b^0 \times b^1 \times \dots$ is a regular Morse sequence and $\lambda_i \leq r$, then the corresponding Morse shift has rank 2.

It can be asked whether simple spectrum implies finite rank property. The negative answer was published in [5]. It turns out that there is an automorphism with simple spectrum which is not LB. However, recently there appeared a new concept of a generalization of finite rank property, the so-called local rank 1 automorphisms /[6]/. Making use of Morse sequences theory one can prove the following

Theorem 2 /[12]/. Let $x = b \times b \times b \times \dots$ be a "constant" Morse sequence, $|b| = \lambda$ and let τ be an ergodic automorphism with discrete spectrum such that

$$Sp(\tau) = G\{\lambda'^{t} : t \geqslant 0\}, \quad \gcd(\lambda, \lambda') = 1.$$

Then $T \times \tau$ is ergodic, LB with simple spectrum but not of local rank 1.

Here Sp(U) denotes the point spectrum of an automorphism U and $G\{n_t : t \geqslant 0\}$ is the subgroup of roots of unity generated by $\{\exp 2\pi i/n_t : t \geqslant 0\}$.

I.2. The sequence entropy.

In a Dekking's paper there are some formulas for measure-theoretic and topological sequence entropies for continuous substitutions on two symbols /[2]/, that means Morse sequences of the form $x = b \times b \times \dots$. Their generalizations to an arbitrary Morse sequence $x = b^0 \times b^1 \times \dots$ are possible in case

$$h_{\{n_t\}}(T) \qquad \text{and} \qquad h^{top}_{\{n_t\}}(T), \text{ where } n_t = \lambda_0 \cdot \lambda_1 \cdot \dots \cdot \lambda_t.$$

The problem is whether this generalization of entopy has the additivity property. In topological case it was stated by Goodman [3]. Although, in metric case, Kushnirenko [9] gave the formula $h_A(T \times T') = h_A(T) + h_A(T')$, but the proof included did not seem to prove it exactly. In fact, for a suitable choice of two Morse sequences one can prove

Theorem 3 /[13]/. The formulas $h_A(T\times T') = h_A(T) + h_A(T')$ and $h_A^{top}(T\times T') = h_A^{top}(T) + h_A^{top}(T')$ fail to be true for a sequence A of the form $A = \{k^t\}$.

I.3. The lifting problems.

Let $\tau : (X, \mathcal{B}, \mu) \rightleftarrows$ be an ergodic automorphism with rational discrete spectrum, i.e. $Sp(\tau) = G\{n_t : t \geq 0\}$ for some sequence $\{n_t\}$, $n_t | n_{t+1}$.

Let $\Theta : X \to Z_2 = \{0,1\}$ be measurable. Then a Z_2-extension τ_Θ of τ is defined by

/4/ $$\tau_\Theta : (X\times Z_2, \tilde{\mu}) \rightleftarrows$$
$$\tau_\Theta(x,i) = (\tau x, \Theta(x)+i),$$

where $\tilde{\mu} = \mu\times\nu$, $\nu(i) = 1/2$, $i=0,1$.

We are interested in the following problems:

A/ Is the centralizer of τ, $C(\tau)=\{S: (X, \mathcal{B}, \mu) \rightleftarrows, S\tau = \tau S\}$ lifting to the centralizer of τ_Θ ?

We mean here the following: Is there for any $S \in C(\tau)$ an $\hat{S} \in C(\tau_\Theta)$ such that $\hat{S}$ considered on the measurable partition η, $A \in \eta$ iff $A = \{(x,0),(x,1)\}$, is isomorphic to S?

B/ Is any factor τ' of τ with an infinity group of eigenvalues lifting to a factor $\hat{\tau}'$ with partly continuous spectrum of τ_Θ such that $Sp(\hat{\tau}') = Sp(\tau)$?

The negative answer to these problems follows from the next

Theorem 4 /[14],[15]/. If $x = b^0 \times b^1 \times \ldots$ is a regular Morse sequence then

$$C(T) = \{T^i \sigma^j : i \in Z, \ j=0,1\},$$

where $\sigma y = \tilde{y}$, $y \in X(x)$. Moreover, any proper factor of T has discrete spectrum.

II. Combinatorial approach to ergodic theory.

Let $x = b^0 \times b^1 \times \ldots$ be a Morse sequence. Then, from metric point of view x is an ergodic Z_2-extension of $\tau : (X, \mathcal{B}, \mu) \rightleftarrows$, where $Sp(\tau) = G\{n_t : t \geq 0\}$ and n_t is defined in I.2.

Take $\tau : (X, \mathcal{B}, \mu) \rightleftarrows$ to be an ergodic automorphism with rational pure point spectrum, $Sp(\tau) = G\{n_t : t \geq 0\}$. Now, consider the class $\mathcal{L}$ of all Z_2-extensions of τ

/5/ $$\mathcal{L} = \{\tau_\Theta : \ \Theta : X \to Z_2 \text{ is measurable}$$

endowed with the topology

/6/ $$d(\tau_\Theta, \tau_{\Theta'}) = \tilde{\mu}\{(x,i) : \tau_\Theta(x,i) \neq \tau_{\Theta'}(x,i)\}.$$

In 1981 J. Kwiatkowski stated the question whether the class of Morse sequences is typical in the class of Z_2-extensions of automorphisms with rational discrete spectra. The precise answer is the following

Theorem 5 /[15]/. The class of all Z_2-extensions from $\mathcal{E}$ which are measure-theoretic isomorphic to Morse sequences is of the second category in $\mathcal{E}$.

But, what seems to be more important, analysing the proof one can prove more. Namely, we exhibit some connections with Katok-Stepin theory of odd approximation of Z_2-extensions [7].

Theorem 6 /[15]/. If $\tau_\Theta \in \mathcal{E}$ and $\Theta^{-1}(1)$ is oddly approximated with a speed $o(1/n^2)$ then there is a Morse sequence x such that τ_Θ and x are measure-theoretic isomorphic.

References

[1] Baxter,J.R., A class of ergodic transformations having simple spectrum, Proc. Amer. Math. Soc. 24(1970),275-279.

[2] Dekking,F., Combinatorial and statistical properties of sequences generated by substitutions, Thesis 1980.

[3] Goodman,T.N.T., Topological sequence entropy, Proc. Lon. Math. Soc. 29(1974),331-350.

[4] del Junco,A., A transformation with simple spectrum which is not rank one, Can. J. Math. 28(1977),655-663.

[5] Ferenczi,S., Systèmes de rang un gauche, Ann. Inst. H. Poincaré 21(1985)

[6] -, Systèmes de rang un localement, Ann. Inst. H. Poincaré 20(1984)

[7] Katok,A.B., Stepin,A.M., Approximation in ergodic theory, Usp. Mat. Nauk 22(1967),81-105.

[8] Keane,M., Generalized Morse sequences, ZfW 19(1968),335-353.

[9] Kushnirenko,A:G., On metric invariant of entropy type, Usp. Mat. Nauk 22(1967),53-61.

[10] Kwiatkowski,J., Spectral isomorphism of Morse dynamical systems, Bull. Pol. Acad. Sc. 29(1980),105-114.

[11] Lemańczyk,M., The rank of regular Morse dynamical systems, ZfW 70(1985),33-48.

[12] -,Sikorski,A., A class of not local rank 1 transformations arising from continuous substitutions, to appear in ZfW

[13] -, The sequence entropy for Morse shifts and some counter-examples, Studia Math. 82(1985),221-241.

[14] Lemańczyk,M., The centralizer of Morse shifts, Ann. Sc. Univ. Clermont-Ferrand 87(1985),43-56.

[15] -, Z_2-extensions over rational pure point spectrum, category and homomorphisms, to appear