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ERGODIC PROPERTIES OF MORSE SEQUENCES

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To my parents

This dissertation was
written in Institute of
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The sequence entropy for Morse shifts

and some counterexamples

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I. Introduction The present thesis deals with a class of infinite 0-1 sequences called commonly Morse sequences.

Consider the best known Morse sequences

x=01×01×...

It arises in the following way. We first write down 01 and then at each succeeding step write the mirror image / i.e. we interchange 0 and 1 / of the complete previous production to the right of the same. So

x=01|10|1001|10010110|1001011001101001|...

This sequence has firstly appeared in Thue's paper [T] in 1906 and has been again rediscovered by Morse [Mo] in 1921. The history of this sequence to 1967 was described in [Hed]. In connection with the possibility of an unending game of chess, some problem in semi-groups and some combinatorial nature this sequence has appeared in [Hed-Mo], 1944 and [Ar], 1937.

Some Morse sequences appear in group theory in connection with Burnside problem / see [Dej] /.

In the 50'th and 60'th some Morse sequences were studied in [Got-Hed 1,2], [Got] as examples in topological dynamics of sequences generating minimal sets.

Some example of Morse sequences / and some related class called substitutions [Dek] / were examined in the context of other areas of mathematics such as combinatorics [Dek], [Ja], automata theory [Cob2], formal language theory [Hev-Ros] and other [Chr-Kam-Me-Ra]. For instance consider the following functional equation

$F(z^2)=F(z)/(1-z) \; -z/[(1-z)(1-z^2)], \quad |z|<1, \; F(0)=0$

It turns out that the unique solution is the function $f(z)=\sum_{n\geq 0} x[n]\, z^n$, where $x[n]$ is the n-th symbol in x / [Cob1] /.

As an object of studying in ergodic theory Morse sequences have been become since Keane's paper [Ke1] 1968 Some smaller class of 0-1 sequences including x=01×01×... was introduced to ergodic theory by Kakutani in [Kak] / here, they shall be called Kakutani sequences /. It turned out that the class of dynamical systems arising from Morse sequences is very interesting in ergodic theory. For instance, at once they provided positive answer to Jacob's question whether any infinite group of roots of unity can occur as eigenvalue group of strictly ergodic system. The Morse sequence x=01×01×... appeared in [Ju2],1977 as a counterexample and other Morse sequences in [Dek] ,1980.

In the thesis our main purpose is to discover some ergodic properties of dynamical systems induced by Morse sequences and to apply them to many questions stated in literature on ergodic theory.

The second kind of problems connected with our sequences is the problem of spectral and metric isomorphy. It has been originated since Kwiatkowski's papers [Kw 1,2]. Recently the problem of metric classification was completely solved by Kwiatkowski [Kw3] and Rojek [Roj] .

Taking into consideration the results of the thesis with the results mentioned above and other ones [Ke2] [Bu] [Ma 1,2] it seems that the class of Morse shifts is one of the best discovered class of shifts with zero entropy.

Our paper consists of five parts.

The first part provides introductory definitions, and elementary properties of Morse dynamical systems.

The rest of the thesis is devoted to solve four different problems. Since each of the parts has an introduction we only give a brief description of Chapters I-IV.

Chapter I is concerned with the rank of Morse shifts. We prove that the rank of such systems is less or equal two and give some characterization of these sequences with the rank one. Our main result shows that the class of regular Morse sequences constructed on a finite number of blocks has the rank two. This generalizes the results [Ju2], [Nü]. We answer also del Junco's question about some characterization rank one shifts.

Chapter II treats the centralizer of Morse systems. Our main goal is to construct a class of automorphisms with "untypical" centralizer, i.e. the centralizer must be countable but not trivial, $C(T) \neq \{T^i : i \in Z\}$. We argue also that any cyclic group can be realisated as the centralizer of some ergodic shift. The coalescence problem of some skew product is also considered.

Chapter III is mainly concerned with calculating sequence entropy with respect to special class of sequences. We develop methods of computing the sequence entropies / topological and measure-theoretical/ for Morse shifts / originated by Dekking's work [Dek] / . We negatively answer Saleski's question [S] and Goodman's one [Goodm]. Our theory applied to Kakutani sequences shows that lemma of Kušnirenko $h_A(T\times T') = h_A(T) + h_A(T')$ is not valid. In a November of 1984 letter, D. Newton communicated me that the Example 3.7 was the first he knew. Other results of such a nature are obtained. As an application we get that any Morse sequence has no weakly mixing factors. This fact allows to show all factors for certain type of Morse sequences. The problem of disjointness in our class is also considered.

Chapter IV / written with M.K. Mentzen / deals with the problem of metric isomorphy in the class of some generalisation of Morse sequences [Ma 1,2] . This class is called G-symbolic Morse sequences, where G is a finite abelian group. Our main result shows that if G and G' are not isomorphic then any continuous Morse sequence over G is not isomorphic to any continuous Morse sequence over G'.

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The results of this thesis will be published in the following papers

"The rank of regular Morse dynamical systems", Z. Wahr. verw. Geb. 24/2/1985

"The centralizer of Morse shifts", Ann. de l' Univ. de Clermont no 4, 1985

" The sequence entropy for Morse shifts and some counterexamples", Studia Math. 82 /3/

"Generalized Morse sequences on n-symbols and m-symbols are not isomorphic", Bull l'Acad Pol. 5-6, 1985

II. Definitions and elementary properties of Morse sequences We start with introducing a bit of terminology Let $A=\{0,1\}$. Each element $B=(b_0,\dots,b_{k-1})\in A^k$ will be called a block, k is the length of B and we denote it by $|B|$. Denote $B[i,j]=(b_i,b_{i+1},\dots,b_j)$, $B[i,i]=B[i]$. The block $\tilde{B}=(\tilde{b}_0,\dots,\tilde{b}_{k-1})$ is defined by setting $\tilde{b}_i=0$ if $b_i=1$ and $\tilde{b}_i=1$ if $b_i=0$. Let $C=(c_0,\dots,c_{m-1})$ be another block. Then the product $B\times C$ is defined by $B\times C=B^{c_0}B^{c_1}\dots B^{c_{m-1}}$, where $B^0=B$, $B^1=\tilde{B}$.

Let $|B|=|C|=k$ then

/1/ $d(B,C)=\frac{1}{k}\operatorname{card}\{0\leq i\leq k-1:\ B[i]\neq C[i]\}$

Let B,C be blocks then

/2/ $fr(B,C)=\operatorname{card}\{i:\ 0\leq i\leq |C|-|B|,\ C[i+|B|-1]=B\}$

/3/ B appears in C at i within $\delta>0$ if $d(B,C[i+|B|-1])<\delta$

If $d(B,C[i+|B|-1])=0$ we say simply B appears in C at i. /The above notations can be easily extended to blocks over any finite alphabet A/

Now, let $b^0,b^1,b^2,\dots$ be finite blocks of lengths at least two beginning with zero and put

/4/ $x=b^0 \times b^1 \times b^2 \times \ldots$

We set $\lambda_i=|b^i|$, $r_i=\min\left(\frac{1}{\lambda_i} fr(0,b^i), \frac{1}{\lambda_i} fr(1,b^i)\right)$, $i=0,1,\ldots$

<u>Definition 0.1</u> /[Ke1]/ The sequence x defined in /4/ is said to be a Morse sequence if

/i/ infinitely many of the b^i's are different from 0...0,

/ii/ infinitely many of the b^i's are different from 01...010,

/iii/ $\sum_{i \geqslant 0} r_i = \infty$

/ obviously /i/ follows from /iii/ /

If x is a Morse sequence and $\lambda_i=2$, $i \geqslant 0$ then x is said to be a Kakutani sequence.

If $x=b^0 \times b^1 \times \ldots$ is a Morse sequence then one can find an almost periodic point $\omega \in X=\{0,1\}^Z$ such that $\omega[k]=x[k]$ $k \geqslant 0$ /[Ke1]/.

Put $\mathcal{O}_x=\overline{\{T^i\omega : i \in Z\}}$, where T is the shift transformation $T((y_i)_{i\in Z})=(z_i)_{i\in Z}$, $z_i=y_{i+1}$, $i \in Z$.

It is known $(\mathcal{O}_x,T)$ is strictly ergodic /[Ke1]/. The unique T-invariant measure / ergodic / we shall denote by μ_x and the system $\theta(x)=(\mathcal{O}_x,\mathcal{B}_x,T,\mu_x)$ will said to be a Morse dynamical system / Morse shift /.

In the sequel all properties of $\theta(x)$ will be called properties of x.

Denote by σ the mirror map on $\mathcal{O}_x$, i.e. $\sigma(y)=\tilde{y}$, $\tilde{y}[i]=\widetilde{y[i]}$, $i \in Z$, $y \in \mathcal{O}_x$. Then $\sigma: \mathcal{O}_x \circlearrowleft$, $\sigma T=T\sigma$ and by strictly ergodicity of $\mathcal{O}_x$ σ preserves μ_x. So, from ergodic theorem we have

$$\mu_x(B)=\mu_x(\tilde{B})=\lim_{n\to\infty}\frac{1}{n}fr(B,x[0,n-1]) \text{ for every block } B$$

Kwiatkowski in [Kw2] has found out the structure of $\mathcal{O}_x$ / the set $X(x)$ described there is contained in $\mathcal{O}_x$ and the set $\mathcal{O}_x \setminus X(x)$ is countable /. Namely, let

/5/ $\quad C_t=\{D_i^{n_t}(j),\ i=0,\dots,n_t-1,\ j=0,1\}$

/6/ $\quad D_i^{n_t}(j)=\{y\in\mathcal{O}_x:\ y[-i+kn_t,-i+(k+1)n_t-1]=c_t^{j_k},\ k\in Z\}$ $\quad j_0=j$

where

$$c_t=b^0\times b^1\times\dots\times b^t,\quad n_t=\lambda_0\cdot\lambda_1\cdot\dots\cdot\lambda_t,\quad t\geqslant 0$$

Put

/7/ $\quad D_i^{n_t}=D_i^{n_t}(0)\cup D_i^{n_t}(1)$

Then $D^{n_t}=(D_0^{n_t},D_1^{n_t},\dots,D_{n_t-1}^{n_t})$ is a partition of $\mathcal{O}_x$ into open and closed subsets. Moreover for every $y\in\mathcal{O}_x$ and every $t\in N$ there is only one i, $0\leqslant i\leqslant n_t-1$ such that $y[-i+kn_t,-i+(k+1)n_t-1]=c_t$ or $\tilde{c}_t$ for every $k\in Z$.

Let $Q=(Q_0,Q_1)$ be the zero time partition of $\mathcal{O}_x$, i.e. $Q_i=\{y\in\mathcal{O}_x:\ y[0]=i\}$, $i=0,1$. The partition Q is a generator of $\mathcal{O}_x$ called the natural generator.

Denote

$$n_m^t=\lambda_t\cdot\dots\cdot\lambda_{t+m},\quad c_m^t=b^t\times\dots\times b^{t+m},\quad t,m\geqslant 0,$$

$$n_m^0=n_m,\quad c_m^0=c_m$$

From Definition 0.1 it follows that the sequences $x_t=b^t\times b^{t+1}\times\dots$, $t\geqslant 0$ are also Morse sequences. We denote $\mu_{x_t}=\mu_t$, c_t and $\tilde{c}_t$ will be called t-symbols.

Let

$$p_0^t=\mu_t(00)+\mu_t(11)=2\mu_t(00)$$

/8/

$$p_1^t=\mu_t(01)+\mu_t(10)=2\mu_t(01)$$

Following [Kw2] a representation of blocks $(b^t)_{t=0}^{\infty}$ of a Morse sequence x is called regular provided there exists $\varrho > 0$ such that

/9/ $\varrho < p_0^t,\ p_1^t < 1-\varrho \quad,\ t \geq 0$

If no confusion can arise we will say that $x=b^0\times b^1\times \dots$ is a regular Morse sequence.

<u>Remark 0.1</u> /[Kw2]/ If we assume $\sup_{i\in N} \lambda_i = \lambda < \infty$ then x is regular iff $\sup\{j\in N:(\exists\, t\in N)\, b^t=0\dots0,\dots,b^{t+j}=0\dots0\}<\infty$ and $\sup\{j\in N:(\exists\, t\in N)\, b^t=01\dots010,\dots,b^{t+j}=01\dots010\}<\infty$

If $x=b^0\times b^1\times \dots$ is a Morse sequence then

/10/ $L^2(\theta_x,\mu_x)=L_0 \oplus L_1$

where

$L_0=\{f\in L^2(\theta_x,\mu_x):\ f\sigma=f\}\ ,\ L_1=\{f\in L^2(\theta_x,\mu_x):\ f\sigma=-f\}$

The unitary operator $U_T:L^2(\theta_x,\mu_x)\circlearrowleft,\ U_T(f)=f\circ T$ has discrete spectrum with the group of eigenvalues $G\{n_t:t\geq 0\}$ on L_0 /[Ke1]/. Here $G\{n_t:t\geq 0\}$ denotes the subgroup of roots of unity generated by $\exp(2\pi i/n_t)$, $t\geq 0$. If there is no eigenvalue on L_1 then x is said to be continuous. Theorem 9 from [Ke1] decides whether given Morse sequence is continuous or not. In particular any regular Morse sequence is continuous and any Morse sequence $x=b^0\times b^1\times \dots$, λ_t are even, $t\geq 0$ is continuous.

THE RANK OF MORSE DYNAMICAL SYSTEMS

I. Introduction The rank of an automorphism has been introduced by Chacon in [Cha] , but some general properties of automorphisms with the rank one have been investigated earlier. In particular Baxter in [Ba] has proved that if an automorphism has the rank one then it has a simple spectrum. The converse theorem is not true. Del Junco in [Ju2] has shown that the well-known Morse sequence $x=01^{\times}01^{\times}\ldots$ has a simple spectrum but it has the rank two. The rank of an automorphism is an isomorphism invariant. Moreover, it is proved in [Or-Ru-We] that if an automorphism has finite rank then it is loosely Bernoulli. In [Nü] there is a proof of the fact that any Morse sequence is loosely Bernoulli. This and the above give us the problem to calculate the rank of an arbitrary Morse sequence.

From results [Kw2] it follows that the rank of an arbitrary Morse sequence is less or equal 2.

The main result of the Chapter is the following. If $x=b^{0}\times b^{1}\times\ldots$ is a regular Morse sequence and the set of blocks $\{b^{0},b^{1},\ldots\}$ is finite then x has the rank 2. However, there exist regular Morse sequences with the rank one. If x is a Kakutani sequence then x has the rank two iff it is regular.

We also consider the rank of substitutions of constant length on two symbols [Deh] . We show that if Θ is such nonperiodic substitution then either Θ has discrete spectrum and consequently the rank one or Θ is a continuous rank two substitution.

We also negatively answer del Junco's question about the following characterization rank one transformations.

A process (T,Q) has the rank one iff for every $\delta > 0$

/1/ there are infinitely many n and n-strings ξ_0 of positive probability such that $nP\{\xi : d(\xi_0, \xi)<\delta\}>1-\delta$.

We provide also some class of zero entropy, LB transformations with infinite rank.

II. Definitions Let $(X, \mathcal{B}, \mu)$ be a Lebesgue space and $\tau : X\circlearrowleft$ be an automorphism. A finite set $Q=\{Q_\gamma, \gamma\in\Gamma\}$ is called a semi-partition of X if $Q_\gamma \in \mathcal{B}$ and $Q_\gamma \cap Q_{\gamma'}=\emptyset$, $\gamma\neq\gamma'$. We put $|Q|=\bigcup_{\gamma\in\Gamma} Q_\gamma$. Then Q is a partition of X if $|Q|=X$.

Let Q be a partition of X. For $x\in X$ and $n\in N$ Q-n-name of x is the block $(\xi(0),\dots,\xi(n-1))\in\Gamma^n$ such that $\xi(i)=\gamma$ iff $\tau^i(x)\in Q_\gamma$. Every $\xi\in\Gamma^n$ will be called an n-string.

Remark 1.1 The process (τ, Q) induces a measure P on Γ^Z given by the formula

$$P(\xi)=\mu(\{x\in X: \text{Q-n-name of } x \text{ is } \xi\})$$

where $\xi\in\Gamma^n$.

Let Q,R be semi-partitions of X. We write $Q\leq R$ iff $|Q|=|R|$ and every atom of Q is a union of atoms of R.

For $Q=\{Q_\gamma : \gamma\in\Gamma\}$, $R=\{R_\gamma : \gamma\in\Gamma\}$ we define

/2/ $$\varrho(Q,R)=\sum_{\gamma\in\Gamma}\mu(Q_\gamma \vartriangle R_\gamma)$$

<u>Definition 1.1</u> A semi-partition $S=(S_0,\dots,S_{n-1})$ is called a τ-stack if $\tau S_i=S_{i+1}$, $i=0,\dots,n-2$. S_0 is called the base and n the height of S.

<u>Definition 1.2</u> An automorphism τ has the rank at most n / and we write $r(\tau)\leq n$ / if for every partition Q and $\delta>0$ there are semi-partitions Q' and R such that

/3/ $$\varrho(Q,Q')<\delta\ ,\quad Q'\leq R$$

and R is a disjoint union of n τ-stacks. τ has the rank n provided n is the smallest positive integer such that $r(\tau)\leq n$.

III. <u>A characterization the rank one dynamical systems.</u>

Let ξ_0 be an r-string and $\delta>0$.

<u>Definition 1.3</u> / [Ju2] / We say that an n-string η is a δ-ξ_0-string if η has a form

$$\eta=\varepsilon_0\,\xi_1\,\varepsilon_1\dots\xi_l\,\varepsilon_l \quad\text{and}\quad \xi_i=\xi_0,\quad i=1,\dots,l,\ \sum_0^l|\varepsilon_i|<\delta n$$

<u>Theorem 1.1</u> / [Ju2] / Let τ be an automorphism of a Lebesgue space $(X,\mathcal{B},\mu)$. Then $r(\tau)=1$ iff for each partition Q and $\delta>0$ there is a ξ_0-r-string and $M>0$ such that for all $n\geq M$

$$P(\{\xi\text{-n-string}:(\exists\eta\text{-}\delta\text{-}\xi_0\text{-string})\ d(\eta,\xi)<\delta\})>1-\delta.$$

Moreover, we can assume that $P(\xi_0)>0$ and r is arbitrarily large.

We will use a modification of Theorem 1.1. For this purpose we change Definition 1.3.

Definition 1.4 Let ξ_o be an r-string and $\varepsilon, \delta > 0$. We say that an n-string η has an ε-structure of a δ-ξ_o-string iff

/4/ $\eta = \varepsilon_o \xi_1 \varepsilon_1 \dots \xi_l \varepsilon_l$, $d(\xi_i, \xi_o) < \varepsilon, 1 \le i \le l, \sum_0^l |\varepsilon_i| < \delta n$

Remark that if η has a 0-structure of a δ-ξ_o-string then η is a δ-ξ_o-string.

Theorem 1.2 Let τ be an automorphism of a Lebesgue space. Then $r(\tau) = 1$ iff for each partition Q and $\delta > 0$ there is a ξ_o-r-string and $M > 0$ such that for all $n \geqslant M$

/5/ $P(\{\xi$-n-string: ξ has a δ-structure of a δ-ξ_o-string$\}) > 1 - \delta$.

Moreover, we may assume that $P(\xi_o) > 0$ and r is arbitrarily large.

Proof It is easy to verify that Theorem 1.1 is a corollary of Theorem 1.2. Conversely, we have an inclusion $\{\xi$-n-string: $(\exists \eta$-δ^2-ξ_o-string$) d(\eta, \xi) < \delta^2\} \subseteq$ $\subseteq \{\xi$-n-string: ξ has a 2δ-structure of a 2δ-ξ_o-string$\}$. This completes the proof.

Remark 1.2 Inequality $\bar{d}((\tau, Q), (\tau, R)) < \varrho(Q, R)/[O_r]$ / implies that $r(\tau) = 1$ iff the condition /5/ is valid for a sequence of generating partitions.

IV. Morse dynamical systems with the rank one. In this section we show that for every Morse sequence x, $r(x) \leq 2$. Moreover, we give a characterization of sequences with the rank one.

It is known that $\{C_t\}_{t=0}^{\infty}$ is a generating sequence of partitions for every Morse sequence $x=b^0 \times b^1 \times \ldots$ / [Kw2] / It is clear that for every t, C_t is a disjoint union of two T-stacks of the height n_t. Then by Remark 1.2

Theorem 1.3 If $x=b^0 \times b^1 \times \ldots$ is a Morse sequence them $r(x) \leq 2$.

Let A, B be strings and C be a block. Then by $\{A,B\} * C$ we mean a string obtained from C by substituting A instead of zero and B instead of one. It is clear that if A is a block then $\{A,\tilde{A}\} * C = A \times C$.

Denote $D_k^{n_t}(0) = C_k^t$, $D_k^{n_t}(1) = C_{n_t+k}^t$, $k=0,\ldots,n_t-1$ and we get $C_t = (C_0^t,\ldots,C_{2n_t-1}^t)$.

Remark 1.3 By the construction of the space $X(x)$ we obtain that Q-n-names are exactly the sectors of x and C_t-n-names are exactly the sectors of $\{(0,\ldots,n_t-1),(n_t,\ldots,2n_t-1)\} * x_{t+1}$.

Remark 1.4 Let $x=b^0 \times b^1 \times \ldots$ be a Morse sequence. For an arbitrary $t \in N$ there exist numbers $M_t, N_t \in N$ such that

/6/
$$M_t = \sup\{|B| : B=0\ldots0 \text{ and } B \text{ appears in } x_t\}$$
$$N_t = \sup\{|B| : B=01\ldots01 \text{ and } B \text{ appears in } x_t\}$$

Indeed, if for instance $M_{t_o}=\infty$ for some $t_o\in N$ then $c_m^{t_o}=0\ldots0$ for any m because x_{t_o} is an infinite concatenation of $c_m^{t_o}$ and $\tilde{c}_m^{t_o}$. Hence x_{t_o} is not a Morse sequence.

If a block $c_t\tilde{c}_t\tilde{c}_t$ / $\tilde{c}_t\tilde{c}_tc_t, c_tc_t\tilde{c}_t, \tilde{c}_tc_tc_t$ / appears in x at i then $n_t|i$. Indeed, if $n_t\nmid i$ then $c_t[0]=c_t[i]$ and $c_t[0]=\tilde{c}_t[i]$ for some $i>0$ what is impossible. From this and by /6/ we see that if $\omega=c_t{}^{\times}\nu$ with $|\nu|\geqslant\max(M_{t+1},N_{t+1})+1$ then $i=kn_t$ for some $k\in N$ whenever ω appears in x at i.

<u>Theorem 1.4</u> Let $x=b^0{}_\times b^1{}_\times\ldots$ be a Morse sequence. Then $r(x)=1$ iff

/7/ $(\forall\delta>0)\ (\exists\ \xi_0\text{-r-string})\ (\exists\ t\in N)\ [c_t \text{ and } \tilde{c}_t \text{ have a } \delta\text{-structure of } \delta\text{-}\xi_0\text{-string and } \mu_x(\xi_0)>0]$

<u>Proof</u> <u>Necessity</u> Applying Theorem 1.2 to the natural generator and by Remark1.3 we obtain that there is an r-string ξ_0 such that $\mu_x(\xi_0)>0$ and the measure of all m-strings / for m large enough / having a δ-structure of a δ-ξ_0-string is greater than $1-\delta$. Hence for t large enough there exists a ξ-n_t-string having a δ-structure of a δ-ξ_0-string such that so is $\tilde{\xi}$ / it is a consequence of the equality $\mu_x(\eta)=\mu_x(\tilde{\eta})$ for each block η /. Since $\mu_x(\xi)>0$ it holds

/8/ $\xi=c_tc_t[i,i+n_t-1]$ or $\xi=c_t\tilde{c}_t[i,i+n_t-1]$ or $\xi=\tilde{c}_tc_t[i,i+n_t-1]$ or $\xi=\tilde{c}_t\tilde{c}_t[i,i+n_t-1]$

It is sufficient to verify /8/ when $\xi=c_tc_t[i,i+n_t-1]$.

Notice that if ξ has a δ-structure of a δ-ξ_0-string then $\xi\xi$ is one.

Assume that $r/n_t < \delta$.

Then $c_t = \xi_k[p,r-1]\varepsilon_k \dots \xi_1 \varepsilon_1 \varepsilon_o \xi_1 \dots \varepsilon_{s-1}\xi_s[0,u]$ where $0 \leq p,u \leq r-1$.

Denoting $\varepsilon'_o = \xi_k[p,r-1]\varepsilon_k$, $\varepsilon'_{l-k} = \varepsilon_l \varepsilon_o$, $\varepsilon'_{s+l-k} = \varepsilon_{s-1}\xi_s[0,u]$ we get a representation of c_t as follows

$$c_t = \varepsilon'_o \xi'_1 \dots \varepsilon'_{s+l-k} \xi'_{s+l-k}$$

Hence c_t has a 4δ-structure of a 4δ-ξ_o-string.

Now, we repeat the above reasoning for $\tilde{c}_t$ and we obtain /7/.

<u>Sufficiency</u> First we prove that if $x = b^0 x\, b^1 x \dots$ satisfies /7/ then $x_1 = b^1 x\, b^2 x \dots$ satisfies /7/.

Let $k = \max(N_1, M_1) + 1$ and δ satisfies $0 < \delta < 1/2kn_o$. Then there is an r-string ξ_o such that $\mu_x(\xi_o) > 0$ and c_t and $\tilde{c}_t$ have a $\delta/2$-structure of a $\delta/2$-ξ_o-string. Assume that $2\lambda_o/r < \delta/2$.

Trowing away no more than $2\lambda_o$ symbols on the left and on the right of ξ_o we can assume that $\xi_o = b^0 x\, \xi'_o$ and c_t and $\tilde{c}_t$ have a δ-structure of a δ-ξ_o-string. Thus

$$/9/ \quad c_t = \varepsilon_o \xi_1 \varepsilon_1 \dots \xi_l \varepsilon_l, \quad \sum_0^l |\varepsilon_i| < \delta n_t, \quad d(\xi_i, \xi_o) < \delta, \quad 1 \leq i \leq l.$$

We divide the block ξ_1 in sequence of blocks of the lengths λ_o. Next, we partition these blocks into succesive groups such that each group has k blocks, except may be the last group. In the same way we can divide and partition the block ξ_o. Then there is a group of k blocks in ξ_1 equal to the corresponding group in ξ_o. Indeed, in the other case

$$d(\xi_o,\xi_1) \geq \frac{1}{r}\left(\frac{r}{kn_o}-1\right)=\frac{1}{kn_o}-\frac{1}{r} > \frac{1}{2kn_o}$$

and we get a contradiction to /9/. By assumption the above group is a concatenation of O-symbols. In addition n_o divides $|\varepsilon_o|+1$.

In the same way we see that ξ_i is a concatenation of O-symbols and n_o divides $|\varepsilon_o\xi_1\varepsilon_1\ldots\xi_{i-1}\varepsilon_{i-1}|+1$, $i=2,\ldots,l$. Then $\xi_i=b^{O}\times\xi'_i$ and $\varepsilon_i=b^{O}\times\varepsilon'_i$ or $|\varepsilon_i|=0$, $i=1,\ldots,l$.

By /9/ it follows that $c^1_t=\varepsilon'_o\xi'_1\ldots\xi'_l\varepsilon'_l$, $\sum_0^l|\varepsilon'_i|<\delta n^1_t$, $d(\xi'_o,\xi'_i)<\delta$, $i=1,\ldots,l$.

Similarly $\tilde{c}^1_t$ has a δ-structure of a δ-ξ'_o-string.

Repeating the above considerations we obtain that for each $t\in N$ the Morse sequence x_t satisfies /7/.

Now, we show that if the condition /7/ is valid for all x_t then the condition /5/ holds.

Since $\{C_t\}_{t\geq 0}$ is a generating sequence it remains to verify this condition for C_t, $t\geq 0$ / see Remark 1.2 /.

Let $t\in N$ and $\delta>0$. By /7/ there is $K\in N$ and an r-string ξ_o such that c^{t+1}_K and $\tilde{c}^{t+1}_K$ have a δ-structure of a δ-ξ_o-string. In addition one may assume

$$r/n^{t+1}_K<\delta \quad\text{and}\quad 2<\delta(n^{t+1}_K+2)$$

It is easy to see that every block of the length n^{t+1}_K contained in x_{t+1} has a 4δ-structure of a 4δ-ξ_o-string Denote $\eta_o=\{(0,1,\ldots,n_t-1),(n_t,n_t+1,\ldots,2n_t-1)\}*\xi_o$. According to Remark 1.3 we obtain C_t-$n_t(n^{t+1}_K+2)$-names have a 5δ-structure of 5δ-ξ_o-string. It makes our proof complete.

Remark 1.5 From Theorem 1.2 it follows that the condition /7/ one can formulate as follows

/10/ $(\forall \delta>0)(\forall M\in N)(\exists \xi_o\text{-r-string},\ r\geqslant M)(\exists t_o\in N)(\forall t\geqslant t_o)$

$[c_t$ and $\tilde{c}_t$ have a δ-structure of a δ-ξ_o-string, $\mu_x(\xi_o)>0]$

Corollary 1.1 $r(x)=1$ iff for every $t\in N$, $r(x_t)=1$.

Corollary 1.2 $r(x)=1$ iff there exists a representation $x=\beta^0\times\beta^1\times\ldots$ such that

/11/ $(\forall\delta>0)(\exists\xi_o\text{-r-string})(\exists t\in N)$ $[\beta^t$ and $\tilde{\beta}^t$ have a δ-structure of a δ-ξ_o-string and $\mu_x(\xi_o)>0]$.

Proof Let $x=b^0\times b^1\times\ldots$ and assume that $r(x)=1$. Let $\{\delta_n\}_{n\geqslant 0}$ be a sequence of real numbers such that $\delta_n\searrow 0$. By induction we define a sequence of blocks $\{\beta^i\}_{i\geqslant 0}$. First, we use /7/ for δ_o and x. As a consequence we get t_o such that c_{t_o} and $\tilde{c}_{t_o}$ have a δ_o-structure of a δ_o-ξ_o-string. We set $\beta^0=c_{t_o}$. Next using /7/ for δ_1 and x_{t_o+1} we have t_1 and ξ_1 such that $c^{t_o+1}_{t_1}$ and $\tilde{c}^{t_o+1}_{t_1}$ have a δ_1-structure of a δ_1-ξ_1-string. Putting $\beta^1=c^{t_o+1}_{t_1}$ we can again use /7/ for δ_2 and $x_{t_o+t_1+1}$. Proceeding in the same way we obtain a sequence of blocks $\{\beta^i\}_{i\geqslant 0}$ such that $x=\beta^0\times\beta^1\times\ldots$ and this representation satisfies /11/.

Example 1.1 There are regular Morse shifts with the rank one.

Let $x=b^0\times b^1\times\ldots$ where $b^i=\beta^i 01$, $\beta^i=00\times\overbrace{01\ldots 01}^{2i}$. It is clear x is a regular Morse shift. By Corollary 1.2 $r(x)=1$.

<u>Example 1.2</u> The rank of nonregular Kakutani sequences.

Let $x=b^0\times b^1\times\ldots$ be a Kakutani sequence such that

$$\sup\{k\in N: (\exists\, i\in N)\ b^i=b^{i+1}=\ldots=b^{i+k}=00\}=\infty$$

Then $r(x)=1$. Indeed, we can determine the groups $\{b^0,b^1,\ldots,b^{i_1}\}$, $\{b^{i_1+1},b^{i_1+2},\ldots,b^{i_2}\},\ldots$ such that $b^{i_k}=00$, $b^{i_k+1}=01,\ldots,b^{i_{k+1}}=00$, $k=1,2,\ldots$. Putting $\beta^k=b^{i_k}\times\ldots\times b^{i_{k+1}}$, $k\geqslant 1$, $\beta^0=b^0\times\ldots\ b^{i_1-1}$ we have a representation $x=\beta^0\times\beta^1\times\ldots$ that satisfies /11/.

V. <u>The rank of regular Morse sequences</u> In the following we shall assume that a Morse sequence has a form

/12/ $\quad x=b^0\times b^1\times\ldots\ ,\ \sup\limits_{i\in N}\lambda_i=\lambda<\infty$ where $\lambda_i=|b^i|$.

In this section we prove that a dynamical system given by a regular Morse sequence satisfying /12/ has the rank 2. We show that a Kakutani sequence has the rank 2 iff it is regular.

Put $\quad K=\max\{\sup\limits_{t\in N}M_t\ ,\ \sup\limits_{t\in N}N_t\}$

Remark 0.1 asserts that x is regular iff $K\in N$.

<u>Lemma 1.1</u> x is regular iff

/13/ $(\exists\,\delta>0)(\exists\, L>0)(\forall\,\eta\text{-string})(\forall\, t\in N)$[if η is a concatenation of L t-symbols and η appears in x at i within δ then n_t divides i and η appears in x at i].

<u>Proof</u> Let x be a regular Morse sequence. We shall show that /13/ holds for $L=K+1$ and $\delta=1/\lambda L$. The proof goes by induction on t.

Let t=0. Assume that $\eta=b^0 \times \mu$ and $|\mu|=L$. It is obvious that $d(\eta,x[i,i+Ln_0-1])=0$ or $d(\eta,x[i,i+Ln_0-1])\geqslant\frac{1}{\lambda L}$

Assume /13/ is valid for some $t\in N$. Let $\xi_1,\ldots,\xi_L$ be (t+1)-symbols and let $\eta=\xi_1\xi_2\ldots\xi_L$ appears in x at i within δ. It is clear that $\xi_j=\xi'_j \times b^{t+1}$, $j=1,\ldots,L$, where ξ'_j are t-symbols. Therefore the string η is a concatenation of $\lambda_{t+1}L$ t-symbols. Dividing succesively η on λ_{t+1} groups of t-symbols in such a way that each group has L t-symbols we obtain strings $\eta_1,\ldots,\eta_{t+1}$. If $d(\eta_j,x[i+\sum_1^{j-1}|\eta_s|,i+\sum_1^{j}|\eta_s|])\geqslant\delta$ for each $j=1,\ldots,\lambda_{t+1}$ then

$$d(\eta,x[i,i+Ln_{t+1}-1])=\frac{1}{\lambda_{t+1}}\sum_1^{\lambda_{t+1}} d(\eta_j,x[\ldots,\ldots])\geqslant\delta$$

Thus there is j, $1\leqslant j\leqslant\lambda_{t+1}$ such that η_j appears in x within δ. Then by the induction assumption n_t divides the number $i+\sum_1^{j-1}|\eta_s|$ and hence n_t divides i. Therefore we have if $d(\eta,x[i,i+Ln_{t+1}-1])\neq 0$ then $d(\eta,x[i,i+Ln_{t+1}-1])\geqslant$ $\geqslant 1/L\lambda_{t+1}\geqslant 1/\lambda L$. Since $\delta=1/\lambda L$ we get $d(\eta,x[i,i+Ln_{t+1}-1])=0$. By Remark 1.4 n_{t+1} divides i and necessity is established.

Sufficiency of the condition /13/ is a simple consequence of Remark 0.1.

<u>Lemma 1.2</u> Let $x=b^0\times b^1\times\ldots$ satisfies /12/. If x is regular Morse sequence and $r(x)=1$ then

/14/ $(\exists\,\xi\text{-string})(\forall m\in N)(\exists\, t\geqslant m)(\exists\, i\in N)\; x_t[i,i+m|\xi|-1]=\underbrace{\xi\xi\ldots\xi}_{m}$

<u>Proof</u> Let $m\in N$ and we choose $t'_0\in N$ such that $t'_0\geqslant m$. We define r' from the condition

/15/ $\qquad L+1 < r'/n_{t'_o} < \lambda(L+1)$

where L is defined by Lemma 1.1.

From Lemma 1.1 there exists a positive δ' satisfying /13/.

Let

/16/ $\qquad 0 < \delta < \min(1/(4\lambda(L+1)m+1), \delta'/2)$

Using Theorem 1.2 and Remark 1.5 for δ and r' we get a ξ_o-r-string and $t \in N$ such that

/17/ $\qquad r \geqslant r', \ t \geqslant \max(t'_o, \log_2((m+1)r/(1-\delta)) - 1)$

/18/ $c_t = \varepsilon_o \xi_o^1 \varepsilon_1 \ldots \xi_o^l \varepsilon_l, \ \sum_0^l |\varepsilon_i| < \delta n_t, \ d(\xi_o^i, \xi_o) < \delta, i=1,\ldots,l$

and $\tilde{c}_t$ has also a δ-structure of a δ-ξ_o-string. Using /15/ and /16/ we shall define $t_o \geqslant t'_o$ satisfying the condition

/19/ $\qquad L+1 \leqslant r/n_{t_o} \leqslant \lambda(L+1)$

If $r/n_{t'_o} \leqslant \lambda(L+1)$ then $t_o = t'_o$. If $r/n_{t'_o} > \lambda(L+1)$ then $r/n_{t'_o+1} > \lambda(L+1)/\lambda_{t'_o+1} > \lambda(L+1)/\lambda = L+1$. If $r/n_{t'_o+1} \leqslant \lambda(L+1)$ then $t_o = t'_o + 1$. Otherwise we repeat the above considerations for $t_o = t'_o + 2$ and in this manner we obtain t_o satisfying /19/.

Now, we show that there exists a set $\{i, i+1, \ldots, i+m\}$ such that for every j, $i \leqslant j \leqslant i+m$ it holds

/20/ $\qquad |\varepsilon_j| < \frac{1}{2} n_{t_o}$

Suppose that each sequence $\{\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_{i+m}\}, 1 \leqslant i \leqslant l-m$ has an ε_j with $|\varepsilon_j| > \frac{1}{2} n_{t_o}$, $j=i,\ldots,i+m$.

By /17/ we have

$$[l/(m+1)] \geqslant 1$$

We then obtain $\quad \sum_0^l |\varepsilon_i| \geqslant [l/(m+1)] \, n_{t_o}/2$

Hence

$$\delta n_t \geqslant [l/(m+1)] \, n_{t_o}/2$$

and next

$$[1/(m+1)]\,|\xi_o|/2 \leq [1/(m+1)]\,\lambda(L+1)\,n_{t_o}/2 \leq \lambda(L+1)\,\delta\,n_t$$

Further we have

$$[1/(m+1)]\; r/2 \leq \lambda(L+1)\,\delta\,n_t$$

By /18/

$$lr \geqslant (1-\delta)\,n_t$$

Then

$$[1/(m+1)]\,(1-\delta)\,n_t/2l \leqslant \lambda(L+1)\,\delta\,n_t$$

what implies

$$(1-\delta)/\delta \leqslant 2l\,\lambda(L+1)/[1/(m+1)] \leqslant 2l\,\lambda(L+1)2m/l \leqslant$$
$$\leqslant 4m\,\lambda(L+1)$$

The later inequality is impossible in view of /16/. Then there exists a set $\{\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_{i+m}\}$ such that for every j, $i \leqslant j \leqslant i+m$ /20/ is valid.

By /19/ $\xi_o = w_1 \eta_o w_2$ where $\eta_o = c_{t_o}^{\times}/\mu_o$, $L \leqslant |\mu_o| < \lambda(L+1)$ and w_1 is the end of a t_o-symbol and w_2 is the beginning of a t_o-symbol and

/21/ $\quad |w_1| + |w_2| < 2n_{t_o}$

Now, we shall construct ξ_m such that

/22/ $\quad x_{t_o+1}[s, s+m|\xi_m|-1] = \underbrace{\xi_m \xi_m \cdots \xi_m}_{m}$

for some $s \in N$.

The following cases are possible

1^o $|w_1| + |w_2| < \frac{1}{2}n_{t_o}$, 2^o $\frac{1}{2}n_{t_o} \leqslant |w_1| + |w_2| < \frac{3}{2}n_{t_o}$,

3^o $\frac{3}{2}n_{t_o} \leqslant |w_1| + |w_2| < 2n_{t_o}$

<u>Case 1</u> Let $j \in \{i, \ldots, i+m-1\}$. Then by /20/ $|\varepsilon_{j+1}| < \frac{1}{2}n_{t_o}$ and by Lemma 1.1 and /16/ we obtain

$$\xi_o^j = w_1^j \eta_o w_2^j \text{ where } |w_k^j| = |w_k|,\ k=1,2,\ j=i,\ldots,i+m-1.$$

From Lemma 1.1 it follows that η_o appears in x only in places being a multiple of n_{t_o}. Then

$$|w_2^j|+|w_1^{j+1}|+|\varepsilon_j|<n_{t_o}$$

what implies

$$|\varepsilon_j|=|w_2^j|=|w_1^{j+1}|=0,\ j=i,\dots,i+m-1$$

so $\xi_o=\eta_o$

We set $\xi_m=\mu_o$.

Case 2 Let $j\in\{i,\dots,i+m-1\}$. Then

$$|w_2^j|+|w_1^{j+1}|+|\varepsilon_j|<2n_{t_o} \quad \text{and therefore}$$

$$|w_2^j|+|w_1^{j+1}|+|\varepsilon_j|=n_{t_o},\ j=i,\dots,i+m-1.$$

Let us suppose that for some j, $i\leqslant j\leqslant i+m-2$

$$w_2^j\,\varepsilon_j w_1^{j+1}=c_{t_o} \quad \text{and} \quad w_2^{j+1}\,\varepsilon_{j+1}w_1^{j+2}=\tilde{c}_{t_o}$$

If for instance $|w_2|>\frac{1}{4}n_{t_o}$ then

$d(\xi_o,\xi_o^j)\geqslant\frac{1}{4}n_{t_o}/\lambda(L+1)\,n_{t_o}=1/4\,\lambda(L+1)$ or $d(\xi_o,\xi_o^{j+1})\geqslant 1/4\,\lambda(L+1)$

But it is a contradiction to /16/.

If $w_2^j\,\varepsilon_j w_1^{j+1}=c_{t_o}$ for $j=i,\dots,i+m-1$ then we set $\xi_m=\mu_o^0$

Otherwise $\xi_m=\mu_o^1$.

Case 3 We have $\frac{3}{2}n_{t_o}<|w_2^j|+|w_1^{j+1}|+|\varepsilon_j|<\frac{5}{2}n_{t_o}$,

$j=i,\dots,i+m-1$. Hence

$$|w_2^j|+|w_1^{j+1}|+|\varepsilon_j|=2n_{t_o}$$

Let $w_2^j\,\varepsilon_j w_1^{j+1}=c_{t_o}\times\nu_j$, $w_2^{j+1}\,\varepsilon_{j+1}w_1^{j+2}=c_{t_o}\times\nu_{j+1}$

where $|\nu_j|=|\nu_{j+1}|=2$ for some $j=i,\dots,i+m-2$.

As in Case 2 we show that $\nu_j=\nu_{j+1}=\nu$ and we set

$\xi_m=\mu_o^\nu$.

Let us observe that in each of the cases $1^0,2^0,3^0$

$|\xi_m|\leqslant|\mu_o|+2\leqslant\lambda(L+1)+2$

We can repeat the above considerations for infinitely many of m. As a consequence we get an infinite set of positive integers and strings ξ_m. Since $\{|\xi_m|\}$ is bounded we can find a common string ξ satisfying /14/. In addition the string ξ satisfies

/23/ $\xi \leq \lambda(L+1)+2$.

In the sequel we will need some properties concerning blocks contained in [Kw2].

Let A be a block / we assume that every block starts with 0 /

<u>Definition 1.5</u> A number i is called

/a/ a period of A iff $A=AA[i,i+|A|-1]$,

/b/ a mirror period of A iff 2i is aperiod of $A\tilde{A}$.

The smallest period of A we denote by $p(A)$ and the smallest mirror period of A by $m(A)$.

Results of [Kw2] follow. If A,B are blocks then

$$/24/ \qquad p(A^{*}B)=\begin{cases} p(A) & \text{if } B=0\ldots0 \\ 2m(A) & \text{if } B=01\ldots01 \\ p(B)\cdot|A| & \text{otherwise} \end{cases}$$

If A appears in $\tilde{A}\tilde{A}$ at j, $1 \leq j \leq |A|-1$ then there exists a block B such that A has a form

$$/25/ \qquad A=B\tilde{B}\ldots B\tilde{B} \quad \text{and} \quad |B|=j.$$

In addition if $j < p(A)$ then

/26/ 2j is the smallest period of A.

As a simple consequence of /24/ and /26/ we have

<u>Lemma 1.3</u> Let A,B be blocks. Assume that

$$/27/ \qquad A^{*}B=C\tilde{C}\ldots C\tilde{C}, \quad |C| < |A|.$$

Then B=0...0 or B=01...01.

<u>Theorem 1.5</u> Let $x=b^{0_x}b^{1_x}\ldots$ be a regular Morse sequence with /12/. Then $r(x)=2$.

<u>Proof</u> Let us assume that $r(x)=1$. Let ξ be an r-string defined in Lemma 1.2 and L be the number defined in Lemma 1.1.

Fix $m_0 \in N$ and put $m=m_0$. From Lemma 1.2 it follows that there is $t_m \geqslant m$ such that $x_{t_m}[i,i+m|\xi|-1]=\underbrace{\xi\ldots\xi}_{m}$

/23/ follows $|\xi|\leqslant\lambda(L+1)+2$

Let

/28/ $$t^{(m)}=\left[\log_\lambda\left(m|\xi|/(L+4)\right)-1\right]$$

For each t, $t\leqslant\log_\lambda(m/(L+4))-1$ it holds

$m|\xi|-3\lambda^{t+1}/\lambda^{t+1}\geqslant L+1$ so $\left[m|\xi|-2n_t^{t_m}/n_t^{t_m}\right]\geqslant L+1$.

The later inequality means one of the following blocks

/29/ $c_t^{t_m}\tilde{c}_t^{t_m}\tilde{c}_t^{t_m}$, $\tilde{c}_t^{t_m}\tilde{c}_t^{t_m}c_t^{t_m}$, $c_t^{t_m}c_t^{t_m}\tilde{c}_t^{t_m}$, $\tilde{c}_t^{t_m}c_t^{t_m}c_t^{t_m}$

appears in x_{t_m} between i and $i+m|\xi|-1$.

Then for $t=t^{(m)}$ /29/ is satisfied.

From /28/ it follows that

/30/ $$\lim_{m\to\infty} t^{(m)}=\infty$$

Suppose that the block $c_t^{t_m}\tilde{c}_t^{t_m}\tilde{c}_t^{t_m}$ / with $t=t^{(m)}$ / appears in x_{t_m} between i and $i+m|\xi|-1$. Then

$$c_t^{t_m}=\xi[s_1,|\xi|-1]\,\xi\ldots\xi\,\xi[0,s_2]$$

Moreover

$$\tilde{c}_t^{t_m}\tilde{c}_t^{t_m}=\xi[\bar{s}_1,|\xi|-1]\,\xi\ldots\xi\,\xi\,[0,\bar{s}_2]$$

where $0\leqslant s_1,s_1,s_2,s_2\leqslant|\xi|-1$.

Hence

/31/ , $c_t^{t_m}=\tilde{c}_t^{t_m}\tilde{c}_t^{t_m}[k,k+n_t^{t_m}-1]$ for some $k\in N$.

Now, we shall use Lemma 1.3. Assume that k is the smallest natural number satisfying /31/. Then

/32/ $\quad k \leqslant 2|\xi|$

Let $t_1^{(m)} \in N$ be such that

/33/ $\quad k < n_{t_1^{(m)}}^{t_m} < 2\lambda|\xi|$

Put
$$A = c_{t_1^{(m)}}^{t_m}, \quad B = b^{t_1^{(m)}+t_m+1} \times \dots \times b^{t+t_m}$$

Lemma 1.3 follows
$$b^{t_1^{(m)}+t_m+1} = 01\dots010, \dots, b^{t+t_m-1} = 01\dots010, b^{t+t_m} = 01\dots01$$
or
$$b^{t_1^{(m)}+t_m+1} = 01\dots01, b^{t_1^{(m)}+t_m+2} = 0\dots0, \dots, b^{t+t_m} = 0\dots0$$
or
$$b^{t_1^{(m)}+t_m+1} = 0\dots0, \dots, b^{t+t_m} = 0\dots0 .$$

We have
$$x = c_{t_1^{(m_0)}+t_{m_0}} \times \overbrace{b^{t_1^{(m_0)}+t_{m_0}+1} \dots b^{t^{(m_0)}+t_{m_0}}}^{\vartheta_{m_0}} \times x_{t_{m_0}+t^{(m_0)}+1}$$

We repeat the above procedure putting
$$x = x_{t_{m_0}+t^{(m_0)}+1}, \quad m = 2m_0$$

Then
$$x = c_{t_1^{(m_0)}+t_{m_0}} \times \overbrace{b^{t_1^{(m_0)}+t_{m_0}+1} \times \dots \times b^{t^{(m_0)}+t_{m_0}}}^{\vartheta_{m_0}} \times c_{t_{2m_0}}^{t^{(m_0)}+t_{m_0}+1+t_1^{(2m_0)}} \times$$
$$\times \overbrace{b^{t^{(m_0)}+t_1^{(2m_0)}+t_{m_0}+t_{2m_0}+2} \times \dots \times b^{t^{(m_0)}+t^{(2m_0)}+t_{m_0}+t_{2m_0}+1}}^{\vartheta_{2m_0}}$$
$$\times x_{t^{(m_0)}+t^{(2m_0)}+t_{m_0}+t_{2m_0}+2}$$

We call this procedure infinitely many times by the above schema. By /30/ and /33/ $\lim_{s\to\infty} \vartheta_{sm_0} = \infty$. Hence x is not regular. This completes the proof of the Theorem.

From the above Theorem and Example 1.2 we get the following.

Corollary 1.3 A Kakutani sequence x is of the rank one iff it is not regular.

VI. Applications As the main application we will show that our considerations thorougly solve the problem of the rank in the class of nonperiodic substitutions of constant length on two symbols. For definition and notation we refer to [Dek] .

There are two kinds of them

/i/ continuous substitution if $\theta(0)=\widetilde{\theta(1)}$ or equivalently if $c(\theta)=2$

/i/ discrete substitution otherwise / $c(\theta)=1$ /

It is not difficult to observe that if $b=\theta^2(0)$ and $x=b\times b\times\ldots$, then x is a Morse sequence and the dynamical system arising from θ is isomorphic to the Morse sequence x. In addition if $c(\theta)=1$ then θ has discrete spectrum, so from the result [Ju 1] it follows that θ has the rank one.

Corollary 1.4 Let θ be a nonperiodic substitution of the constant length on two symbols. Then $r(\theta)=c(\theta)$.

Example 1.3 The answer to Del Junco's question [Ju 2].
Put $b=00110011$, $\lambda=|b|$ and let $x=b\times b\times\ldots$ be the substitution determined by b. Applying Lemma 1.1 we see that for $t_0\geq 0$, $fr(c_{t_0},c_t)=fr(b,c_{t-t_0+1})$ for t large enough. Therefore

$\frac{1}{\lambda^t}\, fr(c_{t_o}, c_t) = \frac{1}{\lambda^{t_o-1}}\, \frac{1}{\lambda^{t-t_o+1}}\, fr(b, c_{t-t_o+1})$ and letting $t \to \infty$ we get

/34/ $\mu_x(c_{t_o}) = \frac{1}{\lambda^{t_o-1}}\, \mu_x(b)$.

Now, we compute $\mu_x(b)$.

Let us observe that $fr(b, b*b) = fr(b, \widetilde{b*b}) = 10$ and $x = b*b*x$, so

$fr(b, c_t) = fr(b, b*b*c_{t-2}) = \lambda^{t-2} \cdot 10 + 2fr(01, c_{t-2}) + fr(10, c_{t-2})$

Hence

$\mu_x(b) = \frac{10}{64} + 3\mu_x(01) = \frac{10}{64} + \frac{3}{64}\,\frac{2}{9} = \frac{1}{6}$

From /34/

/35/ $\mu_x(c_{t_o}) = \frac{8}{6}\; 1/|c_{t_o}|$ for every $t_o \geq 0$

In this way we obtain a process (T, Q)/ Q is the natural generator / satisfying /1/ which is not rank one. This answers Del Junco's question [Ju2] whether any process satisfying /1/ is of rank one.

<u>Example 1.4</u> Examples of infinite rank, LB shift with zero entropy.

Let $x = 01*01*\ldots$ and let T be the shift on $\mathcal{O}_x$.

The following properties can be deduced from [Ju2].

I: $(\forall n \geq 8)(\forall \xi : |\xi| = n)\quad \mu_x(\xi) \leq \frac{11}{12}\,\frac{1}{n}$

II: $(\exists \delta_o > 0)\ (\forall\ 8 \leq L \leq 15)(\forall t \geq 0)$ [if $\xi = c_t * \eta$, $|\eta| = L$ appears in x at i within δ_o then i is a multiple of n_t and ξ appears in x at i

III: / Some criterion to be rank m transformation /

$r(\tau) \leq m$ iff $(\forall \delta > 0)(\forall\ Q\text{-partition})(\exists\ \xi_1, \ldots, \xi_m\text{-strings})(\exists\ M > 0)$ $(\forall n \geq M)\ P\{\xi\text{-n-string} : \xi$ has a δ-structure of a

δ-$\xi_1,\ldots,\xi_m$-string$\}>1-\delta$

$/\eta=\varepsilon_0\xi_1\varepsilon_1\ldots\xi_l\varepsilon_l$ has a δ-structure of a δ-$\xi_1\ldots\xi_m$--string if $d(\xi_i,\xi_{k_i})<\delta$ for i, $1\le i\le l$ and some $1\le k_i\le m$ and $\sum_0^l|\varepsilon_i|<\delta n/$

Let us now take $q\in N$, $(q,2)=1$ and denote by (Z_q,ρ_q,μ_q) the group of integer mod q equipped with uniforme measure and $\rho_q(i)=i+1$ /mod q/

We will sketch that $r(\rho_q\times T)>q$. Let us suppose on the contrary, $r(\rho_q\times T)\le q$ and put / for definiteness / $r(\rho_q\times T)=q$. Take $\hat{Q}=Q\times\{0,1,\ldots q-1\}$ and denote $(Q_k,i)=\hat{Q}_{2i+k}$ $0\le i\le q-1$, $k=0,1$. Let $\delta>0$ be so small as we need. From III we have $\xi_1,\ldots,\xi_q$. There must exist $y\in X(x)$ and $M>0$ such that all $(y,2i)[0,n]$, $0\le i\le q-1$ are approximated by $\xi_1,\ldots,\xi_q$ for all $n\ge M$. We observe that if ξ_k appears in $(y,2i)$ at j within δ then ξ_k cannot appear in another $(y,2l)$, $l\ne i$ at j within δ. Moreover, we see that the sum of places of occuring of all ξ_i's $i=1,\ldots,q$ on each $(y,2i)[0,n]$ is within δ from n, so there is ξ_{i_0} such that the sum of places of occuring ξ_{i_0} on all $(y,2i)[0,n]$ is within δ from n. Going back to Q and y we get that there is $\bar{\xi}_{i_0}$ such that it appears in $y[0,n]$ within δ and the sum of places is within δ from n. According to II we can assume $\bar{\xi}_{i_0}$ appears exactly and using ergodic theorem / for possibly larger n / we have $\mu_x(\bar{\xi}_{i_0})>(1-\delta)/(\bar{\xi}_{i_0})$ and a contradiction to I.

Let $y=\beta^0\times\beta^1\times\ldots$ be another Morse sequence and $(n_t,2)=1$ $t\ge 0$. Then from the above easily follows that the product of corresponding shifts is ergodic and of infinite rank. Moreover, from [Or-Ru-We] it is not difficult to see that finite product of Morse sequences /i.e. of corresponding shifts / is LB whenever it is ergodic.

THE CENTRALIZER OF MORSE SHIFTS

I. <u>Introduction</u> Let $(X, \mathcal{B}, \mu)$ be a Lebesgue space and T an invertible transformation of $(X, \mathcal{B}, \mu)$. By $C(T)$ we mean the centralizer of T i.e. the group of all automorphisms S of (X, μ) with $ST=TS$.

The centralizer is an important invariant in ergodic theory. It can state some ergodic properties of T. In particular, knowing $C(T)$ we can usually answer whether T has roots or T is embeddable in measurable flows. Moreover, if P is a finite generator of T then SP, $S \in C(T)$ are the only generators with the same finite distributions as P.

There are some direct reasons to compute the centralizers of Morse shifts. As we shall see in Section IV the property to have an uncountable centralizer is a typical one in the class of all automorphisms acting in a fixed Lebesgue space. On the other hand examples of automorphisms with the trivial centralizer i.e. $C(T)=\{T^i: i \in Z\}$ are well-known / mixing rank one, minimal self-joining automorphisms [Ru], [Ju 3] /. Our main theorem / Theorem 2.1 / provides a large class of automorphisms with countable but not trivial centralizer.

Consider Morse dynamical systems as examples in topological dynamic /[Me 1]/. We see that their topological properties are usually common for all Morse sequences /[Ma1],[Gov-Ke]/. In particular the group of all homeomorphisms of $\mathcal{O}_x$ commuting with the shift, $C^{top}(x)$, is equal to $\{T^i\sigma^j: i\in Z,\ j=0,1\}$. It is interesting to known whether $C^{top}(x)=C(x)$ or not. Rather surprisingly it turns out that in our class the answer can be negative as well as positive.

For the class of all ergodic automorphisms with $\exp(2\pi i/n_t)$ in the point spectrum we introduce some number $d^{\{n_t\}}(\tau)$ and prove that if it is finite then τ is coalescent.

II. <u>Coalescence</u> Let $(X,\mathcal{B},\mu)$ be a Lebesgue space. We say an automorphism $\tau: X\circlearrowleft$ is coalescent if every endomorphism of (X,μ) commuting with τ is necessarily invertible [Ne].

Consider the class of all ergodic automorphisms τ of $(X,\mathcal{B},\mu)$ for which $Sp(\tau)\subset G\{n_t: t\geqslant 0\}$ where $Sp(\tau)$ is the group of all eigenvalues of unitary operator U_τ defined in the following way $U_\tau(f)=f\circ\tau$. Here $G\{n_t: t\geqslant 0\}$ denotes the group generated by $\{\exp(2\pi i/n_t)\}$, $n_t=\lambda_0\cdot\ldots\cdot\lambda_t$, $\lambda_t\geqslant 2$, $t\geqslant 0$. Let us notice that $\exp(2\pi i/n_t)\in Sp(\tau)$ iff there is a n_t-stack for τ i.e. a partition $\{A,\tau A,\ldots,\tau^{n_t-1}A\}$ of X /[Bl-Fr]/. Moreover, it is not difficult to verify that ergodicity of τ implies that there is only one / reordering if necessary elements of another n_t-stack / n_t-stack for τ, so we denote it by $D^{n_t}=\{D_0^{n_t},\dots,D_{n_t-1}^{n_t}\}$. In addition, if $n_t | n_{t+1}$ then $D^{n_t}\leq D^{n_{t+1}}$ so for our τ we get a sequence of partitions $D^{n_0}\leq D^{n_1}\leq\dots$. Let $D=\{D_i\}_{i\in I}$ be the limit partition. We assert card D_i is a constant number for all $i\in I$ a.e.μ /i.e. either card $D_i=\infty$, $i\in I$ or card $D_i=m$ for some natural m /. Indeed, D is τ-invariant and measurable, so our claim easily follows from [Ab-Roh] Put $d^{\{n_t\}}(\tau)=$card D_i, $i\in I$. Let us observe that $d^{\{n_t\}}(\tau)$ is an invariant of isomorphy.

<u>Proposition 2.1</u> If $d^{\{n_t\}}(\tau)$ is finite then τ is coalescent.

<u>Proof</u> Let $\tau:(X,\mathcal{B},\mu)\circlearrowleft$ and ξ be any τ-invariant and measurable partition of X and let $f:X\longrightarrow X/\xi$ be canonical map. It is sufficient to show /[Kam] / that if $(\tau,X,\mathcal{B},\mu)$ and $(\tau/\xi, X/\xi, \mathcal{B}/\xi, \mu/\xi)$. are isomorphic then ξ is equal to the partition into points.

So, let us suppose it. Thus there is the sequence $\{\bar{D}^{n_t}\}\nearrow\bar{D}$ of n_t-τ/ξ-stacks and $d^{\{n_t\}}(\tau)=d^{\{n_t\}}(\tau/\xi)$. Let $\bar{D}_i$ be any "typical" atom from $\bar{D}$. Therefore $\bar{D}_i=\bigcap_{t\geq 0}\bar{D}^{n_t}_{i_t}$ so $f^{-1}(\bar{D}_i)=\bigcap_{t\geq 0}f^{-1}(\bar{D}^{n_t}_{i_t})=\bigcap_{t\geq 0}D^{n_t}_{j_t}=D_j\in D$ because the preimage carries n_t-stacks into n_t-stacks. Hence f cannot stick together points as soon as they belong to the same atome D_j / because of card $\bar{D}_i=$card D_j /, so $f^{-1}(\mathcal{B}/\xi)$ contains σ-algebra generated by n_t-τ-stacks, $t\geq 0$. We get $\xi\geq D$, so ξ must be equal to the partition into points.

Remark 2.1 / [Kw2] / For any Morse sequence $d^{\{n_i\}}(x)=2$.

Remark 2.2 If $d^{\{n_i\}}(\tau)=\infty$ then τ need not be coalescent. For instance if τ is a Morse shift and τ' any Bernoulli automorphism then $\tau\times\tau'$ cannot be coalescent / [Kam], [Ne] /.

Remark 2.3 If $d^{\{n_i\}}(\tau)<\infty$ then τ need not be unitarily coalescent / [Ne] /. In [Hel-Pa], [Pa] examples of τ with $d^{\{n_i\}}(\tau)=2$ and τ has a component with countable Lebesgue spectrum are constructed.

III. Centralizer and simple spectrum In this section we formulate and prove some characterization of automorphisms having simple spectra that we need in the following.

Proposition 2.2 Let $\tau:(X,\mu)\circlearrowleft$ be an ergodic automorphism of a Lebesgue space. Then U_τ has a simple spectrum iff the unitary centralizer of τ, $C^{unit}(\tau)=\{V:L^2(X,\mu)$, V is unitary, $VU=UV\}$ is abelian.

Proof If U_τ has a simple spectrum then every unitary operator V, $VU_\tau=U_\tau V$ is a function of τ i.e. there exists a bounded function f such that $V=f(\tau)=\int_{U_\tau} f\,dE$, where E is the spectral measure of U_τ. Let $V'\in C^{unit}(\tau)$ then $V'=f'(\tau)$. Hence $VV'=\int_{U_\tau} f\,dE\int_{U_\tau} f'\,dE=\int_{U_\tau} ff'\,dE=V'V$ / [Hal 1] /.

Now, suppose τ does not have simple spectrum. Then there are $f_1, f_2 \in L^2(X,\mu)$ such that $L^2(X,\mu) = B_1 \oplus B_2 \oplus C$, where B_i is the cyclic space generated by f_i, i.e. $B_i = \mathrm{span}(U_\tau^j f_i,\ j \in Z)$, $i=1,2$, C is U_τ-invariant and there exists $U_1 : B_1 \longrightarrow B_2$ which is unitary and $U_1 \circ U_\tau|_{B_1} = U_\tau|_{B_1} \circ U_1$ / [Hal 1] /. We define two unitary operators on $L^2(X,\mu)$ setting

$$\begin{array}{lll} V(b_1) = U_1(b_1) & V'(b_1) = U_\tau(b_1) & b_1 \in B_1 \\ V(b_2) = U_1^{-1}(b_2) & V'(b_2) = b_2 & b_2 \in B_2 \\ V(c) = c & V'(c) = c & c \in C \end{array}$$

It is easy to see that $V, V' \in C^{\mathrm{unit}}(\tau)$ but $VV' \neq V'V$. Indeed, if $VV' = V'V$ then $U_\tau|_{B_1}$ and $U_\tau|_{B_2}$ are identities and a contradiction to ergodicity of τ.

Is is known that every Morse sequence has a simple spectrum / [Kw 1] /. Combining this with Proposition 2.2 we have obtained

<u>Corollary 2.1</u> For any Morse sequence x, C(x) is abelian.

IV. <u>A class of Morse sequences with uncountable centralizer</u>. In this section we give a class of Morse sequences with uncountable centralizer. We also provide some arguments that the property to have an uncountable centralizer is a typical one.

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue space and τ be an ergodic automorphism of (X,μ). Let us consider the group $\mathcal{S}$ of all automorphisms $S: (X,\mu) \mathcal{Q}$ with the weak topology $\mathcal{W}$

/ [Hal 2] / defined in the following way

$$S_n \xrightarrow[w]{} S \text{ iff } \mu(S_n E \vartriangle SE) \xrightarrow[n]{} 0 \text{ for every } E \in \mathcal{B}.$$

Now, we recall some known results on the weak topology

/1/ $(\mathcal{S}, \mathcal{W}, \circ)$ is a topological group / [Hal 2] /.

/2/ $(\mathcal{S}, \mathcal{W})$ is completely metrizable / [Hal 2] /.

/3/ $S_n \xrightarrow[w]{} S$ iff $U_{S_n} \Longrightarrow U_S$ i.e. $\|U_{S_n} f - U_S f\| \xrightarrow[n]{} 0$ / [Hal 2] /.

/4/ $C(\tau)$ is a closed set in $\mathcal{W}$.

/5/ If $\tau^{i_t} \xrightarrow[w]{} S$, $i_t \nearrow \infty$ then $S \in C(\tau)$, $\tau^{i_t - i_{t-1}} \xrightarrow[w]{} id$ and $C(\tau)$ is a perfect set, so from the Baire's property $C(\tau)$ is uncountable / [Kat-Si-St] /.

We let $\mathcal{S}^1$ denote the class of all $S \in \mathcal{S}$ with $S^{i_t} \xrightarrow[w]{} id$ for some sequence $i_t \nearrow \infty$. Then $\mathcal{S}^1$ contains a dense G_δ set of automorphisms of (X, μ). Indeed, if S admits a cyclic approximation with speed $o(1/n)$ then $U_S^{i_t} \Longrightarrow id$ for some sequence $\{i_t\}$ and moreover the class of all automorphisms admitting a cyclic approximation with a fixed speed contains a dense G_δ set / [Kat-St] /. So, we have proved the property to have an uncountable centralizer is a typical one in $\mathcal{W}$. Let us observe that the class $\mathcal{S}^1$ is closed under taking factors, so if $S \in \mathcal{S}^1$ then S does not have mixing factors, in particular $h(S)=0$. But a stronger fact is true. If $S \in \mathcal{S}^1$ then S is disjoint from all mixing transformations [Wa].

Now, we are able to show there are Morse sequences with uncountable centralizer.

<u>Proposition 2.3</u> Let $x = b^0 \times b^1 \times \ldots$ be a Morse sequence. If $\lim p_0^t = 0$ then $C(x)$ is uncountable.

Proof We will prove that x admits a cyclic approximation with speed $o(1/n)$.

We have $C_t=\{D_i^{n_t}(j): i=0,\dots,n_t-1,\ j=0,1\},\ t\geq 0$. From [Kw] it follows that $C_t \nearrow \mathcal{E}$

We define a cyclic approximation putting

$$S_t D_i^{n_t}(j)=D_{i+1}^{n_t}(j) \quad i=0,\dots,n_t-2,\ j=0,1 \tag{6}$$
$$S_t D_{n_t-1}^{n_t}(j)=D_o^{n_t}(1-j)$$

Now, we wish to estimate $A_t=\sum_{j=0}^{1}\sum_{i=0}^{n_t-1}\mu_x\left(TD_i^{n_t}(j)\,\Delta\, S_t D_i^{n}(j)\right)$

We then get $A_t=2\mu_x\left(TD_{n_t-1}^{n_t}(0)\,\Delta\, S_t D_{n_t-1}^{n_t}(0)\right)\leq$
$\leq\frac{2}{n_{t+1}}\left(fr(00,b^{t+1})+fr(11,b^{t+1})\right)=$
$\frac{2}{n_t}\left(fr(00,b^{t+1})+fr(11,b^{t+1})\right)/\lambda_{t+1}\leq 2p_o^{t+1}\frac{1}{n_t}$
so $A_t=o(1/2n_t)$. Therefore x admits desired cyclic approximation.

V. The measure-theoretic centralizer of regular Morse sequences. This section is devoted to prove the main result of the chapter

Theorem 2.1 Let $x=b^0\times b^1\times\dots$ be a regular Morse sequence satisfying /9/ and let $S\in C(x)$. Then $S=T^i\sigma^j$ for some $i\in Z$, $j=0,1$.

We start with presenting our main tool / Proposition 2.4 / nedeed in proving of Theorem 2.1.

Let $x=b^0\times b^1\times\dots$ be a Morse sequence.

A measurable function $\varphi: X=\{0,1\}^Z\to$ is said to be a code of lenght k if

/i/ $\varphi T = T\varphi$,

/ii/ $\varphi(y)[0]$ depends only on $y[-k,k]$, i.e. if $y[-k,k] = y'[-k,k]$ then $\varphi(y)[0] = \varphi(y')[0]$, y, y' a.e μ_x,

/iii/ k is the smallest natural number satisfying /ii/ and we denote it by $|\varphi|$.

The following Proposition establishes a list of properties of finite codes that we will need.

<u>Proposition 2.4</u> /a/ Let φ be finite code. Then for a.e. $y, y' \in \mathcal{O}_x$ if $y[-|\varphi|+t, t+|\varphi|] = y'[-|\varphi|+u, u+|\varphi|]$ then $\varphi(y)[t] = \varphi(y')[u]$.

/b/ Let $S \in C(x)$ and $\delta > 0$. There is a finite code φ such that

/7/ $d(Sy, \varphi y) < \delta$ $\quad$ / $d(z,z') = \lim_{m\to\infty} d(z[-m,m], z'[-m,m])$ /

/8/ $d(\varphi y, \varphi \tilde{y}) > 1-2\delta$ $\quad$ for a.e. $y \in \mathcal{O}_x$.

<u>Proof</u> The proof is straightforward and we use only ergodic theorem.

Now, let $x = b^0 \times b^1 \times \ldots$ be a regular Morse sequence with

/9/ $\sup_{t\in N} \lambda_t = \lambda < \infty$.

In the sequel we will need some facts of combinatorial nature. Let $S \in C(x)$ and let $\delta > 0, L > 0$ be determined by Lemma 1.1. Let us take $\varepsilon > 0$ and assume $\varphi : X^{\mathbb{Z}}$ is a code of length k so that

/10/ $d(\varphi y, Sy) < \varepsilon$ $\quad$ for a.e. $y \in \mathcal{O}_x$.

Fix $y \in \mathcal{O}_x$ for which /10/ holds. Next we find $t \in N$ so large that

/11/ $k/n_t < \varepsilon/2$,

/12/ $d(e_t, \hat{e}_t) > 1-3\varepsilon$ where e_t / $\hat{e}_t$/ is the code c_t / $\tilde{c}_t$ / via φ i.e. $|e_t| = n_t - 2k$, $e_t[j] = \varphi(c_t[k+j, 2k+j-1])$ $j=0,\dots,n_t-2k-1$,

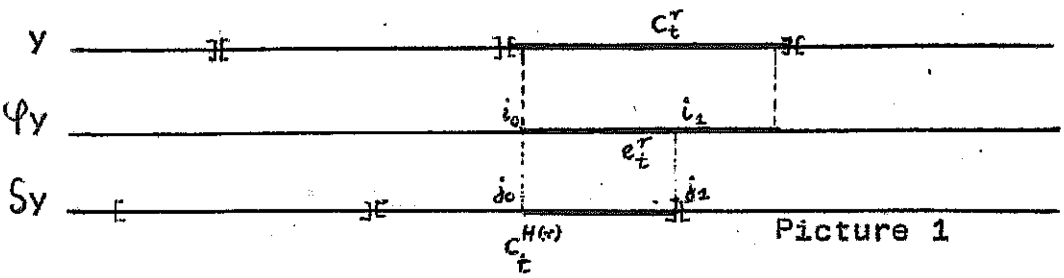
/13/ $(\forall\ m \geq n_t)$ $d(\varphi y[-m,m], Sy[-m,m]) < \varepsilon$.

Assume in addition

/14/ $y \in D_u^{n_t}$

Now we shall define some map $H:\{0,1\}^{\mathcal{Q}}$ in the following way $d(e_t^r[i_o,i_1], c_t^{H(r)}[j_o,j_1]) = \min\{d(e_t^r[i_o,i_1], c_t[j_o,j_1]),$
$d(e_t^r[i_o,i_1], \tilde{c}_t[j_o,j_1])\}$

where $e_t^r = e_t$ if $r=0$ or $\hat{e}_t$ otherwise and $|i_o - i_1| =$
$= |j_o - j_1| \geq \frac{1}{2}|e_t|$ / see Picture 1 /.



Picture 1

Let us observe that

/15/ $d(e_t^r[i_o,i_1], c_t^{H(r)}[j_o,j_1]) < 20\varepsilon$, $r=0,1$

Indeed, otherwise we would have $d(e_t^r[i_o,i_1], c_t^s[j_o,j_1]) \geq 20\varepsilon$ $s=0,1$. Cpose a sector of y, say $y[-m,m]$, $m > n_t$ so that

/16/ $y[-m,m]$ consists of p t-symbols and this sector contains at least $(\frac{1}{2}-\varepsilon)p$ of c_t^r, $r=0,1$ calculated only in the places of the form $-u+vn_t$, $v \in Z$.

To see /16/ it is sufficient to use ergodic theorem and the fact that $\mu_t(r) = \frac{1}{2}$ for every $r=0,1$, $t=0,1,\dots$.

Hence $\varepsilon > d(\varphi y[-m,m], Sy[-m,m]) \geq (\frac{1}{2}-\varepsilon)\ p20\varepsilon\frac{1}{2}|e_t^r|/(2m+1) \geq$
$\geq (\frac{1}{2}-\varepsilon)\,p10\varepsilon|e_t^r|/pn_t = 5\varepsilon(1-2\varepsilon)(1-2k/n_t) \geq 5\varepsilon(1-2\varepsilon)(1-\varepsilon) \geq \varepsilon$

a contradiction.

Now, we show $H:\{0,1\}^{\mathbb{Z}}$ is one-to-one. Indeed, let us suppose $H(0)=H(1)$. Then

$d(e_t[i_o,i_1],\hat{e}_t[i_o,i_1])\leq d(e_t[i_o,i_1],c_t^{H(0)}[j_o,j_1])+$
$d(\hat{e}_t[i_o,i_1],c_t^{H(1)}[j_o,j_1])<40\,\varepsilon$.

But from /12/

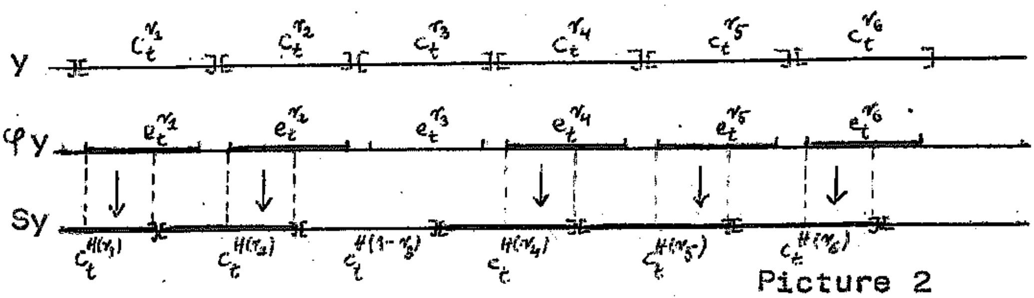
$d(e_t[i_o,i_1],\hat{e}_t[i_o,i_1])\geq(1-3\varepsilon)\frac{1}{2}|e_t^r|/|e_t^r|\geq\frac{1}{2}-2\varepsilon$,

a contradiction.

At present, we estimate $d(e_t^r[i_o,i_1],c_t^{H(1-r)}[j_o,j_1])$. We have $d(e_t^r[i_o,i_1],e_t^{1-r}[i_o,i_1])<d(e_t^r[i_o,i_1],c_t^{H(1-r)}[j_o,j_1])$ $+\,d(c_t^{H(1-r)}[j_o,j_1],e_t^{1-r}[i_o,i_1])$. Hence

/17/ $\qquad d(e_t^r[i_o,i_1],c_t^{H(1-r)}[j_o,j_1])\geq\frac{1}{2}-22\varepsilon$

Let us consider again the sector $y[-m,m]$ satisfying /16/ and we match by arrow e_t^r with $c_t^{H(r)}$ /Picture 2/.



Picture 2

We wish to estimate the number R of e_t^r, r=0,1 without arrows. We have

$d(\varphi y[-m,m],Sy[-m,m])\geq R(\frac{1}{2}-22\varepsilon)\,\frac{1}{2}|e_t^r|/pn_t$, therefore

/18/ $\qquad R<8\varepsilon p$

<u>Proof of Theorem 2.1</u> From invertibility of H we have

/19/ $\qquad c_t^{H(r)}=c_t^{H(0)+r}\qquad$ for r=0,1

Take now $T^s y^{H(0)}$ where $T^s y[-u+in_t,-u+(i+1)n_t-1]$ is always t-symbol. Then $d\left(T^{s}y^{H(0)}[-m+s,m-s], Sy[-m+s,m-s]\right) <$ $< \frac{R}{p} < 8\varepsilon$. Find the greatest t_o such that $y[-m,m]$ contains L t_o-symbols, where L is defined by Lemma 1.1. Using the condition of boundness of $\{\lambda_t\}$ we get $t_o \to \infty$ whenever $p \to \infty$. So choosing ε as small as we need and applying Lemma 1.1 we obtain $T^{s}y^{H(0)}[v,v+Ln_{t_o}-1]=Sy[v,v+Ln_{t_o}-1]$ for some $v \in Z$. Letting $p \to \infty$ we get at once $T^{s}\sigma^{H(0)}y=Sy$. Let us set $A_h^s=\{y \in \mathcal{O}_x :\ Sy=T^s\sigma^{H(0)}y\}, h=0,1$ $s \in Z$. So there exists A_h^s with $\mu_x(A_h^s)>0$. But A_h^s is T-invariant and ergodicity of T forces A_h^s to have full measure. Finally $S=T^s\sigma^h$.

<u>Corollary 2.2</u> For every regular Morse sequence with /9/ there are no roots of the shift induced by x.

<u>Corollary 2.3</u> For every regular Morse sequence with /9/ $C(T) \neq \overline{\{T^i :\ i \in Z\}}^{w}$.

<u>Proof</u> In the case of the equality $C(T)$ is uncountable.

VI. <u>Final remarks</u> Let us consider the class of all nonperiodic substitutions on two symbols of constant lenght / see Chapter 1 /. Their topological centralizer was calculated in [Cov] . It is equal to $\{T^k :\ k \in Z\}$ for discrete substitutions and $\{T^k\sigma^j :\ k \in Z,\ j=0,1\}$ for continuous substitutions /here σ is again the mirror map/

Now, we are able to show measure-theoretic centralizer for such a Θ. Let Θ be a discrete substitution. Then

Θ may be considered from the measure-theoretic point of view as a discrete, ergodic dynamical system with $Sp(\Theta)=G\{\lambda^t: t\geq 0\}$. From [Os] it follows that $C(\Theta)=\mathrm{End}\{G\{\lambda^t: t\geq 0\}\}$. It is easy to see that the later group is equal to the λ-adic integers.

Let Θ be a continuous substitution. Then the dynamival system arising from Θ is equal to $(\mathcal{O}_x, T, \mu_x)$ where $x=B\times B\times\ldots$ is a Morse sequence / see §1.6 /. So from Theorem 2.1 $C^{top}(\Theta)=C(\Theta)=\{T^i\sigma^j: i\in Z,\ j=0,1\}$.

Consider the class of Morse sequences over a fixed finite abelian group G / see Chapter 4 /. Let $x=b^0\times b^1\times\ldots$ be such a one. Let us call it regular if $\sup_{t\in N} s_t=s<\infty$

$$s_t=\sup_{g\in G}\{|B|: B=0\sigma_g(0)\ldots\sigma_{|g|-1}(0)0\sigma_g(0)\ldots 0\sigma_g(0)\ldots \ldots\sigma_{mg}(0) \text{ and } B \text{ appears in } x_t\}$$

$t\geq 0$, $|g|$ denotes the order of g and $\sigma_g(i)=i+g$, $i,g\in G$. If $\{\lambda_t\}$ is bounded then Lemma 1.1 holds for these regular Morse sequences over G. The concept of finite code $\varphi: G^{Z}\to \mathcal{L}$ and Proposition 2.4 go as in Section V. Let us assume $S\in C(x)$ and in addition $S\sigma_g=\sigma_g S$ for every $g\in G$. Repeating considerations of Section V we see that the only formula which is not quite clear is the following $H(g)=H(0)+g$, $g\in G$. To prove it we take p as in /16/. There must exist an $i_0\in Z$ such that $\sigma_g(y)[-u+i_0 n_t, -u+(i_0+1)n_t-1]=\sigma_g(c_t)$ with an arrow for every $g\in G$ / it is a simple consequence of /16/,/18/ and $\sigma_g S=S\sigma_g$ /. This proves that if $S\in C(x)$, $S\sigma_g=\sigma_g S$ $g\in G$ then $S=T^i\sigma_g$ for some $i\in Z$, $g\in G$. To get $S\sigma_g=\sigma_g S$

it is sufficient to know that $(\mathcal{O}_x, T, \mu_x)$ has a simple spectrum. In general it is still unknown whether they have simple spectra or not. Recently Kwiatkowski has communicated me that he knows examples of Morse sequences / over any cyclic group / of the form $x = B \times B \times \ldots$ having simple spectra, but there are some Morse sequences of the above form which do not have simple spectra [Goods] .

THE SEQUENCE ENTROPY FOR MORSE SHIFTS AND SOME COUNTEREXAMPLES

I. Introduction A new metric invariant called the sequence entropy / or A-entropy / has been introduced in [Ku] . Kušnirenko has proved that T has discrete spectrum if and only if $h_A(T)=0$ for every sequence A. He has used the sequence entropy to distinguish between two spectrally isomorphic and zero entropy transformations.

From [Kr-Ne] it follows that if $h(T) > 0$ then $h_A(T)=K(A)\cdot h(T)$, where $K(A)$ does not depend on T. That is it why the sequence entropy is uninteresting as a new metric invariant in the class of automorphisms with positive entropy.

Dekking in [Dek] has considered $\{1^n\}_{n=0}^{\infty}$-entropy for continuous substitutions. It is easy to see that the continuous substitution in normal form / i.e. $0 \longmapsto b$ and the block b starts with zero / is just the Morse sequence $x=b\times b\times \ldots$.

Lots of authors / [Dek], [Goodm], [Ku], [S] / were interested in the following problem. Which of the well-known properties of entropy are valid for sequence entropy? In the present chapter we give some list of properties of entropy that do not hold for sequence entropy. Needed examples arise from the class of Morse shifts. The way of computing the sequence entropy for the sequence $\{n_t\}_{t=0}^{\infty}$ is an easy adaptation of Dekking's considerations in [Dek]. However, calculating $\{n_t\}_{t=0}^{\infty}$-entropy for Morse shifts is important for two reasons. First, the class of Morse sequences is larger than the class of continuous substitutions and we get automorphisms with $\{n_t\}_{t=0}^{\infty}$-entropy equal to zero. Second, we get a possibility to control the size of $\{n_{t_k}\}_{k=0}^{\infty}$-entropy for any subsequence $\{n_{t_k}\}_{k=0}^{\infty}$ of $\{n_t\}_{t=0}^{\infty}$.

In the second section we recall some properties of Morse sequences needed in the following.

In the third section we provide formulas for $\{n_t\}_{t=0}^{\infty}$-entropy and $\{n_t\}_{t=0}^{\infty}$-topological entropy.

In the forth section we deal with zero $\{n_t\}_{t=0}^{\infty}$-entropy We give some condition equivalent to have zero $\{n_t\}_{t=0}^{\infty}$-entropy. For a given Morse sequence $x=b^0\times b^1\times\ldots$ by $M(x)$ we mean the class of all Morse sequences $y=\beta^0\times\beta^1\times\ldots$ with $|\beta^i|=|b^i|$, $i\geqslant 0$. The $\{n_t\}_{t=0}^{\infty}$-entropy gives a natural equivalence relation $SE(x)$ on $M(x)$ such that $y\in SE(x)$ provided $h_{\{n_{t_k}\}}(y)=h_{\{n_{t_k}\}}(x)$ for every subsequence $\{n_{t_k}\}_{k=0}^{\infty}$ of $\{n_t\}_{t=0}^{\infty}$. We compare this relation with the relation on $M(x)$ introduced in [Kw 1] and show that usually $SE(x)$ consists of continuum spectrally nonisomorphic Morse sequences.

In the fifth section we present a list of examples. We show that the formulas $h_A(T^k)=kh_A(T)$, $h_A^{top}(T^k)=kh_A^{top}(T)$, $k\geqslant 2$ need not hold. Moreover, presented examples give

the negative answer to Saleski's question in [S] .
In [Dek] Dekking showed that A-entropy does not depend monotonically on A i.e. it is possible $h_A(T) < h_B(T)$ when A is a subsequence of B. We give analogous example for topological entropy. Goodman in [Goodm] showed that for every sequence A and for every homeomorphism τ of a compact metric space

$$h_A^{top}(\tau) \geqslant \sup_{\mu \in M} h_A^{\mu}(\tau)$$

Here M denotes the collection of all τ-invariant Borel probability measures, with the equality in the case $h^{top}(\tau) > 0$. Goodman gave an example with $h_A^{top}(\tau) = \log 2$ and $\sup_{\mu \in M} h_A(\tau) = 0$, where τ had discrete spectrum. We prove that it is possible even for τ having a continuous part in its spectrum. Since the variational principle need not be valid we ask the following. Does the equality $h_A(T) = h_A(\tau)$ imply $h_A^{top}(T) = h_A^{top}(\tau)$ and vice versa? Here, by (X,T,μ) , (Y,τ,ν) we mean strictly ergodic systems. It turns out that in both cases the answer is negative. It can even happen that topological sequence entropy can distinguish between two strictly ergodic and measure-theoretically isomorphic dynamical systems. For topological sequence entropy we have $h_A^{top}(T) \leqslant h_A^{top}(T) + h_A^{top}(\tau)$. Goodman in [Goodm] showed $h_A^{top}(T \times T) = 2h_A^{top}(T)$ and he asked whether $h_A^{top}(T \times \tau) = h_A^{top}(T) + h_A^{top}(\tau)$. We provide an example for which the above formula fails. In Example6 we compute $\{2^t\}_{t=0}^{\infty}$-entropy for Kakutani sequences. We can reduce this problem to considering some continuous transformation on unit interval and its time averages. In particular for "typical" Kakutani sequences this entropy is constant and supremum of values of $\{2^t\}_{t=0}^{\infty}$-entropy is obtained for Morse sequence $x=01^{\times}01^{\times}\ldots$. In the above class of sequences we find / Example 7 / two automorphisms T and τ for which the formula $h_A(T\times\tau)=h_A(T)+h_A(\tau)$ is false. This shows that Lemma 4 in [Ku] is not valid. It can be constucted the counterexamples in Example 5 and 7 owing to the simple fact that the sequence entropies calculated in them are the upper limits, and not limits, of suitable sequences. We also consider the Abramov's formula for skew product /[Ab-Roh]/ and show that some generalization to sequence entropy is impossible. Finally, we prove that no Morse shifts have weakly mixing factors.

In the sixth section we solve the problem of disjointness in the class of all Morse sequences and give all factors for some special class of Morse shifts.

II. <u>Notations</u> Let $(X,\mathcal{B},\mu)$ be a compact metric space with a Borel measure μ and let $T:X\circlearrowleft$ be a homeomorphism preserving μ .

Let $A=\{n_t\}_{t=0}^{\infty}$ be an infinite sequence of natural numbers. Denote by $\mathcal{Z}$ the collection of measurable partitions of X with finite entropy and by $\mathcal{C}$ the collection of all open covers of X.

The sequence entropy / or A-entropy / of T with respect to A is defined by

$$h_A(T,Q)=\limsup_{t\in N} \frac{1}{t} H(T^{-n_0}Q\vee\ldots\vee T^{-n_{t-1}}Q),\ Q\in \mathcal{Z}$$

$$h_A(T)=\sup_{Q\in\mathcal{Z}} h_A(T,Q)$$

Let $\alpha\in\mathcal{C}$ and denote by $N(\alpha)$ the minimal cardinality of any subcover of α. Then the topological sequence entropy of T with respect to A is given by

$$h_A^{top}(T,\alpha) = \limsup_{t\in N} \frac{1}{t} \log N(T^{-n_0}\alpha\vee\ldots\vee T^{-n_{t-1}}\alpha),\ \alpha\in\mathcal{C}$$

$$h_A^{top}(T)=\sup_{\alpha\in\mathcal{C}} h_A^{top}(T,\alpha)$$

It is well-known that if $\{Q_k\}_{k=0}^{\infty}$ is a sequence of open partitions of X and $Q_k\nearrow\varepsilon$ then

/1/ $h_A^{top}(T)=\lim_{k\to\infty} h_A^{top}(T,Q_k)$, $\quad h_A(T)=\lim_{k\to\infty} h_A(T,Q_k)$

Let Q be a partition of X. Denote

$Q_k=Q\vee T^{-1}Q\vee\ldots\vee T^{-k+1}Q$,

$Q^k=T^{-n_0}Q\vee\ldots\vee T^{-n_{k-1}}Q$,

$Q_k^m=(Q_k)^m$, $k,m\geqslant 1$.

For technical reasons we slightly change some notations defined in §0.2, namely

Remark 3.1 If $x=b^0\times b^1\times\ldots$ is a Morse sequence then we denote $c_t=b^0\times\ldots\times b^{t-1}$, $n_t=|c_t|=\lambda_0\cdot\ldots\cdot\lambda_{t-1}$, $t\geqslant 1$ $n_0=1$, $c_m^t=b^t\times b^{t+1}\times\ldots\times b^{t+m-1}$, $n_0^t=1$, $n_m^t=\lambda_t\cdot\ldots\cdot\lambda_{t+m-1}$, $m>0$ and $A_t=\{n_m^t\}_{m=0}^{\infty}$. We denote by $Q(t)$ the natural generator for x_t, $t\geqslant 1$.

Any nonempty atom of Q_k^m / $m,k\geqslant 1$ / is said to be a general cylinder / for short g.c. /.

Note that any atom of Q_k^m has the form

$\{y \in \mathcal{O}_x : y[n_j, n_j+k-1] = B^j,\ j=0,\dots,m-1\} = T^{-n_0}(B^0) \cap \dots \cap T^{-n_{m-1}}(B^{m-1})$ where B^j is a block with the length k and we denote such an atom by $[B^0:\dots:B^{m-1}]$

It is easy to see that a Q_k^m-block $[B^0:\dots:B^{m-1}]$ is a g.c. iff there exists a natural p such that

$$/2/ \qquad x[p+n_j, p+n_j+k-1] = B^j,\ j=0,\dots,m-1.$$

We also have

$$\mu_x [B^0:\dots:B^{m-1}] = \lim_{t\to\infty} \frac{1}{n_t} \operatorname{fr}([B^0:\dots:B^{m-1}], c_t)$$

Moreover, if $[B^0:\dots:B^{m-1}]$ is a g.c. then $[\tilde{B}^0:\dots:\tilde{B}^{m-1}]$ is g.c. and they have the same measure μ_x.

Remark 3.2 For $t \in N$ we denote by $[B^0:\dots:B^{m-1}]_t$ the atoms for x_t.

Now, we recall some results of Kwiatkowski's paper [Kw1]

Let B,C be blocks with $|B| = |C| = n$, $n \geq 1$ and let

$$s(k,B) = \operatorname{card}\{i : 0 \leq i \leq n-k-1,\ B[i] \neq B[i+k]\},\ k=1,\dots$$

We write $B \overset{s}{=} C$ iff $s(k,B) = s(k,C)$, $k=1,\dots,n-1$.

Theorem 3.1 /[Kw1]/ Let $x = b^0 \times b^1 \times \dots$ and $y = \beta^0 \times \beta^1 \times \dots$ be regular Morse sequences such that $|b^i| = |\beta^i| = \lambda_i$, and $\lambda_i \leq r$, $i \geq 0$. Then $\theta(x)$ and $\theta(y)$ are spectrally isomorphic iff $b^j \overset{s}{=} \beta^j$ for j large enough.

Note that the part "if" of Theorem 3.1 is valid without the assumption of the regularity of x and y

III. $\{n_t\}$-entropy for Morse shifts. In this section we give formulas for $\{n_t\}$-topological and $\{n_t\}$-entropy for a Morse sequence $x=b^0 \times b^1 \times \ldots$.

We start with formulating the following

Throrem 3.2 $h_A^{top}(x)=h_A^{top}(x,Q)$, $h_A(x)=h_A(x,Q)$.

The proof of Theorem 3.2 can be obtained as in Dekking's paper [Dek] . We use the following formulas

/3/ $x[sn_1, sn_1+n_1-1]=c_1$ or $\tilde{c}_1$, $x[sn_1]=x_1[s]$, $s \geq 1$,

/4/ $[a_0:\ldots:a_m]$ is a g.c. then $[a_1:\ldots:a_m]_1$ is one, $a_i=0,1$, $0 \leq i \leq m$,

/5/ $2N(Q^m(1)) \geq N(Q^{m+1}) \geq N(Q^m(1))$,

/6/ $2^t N(Q^{m-t}(t)) \geq N(Q^m) \geq N(Q^{m-t}(t))$, $m \in N$, $0 \leq t \leq m-1$,

/7/ $N(Q_2^{m+1}) \leq 4mN(Q^{m+1})$, $m \in N$,

/8/ $N(Q_k^m) \leq 4(2^k)^t(m-t+1)N(Q^m)$, $t \in N$, $n_t > k$, $k,m \in N$.

The properties /3/-/8/ are obtained as in [Dek] . Instead of Lemma 3 in [Dek] needed to proving $h_A(x)=h_A(x,Q)$ we use

Remark 3.3 Let $M \in N$. If $y \in D_k^{n_t}$, $t \geq 1$, $k=0,\ldots,n_t-1$ then there is a natural p such that $y[0,M]=x[p,p+M]$ and $p=k$ / mod n_t /.

Indeed, if $y \in D_k^{n_t}$ then $y[0,M]=(c_t \times B)[k,M-k]$ for some block B. The block $c_t \times B$ appears in x. Thus it is not difficult to verify that it appears in x at a place being a multiple of n_t. Hence, there is j, $j=sn_t$ for some $s \in N$ such that $x[j, j+|B|n_t-1]=c_t \times B$.

Now, we are in a position to give formulas for $\{n_t\}$-topological entropy.

Take $i \in N$. We define by induction a finite sequence of natural numbers $\{m^{(i)}\}_{m=0}^{i}$ as follows

0° $m^{(i)} = m_0^{(i)} + m_1^{(i)}$

1° $0^{(k)} = 1,\ k=0,1$

2° If $m < i$ then

/9/
$$(m+1)_0^{(i)} = \begin{cases} m^{(i)} & \text{if } b^{i-(m+1)} \neq 01\ldots01, 01\ldots010 \\ m_1^{(i)} & \text{if } b^{i-(m+1)} = 01\ldots01 \\ m_0^{(i)} & \text{if } b^{i-(m+1)} = 01\ldots010 \end{cases}$$

$$(m+1)_1^{(i)} = \begin{cases} m^{(i)} & \text{if } b^{i-(m+1)} \neq 0\ldots0 \\ m_1^{(i)} & \text{if } b^{i-(m+1)} = 0\ldots0 \end{cases}$$

and proceeding as Dekking in [Dek] we get

<u>Theorem 3.3</u> $h_A^{top}(x) = \limsup_{i \in N} \log(i^{(i)}/i)$

At present, we will deal with measure-theoretic entropy. Firt, introduce some notations.

For a given Morse sequence $x = b^0 \times b^1 \times \ldots$ we denote

$h_t(00,11) = fr(00,b^t) + fr(11,b^t)$, $h_t(01,10) = fr(01,b^t) + fr(10,b^t)$

$$\pi_0^t = \frac{1}{\lambda_t} h_t(00,11)$$

$$\pi_1^t = \frac{1}{\lambda_t} h_t(01,10)$$

/10/ $$\pi_{00}^t = \frac{1}{\lambda_t}\left(fr(00,b^t0) + fr(11,b^t0)\right)$$

$$\pi_{10}^t = \frac{1}{\lambda_t}\left(fr(00,b^t1) + fr(11,b^t1)\right)$$

$$\pi_{k1}^t = 1 - \pi_{k0}^t,\quad k=0,1,\quad t \geq 0.$$

In this way we have defined a sequence of matrices

$$\pi^t = \begin{bmatrix} \pi_{00}^t & \pi_{01}^t \\ \pi_{10}^t & \pi_{11}^t \end{bmatrix}$$

Let us observe that for every $t \geqslant 0$

$$/11/ \quad p_o^t = \pi_o^t + \begin{cases} \frac{1}{\lambda_t} \; p_o^{t+1} & \text{if } b^t[\lambda_t - 1] = 0 \\ \frac{1}{\lambda_t} \; p_1^{t+1} & \text{if } b^t[\lambda_t - 1] = 1 \end{cases}$$

This gives the following useful formula

$$/12/ \qquad p_o^t = \pi_{00}^t \pm \frac{1}{\lambda_t} \; p_1^{t+1}$$

Going back to /11/ let us put there the factor $1 = p_o^{t+1} + p_1^{t+1}$ before π_o^t.

We then obtain

$$p_o^t = \begin{cases} (\pi_o^t + 1/\lambda_t)\, p_o^{t+1} + \pi_o^t \; p_1^{t+1} & \text{if } b^t \; [\lambda_t - 1] = 0 \\ \pi_o^t \; p_o^{t+1} + (\pi_o^t + 1/\lambda_t)\, p_1^{t+1} & \text{if } b^t \; [\lambda_t - 1] = 1 \end{cases}$$

This by definition of π_{ij}^t proves the following

<u>Proposition 3.1</u> $\sum_{j=0}^{1} p_j^{t+1} \pi_{ji}^t = p_i^t$, for any $t \in N$, $i = 0, 1$.

By consecutive substitutions of /11/ into itself for $t = 0, 1, \ldots$ one can obtain the following Kwiatkowski's formulas [Kw2].

$$p_o = \frac{h_o(00,11)}{\lambda_o} + \sum_{\substack{i \geqslant 2 \\ i-1 \in I}} \frac{h_{i-1}(00,11)}{n_i} + \sum_{\substack{i \geqslant 2 \\ i-1 \in II}} \frac{h_{i-1}(01,10)}{n_i}$$

/13/

$$p_1 = \frac{h_o(01,10)}{\lambda_o} + \sum_{\substack{i \geqslant 2 \\ i-1 \in I}} \frac{h_{i-1}(01,10)}{n_i} + \sum_{\substack{i \geqslant 2 \\ i-1 \in II}} \frac{h_{i-1}(00,11)}{n_i}$$

where

$$I = \{ i \geqslant 1 ;\; c_i[n_i - 1] = 0 \}, \quad II = \{ i \geqslant 1 :\; c_i[n_i - 1] = 1 \}.$$

These formulas will be used in our study of Kakutani sequences in Example 6.

Now, applying Proposition 3.1 and proceeding in the same way as Dekking in [Dek] we get.

<u>Theorem 3.4</u> $h_A(x)=\limsup_{t\in N} -\frac{1}{t}\sum_{r=0}^{t-1}\left(\sum_{0\leq i,j\leq 1} p_i^{r+1}\pi_{ij}^r \log \pi_{ij}^r\right)$

IV. <u>Zero-$\{n_t\}$-entropy and properties of $SE(x)$.</u> In this section we answer the following question. When $h_{\{n_t\}}(x)=0$? Next, we introduce the equivalence relation on $M(x)$ / see Introduction / and compare it with the relation introduced in [Kw 1].

Let $x=b^0 x\, b^1 x \dots$ be a Morse sequence and put

/14/ $$r_t^* = -\sum_{0\leq i,j\leq 1} p_i^{t+1}\pi_{ij}^t \log \pi_{ij}^t \quad , \quad t \geqslant 0$$

For every x we will consider the following sequence of numbers.

$$w_t=\begin{cases} \min(p_0^t, p_1^t) & \text{if } b^t\neq 0\dots 0, 01\dots 01, 01\dots 010 \\ \min(p_1^{t+1}, 1/\lambda_t) & \text{if } b^t=0\dots 0, 01\dots 01 \\ \min(p_0^{t+1}, 1/\lambda_t) & \text{if } b^t=01\dots 010. \end{cases}$$

<u>Lemma 3.1</u> For every $\delta>0$ there is $\varepsilon>0$ such that

/*/ if $r_t^* < \varepsilon$ then $w_t<\delta$

/**/ if $w_t < \varepsilon$ then $r_t^*<\delta$ for every $t\in N$.

<u>Proof</u> Let $\delta>0$ and $\varepsilon>0$. We observe that if $b^t\neq 0\dots 0, 01\dots 01, 01\dots 010$ then $r_t^* \geqslant \min_{0\leq i,j\leq 1}(-\pi_{ij}^t\log\pi_{ij}^t)>0$. Therefore $0<-\pi_{ij}^t\log\pi_{ij}^t<\varepsilon$ for some $i,j=0,1$. This means that $0<\pi_{ij}^t<\delta/3$ or $1>\pi_{ij}^t>1-\delta/3$ for suitable $\varepsilon>0$. The later inequality implies $0<\pi_{i'j'}^t<\delta/3$ for some $i',j'=0,1$, so we may assume $0<\pi_{ij}^t<\delta/3$.

It is easy to see that this implies $p_0^t<\delta$ or $p_1^t<\delta$. Indeed from the formulas /13/ it follows that

$$p_o^t \leqslant \frac{3}{\lambda_t} h_t(00,11) \; ; \; p_1^t \leqslant \frac{3}{\lambda_t} h_t(01,10)$$

and

$$\pi_{ij}^t \geqslant \min\left(\frac{1}{\lambda_t} h_t(00,11), \frac{1}{\lambda_t} h_t(01,10)\right).$$

Now, we consider the case $b^t=0\ldots0,01\ldots01$. Then

/15/ $r_t^* = -p_1^{t+1}\left(\frac{1}{\lambda_t}\log\frac{1}{\lambda_t} + \frac{\lambda_t-1}{\lambda_t}\log\frac{\lambda_t-1}{\lambda_t}\right)$

If $r_t^* < \varepsilon$ then

$p_1^{t+1} < \sqrt{\varepsilon}$ or $-\left(\frac{1}{\lambda_t}\log\frac{1}{\lambda_t} + \frac{\lambda_t-1}{\lambda_t}\log\frac{\lambda_t-1}{\lambda_t}\right) < \sqrt{\varepsilon}$. But the later inequality denotes $\lambda_t > 1/\delta$ for suitable $\varepsilon > 0$. The same reasoning for $b^t=01\ldots010$ makes the proof of /*/ complete.

Now, let $\delta > 0$ and $\varepsilon > 0$. If $b^t \neq 0\ldots0,01\ldots01$, $01\ldots010$ then $\pi_{ij}^t < 2p_i^t$, $i,j=0,1$ so $r_t^* < \delta$ whenever $\min(p_o^t, p_1^t) < \varepsilon$ for suitable $\varepsilon > 0$. Let $b^t=0\ldots0$. Then r_t^* is one as in /15/. If $\lambda_t > 1/\varepsilon$ then $r_t^* < \delta$ and if $p_1^{t+1} < \varepsilon$ then also $r_t^* < \delta$ for suitable $\varepsilon > 0$.

Proceeding in the same way in the remain cases we obtain the required result.

<u>Corollary 3.1</u> Let $x=b^0 \times b^1 \times \ldots$ be a Morse sequence. Then $h_{\{n_t\}}(x)=0$ iff $\lim_{t\in B} w_t=0$ and the density of $N \setminus B$ is equal to zero.

<u>Proof</u> It is sufficient to see that $h_A(x)=0$ iff $\lim_{t\to\infty}\frac{1}{t}\sum_{i=0}^{t-1} r_i^*=0$. It is known that this is equivalent to $\lim_{t\in B} r_t=0$ for some subsequence $B \subseteq N$ and the density of $N \setminus B$ is equal to zero. Now, we apply Lemma 3.1.

Corollary 3.2 Let $x=b^0 \times b^1 \times \ldots$ be a Morse sequence. Then x is regular iff for every subsequence $\{n_{t_k}\}$ of $\{n_t\}$ $h_{\{n_{t_k}\}}(x)>0$.

Proof Let us observe that the process of grouping the b^i's gives us a new representation of x: $x=\beta^0 \times \beta^1 \times \ldots$ Moreover, the products of the lengths of succesive β^i's form a subsequence $\{n_{t_k}\}$ of $\{n_t\}$. Notice that the corresponding sequence defined in /8/ §0.2 is a subsequence $(p_0^{t_k}, p_1^{t_k})$ of (p_0^t, p_1^t).

This means the process of grouping the b^i's gives a regular representation of x whenever the representation $\{b^t\}$ was regular. Then by Corollary 3.1 and /9/§0.2 $h_{\{n_{t_k}\}}(x) > 0$.

If x is not regular then there is a subsequence $\{t_k\}$ such that $\min(p_0^{t_k}, p_1^{t_k}) \xrightarrow[k]{} 0$. We group the b^i's into new product $x=\beta^0 \times \beta^1 \times \ldots$ in such a way that $p'^k_i = p^t_i$, $k \geq 0$, $i=0,1$ / p'^k_i are the numbers defined in /8/§.0.2 for $\{\beta^t\}$/. Grouping again if necessary we may assume $b^i \neq 0\ldots0, 01\ldots01, 01\ldots010$. This forces $\{w_t\}$ to converge to zero on a set with the density of its completing equal to zero.

At present, we shall examine the relation SE(x) for a given $x=b^0 \times b^1 \times \ldots$.

Proposition3.2 Let $x=b^0 \times b^1 \times \ldots$ be a regular Morse sequence and $y \in M(x)$, $y=\beta^0 \times \beta^1 \times \ldots$ be spectrally isomorphic to x. Then $y \in SE(x)$.

Proof Let us use for x the symbols p_i^t, π_{ij}^t, r_t^*, for y the symbols $p_i'^t$, $\pi_{ij}'^t$, $r_t'^*$ to denote the numbers defined in /8/§0.2,/10/,/14/.

According to the proof of Theorem 2 in [Kw1] if y and x are spectrally isomorphic then

/16/ $\quad |p_i^t - p_i'^t| < \varepsilon_t,\ t \geq 0,\ i=0,1$ and $\lim_{t\to\infty} \varepsilon_t = 0,$

and the following formulas can be obtained

/17/ $[s(q,b^t) - s(q,\beta^t)][1-(p_1^{t+1}-p_o^{t+1})^2] = \lambda_t \delta_t,\ \delta_t \xrightarrow[t]{} 0,\ 0 \leq q \leq \lambda_t$

We do not assume $\{\lambda_i\}$ is bounded, so we cannot make use of Theorem 3.1.

To compare π_{ij}^t with $\pi_{ij}'^t$ recall / /12/ / that

/18/ $\quad \pi_{00}^t = p_o^t \pm \frac{1}{\lambda_t} p_1^{t+1}\ ,\quad \pi_{00}'^t = p'^t_o \pm \frac{1}{\lambda_t} p'^{t+1}_1$

It is sufficient to show that

/19/ $\quad \lim_{t\to\infty} |r_t^* - r_t'^*| = 0.$

Let us suppose that /19/ fails. Hence, there is $c > 0$ and a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and

/20/ $\quad |r_{t_k}^* - r_{t_k}'^*| \geq c,\ k=0,1,\dots$

Due to /16/,/18/ and /20/ we get $\{\lambda_{t_k}\}$ is bounded. In view of /17/ and by regularity of x $s(\lambda_{t_k}-1,b^t) = s(\lambda_{t_k}-1,\beta^t)$ for k large enough, so if $\pi_{00}^{t_k} = p_o^{t_k} - \frac{1}{\lambda_{t_k}} \cdot p_1^{t_k+1}$ then $\pi_{00}'^{t_k} = p_o'^{t_k} - \frac{1}{\lambda_{t_k}} p_1'^{t_k+1}$ and consequently $\lim_{k\to\infty} |r_{t_k}^* - r_{t_k}'^*| = 0$ and a contradiction to /20/.

From Proposition 3.2 and Corollary 3.2 / or directly from /16/ / it follows

<u>Corollary 3.3</u> If $x=b^0 \times b^1 \times \ldots$ is a regular Morse sequence and $y \in M(x)$ is not regular then x and y are not spectrally isomorphic.

Now, we give a class of examples of Morse sequences x for which SE(x) consists of continuum classes of spectral equivalence.

Let $k \geq 4$ and let B be a block with the length k such that there is C with $|B|=|C|$, $s(1,B)=s(1,C)$, $s(k-1,B)=$ $=s(k-1,C)$ and $s(r,B)\neq s(r,C)$ for some r, $2\leq r\leq k-1$. For instance if k=4 one can put B=0110, C=0010.

Let $x=b^0 \times b^1 \times \ldots$ be a Morse sequence such that the set $\mathcal{J}=\{i\in N:\ b^i=B \text{ and } b^{i+1}\neq 0\ldots0, 01\ldots01, 01\ldots010, \lambda_{i+1}\leq k\}$ is infinite. Let $y=\beta^0\times\beta^1\times\ldots$ be a Morse sequence obtained from x by changing infinitely many if the b^i's $i\in\mathcal{J}$ into C. We observe that $y\in SE(x)$ because the sequences $\{\Pi^t\}$ and $\{p_o^t, p_1^t\}$ are the same for x and y. If $\{\lambda_t\}$ is bounded and x is regular we can use Theorem3.1 to verify that x and y are not spectrally isomorphic. In general, we notice that if $t\in\mathcal{J}$ then

$$1-(p_1^{t+1}-p_o^{t+1})^2 \geq 1-((k-2)/k)^2$$

Assume that x and y are spectrally isomorphic. Thus, from /17/ it follows that $s(q,b^t)=s(q,\beta^t)$, $0\leq q\leq\lambda_{t-1}$ for $t\in\mathcal{J}$ large enough. But this is a contradiction because of $s(r,B)\neq s(r,C)$.

Finally, note that x could be regular or not and in both cases SE(x) contains continuum spectrally nonisomorphic Morse sequences.

We do not know whether Proposition 3.2 is true without the assumption of the regularity of x. However, in general we prove the Morse sequence x^{-1} / defined below / is allways spectrally isomorphic to x and $x^{-1} \in SE(x)$

Let $x=b^0 \times b^1 \times \ldots$ be a Morse sequence.

For every block $B=(b_0,\ldots b_{m-1})$ we define a block B^{-1} as follows

$$B^{-1}=\begin{cases}(b_{m-1},\ldots,b_0) & \text{if } b_{m-1}=0\\(\tilde{b}_{m-1},\ldots,\tilde{b}_0) & \text{if } b_{m-1}=1\end{cases}$$

Put

$$x^{-1}=(b^0)^{-1}\times(b^1)^{-1}\times\ldots$$

It is easy to see that x^{-1} is also a Morse sequence. Now, we define a function $\varphi:X^{\mathcal{Q}}$ by the formula $\varphi(z)=z^{-1}$ for every two-sided sequence $z\in X$, where $z^{-1}[i]=z[-i]$. It is clear φ is continuous, invertible and $\varphi T^{-1}=T\varphi$. Moreover $\varphi(\mathcal{O}_x)=\mathcal{O}_{x^{-1}}$. To see this assume $z\in\mathcal{O}_x$ and $z[-j+vn_t,-j+(v+1)n_t-1]=c_t$ or $\tilde{c}_t$ for every $v\in Z$. Then $\varphi(z)[-(n_t-j)+vn_t,-(n_t-j)+(v+1)n_t-1]=c_t^{-1}$ or $\widetilde{(c_t^{-1})}$. Since $(B\times C)^{-1}=B^{-1}\times C^{-1}$ for all blocks B,C our claim is established. Next, $(\mathcal{O}_x,T^{-1},\mu_x)$ and $(\mathcal{O}_{x^{-1}},T,\mu_{x^{-1}})$ are strictly ergodic so φ must be an isomorphism between them. But for every block B, $B \overset{s}{=} B^{-1}$ so from the remark after Theorem 3.1 (T,μ_x), (T^{-1},μ_x) and $(T,\mu_{x^{-1}})$ are allways spectrally isomorphic. Put in the proof of Proposition 3.2 $y=x^{-1}$. Since $s(\lambda_t-1,b^t)=s(\lambda_t-1,(b^t)^{-1})$ we need not use regularity of x to have $y\in SE(x)$. So, we prove $x^{-1}\in SE(x)$ for every Morse sequence x.

Note however, that Theorem 8 in [Kw2] provides several examples (T, μ_x) is not metrically isomorphic to (T^{-1}, μ_x). Dekking in [Dek] has given such an example for which $h_B(T) \neq h_B(T^{-1})$ for some sequence $B \subseteq N$ / obviously B is not of the form $\{n_t\}$ /.

V. <u>Examples and Remarks</u> In this section we will apply obtained result to answer a few questions concerning sequence entropy.

<u>Example 3.1</u> The formula $h_A(T^k) = kh_A(T)$, $k > 0$ need not hold

/a/ k=2

Let $x = 01\times 00\times 01\times 00\times 00\times 01\times 00\times 00\times 00\times 01\times \ldots$

Next, we group x into new products as follows

I: $x = 01\times(00\times 01\times 00)\times(00\times 01\times 00\times 00)\times(00\times 01\times 00\times 00\times 00)\times\ldots$

II: $x = (01\times 00)\times(01\times 00\times 00)\times(01\times 00\times 00\times 00)\times(01\times 00\times 00\times 00\times 00)\times\ldots$

Assume $\{n_t\}, \{m_t\}$ are respectively the corresponding sequences of the products of lengths of the succesive blocks in I and II.

Thus $m_t = 2n_t$, $t \geq 1$.

We see that the representation of x in I is regular, so $h_{\{n_t\}}(x) > 0$. Moreover, it is clear from Corollary 3.1 that $h_{\{m_t\}}(x) = 0$. In addition we can assume $m_0 = 2$ / see Proposition 3 in [S] /. Hence $h_{\{n_t\}}(T^2) \neq 2h_{\{n_t\}}(T)$. Indeed, from the definition of sequence entropy it follows that $h_{\{kn_t\}}(T, Q) = h_{\{n_t\}}(T^k, Q)$ for every $Q \in \mathfrak{Z}$, $k \geq 1$, so

/21/ $h_{\{kn_t\}}(T)=h_{\{n_t\}}(T^k)$, $k \geq 1$.

Therefore, if $h_{\{n_t\}}(T^2)=2h_{\{n_t\}}(T)$ then $0=h_{\{m_t\}}(T)=h_{\{2n_t\}}(T)=$ $=h_{\{n_t\}}(T^2)=2h_{\{n_t\}}(T)>0$, a contradiction.

/b/ $k \geq 3$

Take $B\neq 0\ldots0, 01\ldots01, 01\ldots010$, $|B|=k$ and put

$x=B\times 00\times B\times 00\times 00\times B\times 00\times 00\times 00\times \ldots$

Next, we take two representations of x

I: $x=(B\times 00)\times(B\times 00\times 00)\times(B\times 00\times 00\times 00)\times \ldots$

II: $x=(B\times 00\times B)\times(00\times 00\times B)\times(00\times 00\times 00\times B)\times\ldots$

As in preceding case one can easily prove that $h_{\{n_t\}}(x)>0$ $h_{\{m_t\}}(x)=0$ and $m_t=kn_t$, $t \geq 1$.

Furthemore, from /21/ $0=h_{\{n_t\}}(T^k)\neq kh_{\{n_t\}}(T)>0$.

Remark 3.4 The above example gives a solution to the following Saleski's question:

Suppose $\{a(n)/b(n)\}_{n=1}^{\infty}$ is bounded away from 0 and ∞. Is it true that $h_A(T,Q)=0$ iff $h_B(T,Q)=0$?

It is not difficult to verify that the examples in Example 3.1 provide the negative answer to that question because of $h_A(T)=h_A(T,Q)$, where Q is the natural generator.

Note however, that the question of Saleski remains open if consider A increasing not quicker than exponentially / i.e. $\limsup_{t\in N} \sqrt[t]{n_t}<\infty$/. If, for instance $A=\{k^t\}$, $k\geq 2$ and $B=\{k^{t+1}\}$ then the answer is positive and moreover $h_A(T^k)=h_A(T)$ for any automorphism T.

Example 3.2 The formula $h_A^{top}(T^k)=kh_A^{top}(T)$, $k \geqslant 2$ need not hold.

We only consider the case k=2.

We have

/22/ $$h^{top}_{\{n'_t\}}(T)=h^{top}_{\{n_t\}}(T) \text{ where } n'_t=n_t \text{ for } t \geqslant 1.$$

Now, let $x=01{\times}00{\times}01{\times}00{\times}\ldots$ and we take two reprpsentations of x

I: $x=(01{\times}00){\times}(01{\times}00){\times}\ldots,$

II: $x=(01{\times}00{\times}01){\times}(00{\times}01){\times}(00{\times}01){\times}\ldots$.

Theorem 3.3 and /9/ imply

$h^{top}_{\{n_t\}}(x)=h^{top}_{\{2^t\}}(01{\times}01{\times}\ldots)=\log(1+\sqrt{5})/2$ / see also [Dek] /

and

$h^{top}_{\{m_t\}}(x)=\log 2$ because each block in this representation is different from 0...0,01...01,01...010.

In addition, $m_t=2n_t$, $t \geqslant 1$ and by /22/ we can assume $m_o=2n_o$, so $h^{top}_{\{n_t\}}(T^2)\neq 2h^{top}_{\{n_t\}}(T)$.

Example 3.4 A-topological entropy does not depend monotonically on A.

In [Dek] Dekking has proved that it is possible $h_A(T)<h_{A'}(T)$ when A is a subsequence of A'. We show that the same is true for topological sequence entropy.

Let $B\neq 0\ldots0,01\ldots01,01\ldots010$ and put

$x=00{\times}B{\times}B{\times}00{\times}B{\times}B{\times}00{\times}\ldots$

Taking $i \in N$ we have three possibilities:

a/ $0^{(i)}=2$

$(3n+1)^{(i)}=2\,(3n)^{(i)}$

$(3n+2)^{(i)}=2\,(3n+1)^{(i)}$

$(3n+3)^{(i)}=(3n+2)^{(i)}+(3n+1)^{(i)}$

b/ $0^{(i)}=2$

$(3n+1)^{(i)}=2\,(3n)^{(i)}$

$(3n+2)^{(i)}=(3n+1)^{(i)}+(3n)^{(i)}$

$(3n+3)^{(i)}=2\,(3n+2)^{(i)}$

c/ $0^{(i)}=2$

$1^{(i)}=3$

$(3n+2)^{(i)}=2\,(3n+1)^{(i)}$

$(3n+3)^{(i)}=2\,(3n+2)^{(i)}$

$(3n+4)^{(i)}=(3n+3)^{(i)}+(3n+2)^{(i)}$

In the case a/ we have $(3n+r)^{(i)}=2^{r+1}6^n$, $r=0,1,2$.
Taking into consideration b/ and c/ we get
$h_{A'}^{top}(x)=(\log 6)/3$.
Next, we group x into new product
$x=00\times(B\times B)\times 00\times(B\times B)\times\ldots$ and we obtain $h_A^{top}(x)=(\log 3)/2$
and consequently $h_A^{top}(x)<h_{A'}^{top}(x)$.

Remark 3.5 Let us observe that x which we have just considered is a continuous substitution $0\longmapsto(00\times B\times B)$. Such shifts were examined by Dekking in [Dek] . It is clear that used here A and A' are different from the sequences in [Dek] .

We are able also to strenghten Dekking's result showing $0=h_A(T)<h_{A'}(T)$. Indeed, putting

$$x=\underbrace{00\times\ldots\times 00}_{i_1}\times\underbrace{B\times\ldots\times B}_{i_2}\times\underbrace{00\times\ldots\times 00}_{i_3}\times\underbrace{B\times\ldots\times B}_{i_4}\times\ldots$$

grouping

$$x=00\times\ldots\times 00\times(B\times\ldots\times B)\times 00\times\ldots\times 00\times(B\times\ldots\times B)\times\ldots$$

and assuming

$$\lim_{k\to\infty} i_k / \sum_{j=1}^{k} i_j = 1$$

one can get

$h_{A'}^{top}(x)$, $h_{A'}(x) > 0$. But from Theorem 3.3 $h_A^{top}(x)=0$, so the more $h_A(x)=0$. However in our example A grows quicker than exponentially in opposite to Dekking's example.

Example 3.4 On the variational principle.

Goodman in [Goodm] has given an example of automorphism τ for that $h_A^{top}(\tau)=\log 2$ and $\sup_{\mu\in M} h_A^{\mu}(\tau)=0$ for some $A\subseteq N$. His τ had discrete spectrum.

We see that for every Morse sequence $x=b^0 \times b^1 \times \ldots$ the set $\mathcal{O}_x$ is strictly ergodic, so $\sup_{\mu\in M} h^{\mu}_{\{n_t\}}(T)=h_{\{n_t\}}(x)$. If x is not regular then we can assume that $b^i \neq 0\ldots0, 01\ldots01, 01\ldots010$ and $h_{\{n_t\}}(x)=0$.

But from Theorem 3.3 it follows that $h^{top}_{\{n_t\}}(x)=\log 2$, therefore we obtain an automorphism x with Goodman's property altough this automorphism has partly continuous spectrum.

Suppose (X,T,μ), (Y,τ,ν) are strictly ergodic systems and $A\subseteq N$. Does $h_A(T)=h_A(\tau)$ imply $h_A^{top}(T)=h_A^{top}(\tau)$ and vice versa?

Let $x=b^0\times b^1\times\ldots$ and $y=\beta^0\times\beta^1\times\ldots$, $y\in M(x)$ be Morse sequences b^i, $\beta^i\neq 0\ldots0, 01\ldots01, 01\ldots010$, x is not regular with $h_{\{n_t\}}(x)=0$ and y is regular. Thus $h_{\{n_t\}}(y)>0$ and $h^{top}_{\{n_t\}}(x)=h^{top}_{\{n_t\}}(y)=\log 2$.

Now, let

$$x=\underbrace{01\ldots01}_{\lambda_0}\times\underbrace{01\ldots01}_{\lambda_1}\times\underbrace{01\ldots01}_{\lambda_2}\times\ldots$$

$$y=\underbrace{01\ldots0100}_{\lambda_0}\ \underbrace{01\ldots0100}_{\lambda_1}\ \underbrace{01\ldots0100}_{\lambda_2}\ \ldots$$

and assume

/23/ $$\sum_{k=0}^{\infty} 1/\lambda_k < \infty$$

Writing

$x=b^0\times b^1\times\ldots$ and $y=\beta^0\times\beta^1\times\ldots$ we see that

/i/ x and y are continuous because λ_k, $k\geqslant 0$ are even,

/ii/ $\sum_{k=0}^{\infty} d(b^k,\beta^k)=\sum_{k=0}^{\infty}\operatorname{card}\{i:b^k[i]\neq\beta^k[i]\}/\lambda_k=\sum_{k=0} 1/\lambda_k<\infty$

Then from a recent Kwiatkowski's result [Kw3] it follows that x and y are metrically isomorphic. From /9/ it follows that $h^{top}_{\{n_t\}}(y)=\log 2$, $h^{top}_{\{n_t\}}(x)=\log(1+\sqrt{5})/2$. In this way we show that the topological sequence entropy can distinguish between two strictly ergodic and metrically isomorphic systems.

Example 3.5 An answer to Goodman's question.

In [Goodm] Goodman has shown that $h^{top}_A(T\times T)=2h^{top}_A(T)$, $h^{top}_A(T\times T')\leq h^{top}_A(T)+h^{top}_A(T')$ and he asked whether $h^{top}_A(T\times T')=h^{top}_A(T)+h^{top}_A(T')$.

Let $x=b^0\times b^1\times\ldots$, $y=\beta^0\times\beta^1\times\ldots$, $y\in M(x)$ be Morse sequences and denote Q,Q' respectively the natural generators for $(\mathcal{O}_x,T,\mu_x)$ and $(\mathcal{O}_y,T,\mu_y)$. Then $Q\times Q'$ is an open partition of $\mathcal{O}_x\times\mathcal{O}_y$. Moreover $(Q\times Q')^m_k=Q^m_k\times Q'^m_k$ and

/24/ $N\,(Q\times Q')^m_k=N(Q^m_k)\cdot N(Q'^m_k)$, $m,k\geqslant 1$.

From /8/ it follows that

$N\,(Q\times Q')^m_k\leq 4(2^k)^t(m-t+1)^2N(Q^m)N(Q'^m)=4(2^k)^t(m-t+1)^2N(Q\times Q')^m$

Hence

$$h^{top}_{\{n_t\}}(T\times T\ ,(Q\times Q')_k)=\lim\sup_{m\in N}\frac{1}{m}\log N\left((Q\times Q')^m_k\right)\leq$$

$$\leq \lim\sup_{m\in N}\frac{1}{m}\log\left\{4(2^k)^t(m-t+1)^2N((Q\times Q')^m)\right\}=h^{top}_{\{n_t\}}(T\times T,Q\times Q')$$

Since $(Q\times Q')_k$ refines $Q\times Q'$ and $(Q\times Q')_k \nearrow \varepsilon$, from /1/ we get $h^{top}_{\{n_t\}}(T\times T,Q\times Q')=h^{top}_{\{n_t\}}(T\times T)$.

Put

$$x=\underbrace{000\times\ \ldots\ \times 000}_{\substack{\text{block 000 is}\\ \text{repeated } i_o\\ \text{times}}}\times\underbrace{011\times\ldots\times 0\overset{\downarrow}{1}1}_{i_1}\times\underbrace{000\times\ldots\times 000}_{i_2}\times\underbrace{011\times\ldots\times 0\overset{\downarrow}{1}1}_{i_3}\times\ldots$$

$$y=\underbrace{011\times\ldots\times 0\overset{\downarrow}{1}1}_{i_o}\times\underbrace{000\times\ldots\times 000}_{i_1}\times\underbrace{011\times\ldots\times 0\overset{\downarrow}{1}1}_{i_2}\times\underbrace{000\times\ldots\times 000}_{i_3}\times\ldots$$

and assume

$$/25/ \qquad \lim_{j\to\infty} i_j/\sum_{s=0}^{j} i_s=1$$

It is not difficult to verify that $h^{top}_{\{3^t\}}(x)=h^{top}_{\{3^t\}}(y)=\log 2$. Indeed, it is sufficient to compute $i^{(i)}$ for the subsequences denoted by arrows and to apply /25/.

We will prove

$$/26/ \qquad h^{top}_{\{3^t\}}(x)=h^{top}_{\{3^t\}}(y)=h^{top}_{\{3^t\}}(T\times T).$$

In order to prove /26/ we shall estimate $i^{(i)}$, $\bar{i}^{(\bar{i})}$ where $\bar{i}^{(\bar{i})}$ denotes the number defined in /9/ for y.

Take $i\in N$. Then there is $k\in N$ and s, $0\leq s\leq i_{k+1}$ such that $i=s+i_k+\sum_{t=0}^{k-1} i_t$. Put $\sum_{t=0}^{k-1} i_t=u_{k-1}$:

From /9/ we get

$i^{(i)}\leq 2^s\ i_k\ 2^{u_{k-1}}$

$\bar{i}^{(\bar{i})}\leq s\ 2^{i_k}\ 2^{u_{k-1}}$ or it is necessary to replace i^i by $\bar{i}^{(\bar{i})}$ and vice versa.

We have

$$\log\left(i^{(i)}\bar{i}^{(\bar{i})}/i\right) \leq \frac{2u_{k-1}}{i}\log 2 + (\log s)/i + (\log i_k)/i +$$
$$+ \frac{s+i_k}{i}\log 2.$$

If i tends to infinity then $\log(i^{(i)}\bar{i}^{(\bar{i})}/i)$ tends to $\log 2$. Therefore /26/ is valid.

<u>Example 3.6</u> $\{2^t\}$-entropy of Kakutani sequences.
Putting b^t=00 or 01, $t \geq 0$ we obtain the class of Kakutani sequences.

From /11/ it follows that in this case

$$/27/ \quad p_1^{t+1} = \begin{cases} \lambda_t(p_1^t - \pi_{01}^t) & \text{if } b^t[\lambda_t - 1] = 0 \\ \lambda_t(\pi_{01}^t - p_1^t) & \text{if } b^t[\lambda_t - 1] = 1 \end{cases}$$

where

$$\pi^t = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{or} \quad \pi^t = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

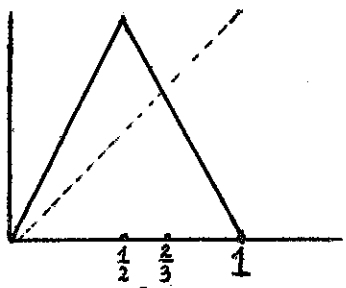
Due to this for every Kakutani sequence and for every $t \geq 0$

$$/28/ \qquad p_1^{t+1} = 2\min(p_1^t, 1-p_1^t).$$

From Theorem 3.4 it follows that

$$/29/ \qquad h_{\{2^t\}}(x) = \log 2 \limsup_{t\in N} \frac{1}{t}\sum_{i=0}^{t-1} p_1^i$$

In connection with /28/ consider the transformation of unit interval into itself $R:[0,1]^{\circlearrowleft}$, $R(d) = 2\min(d, 1-d)$
/ Picture 1 /



Picture 1

Consider also the transformation P mapping every Kakutani sequence x to its p_1.

The formulas /13/ imply that P is one-to-one. Only dyadic rational numbers are not in the image / they would correspond to the sequences with $b^t=00$ for all t sufficiently large, but such sequences have been excluded /.

The correspondence P conjugates left shift S on the space of Kakutani sequences with R. In the measure-theoretic sense P conjugates one-sided Bernoulli shift S with R on [0,1] considered with the Lebesgue measure / which is clearly R-invariant and ergodic /.

By /28/ for almost every x

$$h_{\{2^t\}}(x) = \frac{1}{2} \log 2$$

Let us observe that $h_{\{2^t\}}$ takes its maximal value $\frac{2}{3} \log 2$ at $x_o=01{\times}01{\times}\ldots$. Indeed, we see that x_o is a fixed point under S. Hence $P(x_o)$ must be fixed under R, so $P(x_o)=\frac{2}{3}$ and next we analyze time averages under R, $\frac{1}{t}\sum_{i=0}^{t-1} R^i(d)$. Moreover, we have $h^{top}_{\{2^t\}}(y) \leq h^{top}_{\{2^t\}}(x_o) = \log(1+\sqrt{5})/2$ for every Kakutani sequence y. Indeed, it is a simple consequence of the equality $(m+1)^{(i)} = m^{(i)} + \max(m_o^{(i)}, m_1^{(i)})$ for the sequence $\{i^{(i)}\}$ defined in /9/ for x_o.

<u>Example 3.7</u> The formula $h_A(T{\times}T') = h_A(T) + h_A(T')$ fails.

Let

$$x = \underbrace{\overbrace{01{\times}\ldots{\times}01}^{j_o}{\times}\overbrace{\ldots{\times}01}^{k_o}}_{i_o}{\times}\underbrace{\overbrace{00{\times}\ldots{\times}00}^{j_1}{\times}\overbrace{\ldots{\times}00}^{k_1}}_{i_1}{\times}\underbrace{\overbrace{01{\times}\ldots{\times}01}^{j_2}{\times}\overbrace{\ldots{\times}01}^{k_2}}_{i_2}{\times}\ldots$$

$$y=\underbrace{\overbrace{00\times\ldots\times00}^{j_0}\times\overbrace{\ldots\times00}^{k_0}}_{i_0}\times\underbrace{\overbrace{01\times\ldots\times01}^{j_1}\times\overbrace{\ldots\times01}^{k_1}}_{i_1}\times\underbrace{\overbrace{00\times\ldots\times00}^{j_2}\times\overbrace{\ldots\times00}^{k_2}}_{i_2}\times\ldots$$

and let $\{\varepsilon_s\}_{s\geqslant 0}$ be a sequence of positive numbers such that

/30/ $\lim\limits_{s\to\infty}\varepsilon_s=0$

Put

$$l_s=\sum_{r=0}^{s-1} i_r+j_s$$

and assume

1/ $j_s/l_s \xrightarrow[s]{} 1$

2/ $\sum\limits_{i=0}^{k_s} 1/2^{2i+1} > \frac{2}{3}-\varepsilon_s$

3/ $\sum\limits_{i=k_s+1}^{\infty} 1/2^{i} < \varepsilon_s$

Let us observe that 2/ can be obtained from the simple fact $\sum\limits_{i=0}^{\infty} 1/2^{2i+1}=2/3.$

We have

$$\frac{1}{l_s}\sum_{i=0}^{l_s} p_1^i \geqslant \frac{1}{l_s}\sum_{i=l_s-j_s}^{l_s} p_1^i \geqslant \begin{cases}(\frac{2}{3}-2\varepsilon_s)\frac{j_s}{l_s} & \text{for } x \text{ and } s \text{ even}\\ (\frac{2}{3}-2\varepsilon_s)\frac{j_s}{l_s} & \text{for } y \text{ and } s \text{ odd}\end{cases}$$

Due to /29/ and /30/

$$h_{\{2^t\}}(x)=h_{\{2^t\}}(y)=\frac{2}{3}\log 2$$

Applying considerations from Example 3.5 we see that

$$h^{top}_{\{2^t\}}(T\times T)\leqslant \log 2$$

Therefore from the variational principle

$$h_{\{2^t\}}(T\times T)\leqslant \log 2 < h_{\{2^t\}}(x)+h_{\{2^t\}}(y)\ .$$

<u>Remark 3.6</u> Example 3.7 shows that Lemma 4 in [Ku] is false. We have only $h_A(T\times T)=2h_A(T)$ for every sequence A and automorphism T. Let us observe that here and in

Example 3.5 we make use of the fact that the limits

$$\lim_{t\to\infty} \frac{1}{t} \log N(T^{-n_0}Q \vee \ldots \vee T^{-n_{t-1}}Q)$$

$$\lim_{t} \frac{1}{t} H(T^{-n_0}Q \vee \ldots \vee T^{-n_{t-1}}Q)$$

do not exist.

Example 3.8. On the Abramov's formula.

As a simple consequence of Abramov s formula in [Ab-Roh] we get

Proposition 3.3 Let $T:X\circlearrowleft$ be an ergodic automorphism of a Lebesgue space (X,μ) and let $\tau:Y\circlearrowleft$ be its factor-automorphism via $\varphi:X\longrightarrow Y$. If φ is of finite order i.e. $\operatorname{card}\{\varphi^{-1}y\}$ is finite and constant for a.e. $y\in Y$ then $h(T)=h(\tau)$.

For sequence entropy this Proposition is false. Indeed, take $X=\mathcal{O}_x$ where $x=b^0\times b^1\times\ldots$ is a Morse sequence and $Y=\mathcal{O}_x/\eta$ where η is the partition of $\mathcal{O}_x$ into the sets of the form $\{z,\tilde{z}\}_{z\in\mathcal{O}_x}$. Then $(Y,T/\eta,\mu_x/\eta)$ is an ergodic system with discrete spectrum / [Kw2] /, so all sequence entropies of T/η are equal to zero /[Ku]/.

Let us see that the well-known formula for induced automorphism $h(T_C)=h(T)/\mu(C)$ cannot hold for sequence entropy for ergodic automorphisms with discrete spectra because the family of all induced automorphisms gives / up to isomorphism / all loosely Bernoulli automorphisms with zero entropy / see [Or-Ru-We]/.

<u>Remark 3.8</u> No Morse sequences have weakly mixing factors.

Let T be an automorphism of a Lebesgue space. We say T is bounded if there is $c > 0$ such that $h_A(T) \leq c$ for every sequence $A \subseteq N$.

From [S] easily follows that if T is bounded then T has no weakly mixing factors.

Now, let T be an automorphism and τ be its bounded factor-automorphism via φ. Then, if φ is of finite order then T has no weakly mixing factors. Indeed, it is sufficient to show that T is also bounded. To prove this we use the formula

/31/ $h_A(T,\sigma) \leq h_A(T,\sigma') + H(\sigma|\sigma')$ / [S] /

Let $\sigma = (\sigma_1, \dots \sigma_u)$ be any partition of X. We can constuct a partition $\eta = \{\eta_{i_1 \dots i_n} : i_j \neq i_k$ for $j \neq k$, $j,k = 1,\dots,u\}$ where $\eta_{i_1 \dots i_n} = \varphi \sigma_{i_1} \cap \dots \cap \varphi \sigma_{i_n}$.

Put $\sigma' = \varphi^{-1}\eta$. Then from /31/

$$h_A(T,\sigma) \leq h_A(\tau,\eta) + H(\sigma|\sigma')$$

But each atom of σ' splits into at most n atoms of σ, so we have $h_A(T) \leq h_A(\tau) + \log n$. Now, let $x = b^0 \times b^1 \times \dots$ be a Morse sequence. Since σ-algebra of mirror-invariant sets gives a factor with discrete spectrum via a map of the order two / Example 3.8 / we have proved the result stated above.

VI. <u>Disjointness</u> For the notion of disjointness of automorphisms we refer to [F] .

Two automorphisms of Lebesgue spaces $(X_1, \mathcal{B}_1, \mu_1)$

$(X_2, \mathcal{B}_2, \mu_2)$,respectively, are said to be disjoint if, whenever T_1, T_2 are factors of $T:(X,\mathcal{B},\mu)$ via φ_1, φ_2 respectively then $\varphi_1^{-1}(\mathcal{B}_1)$, $\varphi_2^{-1}(\mathcal{B}_2)$ are independent, i.e. $\mu(\varphi_1^{-1}(B_1) \cap \varphi_2^{-1}(B_2)) = \mu(\varphi_1^{-1}(B_1))\mu(\varphi_2^{-1}(B_2))$, $B_i \in \mathcal{B}_i$, $i=1,2$.

Now, let $x = b^0_x b^1_x \ldots$ be a Morse sequence and $\theta(x) = (\mathcal{O}_x, \mathcal{B}_x, T, \mu_x)$ be a dynamical system determined by x. The proof of the below Theorem is a simplified version of the proof of Theorem I.4 [F] .

Theorem 3.5 An ergodic dynamical system $(Y, \mathcal{B}, \tau, m)$ is disjoint from $\theta(x)$ iff $T \times \tau$ is ergodic.

Proof Sufficiency Let $\theta_\sim(x) = (\mathcal{O}_x/\sim, T/\sim, \mu_x/\sim, \mathcal{B}_x/\sim)$ be the dynamical system with discrete spectrum induced by the partition $\{y, \tilde{y}\}_{y \in \mathcal{O}_x}$. Thus $\theta_\sim(x)$ is ergodic and $Sp(T/\sim) = G\{n_t : t \geq 0\}$ and moreover $T/\sim$ and τ are disjoint. Indeed, we have $Sp(T) \supseteq Sp(T/\sim)$ and $Sp(T) \cap Sp(\tau) = \{1\}$ because of $T \times \tau$ is ergodic. Hence $Sp(T/\sim) \cap Sp(\tau) = \{1\}$, so the spectral measures of $T/\sim$ and τ are singular mod 1 so from [Hah-Pa] it follows that $T/\sim$ and τ are disjoint.

Let μ be any $T \times \tau$ -invariant measure on $\mathcal{O}_x \times Y$ satisfying the following

/i/ $\mu(A \times Y) = \mu_x(A)$ $\quad A \in \mathcal{B}_x$

/ii/ $\mu(\mathcal{O}_x \times B) = m(B)$ $\quad B \in \mathcal{B}$

/iii/ $\mu(A \times B) = \mu_x(A) m(B)$ $\quad A \in \mathcal{B}_x$, $\sigma A = A$, $B \in \mathcal{B}$

We have to show that $\mu = \mu_x \times m$. We define a measure $\mu_\sim$ on $\mathcal{O}_x \times Y$ putting $\mu_\sim(A \times B) = \mu(\sigma A \times B)$, $A \in \mathcal{B}_x$, $B \in \mathcal{B}$ and let $\hat{\mu}(A) = (1/2)(\mu(\hat{A}) + \mu_\sim(\hat{A}))$, $A \in \mathcal{B}_x \otimes \mathcal{B}$.

Let us observe that

/32/ $\mu \ll \hat{\mu}$

/33/ $\hat{\mu}$ is $T\times\tau$ -invariant

Moreover

/34/ $\hat{\mu} \ll \mu_X \times m$

Indeed, we have $\hat{\mu}(A\times B)=\frac{1}{2}\left(\mu(A\times B)+\mu(\sigma A\times B)\right)\leq\mu((A\cup\sigma A)\times B)=$
$=\mu_X(A\cup\sigma A)m(B)\leq \mu_X(A)\cdot m(B) + \mu_X(\sigma A)\ m(B)=2\mu_X(A)m(B)$.
From /32/ and /34/ it follows that $\mu \ll \mu_X\times m$. Since $\mu_X\times m$ is ergodic and μ is $T\times\tau$ -invariant, $\mu=\mu_X\times m$.

<u>Necessity</u> If $T\times\tau$ is not ergodic then there is $m_t\in N$ such that $\exp(2\pi i/m_t)\in Sp(T)\cap Sp(\tau)$ / m_t need not be of the form n_t but any Morse sequence can have only roots of unity in its spectrum /. So T and τ have a common non-trivial factor and therefore they cannot be disjoint / [F] /.

We let $\mathcal{M}$ denote the class of all Morse shifts and $\mathcal{M}^{\perp}$ denote the class of all ergodic dynamical systems which are disjoint from any Morse sequence.

<u>Corollary 3.4</u> $\tau\in\mathcal{M}^{\perp}$ iff there are no roots of unity in $Sp(\tau)$. In particular, weakly mixing, totally ergodic, quasi-discrete spectrum automorphisms contain in $\mathcal{M}^{\perp}$.

<u>Proof</u> It is a simple consequence of the fact that any n_t-root of unity is in point spectrum of some Morse sequences.

Let us return to the factor problem. We have already observed in Remark 3.8 that Morse shifts have no weakly mixing factors / this fact can be also obtain from Corollary 3.4 /. We will show that that fact is sufficient to give all factors for some Morse sequences.

<u>Proposition 3.4</u> Let $x=b^0\times b^1\times\ldots$ be a continuous Morse sequence. Then the only factors with finite point spectrum are rotations ϱ_m on Z_m / p.28 /, $m|n_t$, $t>0$.

<u>Proof</u> Let $\varphi:(\mathcal{O}_x,T,\mu_x)\to(Y,\tau,\mu)$ be a factor-homomorphism and suppose $Sp(\tau)$ is a finite subgroup of $G\{n_t:\ t\geqslant 0\}=Sp(x)$. Therefore $Sp(\tau)$ is the group of all m-roots of unity for some m and there is $t\geqslant 0$ such that $m|n_t$. So we have $mk=n_t$. Write $x=b^0\times b^1\times\ldots$ as $x=c_t\times b^t\times b^{t+1}\times\ldots$ and let B, C be any blocks starting with 0, $|B|=m$, $|C|=k$. Then from [Kw 2] easily follows that $x'=B\times C\times b^t\times b^{t+1}\times\ldots$ is a Morse sequence which is isomorphic to x, so without loss of generality we may assume $x'=x$. From [Ke 1] it follows that

$$/35/\qquad T^m\big|_{D_i^m}\ \text{is isomorphic to}\ x_1=C\times b^t\times b^{t+1}\times\ldots\qquad i=0,\ldots,m-1$$

Since $\exp(2\pi i/m)\in Sp(\tau)$ then there is a m-τ-stack $(A,\tau A,\ldots,\tau^{m-1}A)$, $\bigcup_{i=0}^{m-1}\tau^iA=Y$. Let us consider $\tau^m|_A$. This automorphism is totally ergodic. Indeed, otherwise there is λ, $\lambda^s=1$ for some $s\geqslant 2$ and $\lambda\in Sp(U)$, $U=\tau^m|_A$. So there is $B\subseteq A$ such that $B\cup UB\cup\ldots\cup U^{s-1}B=A$, $(B,\ldots,U^{s-1}B)$ is a U-s-stack. But $(B,\tau B,\ldots,\tau^{sn-1}B)$ is a τ-sn-stack and a contradiction because of $s\geqslant 2$.

We get $\varphi^{-1}(A)=D_i^m$ for some i, $0 \le i \le m-1$, so

$$\begin{array}{ccc} D_i^m & \xrightarrow{\varphi} & A \\ T^m|_{D_i^m} \downarrow & & \downarrow U \\ D_i^m & \xrightarrow[\varphi]{} & A \end{array}$$

where $T^m|_{D_i^m}$ and U are considered with normalized measures $\frac{1}{m}\mu_x$ and $\frac{1}{m}\mu$.

From /35/ it follows that x_1 has U as a factor. Hence $Sp(U)$ can contain only roots of unity. Therefore totally ergodicity of U implies U is weakly mixing. We conclude that U is trivial, so τ is equal to ϱ_m.

<u>Corollary 3.5</u> Let $x=b^0 \times b^1 \times \ldots$ be a continuous Morse sequence and let $\lambda_t=p^{k_t}$ where p is a fixed prime number and $k_t \geqslant 1$. Then the only nontrivial factors of x are

/i/ dynamical system with discrete spectrum $Sp(\tau)=G\{n_t : t \geqslant 0\}$

/ii/ rotations ϱ_{p^s} on Z_{p^s}, $s \geqslant 0$.

<u>Proof</u> Applying Proposition 3.4 we can only consider factor-automorphisms with $Sp(\tau)$ is infinite. Thus by our assumption $Sp(\tau)= Sp(T)$. Then the same reasoning as in the proof of Proposition 2.1 makes our proof complete.

Because of any Kakutani sequence is continuous / $\lambda_t=2$ are even, $t \geqslant 0$ /, we have obtained

<u>Corollary 3.6</u> If $x=b^0 \times b^1 \times \ldots$ is a Kakutani sequence then the only factors are dynamical system with discrete spectrum $Sp(\tau)=G\{2^t : t \geqslant 0\}$ and the rotations on Z_{2^t}, $t \geqslant 0$.

ISOMORPHISM OF G-SYMBOLIC MORSE-SHIFTS

I. Introduction Let G be an abelian group of the order n, $n \geqslant 2$. Consider the space $X=G^Z$ of all two-sided sequences of elements of G and the shift transformation $T: X \circlearrowleft$.

Each element $B=(b_0,\dots,b_{m-1})$, where $b_i \in G$, $i=0,\dots,m-1$ is called a block over G. We put $m=|B|$ and call it the length of B. Denote $B[i,j]=(b_i,b_{i+1},\dots,b_j)$, $B[i,i]=B[i]$ If C is another block over G we define

$$B \times C = \sigma_{c_0}(B)\,\sigma_{c_1}(B)\dots\,\sigma_{c_{k-1}}(B) \qquad C=(c_0,\dots,c_{k-1})$$

where $\sigma_g(B)$ is defined by the formula

$$\sigma_g(B)[j]=B[j]+g \qquad j=0,\dots,m-1,\ g \in G.$$

By the same way we can define $\sigma_g(y)$, $y \in X$.

Let $b^0, b^1, b^2, \dots$ be finite blocks over G with the lengths at least two, beginning with zero and let

/1/ $\quad x=b^0 \times b^1 \times b^2 \times \dots$

Assume that x is nonperiodic and find an almost periodic point $\omega \in X$ with $\omega[k]=x[k]$, $k \geqslant 0$ /for details see [Ma2]/. Let $\mathcal{O}_x=\overline{\{T^k\omega \ : \ k \in Z\}}$.

Definition 4.1 The sequence x defined by /1/ is called a generalized Morse sequence /shortly a Morse sequence/ over G if $\mathcal{O}_x$ is strictly ergodic.

Denote by μ_x the unique T-invariant measure and put $\Theta(x)=(\mathcal{O}_x, T, \mu_x)$. $\Theta(x)$ is called a Morse dynamical system determined by x. If no confusion arises we shall consider properties of x instead of $\Theta(x)$.

Set

/2/ $c_t=b^0\times b^1\times\ldots\times b^t$, $\lambda_t=|b^t|$, $n_t=|c_t|=\lambda_0\cdot\ldots\cdot\lambda_t, t\geq 0$

By Sp(x) we mean the set of all eigenvalues of the unitary operator U_T: $L^2(X,\mu_x)\circlearrowleft$, $U_T(f)=f\circ T$.

We let $G\{n_t, t\geq 0\}$ denote the subgroup of the group of all roots of unity generated by $\{\exp 2\pi i/n_t, t\geq 0\}$. It is known that $G\{n_t, t\geq 0\}\subset Sp(x)$ / see [Ma2]/. If $G\{n_t, t\geq 0\}=Sp(x)$ then x is said to be a continuous Morse sequence over G. There are some conditions deciding whether x is continuous or not / Theorem 14 [Ma2]/

Throughout this section we shall consider continuous Morse sequences.

II. Morse sequences over G and $\bar{G}$ are not isomorphic

We begin with recalling some properties of Morse sequences over G that we need in the following.

Let $x=b^0\times b^1\times\ldots$ be a Morse sequence over G.

/i/ [Ma2] For any $t\geq 0$ there exists k_t such that any block over G with $|B|=k_t$ has the property that whenever $B=x[j,j+k_t-1]=x[m,m+k_t-1]$ then $j=m$ /mod n_t/.

/ii/ [Ma2] $D_k^{n_t}=\{y\in\mathcal{O}_x: y[-k+jn_t,-k+(j+1)n_t-1]=\sigma_{g_j}(c_t)$ $j\in Z$, $t\geq 0$, $k=0,\ldots,n_t-1$. We set $D^{n_t}=\{D_0^{n_t},\ldots D_{n_t-1}^{n_t}\}$

In this way we partition $\mathcal{O}_x$ into open and closed subsets and $T(D^{n_t}_k)=D^{n_t}_{k+1}$, $0\leq k\leq n_t-1$, $T(D^{n_t}_{n_t-1})=D^{n_t}_o$.

/iii/ [M&L] For every $g\in G$, σ_g is a homeomorphism from $\mathcal{O}_x$ into itself which commutes with T and $\sigma_g(D^{n_t}_k)=D^{n_t}_k$, $t\geq 0$, $0\leq k\leq n_t-1$.

Let $G,\bar{G}$ be finite abelian groups. Assume $x=b^0\times b^1\times \ldots$ is a Morse sequence over G and $\bar{x}=\bar{b}^0\times\bar{b}^1\times\ldots$ is a Morse sequence over $\bar{G}$.

Let $|b^t|=\lambda_t$, $|\bar{b}^t|=\bar{\lambda}_t$, $\{n_t\}$, $\{\bar{n}_t\}$ be corresponding sequences defined in /2/.

If x and $\bar{x}$ are isomorphic then we can assume that $n_t|\bar{n}_t$, $t\geq 0$ / grouping $\bar{x}$ into a new product of blocks over $\bar{G}$ if necessary /. Indeed, it is a simple consequence of the equality $Sp(x)=Sp(\bar{x})$ and their contunuoity.

<u>Theorem 4.1</u> If $\Theta(x)$ and $\Theta(\bar{x})$ are isomorphic then G and $\bar{G}$ are isomorphic.

<u>Proof</u> Let us suppose x and $\bar{x}$ are isomorphic. Then we may assume $n_t|\bar{n}_t$, $t\geq 0$.

Let $\varphi:\mathcal{O}_{\bar{x}}\longrightarrow\mathcal{O}_x$ establishe an isomorphism between them.

Fix $t\in N$ and let D^{n_t} and $\bar{D}^{\bar{n}_t}$ be corresponding partitions / resp. n_t in $\mathcal{O}_x$ and $\bar{n}_t$ in $\mathcal{O}_{\bar{x}}$ /. Since φ is an isomorphism $\varphi\bar{D}^{\bar{n}_t}=\{\varphi\bar{D}^{\bar{n}_t}_o,\ldots,\varphi\bar{D}^{\bar{n}_t}_{\bar{n}_t-1}\}$ is a $\bar{n}_t$-stack for T on $\mathcal{O}_x$ so considerations as in Section II of

Chapter 2 show that

/3/ $\quad \varphi\, \bar{D}^{\bar{n}_t} = D^{\bar{n}_t}\ , \ t \geqslant 0.$

Now, let $\bar{g} \in \bar{G}$. Then by /iii/ $\sigma_{\bar{g}} : \bar{D}_k^{\bar{n}_t} \circlearrowleft$, $k=0,\dots,\bar{n}_t-1$, consequently by /3/ $\varphi \sigma_{\bar{g}} \varphi^{-1} : D_k^{\bar{n}_t} \circlearrowleft$, $k=0,\dots,\bar{n}_t-1$ and again by /iii/ $\varphi \sigma_{\bar{g}} \varphi^{-1}$ commutes with T on $\mathcal{O}_x$. In particular $\varphi \sigma_{\bar{g}} \varphi^{-1} : D_j^{n_t} \circlearrowleft$, $j=0,\dots,n_t-1$ because $D^{n_t} \leqslant D^{\bar{n}_t}$.

We get the following

/4/ $\quad \varphi \sigma_{\bar{g}} \varphi^{-1} : D_j^{n_t} \circlearrowleft\ ,\ \bar{g} \in \bar{G},\ t \geqslant 0,\ j=0,\dots,n_t-1$

At present we prove that the only automorphisms h of the space $\mathcal{O}_x$ satisfying hT=Th and $h : D_j^{n_t} \circlearrowleft$ for every $t \geqslant 0$ and $j=0,\dots,n_t-1$ are σ_g, $g \in G$. Indeed, first we see that σ_g is μ_x-preserving transformation in view of /iii/ and by strictly ergodicity of $\mathcal{O}_x$. As an immediate consequence of /i/ is the fact that for every $y \in \mathcal{O}_x$ and every $t \in N$

/5/ there is a unique k, $0 \leqslant k \leqslant n_t - 1$ such that
$y[-k+jn_t, -k+(j+1)n_t-1] = \sigma_{g_j}(c_t)$ for $j \in Z$.

In other words y has a unique representation in the form $y = \dots \sigma_{g_{-1}}(c_t)\, \sigma_{g_0}(c_t)\, \sigma_{g_1}(c_t) \dots$ / see [Ma λ] p.345/

Fix again $t \in N$ and let $y \in D_k^{n_t}$. Then by /ii/

/6/ $\ y[-k+jn_t, -k+(j+1)n_t-1] = \sigma_{g_j}(c_t)\ ,\ j \in Z,\ g_j \in G.$

Since $h(y) \in D_k^{n_t}$ we have

/7/ $\ h(y)[-k+jn_t, -k+(j+1)n_t-1] = \sigma_{\hat{g}_j}(c_t)\ ,\ j \in Z,\ \hat{g}_j \in G.$

In particular

/8/ $\ h(y)[-k, -k+n_t-1] = \sigma_{g_0+g}(c_t)$ for some $g \in G$.

If $y \in D_p^{n_{t+1}}$ then

/9/ $y[-p+jn_{t+1}, -p+(j+1)n_{t+1}-1] = \sigma_{g'_j}(c_{t+1}), j \in Z,\ g'_j \in G$

Because of /5/, /6/ and /9/ we get

/10/ $\quad p \geqslant k$ and $-p = -k + j_o n_t$ for some $j_o \in Z$.

Now, we have $h(y)[-p, -p+n_{t+1}-1] = \sigma_{g'_o+g'}(c_{t+1})$ for some $g' \in G$. We see that the block $h(y)[-p, -p+n_{t+1}-1]$ arises from the block $y[-p, -p+n_{t+1}-1]$ by adding g' to each its element and the block $h(y)[-k, -k+n_t-1]$ from $y[-k, -k+n_t-1]$ by adding g. So from /10/ it follows that $g = g'$.

Repeating these considerations infinitely many times we obtain $h(y) = \sigma_g(y)$.

Let $A_g = \{y \in \mathcal{O}_x : \sigma_g(y) = h(y)\}$ and assume $\mu_x(A_g) > 0$ for some $g \in G$. Then $\mu_x(A_g) = 1$ because A_g is T-invariant. So we have proved that h is of required form.

We conclude that for every $\bar{g} \in \bar{G}$ there is only one $g \in G$ such that $\varphi \sigma_{\bar{g}} \varphi^{-1} = \sigma_g$. Moreover, the order of σ_g equals the order of $\varphi \sigma_{\bar{g}} \varphi^{-1}$ and it is equal to the order of $\bar{g}$. Finally we get a bijection $\bar{g} \longmapsto g$ between elements of $\bar{G}$ and G which preserves the orders of elements. Since $\bar{G}$ and G are finite abelian groups they must be isomorphic.

III. <u>Remarks</u> Let $x = b^0 \times b^1 \times \ldots$ be a continuous Morse sequence over G. Martin in [Ma1] showed that $C^{top}(x) = \{T^k \sigma_g : k \in Z,\ g \in G\}$. In this way he proved that Morse sequences over G are not topologically isomorphic to Morse sequences over $\bar{G}$ whenever G and $\bar{G}$ are not isomorphic. Since $\mathcal{O}_x$ is strictly ergodic Theorem 4.1

is stronger than the above.

Hamachi in [Ham] introduced a group homomorphism $\Psi : C(x) \longrightarrow \mathrm{End}(\mathrm{Sp}(x))$ defined in the following way. If $U \in C(x)$ then the function $f_t(Ud)/f_t(d)$ is T-invariant so by ergodicity of T there is $\Psi_U(1/n_t) \in \mathrm{Sp}(x)$ so that

$$f_t(Ud) = \exp(2\pi i \Psi_U(1/n_t))\, f_t(d).$$

Here we use additional notation regarding Sp(x) as a countable subgroup of $[0,1)$ with addition mod 1 and $f_t = \sum_{k=0}^{n_t-1} \exp(2\pi ik/n_t)\, \chi_{D_k^{n_t}}$ is the eigenfunction corresponding to $1/n_t$.

If the sequence $\{\lambda_t\}_{t=0}^{\infty}$ has the property if a prime p divides λ_{t_0} then p divides infinitely many of λ_t's then some straightforward arguments show that End(Sp(x)) is a torsion-free group. In particular $\Psi(C(x)) \subseteq \mathrm{End}(\mathrm{Sp}(x))$ is one. Therefore all of the torsion elements are carrying by Ψ into zero-homomorphism in End(Sp(x)).

Arguing as in the end of the proof of Theorem 4.1 we get a new proof of Theorem 4.1 in this case. This can be used to prove that if constant Morse sequence $x = b \times b \times \ldots$ over G and $\bar{x} = \bar{b} \times \bar{b} \times \ldots$ over $\bar{G}$ / they need not be continuous when the order of G is not a prime [Ma 2]/ are isomorphic then G and $\bar{G}$ are isomorphic. Such sequences may be obtained from substitution of constant length / [Dek] /.

We see that if $\tau : \mathcal{O}_x \circlearrowleft$ is a n_t-root of T /i.e. $\tau^{n_t} = T$ / then $\tau \in C(x)$ and τ is ergodic. In addition

$$\mathrm{id}_{Sp(x)} = \Psi(T) = \Psi(\tau^{n_t}) = n_t \Psi(\tau)$$

and for $1/n_t \in Sp(x)$ we obtain $1/n_t=0$ what is impossible.

As a simple consequence we have there are no Morse sequences which are embeddable in measurable flows.

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