# On Parreau's theorem on non-mixing transformations

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## Introduction

Assume that $T$ is an ergodic automorphism of a standard probability space $(X, \mathcal{B}, \mu)$. Then $T$ may belong to one of classical classes of automorphisms:

WM=weakly mixing automorphisms,

MM=mildly mixing automorphisms,

K=K-automorphisms, ZE=zero entropy automorphisms

or

D=distal automorphisms, R=rigid automorphisms,

AC=absolutely continuous spectrum automorphisms.

It is easy to see that the first four classes have one common property shared by all members of their COMPLEMENTS, namely they have a non-trivial factor which is completely different form all members of the class, or being more precise they have a non-trivial factor disjoint (in the sense of Furstenberg) from all members of the class:

$T \notin$ WM then $T$ has a non-trivial Kronecker factor,
$T \notin$ MM then $T$ has a non-trivial rigid factor,
$T \notin$ K then $T$ has a non-trivial Pinsker factor,
$T \notin$ ZE then $T$ has a non-trivial Bernoulli factor.

For the next three classes it is difficult to say if such a factor exists (in fact it does not). Let us also see that the existence of such factors as above gives rise to a short formula of type

$$\mathrm{A}=\mathrm{A}^{\perp\perp}$$

where A denotes one of classes WM,MM,K,ZE and $\mathrm{A}^{\perp}$ stands for the class of automorphisms disjoint from all members of A (indeed, $\mathrm{A}\subset\mathrm{A}^{\perp\perp}$ by definition; suppose for example that $T\in \mathrm{ZE}^{\perp\perp}\setminus\mathrm{ZE}$, then $h(T)>0$ and $T$ has a non-trivial Bernoulli factor $B$ which being a K-automorphism belongs to $\mathrm{ZE}^{\perp}$, so $B$ is disjoint with any member of $\mathrm{ZE}^{\perp\perp}$ in particular with $T$, a contradiction).

For other classes:

$$\mathrm{D}\subseteq \mathrm{D}^{\perp\perp}, \mathrm{D}\neq \mathrm{D}^{\perp\perp}$$

(Glasner-Weiss, L.-Parreau)

$$\mathrm{AC}\subseteq \mathrm{AC}^{\perp\perp}, \mathrm{AC}\neq \mathrm{AC}^{\perp\perp}\text{(and the same for R)}$$

(a reason for that for this class is that any class $\mathrm{A}^{\perp\perp}$ is closed under taking factors and group extensions, and AC is not closed under taking group extensions).

Denote by

$$\mathrm{M}=\text{mixing automorphisms.}$$

Where to put this class on the above "picture"? For $T\notin\mathrm{M}$ is there a "characteristic" non-trivial factor extremely different (disjoint) for all mixing automorphisms? Quite surprisingly a few years ago F. Parreau gave the positive answer to this question, and in particular he proved that

$$\mathrm{M}=\mathrm{M}^{\perp\perp}.$$

To see how surprising this result is just imagine a $T$ non-mixing without non-trivial (proper) factors. It must be disjoint from ALL mixing transformations, despite the fact that its maximal spectral type can be “a little bit Rajchman” (even “a little bit Lebesgue”). We will see this and some other consequences in a clear way when Parreau’s result will be presented.

## Joinings, disjointness and Markov operators

Given ergodic systems $T, S$ on standard probability spaces $(X, \mathcal{B}, \mu)$, $(Y, \mathcal{C}, \nu)$ respectively the set of joinings between $T, S$ is denoted by

$$J(T,S) = \begin{array}{l}\text{set of } T \times S\text{-invariant measures}\\ \text{on } X \times Y \text{ with projections } \mu \text{ and } \nu \text{ respectively.}\end{array}$$

$$J(T,S) \supset J^e(T,S) = \text{subset of ergodic joinings.}$$

*Disjointness* in the sense of Furstenberg:

$$T \perp S \ \text{ if } \ J(T,S) = \{\mu \otimes \nu\}.$$

If $\lambda \in J(T,S)$ then it determines a Markov operator $\Phi_\lambda : L^2(X, \mathcal{B}, \mu) \to L^2(Y, \mathcal{C}, \nu)$,

$$(1) \qquad \int_Y \Phi_\lambda(f) g \, d\nu = \int_{X \times Y} f \otimes g \, d\lambda, \ \Phi_\lambda \circ T = S \circ \Phi_\lambda$$

(here $T$ or $S$ stand for the associated Koopman operators acting on the underlying $L^2$-spaces, $f \mapsto f \circ T$, $g \mapsto g \circ S$ – we will also use notation $U_T$, $U_S$). That is $\Phi_\lambda$ is doubly stochastic: $\Phi_\lambda(1) = \Phi_\lambda^*(1) = 1$, and the image via $\Phi_\lambda$ of a non-negative function is non-negative. Moreover (up to some abuse of notation)

$$\Phi_\lambda(f) = E^\lambda(f|Y).$$

In fact (1) establishes a 1-1 correspondence between joinings and (intertwining) Markov operators.

**Notation for spectral measures** $\sigma_{T,f}$ or $\sigma_f$ if $T$ is understood (recall that $\widehat{\sigma}(n) = \int_X f \circ T^n \cdot \overline{f} \, d\mu$ for $n \in \mathbf{Z}$).

## A factor determined by a joining

Suppose that $T$ ($S$) is an automorphism of $(X, \mathcal{B}, \mu)$ ($(Y, \mathcal{C}, \nu)$) and let $\rho \in J(T, S)$. Then $\Phi_\rho : L^2(X, \mathcal{B}, \mu) \to L^2(Y, \mathcal{C}, \nu)$ and also $\Phi_\rho : L^\infty(X, \mathcal{B}, \mu) \to L^\infty(Y, \mathcal{C}, \nu)$. Let

$$F(\rho) = \overline{\text{span}}\{\Phi_\rho(f_1) \cdot \ldots \cdot \Phi_\rho(f_k) : \; f_i \in L^\infty(X, \mathcal{B}, \mu), k \geq 1\}.$$

Then $F(\rho) \subset L^\infty(X, \mathcal{B}, \mu)$ is an algebra of functions.

**Lemma 1** *There exists an $S$-invariant $\sigma$-algebra $\mathcal{C}_\rho \subset \mathcal{C}$ (a factor of $S$) such that*

$$\overline{F(\rho)}^{L^2} = L^2(\mathcal{C}_\rho).$$

**Theorem 1** [L.-Parreau-Thouvenot, Glasner-Thouvenot-Weiss (2000)]
The factor $S|_{\mathcal{C}_\rho}$ is also a factor, say $\mathcal{B}_\infty(\rho)$, of an infinite self-joining $\rho^\infty|_{X^\infty}$ of $T$.

**Construction** We define $\rho^\infty$ on $(X^\infty \times Y, \mathcal{B}^\infty \otimes \mathcal{C})$ is the (infinite) relative product

$$\int_{X^\infty \times Y} \Pi_{i=1}^n f_i(x_i) g(y) \, d\rho^\infty(\underline{x}, y) := \int_Y \Pi_{i=1}^n (\Phi_\rho f_i) \, g \, d\nu,$$

where $\underline{x} = (x_i)$. Then we have

$$\left( \begin{array}{ll} \;X \times X \times X \times \ldots & \\ (\rho \;\mid\; \rho \mid \;\; \rho \mid \ldots) & , \mathcal{B}^\infty \otimes \mathcal{C}, \rho^\infty \\ \qquad\quad \times Y & \end{array} \right)$$

and modulo $\rho^\infty$

$$\mathcal{B}_\infty(\rho) \otimes \{\emptyset, Y\} \overset{\rho^\infty}{=} (\{\mathcal{B}^\infty \otimes \{\emptyset, Y\}) \cap (\{\emptyset, X^\infty\} \otimes \mathcal{C}) \overset{\rho^\infty}{=} \{\emptyset, X^\infty\} \otimes \mathcal{C}_\rho$$

which should be seen as the definition of $\mathcal{B}_\infty(\rho)$ (in what follows $\mathcal{B}_\infty(\rho)$ will be seen also as a factor of $Y$).

**Remark 1** When $T$ and $S$ are ergodic, then the factor $S|_{\mathcal{C}_\rho}$ is ergodic as well, and it is a factor of a.e. ergodic component of $\rho^\infty$. In other words $S$ and a certain infinite ergodic self-joining of $T$ have a non-trivial common factor (once they are non-disjoint, i.e. $\rho \neq \mu \otimes \nu$).

Denote by $M(\mathbf{T})$ the Banach space of complex measures on $\mathbf{T}$ and let $M^+(\mathbf{T})$ stand for the subset of positive measures. Let $M_0 \subset M^+(\mathbf{T})$ be an ideal ($\nu * \kappa \in M_0$ whenever $\nu \in M_0$ and $\kappa \in M^+(\mathbf{T})$).

**Examples:** Rajchman measures, absolutely continuous measures.

Assume that $T$ is an ergodic automorphism of $(X, \mathcal{B}, \mu)$ and let $\rho \in J_2(T)$. Notice that $\overline{\mathrm{Im}\,\Phi_\rho} \subset L^2(X, \mathcal{B}, \mu)$ is a closed $T$-invariant subspace.

**Theorem 2 (Parreau's Theorem)** *If the maximal spectral type of $T$ on $\overline{Im\,\Phi_\rho}$ is singular with respect to all measures from $M_0$ then for each ergodic automorphism $S$ (of $(Y, \mathcal{C}, \nu)$) whose maximal spectral type is in $M_0$ we have*

$$T|_{\mathcal{B}_\infty(\rho)} \perp S.$$

**Proof.**

(sketch) Take $\eta \in J(T|_{\mathcal{B}_\infty(\rho)}, S)$. We want to show $\eta = (\mu|_{\mathcal{B}_\infty(\rho)}) \otimes \nu$. Let $\widehat{\eta} \in J(T, S)$ be the relatively independent extension of $\eta$ to a joining between $T$ and $S$. Take the relative product $(X^\infty \times X, \rho^\infty)$ and $(X \times Y, \widehat{\eta})$ relative with respect to the factor $(X, \mu)$

$$((X^\infty \times X) \times_X (X \times Y), \rho^\infty \otimes_X \widehat{\eta})$$

It is enough to show that, relative to the measure $\rho^\infty \otimes_X \hat{\eta}$ the factors $\mathcal{B}^\infty \otimes \{\emptyset, X\} \otimes \{\emptyset, Y\}$ and $\{\emptyset, X^\infty \times X\} \otimes \mathcal{C}$ are independent.

The proof is a clever induction. Indeed, notice that $(T^\infty, \rho^\infty|_{X^\infty})$ has a natural structure of an inverse limit: $\mathcal{D}_n \otimes \{\emptyset, X\}$ $(n \geq 1)$, where

$$\mathcal{D}_n = \underbrace{(\mathcal{B} \otimes \ldots \otimes \mathcal{B})}_{n} \otimes (\{\emptyset, X\} \otimes \ldots).$$

We have

$$\int_{X^\infty \times X \times Y} f_1(x_1) g(y)\, d\rho^\infty \otimes_X \hat{\eta}(\underline{x}, x, y) =$$

$$\int E(\ldots|X)\, d\mu = \int E(f_1|X) E(g|X)\, d\mu = \int_X \Phi_\rho(f_1) \Phi^*_\eta(g)\, d\mu.$$

Now $\sigma_{T,\Phi^*_\eta(g)} \ll \sigma_{S,g}$ while $\Phi_\rho(f_1) \in \mathrm{Im}\,\Phi_\rho$ so we have "a lot of orthogonality" to have independence. Then one verifies that

$$\int_{X^\infty \times X \times Y} f_1 \otimes \ldots \otimes f_n \otimes f_{n+1} \otimes g\, d(\rho^\infty \otimes_X \hat{\eta}) =$$

$$\int_{X^\infty \times Y \times X} f_1 \otimes \ldots \otimes f_n \otimes g \otimes \Phi_\rho(f_{n+1})\, d(\rho^\infty \otimes_X \hat{\eta}).$$

By the induction assumption the spectral measure of $f_1 \otimes \ldots \otimes f_n \otimes g$ will be the convolution of $\sigma_{f_1 \otimes \ldots \otimes f_n}$ with $\sigma_{S,g}$ so belongs to $M_0$, hence orthogonality. □

Let us pass to a "desired" consequence...

**Lemma 2** *Let $U$ be a unitary operator on a separable Hilbert space $H$. Suppose that $n_k \to \infty$ and $U^{n_k} \to V$ weakly in $L(H)$. Then the maximal spectral type of $U$ on $\overline{Im(V)}$ is singular with respect to any Rajchman measure.*

Finally just recall that $T$ is mixing if and only if $U_T^n \to 0$ weakly on the space $L_0^2(X, \mathcal{B}, \mu)$. Therefore if $T$ is not mixing, in the weak closure of powers of $T$ (seen as Markov operators) we will always find a non-trivial Markov operator.

**Corollary 1 (Parreau)** *Assume that $T$ is an ergodic automorphism of $(X, \mathcal{B}, \mu)$. Let $n_k \to \infty$ and $U_T^{n_k} \to \Phi_\rho$ in the space of Markov operators. Then for every mixing automorphism $S$ we have*

$$T|_{\mathcal{B}_\infty(\rho)} \perp S.$$

*In particular, if $T$ is not mixing then it has a non-trivial factor disjoint from all mixing transformations.*

As we indicated above, a consequence of such a corollary is the following formula

$$M = M^{\perp\perp}.$$

Let us pass now to other consequences...

If $T$ is mixing and $R$ is not, then clearly the product automorphism $T \times R$ is not mixing. A direct consequence of Corollary 1 is the following.

**Corollary 2** *Let $T$ be mixing and let $\widetilde{T}$ be its ergodic extension. If $\widetilde{T}$ is not mixing then $\widetilde{T}$ has a non-trivial product factor $T \times R$, where $R$ is not mixing (in fact, $R \perp M$). In particular for $\tilde{T}$ we can take any ergodic self-joining of $T$.*

As we have already mentioned, another natural ideal of measures is the set of absolutely continuous measures. Let us see now what are consequences of Parreau's Theorem in this category.

**Corollary 3** *Assume that $T$ is an ergodic automorphism of $(X, \mathcal{B}, \mu)$. Assume that there exists a non-trivial automorphism $S$ (defined on $(Y, \mathcal{C}, \nu)$), which has a singular spectrum and such that $T \not\perp S$. Then $T$ has a non-trivial factor, which is disjoint with any automorphism having absolutely continuous spectrum.*

**Proof.**

Let $\Psi : L^2(X, \mathcal{B}, \mu) \to L^2(Y, \mathcal{C}, \nu)$ be a non-trivial Markov operator, $\Psi \circ U_T = U_S \circ \Psi$. Then $\Phi = \Psi^* \circ \Psi$ is still non-trivial and moreover $\sigma_{\Psi^* \circ \Psi(f)} \ll \sigma_{\Psi(f)}$, whence the spectral measure $\sigma_{\Phi(f)}$ is singular. It follows that spectral measures of all functions belonging to $\overline{Im(\Phi)}$ are singular. □

**Corollary 4** *Let $T$ be an automorphism with a Lebesgue spectrum and let $\widetilde{T}$ be its ergodic extension. Then either $\widetilde{T}$ is disjoint from any automorphism with a singular spectrum or $\widetilde{T}$ has a product factor $T \times R$, where $R \perp$ Lebesgue spectrum. In particular as $\tilde{T}$ we can take any ergodic self-joining of $T$.*

**Remark 2** We recall that we can have an automorphism with Lebesgue spectrum and its ergodic group extension whose spectrum is no longer absolutely continuous – there is a singular component in the spectrum (this ergodic extension will be still disjoint with any automorphisms having a singular spectrum). This is a classical example of the even factor with Lebesgue spectrum for a zero entropy Gaussian automorphism.

## Automorphisms with "big factors"

Given an ergodic automorphism $T$ of $(X, \mathcal{B}, \mu)$, for the purpose of this lecture, let us call a factor $T_1$ of it a "big" factor if for every $S$

$$T_1 \perp S \;\; \Rightarrow \;\; T \perp S.$$

We are interested in $T$ so that all its non-trivial factors are "big": prime automorphisms (those without any non-trivial proper factors), in particular MSJ-automorphisms (Rudolph); but also those $T$ for which all non-trivial factors are relatively distal: simple automorphisms (Veech, del Junco-Rudolph), automorphisms with Ratner's property (Ratner – horocycle automorphisms, Thouvenot, Ryzhikov, Frączek-L.), distally simple automorphisms (Ryzhikov-Thouvenot, del Junco-L.).

**Corollary 5** *Assume that $T$ has only "big" non-trivial factors.*
*(i) Either $T$ is mixing or $T \perp M$.*
*(ii) Either $T$ $\perp$singular spectrum or $T$ $\perp$Lebesgue spectrum.*

**Remark 3** It seems to be an open question whether one can find a non-trivial $T$ which is at the same time disjoint from all automorphisms with singular spectrum and disjoint from all automorphisms with Lebesgue spectrum.

## L-ideals and disjointness of weakly mixing automorphisms

Another direction of exploiting Parreau's Theorem is to look at so called $L$-ideals $I$, i.e. we additionally require $I \subset M^+(\mathbf{T})$ is closed and

$$\sigma \in I,\ M^+(\mathbf{T}) \ni \eta \ll \sigma \ \Rightarrow\ \eta \in I.$$

**Corollary 6** *Assume that there exists $\mu \otimes \mu \neq \rho \in J(T)$ such that*

$$\Phi_\rho(\{f \in L^2_0(X, \mathcal{B}, \mu) :\ \sigma_{T,f} \in I\}) = 0.$$

*Then for each $S$ whose maximal spectral type belongs to $I$ there exists a non-trivial factor of $T$ disjoint with $S$.*

Let us look at the following example. Let $(n_i)$ be an increasing sequence of natural numbers. Set

$$I = I((n_i)) = \{\nu \in M^+(\mathbf{T}); \widehat{\nu}(n_i + k) \to 0 \text{ for each } k \in \mathbf{Z}\}.$$

Then $I$ is an $L$-ideal. For Koopman operators, any sequence $(n_i)$ for which $U_T^{n_i} \to 0$ weakly on $L^2_0(X, \mathcal{B}, \mu)$ is (naturally) called a *mixing* time.

**Corollary 7** *If two weakly mixing automorphisms $T$ and $S$ do not have the same mixing times then one of them has a non-trivial factor disjoint with the remaining automorphism.*

## Multipliers and Parreau's factor

If $\mathcal{S}$ is a certain family of ergodic automorphisms of a fixed standard probability Borel space $(X, \mathcal{B}, \mu)$ (automorphisms are considered up to isomorphism). A *multiplier* of $\mathcal{S}$ is any ergodic automorphism $T$ such that for each $S \in \mathcal{S}$,

$$(T \times S, \rho) \in \mathcal{S}$$

for an arbitrary $\rho \in J^e(T, S)$. The family of all multipliers of $\mathcal{S}$ will be denoted by $\mathcal{M}(\mathcal{S})$. If the one-point system is in $\mathcal{S}$ then $\mathcal{M}(\mathcal{S}) \subset \mathcal{S}$. In what follows $\mathcal{S}$ will be of the form $\mathrm{A}^{\perp}$: i.e. $T \in \mathcal{M}(\mathrm{A}^{\perp})$ if for each $R$ disjoint from all members of A

$$(T \times R, \rho) \perp S$$

for arbitrary $\rho \in J^e(T, R)$ and $S \in \mathrm{A}$. Clearly, $\mathcal{M}(\mathrm{A}^{\perp}) \subset \mathrm{A}^{\perp}$.

Recall that spectral disjointness implies disjointness. Here is a simple spectral criterion of being a multiplier.

**Proposition 1** *Assume that $T$ is an ergodic automorphism of $(X, \mathcal{B}, \mu)$. Assume moreover that $M_0 \subset M^+(\mathbf{T})$ is an ideal. If the maximal spectral type of $U_T$ on $L^2_0(X, \mathcal{B}, \mu)$ is singular with respect to any measure in $M_0$ then for an arbitrary ergodic automorphism $S$ of $(Y, \mathcal{C}, \nu)$ whose maximal spectral type (on $L^2_0(Y, \mathcal{C}, \nu)$) belongs to $M_0$,*

$$T \in \mathcal{M}(\{S\}^{\perp}).$$

**Proof.**

Take $\lambda \in J(T, R, S)$, where $R$ is an ergodic automorphism of $(Z, \mathcal{D}, \rho)$ and $R \perp S$. Fix $f \in L^{\infty}(X, \mathcal{B}, \mu)$, $g \in L^{\infty}(Z, \mathcal{D}, \rho)$ and $h \in L^{\infty}(Y, \mathcal{C}, \nu)$. We need to show that

$$\int_{X \times Z \times Y} f \otimes g \otimes h \, d\lambda = \int_{X \times Z} f \otimes g \, d\lambda|_{X \times Z} \cdot \int_Y h \, d\nu.$$

Suppose $\int_Y h\,d\nu = 0$. Then from disjointness $R$ and $S$ it follows that

$$\sigma_{(T\times R\times S,\lambda),1\otimes g\otimes h} = \sigma_{(R\times S,\rho\otimes\nu,g\otimes h)} = \sigma_{R,g} * \sigma_{S,h},$$

that is, this spectral measure is absolutely continuous with respect to a measure from $M_0$. It follows from the assumption that the spectral measure of $f\otimes 1\otimes 1$ (with respect to $(T\times R\times S,\lambda)$) is then singular with respect to the spectral measure of $1\otimes g\otimes h$, whence orthogonality and this completes the proof. $\square$

**Corollary 8** *Any automorphism with singular spectrum is a multiplier of the class $AC^{\perp}$.*

**Remark 4** Notice that by a result of Smorodinsky-Thouvenot there are automorphisms $T$ with Lebesgue spectrum such that $T\vee T$ has a non-trivial factor with singular spectrum. However, given $T$ with singular spectrum and considering ergodic self-joinings $T\vee T$ we cannot produce a factor with a Lebesgue spectrum.

With a little bit more effort one can prove in Parreau's Theorem the following assertion (without changing the assumption)

$$T|_{\mathcal{B}_\infty(\rho)} \in \mathcal{M}(S^{\perp}),$$

that is, if $R\perp S$ then any ergodic joining

$$T|_{\mathcal{B}_\infty(\rho)} \vee R \perp S.$$

Now, given an ergodic automorphism $T$ and a certain family $\mathcal{S} = \mathrm{A}^{\perp}$ consider

$$\{\mathcal{A}\subset\mathcal{B};\ T^{-1}\mathcal{A} = \mathcal{A} \text{ and } T|_{\mathcal{A}} \in \mathcal{M}(\mathcal{S})\}.$$

This family is closed under taking inverse limits, it is then closed under taking joinings.

**Corollary 9** *For each ergodic automorphis $T$ there exists a biggest factor $\mathcal{A} \subset \mathcal{B}$ such that $T|_{\mathcal{A}} \in \mathcal{M}(A^{\perp})$.*

**Corollary 10** *If $T$ is ergodic but not mixing then the biggest factor of it being a multiplier of $M^{\perp}$ is non-trivial.*

For non-mixing transformations such a factor I propose to call Parreau factor.

**Remark 5** One might hope that Parreau factor is an analog of something like Kronecker factor or the maximal distal factor in the non-weak mixing case. However this is not the case. We have

D$\subset \mathcal{M}(\mathrm{WM}^{\perp}) \subset$WM$^{\perp}$ and the inclusions are STRICT!

(Glasner-Weiss, L.-Parreau).

It is an open problem whether each ergodic automorphism in $M^{\perp}$ is a multiplier of this class.