EXACTNESS OF ROKHLIN ENDOMORPHISMS AND WEAK MIXING OF POISSON BOUNDARIES.

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Abstract. We give conditions for the exactness of Rokhlin endomorphisms, apply these to random walks on locally compact, second countable topological groups and obtain that the action on the Poisson boundary of an adapted random walk on such a group is weakly mixing.

§0 Introduction

Rokhlin endomorphisms.

By a *non-singular endomorphism* we mean a quadruple $(X,\mathcal{B},m,T)$ where $(X,\mathcal{B},m)$ is a standard probability space and $T:X_0\to X_0$ is a measurable transformation of $X_0\in\mathcal{B},\ m(X_0)=1$ satisfying $m(T^{-1}A)=0\ \Leftrightarrow\ m(A)=0\ \ (A\in\mathcal{B})$. As in [Ro2], the endomorphism T is called *exact* if $\mathfrak{T}(T):=\bigcap_{n=0}^{\infty}T^{-n}\mathcal{B}\overset{m}{=}\{\emptyset,X\}$.

Let $\mathbb{G}$ be a locally compact, Polish topological (LCP) group.

By a *non-singular* $\mathbb{G}$*-action* on a probability space $(Y,\mathcal{C},\nu)$ we mean a measurable homomorphism $S:\mathbb{G}\to\text{Aut}\,(Y)$ where $\text{Aut}\,(Y)$ denotes the group of invertible, non-singular transformations of Y equipped with its usual Polish topology. The action S is called *probability preserving* if each S_g preserves ν.

Given a non-singular endomorphism $(X,\mathcal{B},m,T)$, a non-singular $\mathbb{G}$-action $S:\mathbb{G}\to\text{Aut}\,(Y)$ of a LCP group $\mathbb{G}$ and a measurable function $f:X\to\mathbb{G}$, we consider the *Rokhlin endomorphism* $\widetilde{T}=\widetilde{T}_{f,S}:X\times Y\to X\times Y$ defined by

$$\widetilde{T}(x,y):=(Tx,S_{f(x)}y).$$

Rokhlin endomorphisms first appeared in [Ro1] (see also [AR]). We give conditions for their exactness (theorem 2.3).

These conditions are applied to random walk endomorphisms. Meilijson (in [Me]) gave sufficient conditions for exactness for random walk endomorphisms over $\mathbb{G}=\mathbb{Z}$. We clarify Meilijson's theorem proving a converse (proposition 4.2), extend it to LCP Abelian groups (theorem 4.1), characterize the exactness of the Rokhlin endomorphism for a steady random walk (theorem 4.5) and obtain that the group action on the Poisson boundary (see §4) of an adapted (i.e. globally supported) random walk is weakly mixing (proposition 4.4).

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Tools employed include the ergodic theory of "associated actions" (see §1), and the boundary theory of random walks (see §4).

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§1 Associated actions

For a non-singular endomorphism $(Z, \mathcal{D}, \nu, R)$ set

- $\mathcal{I}(R) := \{A \in \mathcal{D} : \ R^{-1}A = A\}$ – the *invariant* σ-algebra, and
- $\mathcal{T}(R) := \bigcap_{n=0}^{\infty} R^{-n}\mathcal{D}$ – the *tail* σ-algebra.

Let $(X, \mathcal{B}, m, T)$ be a non-singular endomorphism. Let $\mathbb{G}$ be a locally compact, Polish topological (LCP) group, let $f : X \to \mathbb{G}$ be measurable.

There are two *associated (right)* $\mathbb{G}$-*actions* arising from the invariant and tail σ-algebras of T_f, which are defined as follows:

- define the (left) *skew product* endomorphism $T_f : X \times \mathbb{G} \to X \times \mathbb{G}$ by $T_f(x, g) := (Tx, f(x)g)$ and fix $\mathbb{P} \in \mathcal{P}(X \times \mathbb{G}),\ \mathbb{P} \sim m \times m_{\mathbb{G}}$;
- for $t \in \mathbb{G}$, define $Q_t : X \times \mathbb{G} \to X \times \mathbb{G}$ by $Q_t(x, g) := (x, gt^{-1})$, then $Q_t \circ T_f = T_f \circ Q_t$.

The associated invariant action.

The *invariant factor* of $(X \times \mathbb{G}, \mathcal{B}(X \times \mathbb{G}), \mathbb{P}, T_f)$ is a standard probability space $(\Omega, \mathcal{F}, P) = (\Omega_{\mathcal{I}}, \mathcal{F}_{\mathcal{I}}, P_{\mathcal{I}})$ equipped with a measurable map $\pi : X \times \mathbb{G} \to \Omega$ such that $\mathbb{P} \circ \pi^{-1} = P,\ \pi \circ T_f = \pi$ and $\pi^{-1}\mathcal{F} = \mathcal{I}(T_f)$.

Since $Q_t\mathcal{I}(T_f) = \mathcal{I}(T_f)$ (because $T_f \circ Q_t = Q_t \circ T_f$),

- $\exists$ a P-non-singular $\mathbb{G}$-action $\mathfrak{p} : \Omega \to \Omega$ so that $\pi \circ Q = \mathfrak{p} \circ \pi$.

Proposition 1.1. *$(\Omega, \mathcal{F}, P, \mathfrak{p})$ is ergodic iff $(X, \mathcal{B}, m, T)$ is ergodic.*

Proof. If $(X, \mathcal{B}, m, T)$ is ergodic, then so is $(X \times \mathbb{G}, \mathcal{B}(X \times \mathbb{G}), \mathbb{P}, \langle T_f, Q\rangle)$ where $\langle T_f, Q\rangle$ denotes the $\mathbb{Z}_+ \times \mathbb{G}$ action defined by $(n, t) \mapsto T_f^n \circ Q_t \in \text{Aut}\,(X \times \mathbb{G})$. Any $\mathfrak{p}$-invariant, measurable function on Ω lifts by π to a $\langle T_f, Q\rangle$-invariant, measurable function on $X \times \mathbb{G}$, which is $\mathbb{P}$-a.e. constant.

Conversely, any T-invariant function on X lifts to a T_f-invariant function on $X \times \mathbb{G}$ which is also Q-invariant and thus the lift of a $\mathfrak{p}$-invariant, measurable function on Ω. If $(\Omega, \mathcal{F}, P, g)$ is ergodic this function is constant (a.e.). $\square$

The non-singular $\mathbb{G}$-action $(\Omega, \mathcal{F}, P, \mathfrak{p})$ is called the *invariant-* or *Poisson* $\mathbb{G}$-*action associated to* (T, f) and denoted $\mathfrak{p} = \mathfrak{p}(T, f)$.

This action is related to the *Mackey range* of a cocycle (see [Zi2] and §3), and the *Poisson boundary* of a random walk (see §4).

The associated tail action.

The *tail factor* of $(X \times \mathbb{G}, \mathcal{B}(X \times \mathbb{G}), \mathbb{P}, T_f)$ is a standard probability space $(\Omega, \mathcal{F}, P) = (\Omega_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}, P_{\mathcal{T}})$ equipped with a measurable map $\pi : X \times \mathbb{G} \to \Omega$ such that $\mathbb{P} \circ \pi^{-1} = P,\ \pi^{-1}\mathcal{F} = \mathcal{T}(T_f)$.

Since $Q_t\mathcal{T}(T_f) = \mathcal{T}(T_f)$ (because $T_f \circ Q_t = Q_t \circ T_f$),

- $\exists$ a P-non-singular $\mathbb{G}$-action $\tau : \Omega \to \Omega$ so that $\pi \circ Q = \tau \circ \pi$.

Proposition 1.2. *$(\Omega, \mathcal{F}, P, \tau)$ is ergodic iff $(X, \mathcal{B}, m, T)$ is exact.*

Proof. Suppose first that $(X, \mathcal{B}, m, T)$ is exact. Any τ-invariant, measurable function $F : \Omega \to \mathbb{R}$ lifts by π to a Q-invariant, $\mathcal{T}(T_f)$-measurable function $\overline{F} : X \times \mathbb{G} \to \mathbb{R}$. In particular $\exists\,!\ \overline{F}_n : X \times \mathbb{G} \to \mathbb{R}$ $(n \geq 0)$ measurable so that $\overline{F} = \overline{F}_n \circ T_f^n$. It follows from $Q_t \circ T_f = T_f \circ Q_t$ that each $\overline{F}_n$ is Q-invariant, whence $\exists\ \hat{F}_n : X \to \mathbb{R}$ measurable so that $\overline{F}_n(x, y) = \hat{F}_n(x)$ for $\mathbb{P}$-a.e. $(x, y) \in X \times \mathbb{G}$. Thus $\overline{F} = \overline{F}_0 = \hat{F}_0$ is $\mathcal{T}(T)$-measurable, whence constant $\mathbb{P}$-a.e. by exactness of T.

Conversely, any $\mathcal{T}(T)$-measurable function on X lifts to a $\mathcal{T}(T_f)$-measurable function on $X \times \mathbb{G}$ which is also Q-invariant and thus the lift of a τ-invariant, measurable function on Ω. If $(\Omega, \mathcal{F}, P, \tau)$ is ergodic this function is constant (a.e.). ☐

The non-singular $\mathbb{G}$-action $(\Omega, \mathcal{F}, P, \tau)$ is called the *associated tail* $\mathbb{G}$*-action* (of (T, f)) and denoted $\tau = \tau(T, f)$. As with the Poisson action, the tail action is related to the Mackey range of a cocycle, and also to the *tail boundary* of a random walk (see §4).

§2 Conditions for exactness and a construction of Zimmer

We begin with a proposition generalising Zimmer's construction (in [Zi1]) of a $\mathbb{G}$-valued cocycle over an ergodic, probability preserving transformation with a prescribed ergodic, non-singular $\mathbb{G}$-action as Mackey range.

Proposition 2.1.

Suppose that $\mathbb{G}$ is a LCP group,
that $(X, \mathcal{B}, m, T)$ is a non-singular endomorphism, and suppose that $f : X \to \mathbb{G}$ is measurable.

Let S be a non-singular $\mathbb{G}$-action on a probability space $(Y, \mathcal{C}, \nu)$ and define $\widetilde{T} = \widetilde{T}_{f,S} : X \times Y \to X \times Y$ by

$$\widetilde{T}(x, y) := (Tx, S_{f(x)}y);$$

then

$$\mathfrak{p}(\widetilde{T}, f) \cong \mathfrak{p}(T, f) \times S,\ \&\ \tau(\widetilde{T}, f) \cong \tau(T, f) \times S.$$

Proof.

Define $\pi : X \times Y \times \mathbb{G} \to X \times \mathbb{G} \times Y$ by $\pi(x, y, g) := (x, g, S_{g^{-1}}y)$. Evidently π is a bimeasurable bijection.

Fix $p \in \mathcal{P}(\mathbb{G}),\ p \sim m_{\mathbb{G}}$. A calculation shows that

$$(m \times \nu \times p) \circ \pi^{-1} \sim m \times p \times \nu, \tag{1}$$

indeed

$$\frac{d(m \times \nu \times p) \circ \pi^{-1}}{d(m \times p \times \nu)}(x, g, y) = \frac{d\nu \circ S_g}{d\nu}(y) =: D(g, y).$$

Next, we claim that

$$\pi \circ \widetilde{T}_f = (T_f \times \mathrm{Id}|_Y) \circ \pi. \tag{2}$$

To see this,

$$\pi \circ \widetilde{T}_f(x, y, g) = \pi(Tx, S_{f(x)}y, f(x)g) = (Tx, f(x)g, S_{(f(x)g)^{-1}}S_{f(x)}y)$$
$$= (Tx, f(x)g, S_{g^{-1}}y) = T_f \times \mathrm{Id} \circ \pi(x, y, g)$$

It follows from (2) that

$$\mathcal{I}(\widetilde{T}_f) = \pi^{-1}(\mathcal{I}(T_f \times \text{Id})), \;\; \mathcal{T}(\widetilde{T}_f) = \pi^{-1}(\mathcal{T}(T_f \times \text{Id})).$$

Now, in general

$$\mathcal{I}(T_f \times \text{Id}) = \mathcal{I}(T_f) \otimes \mathcal{B}(Y), \;\; \mathcal{T}(T_f \times \text{Id}) = \mathcal{T}(T_f) \otimes \mathcal{B}(Y) \quad \text{mod } \; m \times p \times \nu$$

and so

$$\mathcal{I}(\widetilde{T}_f) = \pi^{-1}(\mathcal{I}(T_f) \otimes \mathcal{B}(Y)), \;\; \mathcal{T}(\widetilde{T}_f) = \pi^{-1}(\mathcal{T}(T_f) \otimes \mathcal{B}(Y)) \quad \text{mod } \; m \times p \times \nu.$$

Now let $(\Omega_i, \mathcal{F}_i, P_i)$ and let $(\widetilde{\Omega}_i, \widetilde{\mathcal{F}}_i, \widetilde{P}_i)$ $(i = \mathfrak{p}, \; \tau)$ be the invariant or tail factors of $(X \times \mathbb{G}, m \times p, T_f)$ and $(X \times Y \times \mathbb{G}, m \times \nu \times p, \widetilde{T}_f)$ respectively, according to the value of $i = \mathfrak{p}, \; \tau$. By (1) and (2), π induces a measure space isomorphism of $(\widetilde{\Omega}_i, \widetilde{P}_i)$ with $(\Omega_i \times Y, P_i \times \nu)$.

Denoting the associated $\mathbb{G}$-actions by $\widetilde{Q}_t(x,y,g) := (x,y,gt^{-1})$ and $Q_t(x,g) := (x, gt^{-1})$, we note that

$$\pi \circ \widetilde{Q}_t = (Q_t \times S_t) \circ \pi.$$

The proposition now follows from this. □

Corollary 2.2.
1) $\widetilde{T}$ is ergodic iff $\mathfrak{p}(T,f) \times S$ is ergodic.
2) $\widetilde{T}$ is exact iff $\tau(T,f) \times S$ is ergodic.
3) If both T_f and S are ergodic, then $\widetilde{T}$ is ergodic and $\mathfrak{p}(\widetilde{T}, f) \cong S$.
4) If T_f is exact and S is ergodic, then $\widetilde{T}$ is exact and $\tau(\widetilde{T}, f) \cong S$.

Proof. Parts 1) and 2) follow from propositions 1.1 and 1.2. Parts 3) and 4) follow from these and form essentially Zimmer's construction. □

§3 Locally invertible endomorphisms

In this section, we obtain additional results for a non-singular, exact endomorphism $(X, \mathcal{B}, m, T)$ of a standard measure space which is *locally invertible* in the sense that $\exists$ an at most countable partition $\alpha \subset \mathcal{B}$ so that $T : a \to Ta$ is invertible, non-singular $\forall \; a \in \alpha$. Under the assumption of local invertibility, the associated actions of §1 are *Mackey ranges of cocycles* (as in [Zi2]). See proposition 3.2 (below).

As in [S-W], we call an ergodic, non-singular $\mathbb{G}$-action $(X, \mathcal{B}, m, U)$ *properly ergodic* if $m(U_{\mathbb{G}}(x)) = 0 \; \forall \; x \in X$ and call a properly ergodic, non-singular $\mathbb{G}$-action $S : \mathbb{G} \to \text{Aut}\,(Y)$ *mildly mixing* if $U \times S$ is ergodic for any properly ergodic non-singular $\mathbb{G}$-action $(X, \mathcal{B}, m, U)$. As shown in [S-W]:

- there are no mildly mixing actions of compact groups,
- a mildly mixing action of non-compact LCP group has an equivalent, invariant probability. Moreover,
- a probability preserving $\mathbb{G}$-action ($\mathbb{G}$ a non-compact LCP group) $(Y, \mathcal{C}, \nu, S)$ is mildly mixing iff

$$f \in L^2(\nu), \; g_n \in \mathbb{G}, \; g_n \xrightarrow{\mathbb{G}} \infty, \; f \circ S_{g_n} \xrightarrow{L^2(\nu)} f \;\Rightarrow\; f \;\; \text{is constant.}$$

We prove

Theorem 3.1.

Suppose that $(X, \mathcal{B}, m, T)$ is a non-singular, locally invertible, exact endomorphism of a standard measure space, that $\mathbb{G}$ is a LCP, non-compact, Abelian group and that $f : X \to \mathbb{G}$ is measurable.

Either $\widetilde{T}_{f,S}$ is exact for every mildly mixing probability preserving $\mathbb{G}$-action $S : \mathbb{G} \to \text{Aut}\,(Y)$, or $\exists$ a compact subgroup $\mathbb{K} \le \mathbb{G},\ t \in \mathbb{G}$ and $\overline{f} : X \to \mathbb{K},\ g : X \to \mathbb{G}$ measurable so that

$$f = g - g \circ T + t + \overline{f}.$$

Note that the invertible version of this generalizes corollary 6 of [Ru]. The rest of this section is the proof of theorem 3.1.

Tail relations.

Let $(X, \mathcal{B}, m, T)$ be a non-singular, locally invertible endomorphism of a standard probability space. Consider the *tail relations*

$$\mathfrak{T}(T) := \{(x, y) \in X \times X :\ \exists\, k \ge 0,\ T^k x = T^k y\};$$

$$\mathfrak{G}(T) := \{(x, y) \in X_0 \times X_0 :\ \exists\, k,\ \ell \ge 0,\ T^k x = T^\ell y\}$$

where $X_0 := \{x \in X :\ T^{n+k} x \ne T^k x\ \forall\, n,\ k \ge 1\}$. We assume that $m(X \setminus X_0) = 0$ (which is the case if T is ergodic and m is non-atomic) and so $\mathfrak{T}(T) \subset \mathfrak{G}(T)$ mod m. Both $\mathfrak{T}(T)$ and $\mathfrak{G}(T)$ are *standard, countable, m-non-singular equivalence relations* in sense of [F-M] whose invariant sets are given by

$$\mathcal{I}(\mathfrak{G}(T)) = \mathcal{I}(T), \quad \mathcal{I}(\mathfrak{T}(T)) = \mathcal{T}(T)$$

respectively.

Given a LCP group $\mathbb{G}$ and $f : X \to \mathbb{G}$ measurable, define $f_n : X \to \mathbb{G}\ \ (n \ge 1)$ by

$$f_n(x) := f(T^{n-1}x) f(T^{n-2}x) \ldots f(Tx) f(x)$$

and define $\Psi_f : \mathfrak{G}(T) \to \mathbb{G}$ by

$$\Psi_f(x, x') := f_\ell(x')^{-1} f_k(x) \quad \text{for } k,\ \ell \ge 0 \ \text{ such that } \ T^k x = T^\ell x'$$

(this does not depend on the $k,\ \ell \ge 0$ such that $T^k x = T^\ell x'$ for $x, x' \in X_0$).

It follows that $\Psi_f : \mathfrak{G}(T) \to \mathbb{G}$ is a (left) $\mathfrak{G}(T)$-*orbit cocycle* in the sense that

$$\Psi_f(y, z)\Psi_f(x, y) = \Psi_f(x, z)\ \forall\ (x, y),\ (y, z) \in \mathfrak{G}(T).$$

Note that since $\mathfrak{T}(T) \subset \mathfrak{G}(T)\quad$ mod m, the restriction $\Psi_f : \mathfrak{T}(T) \to \mathbb{G}$ is a (left) $\mathfrak{T}(T)$-orbit cocycle.

Mackey ranges of cocycles. Let $\mathcal{R}$ be a countable, standard, non-singular equivalence relation on the standard measure space $(X, \mathcal{B}, m)$ and let $\Psi : \mathcal{R} \to \mathbb{G}$ be a left $\mathcal{R}$-orbit cocycle. It follows from theorem 1 in [F-M], there is a countable group Γ and a non-singular Γ-action $(X, \mathcal{B}, m, V)$ so that

$$\mathcal{R} {=} \mathcal{R}_V := \{(x, V_\gamma x) :\ \gamma \in \Gamma,\ x \in X\}.$$

Let $f(\gamma, x) := \Psi(x, V_\gamma x)\ \ (f = f_{\Psi,V} : \Gamma \times X \to \mathbb{G})$ be the associated *left V-cocycle* (satisfying $f(\gamma\gamma', x) = f(\gamma, V_{\gamma'} x) f(\gamma', x)$)

The *Mackey range* $\mathfrak{R}(V,f)$ (see [Zi2]) is analogous to the invariant action of §1. It is the non-singular $\mathbb{G}$-action of Q $(Q_t(x,y) := (x, yt^{-1}))$ on the invariant factor of $V_f : X \times \mathbb{G} \to X \times \mathbb{G}$ $((V_f)_\gamma(x,g) := (V_\gamma x, f(\gamma,x)g))$.

As before, $\mathfrak{R}(V,f)$ is ergodic iff $(X,\mathcal{B},m,V)$ is ergodic.

It can be shown that $\mathfrak{R}(V, f_{\Psi,V})$ does not depend on V such that $\mathcal{R}=\mathcal{R}_V$ and we define the *Mackey range of* Ψ *over* $\mathcal{R}$ as $\mathfrak{R}(\mathcal{R},\Psi) := \mathfrak{R}(V, f_{\Psi,V})$.

Proposition 3.2.

$$\mathfrak{p}(T,f) = \mathfrak{R}(\mathfrak{G}(T),\Psi_f), \quad \tau(T,f) = \mathfrak{R}(\mathfrak{T}(T),\Psi_f).$$

We also have the following version of proposition 2.1 (whose proof is similar) :

Proposition 3.3.
Suppose that Γ *is a countable group, that* $\mathbb{G}$ *is a LCP group, that* $(X,\mathcal{B},m,V)$ *is an ergodic, non-singular* Γ*-action, and suppose that* $f : \Gamma\times X \to \mathbb{G}$ *is a measurable cocycle.*

Let S *be a non-singular* $\mathbb{G}$*-action on a probability space* $(Y,\mathcal{C},\nu)$ *and define* $\widetilde{V} : \Gamma \to \text{Aut}(X\times Y)$ *by* $\widetilde{V}_\gamma(x,y) := (V_\gamma x, S_{f(\gamma,x)}(y))$*; then*

$$\mathfrak{R}(\widetilde{V},f) \cong \mathfrak{R}(V,f)\times S.$$

Compact reducibility.
Let Γ be a countable group, $\mathbb{G}$ be a LCP group and let $(X,\mathcal{B},m,V)$ be an ergodic, non-singular Γ-action.

We call a measurable V-cocycle $F : \Gamma\times X \to \mathbb{G}$ *compactly reducible* if $\exists\ \mathbb{K} \le \mathbb{G}$ compact, a measurable cocycle $f : \Gamma\times X \to \mathbb{K}$ and $h : X \to \mathbb{G}$ measurable so that $F(\gamma,x) = h(V_\gamma x)^{-1} f(\gamma,x) h(x)$.

Regularity and range of an orbit cocycle. Let $\mathbb{G}$ be a LCP group and let $\mathcal{R} \in \mathcal{B}(X\times X)$ be a standard, countable, non-singular equivalence relation. We call the left $\mathcal{R}$-orbit cocycle $\Psi : \mathcal{R} \to \mathbb{G}$ *compactly reducible* if the associated $f = f_{\Psi,V}$ has this property for some (and hence every) non-singular Γ-action $(X,\mathcal{B},m,V)$ with $\mathcal{R}=\mathcal{R}_V$.

For the rest of the section, we assume that $\mathbb{G}$ is Abelian.

Proposition 3.4. *If the measurable* V*-cocycle* $F : \Gamma\times X \to \mathbb{G}$ *is not compactly reducible, then* $\mathfrak{R}(V,F)\times S$ *is ergodic for any mildly mixing probability preserving* $\mathbb{G}$*-action* $S : \mathbb{G} \to \text{Aut}\,(Y)$*.*

Proof. Suppose that the conclusion fails, then $\mathfrak{R}(V,F)$ is not properly ergodic. It follows from [Zi2], proposition 4.2.24 that $\exists\ \mathbb{H} \le \mathbb{G}$ a closed subgroup which is non-compact by assumption, a measurable cocycle $f : \Gamma\times X \to \mathbb{H}$ and $h : X \to \mathbb{G}$ measurable so that $F(\gamma,x) = f(\gamma,x) + h(x) - h(V_\gamma x)$ and such that $(X\times\mathbb{H}, \mathcal{B}(X\times\mathbb{H}), m\times m_{\mathbb{H}}, V_f)$ is ergodic. In this case, $\mathfrak{R}(V,F)$ is the action of $\mathbb{G}$ on $\mathbb{G}/\mathbb{H}$.

Let $S : \mathbb{G} \to \text{Aut}\,(Y)$ be a mildly mixing probability preserving $\mathbb{G}$-action, then since $\mathbb{H}$ is not compact, $S|_{\mathbb{H}}$ is mildly mixing whence ergodic. Also $\mathfrak{R}(V,F)|_{\mathbb{H}} = \text{Id}$.

To see that $\mathfrak{R}(V,F)\times S$ is ergodic, let F be bounded, measurable and $\mathfrak{R}(V,F)\times S$-invariant. For $h\in\mathbb{H}$, $F\circ(\mathfrak{R}(V,F)\times S)_h(\omega,y)=F(\omega,S_hy)$. By ergodicity of $S|_{\mathbb{H}}$, $\exists\ \overline{F}(\omega)$ so that a.e. $F(\omega,y)=\overline{F}(\omega)$. The function $\overline{F}$ is $\mathfrak{R}(V,F)$-invariant, whence constant. ☐

By proposition 3.4,

Proposition 3.5.

If Ψ is not compactly reducible, then $\mathfrak{R}(\mathcal{R},\Psi)\times S$ is ergodic for any mildly mixing probability preserving $\mathbb{G}$-action $S:\mathbb{G}\to\mathrm{Aut}\,(Y)$.

We now have by propositions 3.2 and 3.5 that

Proposition 3.6.

For a locally invertible, exact endomorphism $(X,\mathcal{B},m,T)$ and $f:X\to\mathbb{G}$ measurable: If Ψ_f is not $\mathfrak{T}(T)$-compactly reducible, then $\tau(T,f)\times S$ is ergodic for any mildly mixing probability preserving $\mathbb{G}$-action $S:\mathbb{G}\to\mathrm{Aut}\,(Y)$.

Proof of theorem 3.1.

The previous propositions show that if Ψ_f is not $\mathfrak{T}(T)$-compactly reducible, then $\widetilde{T}_{f,S}$ is exact for every mildly mixing probability preserving $\mathbb{G}$-action $S:\mathbb{G}\to\mathrm{Aut}\,(Y)$.

We must show that if Ψ_f is $\mathfrak{T}(T)$-compactly reducible then $\exists$ a compact subgroup $\mathbb{K}\le\mathbb{G},\ t\in\mathbb{G}$ and $\overline{f}:X\to\mathbb{K},\ g:X\to\mathbb{G}$ measurable so that

$$f=g-g\circ T+t+\overline{f}.$$

To see this suppose that $\mathbb{K}\le\mathbb{G}$ is a compact subgroup and $g:X\to\mathbb{G}$ is measurable so that the quotient cocycle (under the map $s\mapsto\widetilde{s}:=s+\mathbb{K}\ \ (\mathbb{G}\to\mathbb{G}/\mathbb{K})$) is a coboundary, i.e. $\widetilde{\Psi}_f(x,x')=\widetilde{g}(x)-\widetilde{g}(x')$. Set $h(x):=\widetilde{f}(x)-\widetilde{g}(x)+\widetilde{g}(Tx)$.

We claim that h is $\mathfrak{T}(T)$-invariant, whence constant.

To prove this $\mathfrak{T}(T)$-invariance, (as in proposition 1.2 of [ANS]), note that $T^nx=T^nx'\ \Rightarrow\ h_n(x)-h_n(x')=\widetilde{f}_n(x)-\widetilde{f}_n(x')-\widetilde{g}(x)+\widetilde{g}(x')=0$. Also $T^nx=T^nx'\ \Rightarrow\ T^{n-1}(Tx)=T^{n-1}(Tx')\ \Rightarrow\ h_{n-1}(Tx)=h_{n-1}(Tx')$. Since $h(x)=h_n(x)-h_{n-1}(Tx)$, we have $h(x)=h(x')$. ☐

§4 Random walks

Let $\mathbb{G}$ be a LCP group and let $p\in\mathcal{P}(\mathbb{G})$.

The (left) *random walk on* $\mathbb{G}$ with *jump probability* $p\in\mathcal{P}(\mathbb{G})$ ($\mathtt{RW}(\mathbb{G},p)$ for short) is the stationary, one-sided shift of the Markov chain on $\mathbb{G}$ with transition probability $P(g,A):=p(Ag^{-1})\ \ (A\in\mathcal{B}(\mathbb{G}))$. The random walk $\mathtt{RW}(\mathbb{G},p)$ is said to be *adapted* if $p\in\mathcal{P}(\mathbb{G})$ is *globally supported* in the sense that $\overline{\langle\mathrm{supp}\,(p)\rangle}=\mathbb{G}$.

The random walk $\mathtt{RW}(\mathbb{G},p)$ is isomorphic to the measure preserving transformation

$$(X\times\mathbb{G},\mathcal{B}(X\times\mathbb{G}),\mu_p\times m_{\mathbb{G}},W)$$

where $X=\mathbb{G}^{\mathbb{N}}$, $\mu_p:\mathcal{B}(X)\to[0,1]$ is the product measure $\mu_p:=p^{\mathbb{N}}$ defined by $\mu_p([A_1,\dots,A_n]):=\prod_{k=1}^{n}p(A_k)$ where $[A_1,\dots,A_n]:=\{x\in X:\ x_k\in A_k\ \forall\ 1\le$

$k \le n\}$, $(n \ge 1,\ A_1, \ldots, A_n \in \mathcal{B}(\mathbb{G}))$, $m_{\mathbb{G}}$ is left Haar measure on $\mathbb{G}$ and $W : X \times \mathbb{G} \to X \times \mathbb{G}$ is defined by $W(x, g) := (Tx, x_1 g)$ with $T : X \to X$ being the shift $(Tx)_n := x_{n+1}$.

The *jump process* of the random walk $\mathtt{RW}(\mathbb{G}, p)$ is

$$(X, \mathcal{B}(X), \mu_p, T, f)$$

where $(X, \mathcal{B}(X), \mu_p, T)$ is as above and $f : X \to \mathbb{G}$ is defined by $f(x) := x_1$.

Boundaries.

The *tail boundary* of the random walk $\mathtt{RW}(\mathbb{G}, p)$ is $\tau(\mathbb{G}, p) := \tau(T, f)$, and the *Poisson boundary* is $\mathfrak{p}(\mathbb{G}, p) := \mathfrak{p}(T, f)$ where $(X, \mathcal{B}(X), \mu_p, T, f)$ is the *jump process* of the random walk $\mathtt{RW}(\mathbb{G}, p)$.

These definitions are equivalent with those in [K-V, K] (see also [Fu]) .

Weakly mixing actions.

Let $\mathbb{G}$ be a LCP group. A non-singular $\mathbb{G}$-action $V : \mathbb{G} \to \mathrm{Aut}\,(X, \mathcal{B}, m)$ is called *weakly mixing* if $V \times S$ is ergodic on $X \times Y$ whenever $S : \mathbb{G} \to \mathrm{Aut}\,(Y, \mathcal{C}, \nu)$ is an ergodic, probability preserving $\mathbb{G}$-action.

In case $\mathbb{G} = \mathbb{Z}$, this agrees with the definition given in [ALW]. More generally, in case $\mathbb{G}$ is Abelian, weak mixing of V is equivalent to the condition

$$f \in L^\infty(X, \mathcal{B}, m),\ \gamma \in \widehat{\mathbb{G}},\ f \circ V_g = \gamma(g) f \text{ a.e. } \forall\ g \in \mathbb{G}\ \Rightarrow\quad f \ \text{ is a.e. constant}$$

where $\widehat{\mathbb{G}}$ denotes the dual group of $\mathbb{G}$. For a proof of this equivalence when $\mathbb{G} = \mathbb{Z}$, see §4 of [ALW]. Alternatively, see theorem 2.7.1 in [A] (which can easily be extended to the general Abelian case).

Random walks on Abelian groups. In case $\mathbb{G}$ is Abelian, the tail and Poisson boundaries are the actions of $\mathbb{G}$ on $\mathbb{G}/\mathbb{H}_\tau$ and $\mathbb{G}/\mathbb{H}_{\mathfrak{p}}$ by translation (respectively), where

$$\mathbb{H}_{\mathfrak{p}} := \overline{\langle \mathrm{supp}\, p \rangle},\ \mathbb{H}_\tau := \overline{\langle \mathrm{supp}\, p\ -\ \mathrm{supp}\, p \rangle}.$$

See [D-L]. Here, for $F \subset \mathbb{G}$, $\langle F \rangle$ denotes the minimal subgroup of $\mathbb{G}$ containing F.

Theorem 4.1 (extension of [Me]).

Let $\mathbb{G}$ be a LCP, Abelian group, let $p \in \mathcal{P}(\mathbb{G})$ and let $(X, \mathcal{B}(X), \mu_p, T, f)$ be the jump process of the random walk $\mathtt{RW}(\mathbb{G}, p)$.

1) If $\mathbb{G}/\mathbb{H}_{\mathfrak{p}}$ $(\mathbb{G}/\mathbb{H}_\tau)$ is compact and $S : \mathbb{G} \to \mathrm{Aut}\,(Y)$ is a weakly mixing probability preserving $\mathbb{G}$-action, then $\widetilde{T}_{f,S}$ is ergodic (exact).

2) If $\mathbb{H}_{\mathfrak{p}}$ $(\mathbb{H}_\tau)$ is non-compact and $S : \mathbb{G} \to \mathrm{Aut}\,(Y)$ is a mildly mixing probability preserving $\mathbb{G}$-action, then $\widetilde{T}_{f,S}$ is ergodic (exact).

Proof.

1) It follows from the assumption that $\mathfrak{p}(T, f)$ $(\tau(T, f))$ is an ergodic, probability preserving $\mathbb{G}$-action, whence $S \times \mathfrak{p}(T, f)$ $(S \times \tau(T, f))$ is ergodic whenever S is weakly mixing.

2) The proof is as the proof of proposition 3.4 (above). ☐

Proposition 4.2 (converse of [Me]**).**

Let $p \in \mathcal{P}(\mathbb{Z})$ and let $(X, \mathcal{B}(X), \mu_p, T, f)$ be the jump process of the random walk RW $(\mathbb{Z}, p)$. *Suppose*

$$\mathbb{H} := \overline{\langle \text{supp}\,(p) - \text{supp}\,(p)\rangle} = d\mathbb{Z}$$

with $d \neq \{0\}$ and let S be an ergodic, probability preserving transformation of Y, then $\widetilde{T}_{f,S}$ is exact iff S^d is ergodic.

Proof. Here, $\tau(T, f)$ is the cyclic permutation of d points. The ergodicity of S^d characterizes the ergodicity of $S \times \tau(\mathfrak{T}(T), \psi_f)$. $\square$

Remark. In case $\mathbb{G}$ is LCP, Abelian, non-compact and $\mathbb{H}_i$ is compact ($i = \mathfrak{p}$ or τ), then a Gaussian action of $\mathbb{G}/\mathbb{H}_i$ with Haar spectral measure type is mildly mixing. Lifting this action, we obtain a mildly mixing probability preserving $\mathbb{G}$-action $S : \mathbb{G} \to \text{Aut}\,(Y)$ with $S|_{\mathbb{H}_i} \equiv \text{Id}$. Here, $\widetilde{T}_{f,S}$ is not ergodic or exact (according to whether $i = \mathfrak{p}$ or τ). This is because the action $V \times S$ on $\mathbb{G}/\mathbb{H}_i \times Y$ is not ergodic (where $V_t(g + \mathbb{H}_i) := t + g + \mathbb{H}_i$).

Indeed, if $A \in \mathcal{B}(Y)$, then $S_k A = A \mod \nu\ \forall\ k \in \mathbb{H}_i$. It follows that $B = B(A) := \bigcup_{g \in \mathbb{G}} g\mathbb{H}_i \times gA$ is Lebesgue measurable, $V \times S$-invariant, $m_{\mathbb{G}/\mathbb{H}_i} \times \nu(B(A)) > 0$ for $\nu(A) > 0$ and $B(A) \cap B(A^c) = \emptyset$.

Weak mixing of Poisson boundary.

We show that the Poisson action $\mathfrak{p}(\mathbb{G}, p)$ is weakly mixing when RW $(\mathbb{G}, p)$ is adapted.

Let $p \in \mathcal{P}(\mathbb{G})$ is globally supported and let $(X, \mathcal{B}(X), \mu_p, T, f)$ be the jump process of the random walk RW $(\mathbb{G}, p)$.

Proposition 4.3.

Suppose that $S : \mathbb{G} \to \text{Aut}\,(Y)$ is a probability preserving $\mathbb{G}$-action, then $\widetilde{T}_{f,S}$ is ergodic iff S is ergodic.

Proof of S ergodic $\Rightarrow$ $\widetilde{T}_{f,S}$ ergodic as in [Mo], (see also [ADSZ], proposition 1).

Suppose that $h : X \times Y \to \mathbb{R}$ is bounded, measurable and $\widetilde{T}_{f,S}$-invariant, then $P_{\widetilde{T}_{f,S}} h = h$ where $P_{\widetilde{T}_{f,S}} : L^1(\mu_p \times \nu) \to L^1(\mu_p \times \nu)$ is the predual of

$$F \mapsto F \circ \widetilde{T}_{f,S} \quad (L^\infty(\mu_p \times \nu) \to L^\infty(\mu_p \times \nu))$$

and given by

$$P^n_{\widetilde{T}_{f,S}} F(x, y) = P^n_T(F(\cdot, S^{-1}_{\alpha_n(\cdot)}(y))(x)$$

where $\alpha_n(x) := x_n \ldots x_1$ and $P_T : L^1(\mu_p) \to L^1(\mu_p)$ is the predual of $F \mapsto F \circ T \quad (L^\infty(\mu_p) \to L^\infty(\mu_p))$.

We claim that h is $X \times \mathcal{C}$-measurable.

To see this, note that $\exists\ h_n,\ \sigma(x_1, \ldots, x_n) \times \mathcal{C}$-measurable so that $\|h - h_n\|_1 \to 0$. By independence of $x_1, x_2, \ldots,$

$$\begin{aligned} P_{\widetilde{T}^n_{f,S}} h_n(x, y) &= P^n_T(h_n(\cdot, S^{-1}_{\alpha_n(\cdot)}(y))(x) \\ &= E(h_n(\cdot, S^{-1}_{\alpha_n(\cdot)}(y)) = E(P_{\widetilde{T}^n_{f,S}} h_n | X \times \mathcal{C}). \end{aligned}$$

Thus

$$\begin{aligned}&\|h - E(h|X\times\mathcal{C})\|_1\\ &\le \|P_{\widetilde{T}^n_{f,S}}h - P_{\widetilde{T}^n_{f,S}}h_n\|_1 + \|E(P_{\widetilde{T}^n_{f,S}}h_n|X\times\mathcal{C}) - E(P_{\widetilde{T}^n_{f,S}}h|X\times\mathcal{C})\|_1\\ &\le 2\|h-h_n\|_1 \to 0.\end{aligned}$$

Thus $h = G$ where $G : Y \to \mathbb{R}$ and $G\circ S_g = G$ ν-a.e. $\forall\ g \in \overline{\langle \text{supp}\,(p)\rangle} = \mathbb{G}$ and G (whence h) is a.e. constant by ergodicity of S. □

Proposition 4.4.

$\mathfrak{p}(T,f)$ is weakly mixing in the sense that $\mathfrak{p}(T,f)\times S$ whenever S is an ergodic, probability preserving $\mathbb{G}$-action.

Proof. Let $S : \mathbb{G} \to \text{Aut}\,(Y)$ be an ergodic, probability preserving $\mathbb{G}$-action. By proposition 4.3, $\widetilde{T}_{f,S}$ is ergodic, whence by proposition 2.1, $\mathfrak{p}(T,f)\times S$ is ergodic. □

Remark.

A *fibred system* $(X,\mathcal{B},m,T,\alpha)$ is a non-singular endomorphism $(X,\mathcal{B},m,T)$ which is locally invertible with respect to the at most countable partition $\alpha\subset\mathcal{B}$, which also satisfies $\sigma(\{T^{-n}a :\ n\ge 0,\ a\in\alpha\}) \overset{m}{=} \mathcal{B}$. Propositions 4.3 and 4.4 remain true whenever $(X,\mathcal{B},m,T,\alpha)$ is a probability preserving fibred system, which is *weak quasi–Markov, almost onto* in the sense of [ADSZ] and $f : X \to \mathbb{G}$ is α-measurable with $m\circ f^{-1}$ is globally supported on $\mathbb{G}$.

Aperiodic random walks.

Let $\mathbb{G}$ be a LCP group and suppose that $p\in\mathcal{P}(\mathbb{G})$.

A random walk $\texttt{RW}(\mathbb{G},p)$ is called *steady* (see [K]) if $\mathfrak{p}(\mathbb{G},p) = \tau(\mathbb{G},p)$.

Let $\mathbb{G}$ be a countable group and suppose that $p\in\mathcal{P}(\mathbb{G})$. The random walk $\texttt{RW}(\mathbb{G},p)$ is called *aperiodic* if the corresponding Markov chain is aperiodic. Equivalent conditions for this are

- $p_e^{n*} > 0\ \forall\ n$ large;
- $\exists\ n\ge 1$ so that $\text{supp}\,(p^{n*})\cap\text{supp}\,(p^{(n+1)*}) \ne \emptyset$.

An aperiodic random walk on a countable group is steady. This can be gleaned from [Fo] and théorème 3 in [D] (see also proposition 4.5 in [K]).

Theorem 4.5.

Let $\mathbb{G}$ be a LCP group and suppose that $p\in\mathcal{P}(\mathbb{G})$ is globally supported and that $\texttt{RW}(\mathbb{G},p)$ is steady. Let $(X,\mathcal{B}(X),\mu_p,T,f)$ be the jump process of the random walk RW $(\mathbb{G},p)$ *and let $S:\mathbb{G}\to\text{Aut}\,(Y)$ be a probability preserving $\mathbb{G}$-action, then $\widetilde{T}_{f,S}$ is ergodic iff S is ergodic and in this case $\widetilde{T}_{f,S}$ is exact.*

Proof of S ergodic $\Rightarrow$ $\widetilde{T}_{f,S}$ exact. By proposition 4.4, $\mathfrak{p}(T,f)\times S$ is ergodic, whence by steadiness, $\tau(T,f)\times S$ is ergodic. Thus, $\widetilde{T}_{f,S}$ is exact by corollary 2.2, 2). □

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