ON THE SELF–SIMILARITY PROBLEM FOR ERGODIC FLOWS

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Abstract. Given an ergodic flow $(T_t)_{t\in\mathbb{R}}$ we study the problem of its self-similarities, i.e. we want to describe the set of these $s \in \mathbb{R}$ for which the original flow is isomorphic to the flow $(T_{st})_{t\in\mathbb{R}}$. The problem is examined in some classes of special flows over irrational rotations and over interval exchange transformations. In particular translation flows on translation surfaces are considered, and, in such a case, it is proved that, under the weak mixing condition, the set of self-similarities has Lebesgue measure zero. For von Neumann special flows over irrational rotations given by Diophantine numbers this set is shown to be equal to $\{1\}$ while for horocycle flows a weak convergence in case of some singular (to the volume measure) measures is shown giving rise to some new equidistribution result.

The problem of self-similarity is also studied from the spectral point of view, especially in the class of Gaussian systems.

Contents

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1. Introduction

Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be an ergodic measurable flow on a standard probability Borel space $(X, \mathcal{B}, \mu)$. Given $s \in \mathbb{R} \setminus \{0\}$ by $\mathcal{T}_s$ denote the flow $(T_{st})_{t\in\mathbb{R}}$. Let

$$I(\mathcal{T}) = \{s \in \mathbb{R} \setminus \{0\} : \mathcal{T} \text{ and } \mathcal{T}_s \text{ are isomorphic}\}.$$

If there exists $s \in I(\mathcal{T}) \setminus \{-1, 1\}$, the flow is called *self-similar* with the scale of self-similarity s. Another weaker symptom of self-similarity for flows is the existence of pairs of distinct real numbers t, s for which the automorphisms T_t and T_s are isomorphic.

A natural example of dynamical system which has plenty of self-similarities is the horocycle flow $(\eta_t)_{t\in\mathbb{R}}$ on any finite surface of constant negative curvature M. If $(\gamma_t)_{t\in\mathbb{R}}$ stands for the geodesic flow on M then

$$\gamma_s \circ \eta_t \circ \gamma_s^{-1} = \eta_{e^{-2s}t} \text{ for all } s, t \in \mathbb{R}, \tag{1}$$

and hence every positive number s is the scale of self-similarity for the horocycle flow. This property yields a lot of information on the dynamics of the flow such as Lebesgue spectrum (see Proposition 1.23 [17]) and mixing of all orders (see Theorem 1 [23]). Our first aim is to study further mixing properties which are consequences of the condition (1). The mixing condition for the flow $(\eta_t)_{t\in\mathbb{R}}$ says that

$$(\eta_t)_*\rho \to \mu \text{ weakly as } t \to \infty \tag{2}$$

for every probability measure ρ absolutely continuous with respect to μ. An application of some ideas from [25] to the property (1) gives an opportunity to extend (2) to measures ρ singular with respect to μ (see Theorem 7, Corollary 9 and Theorem 12). As a consequence we obtain a new result concerning equidistribution theory for horocycle flows (see discussion after Corollary 9).

The next subject of the study is the size of the set $I(\mathcal{T})$ and

$$I_{aut}(\mathcal{T}) = \{(s, t) \in \mathbb{R}^2 : T_s \text{ and } T_t \text{ are isomorphic}\}$$

in relation to some dynamical properties of $\mathcal{T}$. For example, if $\mathcal{T}$ has positive and finite entropy then $h_\mu(\mathcal{T}_s) = |s|h_\mu(\mathcal{T}) \neq h_\mu(\mathcal{T})$, and hence $\mathcal{T}_s$ and $\mathcal{T}$ are not isomorphic for $s \in \mathbb{R} \setminus \{-1, 1\}$; similarly $I_{aut}(\mathcal{T}) \subset \{(s, t) : |s| = |t|\}$. In the zero entropy case, of course, there is no universal bound on the size of $I(\mathcal{T})$ because of the horocycle flow. Nevertheless, as it was proved by Ryzhikov in [23], the absence of mixing for $\mathcal{T}$ implies zero Lebesgue measure of $I(\mathcal{T})$ and zero (two-dimensional) Lebesgue measure of $I_{aut}(\mathcal{T})$. Furthermore, if $\mathcal{T}$ is additionally rigid (i.e. $T_{t_n} \to Id$ for some $t_n \to \infty$) then $\mathcal{T}$ and $\mathcal{T}_s$ are disjoint in the sense of Furstenberg for almost every $s \in \mathbb{R}$, and T_s is disjoint from T_t for almost every $(s, t) \in \mathbb{R}^2$ with respect to the Lebesgue measure (see [25]). In this paper we extend the disjointness result (see Theorem 14) to the class of weakly mixing flows for which there exist $t_n \to \infty$, $0 < \lambda \leq 1$ and a probability Borel measure P on $\mathbb{R}$ such that

$$\liminf_{n\to\infty} \mu(T_{t_n}A \cap B) \geq \lambda \int_{\mathbb{R}} \mu(T_sA \cap B)\, dP(s) \text{ for all } A, B \in \mathcal{B}. \tag{3}$$

As a consequence we obtain that for every translation structure on a compact surface with genus greater than one if a direction flow $\mathcal{F}^\theta$ is weakly mixing then the flows $\mathcal{F}^\theta$ and $\mathcal{F}^\theta_s$ are disjoint for almost all $s \in \mathbb{R}$ and the automorphisms F^θ_s and F^θ_t are disjoint for almost all $(s, t) \in \mathbb{R}^2$ (see Corollary 18).

The property (3) turned out to be useful in proving the absence of self-similarity also for some flows on surfaces that arise from quasi-periodic Hamiltonians flows on the torus by velocity changes. More precisely, if (3) holds for $\lambda = 1$ and $\mathcal{T}$ is not rigid or if (3) holds for some $\lambda > 0$ and $\mathcal{T}$ is not partially rigid then $I(\mathcal{T}) \subset \{-1, 1\}$ (see Theorem 22). This result have been used to prove the absence of self-similarity for special flows built over irrational rotations on the circle (or ergodic interval exchange transformations) and under piecewise absolutely continuous functions. For example, if $T : [0, 1) \to [0, 1)$ is an ergodic interval exchange transformation and $f : [0, 1) \to \mathbb{R}^+$ is a piecewise absolutely continuous function with non-zero sum of jumps then the special flow $\mathcal{T}^f$ is not self-similar.

The absence of self-similarity is observed also for special flows built over ergodic rotations on the circle by α satisfying the Diophantine condition

$$|p - q\alpha| \geq \frac{c}{q}, \quad \text{for some } c > 0 \text{ for all } q \in \mathbb{N},\ p \in \mathbb{Z} \tag{4}$$

and under some piecewise constant roof. Such special flows are partially rigid. Here the absence of self-similarities follows from the mild mixing property which has been proved in [12] for some special classes of piecewise constant roof functions.

In Appendix B we study the reversibility problem for special flows built over irrational rotations $Tx = x + \alpha$ on the circle. Recall that a flow $\mathcal{T}$ on $(X, \mathcal{B}, \mu)$ is *reversible* if there exists an automorphism S of $(X, \mathcal{B}, \mu)$ such that $S \circ T_t = T_{-t} \circ S$ for all $t \in \mathbb{R}$ and $S^2 = I$. If the roof function $f : \mathbb{T} \to \mathbb{R}^+$ is symmetric then a simple observation shows that the special flow $\mathcal{T}^f$ is reversible (see Remark 2). In Appendix B we show the absence of reversibility (even disjointess of $(\mathcal{T}^f)_{-1}$ from $\mathcal{T}^f$) form some piecewise absolutely continuous non-symmetric roof functions. More precisely, using elements of Ratner's theory we prove that if α satisfies the Diophantine condition (4) and f has non-zero sum of jumps then the flows $(\mathcal{T}^f)_{-1}$ and $\mathcal{T}^f$ are disjoint. Furthermore, using the minimal self-joining property of such flows (see [11]) we obtain $I(\mathcal{T}^f) = \{1\}$ and the disjointess of T^f_t from T^f_s for distinct real numbers s and t. Recall that the same property has been observed in [14] for some special flows over Chacon transformation.

Take an arbitrary countable multiplicative subgroup $G \subset \mathbb{R}$. The example of weakly mixing flow $\mathcal{T}$ with the minimal self-joining property and such T_t and T_s are disjoint for distinct s and t allows us to construct (the idea of this construction comes from [25]) a self-similar flow $\mathcal{T}^G$ such that $I(\mathcal{T}^G) = G$ and $\mathcal{T}^G_s$ is disjoint from $\mathcal{T}^G$ for all $s \notin G$.

The self-similarity of dynamical systems can be also considered from the spectral point of view. Let us consider spectral version of $I(\mathcal{T})$:

$$SI(\mathcal{T}) = \{s \in \mathbb{R} \setminus \{0\} : \mathcal{T} \text{ and } \mathcal{T}_s \text{ are spectrally isomorphic}\}.$$

Recall that $\mathcal{T}_{-1}$ is always spectrally isomorphic to $\mathcal{T}$, hence $-1 \in SI(\mathcal{T})$. If $\mathcal{T}$ is spectrally self-similar, i.e. $SI(\mathcal{T}) \neq \{-1, 1\}$ and $SI(\mathcal{T})$ has positive Lebesgue measure then $\mathcal{T}$ has pure Lebesgue spectrum (see Proposition 31). On the other side, if $\mathcal{T}$ has singular continuous spectrum then $SI(\mathcal{T})$ has zero Lebesgue measure and $\mathcal{T}_s$ is spectrally disjoint from $\mathcal{T}$ for almost all s. Moreover, T_s and T_t are spectrally disjoint for almost all $(s, t) \in \mathbb{R}^2$.

We construct ergodic flows which are not self-similar in the unitary category. For this purpose, in Section 9, we deal with Gaussian systems which are completely determined by the spectral measure of the underlying Gaussian process. A construction of measures which is supported by a set which emulate the classical Kronecker set yields a Gaussian flow $\mathcal{T}$ with simple spectrum such that $SI(\mathcal{T}) = \{-1, 1\}$ and

$\mathcal{T}_s$ is spectrally disjoint from $\mathcal{T}$ for $s \neq \pm 1$. Moreover, for some countable multiplicative symmetric subgroups $G \subset \mathbb{R}$ a modification of the above construction yields a Gaussian flow $\mathcal{T}^G$ with simple spectrum such that $SI(\mathcal{T}) = G$ and $\mathcal{T}_s$ is spectrally disjoint from $\mathcal{T}$ for $s \notin G$.

2. Adjoint representations of $\mathbb{R}$

Let B be a separable Banach space. Denote by $\mathcal{L}(B)$ the space of all linear bounded operators on B. Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous bounded representation of $\mathbb{R}$ in $\mathcal{L}(B)$, i.e. the map

$$\mathbb{R} \ni t \mapsto U_t x \in B$$

is continuous for every $x \in B$, and the map

$$\mathbb{R} \ni t \mapsto \|U_t\| \in \mathbb{R}$$

is bounded. Let $C := \sup_{t\in\mathbb{R}} \|U_t\|$. Then the dual representation $(U_t^*)_{t\in\mathbb{R}}$ is bounded and $*$-weakly continuous, i.e. the map

$$\mathbb{R} \ni t \mapsto \langle x, U_t^* y^* \rangle \in \mathbb{R}$$

is continuous for every $x \in B$ and $y^* \in B^*$. Let

$$B_U^{\odot} = B^{\odot} = \{x^* \in B^* : \mathbb{R} \ni t \mapsto U_t^* x^* \in B^* \text{ is continuous}\}.$$

$B^{\odot}$ is a closed (U_t^*)–invariant subspace of B^* which is $*$-weakly dense (see [20] Ch.1). Given $x^* \in B^{\odot}$ let $B^{\odot}(x^*)$ stand for the smallest closed (U_t^*)–invariant subspace of $B^{\odot}$ containing x^*. Then $B^{\odot}(x^*)$ is a separable Banach space.

Let $\mathcal{P}(\mathbb{R})$ stand for the space of all Borel probability measures on $\mathbb{R}$. For every $\sigma \in \mathcal{P}(\mathbb{R})$ and $y^* \in B^*$ let $\int_{\mathbb{R}} U_t^* y^* \, d\sigma(t)$ (see [20] Appendix 2) denote the element of B^* determined by

$$\langle x, \int_{\mathbb{R}} U_t^* y^* \, d\sigma(t) \rangle = \int_{\mathbb{R}} \langle x, U_t^* y^* \rangle \, d\sigma(t) \text{ for any } x \in B.$$

Then $\| \int_{\mathbb{R}} U_t^* y^* \, d\sigma(t) \| \leq C \|y^*\|$. Note that if $B' \subset B^*$ is a (U_t^*)-invariant $*$-weakly closed subspace then for every $y^* \in B'$ and $\sigma \in \mathcal{P}(\mathbb{R})$ we have $\int_{\mathbb{R}} U_t^* y^* \, d\sigma(t) \in B'$.

Although the results of this section are formulated for continuous bounded representations of $\mathbb{R}$, as the proofs show they hold also for such representations of $\mathbb{R}^+$. Denote by λ Lebesgue measure on $\mathbb{R}$.

Lemma 1 (cf. [23], Theorem 3)**.** *Suppose that $B^0 \subset B^*$ is a $*$-weakly closed (U_t^*)–invariant subspace of B^* such that*

$$\{x^* \in B^0 : \forall_{t\in\mathbb{R}} \, U_t^* x^* = x^*\} = \{0\}.$$

If $D \subset \mathbb{R}$ is a measurable set with $0 < \lambda(D) < +\infty$ then

$$\frac{1}{\lambda(D)} \int_D U_{rt}^* y^* \, dr \to 0 \ *\text{-weakly, as } t \to +\infty, \text{ for every } y^* \in B^0. \tag{5}$$

Proof. Put $P_t y^* := \frac{1}{\lambda(D)} \int_D U_{rt}^* y^* \, dr$ Since any closed ball in B^* endowed with the $*$-weak topology is a compact metric space and $\|P_t y^*\| \leq C \|y^*\|$, it suffices to check that if $z^* \in B^*$ is a $*$-weak limit of a sequence $(P_{t_n} y^*)$ with $t_n \to +\infty$ then $z^* = 0$.

Since B^0 is (U_t^*)-invariant and $*$-weakly closed, $P_{t_n}y^* \in B^0$ for every $n \in \mathbb{N}$, and hence $z^* \in B^0$. Observe that $U_s^* z^* = z^*$ for every $s \in \mathbb{R}$. Indeed, since

$$U_s^* \circ P_{t_n} y^* = \frac{1}{\lambda(D)} \int_D U_{s+rt_n}^* y^* \, dr = \frac{1}{\lambda(D)} \int_{D+s/t_n} U_{rt_n}^* y^* \, dr,$$

for every $x \in B$ we have

$$\begin{aligned}
&|\langle x, U_s^* \circ P_{t_n} y^* - P_{t_n} y^* \rangle| \\
&= \left| \frac{1}{\lambda(D)} \int_{D+s/t_n} \langle x, U_{rt_n}^* y^* \rangle \, dr - \frac{1}{\lambda(D)} \int_D \langle x, U_{rt_n}^* y^* \rangle \, dr \right| \\
&\leq C \frac{\lambda(D \triangle (D + s/t_n))}{\lambda(D)} \|x\| \|y^*\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
|\langle x, U_s^* z^* - z^* \rangle| &= \lim_{n\to\infty} |\langle x, U_s^* \circ P_{t_n} y^* - P_{t_n} y^* \rangle| \\
&\leq \lim_{n\to\infty} C \frac{\lambda(D \triangle (D + s/t_n))}{\lambda(D)} \|x\| \|y^*\| = 0
\end{aligned}$$

for every $x \in B$. Therefore $z^* \in B^0$ is a fixed vector for the representation (U_t^*), and hence $z^* = 0$. □

Lemma 2 (cf. [25], the proof of Proposition 2)**.** *Suppose that $B^0 \subset B^*$ is a closed (U_t^*)-invariant separable space (in the norm topology) which verifies (5). Then for every sequence $t_n \to +\infty$ we have $\lambda(E^c) = 0$ where*

$$E = \{r \in \mathbb{R} : \forall_{x\in B,\, y^* \in B^0} \liminf_{n\to\infty} Re\langle x, U_{rt_n}^* y^* \rangle \leq 0\}.$$

Proof. Notice that

$$E^c = \{r \in \mathbb{R} : \exists_{x\in B,\, \|x\|\leq 1,\, y^*\in B^0,\, \|y^*\|\leq 1} \liminf_{n\to\infty} Re\langle x, U_{rt_n}^* y^* \rangle > 0\}.$$

Given $\varepsilon > 0$, $N \in \mathbb{N}$, $x \in B$ and $y^* \in B^0$ put

$$D_{\varepsilon,N,x,y^*} := \{r \in \mathbb{R} : \forall_{n\geq N} \, Re\langle x, U_{rt_n}^* y^* \rangle > \varepsilon\}.$$

In view of the $*$–weak continuity of (U_t^*), D_{ε,N,x,y^*} is a G_δ, hence Borel, subset of $\mathbb{R}$. Moreover, $\lambda(D_{\varepsilon,N,x,y^*}) = 0$. Indeed, suppose that $\lambda(D_{\varepsilon,N,x,y^*}) > 0$. Let D be a subset of D_{ε,N,x,y^*} such that $0 < \lambda(D) < +\infty$. Then

$$Re \frac{1}{\lambda(D)} \int_D \langle x, U_{rt_n}^* y^* \rangle \, dr = \frac{1}{\lambda(D)} \int_D Re\langle x, U_{rt_n}^* y^* \rangle \, dr > \varepsilon$$

for all $n \geq N$, which contradicts (5) for $y^* \in B^0$.

Let $(x_{\varepsilon,k})_{k\in\mathbb{N}}$ stand for an $\varepsilon/(2C)$–net of the unit ball in B, $(y_{\varepsilon,k}^*)_{k\in\mathbb{N}}$ stand for an $\varepsilon/(2C)$–net of the unit ball in B^0 and let Δ be a countable set of real positive numbers such that $\inf \Delta = 0$. It suffices to prove that

$$E^c \subset \bigcup_{\varepsilon\in\Delta} \bigcup_{N\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} D_{\varepsilon,N,x_{\varepsilon,k},y_{\varepsilon,l}^*}.$$

Assume that $r \in E^c$. Then there exist $x \in B$ with $\|x\| \leq 1$, $y^* \in B^0$ with $\|y^*\| \leq 1$, $\varepsilon \in \Delta$ and $N \in \mathbb{N}$ such that

$$Re\langle x, U_{rt_n}^* y^* \rangle > 2\varepsilon \text{ for all } n \geq N.$$

Next choose $k, l \in \mathbb{N}$ such that $\|x - x_{\varepsilon,k}\| < \varepsilon/(2C)$ and $\|y^* - y^*_{\varepsilon,l}\| < \varepsilon/(2C)$. Thus

$$|\langle x, U^*_{rt_n} y^*\rangle - \langle x_{\varepsilon,k}, U^*_{rt_n} y^*_{\varepsilon,l}\rangle| \le \|x - x_{\varepsilon,k}\|\|U^*_{rt_n} y^*\| + \|x_{\varepsilon,k}\|\|U^*_{rt_n}(y^* - y^*_{\varepsilon,l})\| < \varepsilon.$$

It follows that $Re\langle x_{\varepsilon,k}, U^*_{rt_n} y^*_{\varepsilon,l}\rangle > \varepsilon$ for all $n \ge N$, and hence $r \in D_{\varepsilon,N,x_{\varepsilon,k},y^*_{\varepsilon,l}}$. □

Lemma 3. *Suppose that $B^0 \subset B^{\odot}$ is a closed (U^*_t)-invariant separable space which verifies (5). Assume that there exist a sequence $t_n \to +\infty$ and a continuous linear operator $P : B^0 \to B^*$ such that $U^*_{t_n} y^* \to Py^*$ $*$-weakly for every $y^* \in B^0$. Then there exists a measurable subset $E \subset \mathbb{R}$ with $\lambda(E^c) = 0$ such that for every $s \neq 0$ and $r \in E$ if $A_r : B^* \to B^*$ is a continuous linear operator such that $A_r(B^0) \subset B^0$ and $A_r U^*_s = U^*_{rs} A_r$ then $U^*_{rs} A_r P y^* = 0$ for every $y^* \in B^0$.*

Proof. An application of Lemma 2 yields the existence of $E \subset \mathbb{R}$ with $\lambda(E^c) = 0$ such that for $r \in E$

$$\liminf_{n\to\infty} Re\langle x, U^*_{rt_n} y^*\rangle \le 0 \text{ for all } x \in B \text{ and } y^* \in B^0. \tag{6}$$

Suppose that $s \neq 0$, $r \in E$ and $A_r U^*_s = U^*_{rs} A_r$. By passing to a subsequence, if necessary, we can assume that the fractional parts $\{t_n/s\} \to \theta \in [0,1]$. Take $x \in B$, $y^* \in B^0$ and $\varepsilon > 0$. In view of the continuity of $t \mapsto U_t x$ and $t \mapsto U^*_t y^*$,

$$\begin{aligned}
&\langle x, U^*_{rt_n} A_r U^*_{s\theta+\varepsilon} y^*\rangle \\
&= \langle x, U^*_{rs[t_n/s]+rs\{t_n/s\}} A_r U^*_{s\theta+\varepsilon} y^*\rangle = \langle x, U^*_{rs\{t_n/s\}} A_r U^*_{s[t_n/s]} U^*_{s\theta+\varepsilon} y^*\rangle \\
&= \langle U_{rs\{t_n/s\}} x, A_r U^*_{t_n} U^*_{s(\theta-\{t_n/s\})+\varepsilon} y^*\rangle \\
&\to \langle U_{rs\theta} x, A_r P U^*_\varepsilon y^*\rangle = \langle x, U^*_{rs\theta} A_r P U^*_\varepsilon y^*\rangle.
\end{aligned}$$

But $A_r U^*_{s\theta+\varepsilon} y^* \in B^0$, hence using (6),

$$Re\langle x, U^*_{rs\theta} A_r P U^*_\varepsilon y^*\rangle \le 0 \text{ for all } x \in B,\ y^* \in B^0,\ \varepsilon > 0,$$

and hence $U^*_{rs\theta} A_r P y^* = 0$. It follows that $U^*_{rs} A_r P y^* = 0$. □

Lemma 4. *Let $B' \subset B^*$ be a closed separable subspace. Assume that $(P_n : B' \to B^*)_{n\in\mathbb{N}}$ is a sequence of continuous linear operators such that $\|P_n y^*\| \le C\|y^*\|$ for all $y^* \in B'$ and $n \in \mathbb{N}$. Then there exist an increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers and a continuous linear operator $P : B' \to B^*$ such that $P_{k_n} y^* \to Py^*$ $*$-weakly for every $y^* \in B'$.*

Proof. Let D be a dense countable subset of B'. Since $\|P_n x^*\| \le C\|x^*\|$ for every $n \in \mathbb{N}$ and $x^* \in D$ and any closed ball in B^* endowed with the $*$-weak topology is a compact metric space, by a diagonalisation argument we can find an increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers such that $(P_{k_n} x^*)_{n\in\mathbb{N}}$ is $*$-weakly convergent for every $x^* \in D$. Let $Px^* \in B^*$ stand for the $*$-weak limit the sequence $(P_{k_n} x^*)_{n\in\mathbb{N}}$ for $x^* \in D$. Note that for every $x \in B$ and $x^*, y^* \in D$ we have

$$|\langle x, Px^* - Py^*\rangle| = \lim_{n\to\infty} |\langle x, P_{k_n} x^* - P_{k_n} y^*\rangle| \le C\|x\|\|x^* - y^*\|.$$

It follows that

$$\|Px^* - Py^*\| \le C\|x^* - y^*\| \text{ for all } x^*, y^* \in D. \tag{7}$$

Furthermore $(P_{k_n} x^*)_{n\in\mathbb{N}}$ is $*$-weakly convergent for every $x^* \in B'$. Indeed, let $(x^*_l)_{l\in\mathbb{N}}$ be a sequence in D such that $\|x^*_l - x^*\| \to 0$ as $l \to \infty$ and $\sum_{l=1}^\infty \|x^*_l - x^*_{l+1}\| < \infty$. From (7), $\sum_{l=1}^\infty \|Px^*_l - Px^*_{l+1}\| < \infty$, and hence $(Px^*_l)_{l\in\mathbb{N}}$ converges to an element $Px^* \in B^*$. Fix $\varepsilon > 0$ and $0 \neq x \in B$. Take $l_0 \in \mathbb{N}$ such that

$\|x^*_{l_0} - x^*\| < \varepsilon/(3C\|x\|)$. Next choose $n_0 \in \mathbb{N}$ such that $|\langle x, P_{k_n}x^*_{l_0} - Px^*_{l_0}\rangle| < \varepsilon/3$ for all $n \geq n_0$. Then for $n \geq n_0$,

$$\begin{aligned}
&|\langle x, P_{k_n}x^* - Px^*\rangle| \\
\leq\ & |\langle x, P_{k_n}x^* - P_{k_n}x^*_{l_0}\rangle| + |\langle x, P_{k_n}x^*_{l_0} - Px^*_{l_0}\rangle| + |\langle x, Px^*_{l_0} - Px^*\rangle| \\
\leq\ & 2C\|x\|\|x^*_{l_0} - x^*\| + |\langle x, P_{k_n}x^*_{l_0} - Px^*_{l_0}\rangle| < \varepsilon.
\end{aligned}$$

It follows that $P_{k_n}x^* \to Px^*$ $*$-weakly for every $x^* \in B'$. It is easy to see that $P : B' \to B^*$ is a linear bounded operator. □

Theorem 5. *Suppose that $B^0 \subset B^{\odot}$ is a closed (U^*_t)-invariant separable space which verifies (5). Suppose that there exists a subset $D \subset \mathbb{R}$ of positive Lebesgue measure such that for every pair $(t,s) \in D \times D$ there exists $A_{t,s} \in \mathcal{L}(B^*)$ with the trivial kernel such that $A_{t,s}(B^0) \subset B^0$ and $A_{t,s}U^*_s = U^*_t A_{t,s}$. Then $U^*_t y^* \to 0$ $*$-weakly, as $|t| \to \infty$, for every $y^* \in B^0$.*

Proof. Suppose, contrary to our claim, that there exists $y^*_0 \in B^0$ such that $U^*_t y^*_0 \nrightarrow 0$ $*$-weakly, as $|t| \to \infty$. Since $\|U^*_t y^*_0\| \leq C\|y^*_0\|$, there exists a sequence $t'_n \to \infty$ such that $U^*_{t'_n} y^*_0$ converges $*$-weakly to a nonzero element. Since B^0 is separable, by Lemma 4, we can assume (passing to a subsequence if necessary) that there exists a non-zero continuous linear operator $P' : B^0 \to B^*$ such that $U^*_{t'_n} y^* \to P'y^*$ $*$-weakly for every $y^* \in B^0$.

Fix a non-zero number $s \in D$ and put $t_n = t'_n - s$. By Lemma 4, we can assume (passing to a subsequence if necessary) that there exists a continuous linear operator $P : B^0 \to B^*$ such that $U^*_{t_n} y^* \to Py^*$ $*$-weakly for every $y^* \in B^0$. Then $U^*_s P = P'$.

Take $r \in (D/s) \cap E$ (see Lemma 3 applied for (t_n) and P above). Set $A_r := A_{rs,s}$. Then $A_r U^*_s = U^*_{rs} A_r$. By Lemma 3, $A_r U^*_s P y^* = U^*_{rs} A_r P y^* = 0$ for every $y^* \in B^0$. Since A_r has the trivial kernel, $P'y^* = U^*_s P y^* = 0$ for every $y^* \in B^0$, which is a contradiction. □

3. Inner self-similarity of $\mathbb{R}$-actions

Let G stand for the Lie group

$$\left\{ \begin{bmatrix} e^s & 0 \\ t & e^{-s} \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

Then $d\nu = e^s\, ds\, dt$ is a left Haar measure of G. Let $S : G \to \mathcal{L}(B)$ be a strongly continuous bounded representation in a separable Banach space B, i.e.

$$S_{g_1 g_2^{-1}} = S^{-1}_{g_2} \circ S_{g_1} \text{ for all } g_1, g_2 \in G,$$

$$G \ni g \mapsto S_g x \in B \text{ is continuous for every } x \in B,$$

$$C := \sup_{g \in G} \|S_g\| < +\infty.$$

Then the dual representation $S^* : G \to \mathcal{L}(B^*)$ is $*$-weakly continuous, bounded and $S^*_{g_1 g_2} = S^*_{g_1} \circ S^*_{g_2}$ for all $g_1, g_2 \in G$. Let

$$B^{\odot}_S = \{x^* \in B^* : G \ni g \mapsto S^*_g x^* \in B^* \text{ is continuous}\}.$$

It is easy to see that $B^{\odot}_S$ is a closed linear subspace.

Remark 1. Note that $B_S^{\odot}$ is not trivial. Indeed, fix $f \in L^1(G,\nu)$ and $y^* \in B^*$. Let $\bar{y}^* = \int_G S_g^* y^* f(g)\, d\nu(g)$, i.e.

$$\langle x, \bar{y}^*\rangle = \int_G \langle x, S_g^* y^*\rangle f(g)\, d\nu(g) \text{ for all } x \in B.$$

Then for every $g_1, g_2 \in G$ we have

$$\begin{aligned} |\langle x, S_{g_1}^*\bar{y}^* - S_{g_2}^*\bar{y}^*\rangle| &= \left|\int_G \langle x, S_{g_1 g}^* y^* - S_{g_2 g}^* y^*\rangle f(g)\, d\nu(g)\right| \\ &= \left|\int_G \langle x, S_{g_1 g}^* y^*\rangle (f(g) - f(g_2^{-1}g_1 g))\, d\nu(g)\right| \\ &\leq C\|x\|\|y^*\| \int_G |f(g) - f(g_2^{-1}g_1 g)|\, d\nu(g). \end{aligned}$$

It follows that

$$\|S_{g_1}^*\bar{y}^* - S_{g_2}^*\bar{y}^*\| \leq C\|y^*\| \int_G |f(g_1^{-1}g) - f(g_2^{-1}g)|\, d\nu(g),$$

and hence the continuity of $g \mapsto S_g^*\bar{y}^*$ is a consequence of the continuity of the regular representation

$$G \ni g \mapsto \Psi_g \in \mathcal{L}(L^1(G,\nu)), \quad \Psi_g f(g') = f(g^{-1}g').$$

Moreover, taking a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^2(G,\nu)$ such that $f_n\, d\nu \to \delta_{\mathrm{I}}$ weakly in the space $\mathcal{P}(G)$ of probability Borel measures on G we can conclude that ${}^{\perp}B_S^{\odot} = B$, and hence $B_S^{\odot}$ is $*$-weakly dense.

Given $y^* \in B_S^{\odot}$ let $B_S^{\odot}(y^*)$ denote the smallest closed S^*-invariant subspace of $B_S^{\odot}$ containing y^*. Then $B_S^{\odot}(y^*)$ is separable.

Let

$$u_t = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \text{ and } a_s = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix}.$$

Note that

$$a_s u_t a_s^{-1} = u_{e^{-2s}t} \text{ for all } s,t \in \mathbb{R}. \tag{8}$$

Let us consider two representations of $\mathbb{R}$ in $\mathcal{L}(B)$ given by $U_t = S_{u_t}$ and $A_s = S_{a_s}$. Since $\mathbb{R} \ni t \mapsto u_t \in G$ and $\mathbb{R} \ni s \mapsto a_s \in G$ are continuous homomorphisms, representations (U_t) and (A_s) are strongly continuous and bounded. Then the dual representations (U_t^*) and (A_s^*) are $*$-weakly continuous and bounded.

Corollary 6. *Suppose that $B^0 \subset B^*$ is a closed S^*-invariant subspace which verifies (5). If $y^* \in B_S^{\odot} \cap B^0$ then $U_t^* y^* \to 0$ $*$-weakly as $|t| \to \infty$.*

Proof. Fix $y^* \in B_S^{\odot} \cap B^0$. Then $B_S^{\odot}(y^*) \subset B^0$ is a closed (S_g^*)-invariant separable subspace. From (8), $A_s^* \circ U_t^* \circ (A_s^*)^{-1} = U_{e^{-2s}t}^*$. Now an application of Theorem 5, for $B^0 := B_S^{\odot}(y^*)$, gives $U_t^* x^* \to 0$ for every $x^* \in B_S^{\odot}(y^*)$. □

Let (X,d) be a compact metric space and let $\phi : G \to Hom(X)$ be a continuous representation of G in the group of homeomorphisms on X. ϕ determines two continuous flows $(\eta_t)_{t\in\mathbb{R}}$ and $(\gamma_s)_{s\in\mathbb{R}}$ on X:

$$\eta_t(x) = \phi_{u_t} x \text{ and } \gamma_s(x) = \phi_{a_s} x.$$

Suppose that (η_t) is uniquely ergodic and let μ be the unique invariant probability measure for (η_t).

Let us consider the representation of G in $\mathcal{L}(C(X))$ given by $S_gf(x) = f(\phi_g x)$.

Denote by $M(X)$ the Banach space of signed real Borel measures on (X, d) equipped with the norm given by the total variation. Let $\mathcal{P}(X) \subset M(X)$ stand for the subset of probability measures. Since $C(X)^* = M(X)$, the dual representation S^* of G in $\mathcal{L}(M(X))$ is given by $S^*_g(\rho) = (\phi_g)_*\rho$, the latter being the image of ρ via ϕ_g. By the unique ergodicity of (η_t), every (U^*_t)-invariant measure $\rho \in M(X)$ is a real multiple of $\mu \in \mathcal{P}(X) : \rho = \rho(X)\mu$.

Theorem 7. *If $\rho \in \mathcal{P}(X) \cap M(X)^{\odot}_S$ then $(\eta_t)_*\rho \to \mu$ weakly as $|t| \to \infty$.*

Proof. Let

$$M_0(X) = \{\tau \in M(X) : \tau(X) = \langle 1, \tau\rangle = 0\}.$$

Clearly, $M_0(X)$ is $*$-weakly closed and (S^*_g)-invariant and as we have already noticed any (U^*_t)-invariant measure $\tau \in M_0(X)$ is equal to $\tau(X)\mu = 0$. By Lemma 1, the space $M_0(X)$ verifies (5)

Suppose that $\rho \in \mathcal{P}(X) \cap M(X)^{\odot}_S$. Then $\rho - \mu \in M(X)^{\odot}_S \cap M_0(X)$. Now an application of Corollary 6, for $B^0 = M_0(X)$, yields

$$(\eta_t)_*\rho - \mu = U^*_t(\rho - \mu) \to 0 \text{ $*$-weakly as } |t| \to \infty.$$

□

Corollary 8. *Let $D \subset G$ be a Borel set such that $0 < \nu(D) < \infty$. Then for every continuous function $f : X \to \mathbb{R}$ and $x \in X$,*

$$\frac{1}{\nu(D)} \int_D f(\eta_t\phi_g x)\, d\nu(g) \to \int_X f\, d\mu \textit{ as } |t| \to \infty.$$

Proof. Fix $x \in X$ and let us consider the probability Borel measure ρ on X determined by

$$\int_X f\, d\rho = \frac{1}{\nu(D)} \int_D f(\phi_g x)\, d\nu(g) \text{ for all } f \in C(X).$$

Since

$$\langle f, \rho\rangle = \int_G \langle f, S^*_g\delta_x\rangle \frac{1_D}{\nu(D)}\, d\nu \text{ for all } f \in C(X),$$

in view of Remark 1, $\rho \in M(X)^{\odot}_S$. Moreover,

$$\langle f, U^*_t\rho\rangle = \langle f \circ \eta_t, \rho\rangle = \frac{1}{\nu(D)} \int_D f(\eta_t\phi_g x)\, d\nu(g)$$

for every $f \in C(X)$. Now an application of Theorem 7 yields

$$\frac{1}{\nu(D)} \int_D f(\eta_t\phi_g x)\, d\nu(g) = \langle f, U^*_t\rho\rangle \to \langle f, \mu\rangle = \int_X f\, d\mu$$

for every $f \in C(X)$. □

Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a discrete subgroup. Then the homogeneous space $X = \Gamma\backslash \mathrm{PSL}(2, \mathbb{R})$ is the unit tangent bundle of a surface M of constant negative curvature. Consider the action $\pi : \mathrm{PSL}(2, \mathbb{R}) \to Hom(X)$ by right translations, i.e. $\pi_g(\Gamma x) = \Gamma xg$ for all $g, x \in \mathrm{PSL}(2, \mathbb{R})$. Assume that $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a lattice, i.e. Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ such that the action π has an invariant finite measure. Let us denote by μ_Γ the unique π-invariant probability measure on X.

Since G is a subgroup of $\mathrm{PSL}(2, \mathbb{R})$, we can consider its subaction $\phi : G \to Hom(X)$. Then the corresponding flows $(\eta_t)_{t\in\mathbb{R}}$ and $(\gamma_s)_{s\in\mathbb{R}}$ are called respectively the horocycle and the geodesic flows on M.

Suppose that the lattice Γ is cocompact, i.e. $\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$ is compact, then the surface M is also compact. In 1973 Furtenberg [13] proved that the horocycle flow has a unique invariant probability measure which is equal to μ_Γ.

Corollary 9. *Assume that $\Gamma\subset\operatorname{PSL}(2,\mathbb{R})$ is cocompact. If $\rho\in\mathcal{P}(\Gamma\backslash\operatorname{PSL}(2,\mathbb{R}))$ and the map*

$$G\ni g\mapsto(\phi_g)_*\rho\in\mathcal{P}(\Gamma\backslash\operatorname{PSL}(2,\mathbb{R}))$$

is strongly continuous then $(\eta_t)_*\rho$ tends weakly (as $|t|\to\infty$) to the unique invariant probability measure for the horocycle flow.

The unique ergodicity of the horocycle flow is equivalent to the equidistribution property of all its orbits, i.e.

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\eta_t x)\,dt\to\int_X f(y)\,d\mu_\Gamma(y)\text{ for any }x\in X\text{ and }f\in C(X).$$

Fix $x\in X=\Gamma\backslash\operatorname{PSL}(2,\mathbb{R})$ and an open and bounded subset $D\subset G$. Let us consider the two-dimensional set $D_x=\{\phi_g x: g\in D\}\subset X$. By Corollary 8, the image $\eta_t(D_x)$ is equidistributed on X as $|t|\to\infty$.

3.1. **Horocycle flow on non-compact finite surfaces.**

Lemma 10. *Let $f:[0,\infty)\to\mathbb{R}$ be a measurable bounded function such that*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(s)\,ds\to\theta.$$

Then for every Borel set $D\subset[0,\infty)$ with finite Lebesgue measure we have

$$\lim_{t\to\infty}\int_D f(st)ds=\theta\lambda(D).$$

Proof. Let

$$\mathcal{D}=\left\{D\in\mathcal{B}([0,\infty)):\lambda(D)<\infty,\ \lim_{t\to\infty}\int_D f(st)ds=\theta\lambda(D)\right\}.$$

By the definition of $\mathcal{D}$,

$$\text{if }D_1\subset D_2\text{ and }D_1,D_2\in\mathcal{D}\text{ then }D_2\setminus D_1\in\mathcal{D}.\tag{9}$$

Moreover,

$$\begin{array}{l}\text{if }(D_n)_{n\in\mathbb{N}}\text{ is a sequence in }\mathcal{D}\text{ such that }\lambda(D_n\triangle D)\to0\text{ as }n\to\infty\\ \text{for some }D\in\mathcal{B}([0,\infty))\text{ then }D\in\mathcal{D}.\end{array}\tag{10}$$

Indeed, fix $\varepsilon>0$ and choose $n_0\in\mathbb{N}$ such that $\lambda(D_{n_0}\triangle D)<\varepsilon/2(\|f\|_\infty+|\theta|)$ and $t_0>0$ such that

$$\left|\int_{D_{n_0}}f(st)ds-\theta\lambda(D_{n_0})\right|<\varepsilon/2\text{ for all }t>t_0.$$

Then for $t>t_0$,

$$\begin{aligned}\left|\int_D f(st)ds-\theta\lambda(D)\right| &\le \left|\int_D f(st)ds-\int_{D_{n_0}}f(st)ds\right|\\ &\quad+\left|\int_{D_{n_0}}f(st)ds-\theta\lambda(D_{n_0})\right|+|\theta||\lambda(D_{n_0})-\lambda(D)|\\ &\le \lambda(D_{n_0}\triangle D)\|f\|_\infty+\varepsilon/2+|\theta|\lambda(D_{n_0}\triangle D)<\varepsilon.\end{aligned}$$

Thus $\int_D f(st)ds \to \theta\lambda(D)$ as $t \to \infty$.

By assumption, $[0,a] \in \mathcal{D}$ for every $a \geq 0$. In view of (9), $\mathcal{D}$ includes every finite interval in $[0,\infty)$. Fix $C > 0$. From (9) and (10), $\mathcal{D} \cap \mathcal{B}([0,C])$ is a λ-system containing the family of all subintervals of $[0,C]$ (this is a π-system). By the Dynkin's lemma, $\mathcal{B}([0,C]) \subset \mathcal{D}$ for every $C > 0$. An application again of (10) yields $D \in \mathcal{D}$ for every Borel set $D \subset [0,\infty)$ with finite Lebesgue measure. □

Lemma 11. *Let $(\eta_t)_{t\in\mathbb{R}}$ be a continuous flow on a locally compact metric space space (X,d). Suppose that there exists a Borel set $X_0 \subset X$ and a Borel probability measure μ_0 on X such that $\mu_0(X_0) = 1$ and*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \varphi(\eta_s x)\, ds = \int_X \varphi(y)\, d\mu_0(y)$$

for every $x \in X_0$ and every continuous bounded function $\varphi : X \to \mathbb{R}$. Then for every $D \in \mathcal{B}(\mathbb{R})$ with $0 < \lambda(D) < \infty$ and every $\mu \in M(X)$ with $|\mu|(X \setminus X_0) = 0$ we have

$$\lim_{t\to\infty} \frac{1}{\lambda(D)} \int_D \int_X \varphi(\eta_{st} x)\, d\mu(x)\, ds = \mu(X) \int_X \varphi(x)\, d\mu_0(x)$$

for every continuous bounded function $\varphi : X \to \mathbb{R}$.

Proof. Since $\frac{1}{t}\int_0^t \varphi(\eta_s x)\, ds \to \int_X \varphi(y)\, d\mu_0(y)$ and $|\frac{1}{t}\int_0^t \varphi(\eta_s x)\, ds| \leq \|\varphi\|_\infty$ for μ-a.e. $x \in X$, by Lebesgue's dominated convergence theorem and Fubini's theorem,

$$\frac{1}{t}\int_0^t \int_X \varphi(\eta_s x)\, d\mu(x)\, ds = \int_X \frac{1}{t}\int_0^t \varphi(\eta_s x)\, ds\, d\mu(x) \to \mu(X) \int_X \varphi(x)\, d\mu_0(x).$$

Putting $f(s) = \int_X \varphi(\eta_s x)\, d\mu(x)$, $\theta = \mu(X)\int_X \varphi(x)\, d\mu_0(x)$ and applying Lemma 10 we complete the proof. □

Assume that $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a non-compact lattice and let consider the horocycle flow $(\eta_t)_{t\in\mathbb{R}}$ on $X = \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$. Then X is a locally compact space and the horocycle flow has periodic orbits. Let $X_0 \subset X$ stand for the set of non-periodic orbits. Dani [4] has shown that every probability ergodic measure invariant with respect to the horocycle flow is either equal to μ_Γ or is supported by a periodic orbit. Moreover, every non-periodic orbit is equidistributed on X (see [5]), i.e. for every $x \in X_0$ and every bounded continuous function $f : X \to \mathbb{R}$ we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\eta_s x)\, ds = \int_X f(x)\, d\mu_\Gamma(x). \tag{11}$$

Let $C_0(X)$ denote the space of the continuous functions on X vanishing at infinity equipped with the supremum norm. Recall that the dual space $C_0^*(X)$ may be identified with $M(X)$ with the total variation norm.

Let B^0 stand for the space all signed measures $\mu \in M(X)$ such that $\mu(X) = 0$ and $|\mu|(X \setminus X_0) = 0$. The subspace $B^0 \subset M(X)$ is closed. Moreover, since the set of periodic orbits $X \setminus X_0$ is $(\phi_g)_{g\in G}$-invariant, B^0 is $(S_g^*)_{g\in G}$–invariant. By (11) and Lemma 11,

$$\frac{1}{\lambda(D)} \int_D U_{rs}^* \mu\, ds \to 0 \ *\text{-weakly}$$

for every $\mu \in B^0$ and every measurable set $D \subset \mathbb{R}$ with $0 < \lambda(D) < \infty$. Suppose that $\mu \in B^0 \cap M(X)^{\odot}$. An application of Corollary 6 gives $U_t^*\mu \to 0$ $*$-weakly as $t \to \infty$. This yields the following.

Theorem 12. *For every lattice* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *and* $\rho \in \mathcal{P}(\Gamma\backslash \mathrm{PSL}(2,\mathbb{R}))$ *if the map*

$$G \ni g \mapsto (\phi_g)_*\rho \in \mathcal{P}(\Gamma\backslash \mathrm{PSL}(2,\mathbb{R}))$$

is strongly continuous and ρ *is supported by the set of non-periodic orbits for the horocycle flow* $(\eta_t)_{t\in\mathbb{R}}$ *on* $\Gamma\backslash \mathrm{PSL}(2,\mathbb{R})$ *then* $(\eta_t)_*\rho \to \mu_\Gamma$ *weakly as* $|t| \to \infty$.

4. Flows and joinings

In this section we briefly put together necessary definitions and some known facts about flows and their joinings. Although definitions and facts are formulated for flows, all of them hold (and will be applied) for automorphisms.

The flow $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ determines a unitary representation, still denoted by $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$, of $\mathbb{R}$ in $\mathcal{U}(L^2(X,\mathcal{B},\mu))$ by the formula $T_t(f) \mapsto f \circ T_t$. Since the flow $\mathcal{T}$ is measurable, the unitary representation $\mathcal{T}$ is continuous. Let $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ be another ergodic flow defined on $(Y,\mathcal{C},\nu)$. By a *joining* between $\mathcal{T}$ and $\mathcal{S}$ we mean any probability $\{T_t \times S_t\}_{t\in\mathbb{R}}$–invariant measure on $(X\times Y, \mathcal{B}\otimes\mathcal{C})$ whose projections on X and Y are equal to μ and ν respectively. The set of joinings between $\mathcal{T}$ and $\mathcal{S}$ is denoted by $J(\mathcal{T},\mathcal{S})$ (simply $J(\mathcal{T})$ where $\mathcal{S} = \mathcal{T}$). The subset of ergodic joinings is denoted by $J^e(\mathcal{T},\mathcal{S})$. Ergodic joinings are exactly extremal points in the simplex $J(\mathcal{T},\mathcal{S})$. Given $\rho \in J(\mathcal{T},\mathcal{S})$ define an operator $\Phi_\rho : L^2(X,\mathcal{B},\mu) \to L^2(Y,\mathcal{C},\nu)$ by requiring that

$$\int_{X\times Y} f(x)g(y)\, d\rho(x,y) = \int_Y \Phi_\rho(f)(y)g(y)\, d\nu(y)$$

for each $f \in L^2(X,\mathcal{B},\mu)$ and $g \in L^2(Y,\mathcal{C},\nu)$. This operator has the following Markov property

$$\Phi_\rho 1 = \Phi_\rho^* 1 = 1 \text{ and } \Phi_\rho f \geq 0 \text{ whenever } f \geq 0. \tag{12}$$

Moreover,

$$\Phi_\rho \circ T_t = S_t \circ \Phi_\rho \text{ for each } t \in \mathbb{R}. \tag{13}$$

In fact, there is a one-to-one correspondence between the set of Markov operators $\Phi : L^2(X,\mathcal{B},\mu) \to L^2(Y,\mathcal{C},\nu)$ satisfying (13) and the set $J(\mathcal{T},\mathcal{S})$ (see e.g. [26] for details). Notice that the product measure corresponds to the Markov operator denoted by $\int$, where $\int(f)$ equals the constant function $\int_X f\,d\mu$. On $J(\mathcal{T})$ we consider the weak operator topology. In this topology $J(\mathcal{T})$ becomes a metrizable compact space which is a Choquet simplex.

We denote by $C(\mathcal{T})$ the *centralizer* of the flow $\mathcal{T}$, this is the group of Borel automorphisms $R : (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ such that $T_t \circ R = R \circ T_t$ for every $t \in \mathbb{R}$. Every $R \in C(\mathcal{T})$ can be considered as a Markov operator. The corresponding self-joining, denoted by μ_R, and is determined by $\mu_R(A \times B) = \mu(A \cap R^{-1}B)$ for $A, B \in \mathcal{B}$. Then μ_R is concentrated on the graph of R and $\mu_R \in J^e(\mathcal{T})$.

Flows $\mathcal{T}$ and $\mathcal{S}$ are called *disjoint* if $J(\mathcal{T},\mathcal{S}) = \{\mu\otimes\nu\}$. Equivalently, the operator $\int$ is the only Markov operator that intertwines T_t and S_t (for each $t \in \mathbb{R}$). Notice that if automorphisms T_t and S_t are disjoint for a certain $t \neq 0$ then the flows $\mathcal{T}$ and $\mathcal{S}$ are disjoint as well.

If $\mathcal{T}_i = (T_t^{(i)})_{t\in\mathbb{R}}$ is a Borel flow on $(X_i,\mathcal{B}_i,\mu_i)$ for $i = 1,\ldots,k$ then by a *k–joining* of $\mathcal{T}_1,\ldots,\mathcal{T}_k$ we mean any probability $(T_t^{(1)} \times \ldots \times T_t^{(k)})_{t\in\mathbb{R}}$–invariant measure on $\left(\prod_{i=1}^k X_i, \bigotimes_{i=1}^k \mathcal{B}_i\right)$ whose projection on X_i is equal to μ_i for $i = 1,\ldots,k$.

Suppose that $\mathcal{T}$ is an ergodic flow on $(X, \mathcal{B}, \mu)$ and $\mathcal{T}_i = \mathcal{T}$ for $i = 1, \ldots, k$. If $R_1, \ldots, R_k \in C(\mathcal{T})$ then the image of μ via the map

$$X \ni x \mapsto (R_1 x, \ldots, R_k x) \in X^k$$

is called an *off-diagonal* joining. Of course, any off-diagonal joining is an ergodic k–self–joining. Suppose that the set of indices $\{1, \ldots, k\}$ is now partitioned into some subsets and let on each of these subsets an off–diagonal joining be given. Then clearly the product of these off–diagonal joinings is a k–self–joining of $\mathcal{T}$.

An ergodic flow $\mathcal{T}$ is said to has *minimal self–joining (MSJ)* if every ergodic k–self–joining is a product of off–diagonal joinings for every $k \in \mathbb{N}$ and $C(\mathcal{T}) = \{T_t : t \in \mathbb{R}\}$.

A flow $\mathcal{T}$ on $(X, \mathcal{B}, \mu)$ is *pairwise independently determined* (PID) if any n-joining ($n \geq 3$) of $\mathcal{T}$ which is pairwise independent, i.e. its projection on the product of any two copies of X in X^n is the product $\mu \otimes \mu$, must be the product measure $\mu^{\otimes n}$ (see [15]). Obviously, every weakly mixing MSJ flow is PID.

Proposition 13 (Ryzhikov [24]). *Suppose that $\mathcal{T}$ is a weakly mixing PID flow and take arbitrary two ergodic flows $\mathcal{S}$ on $(Y, \mathcal{C}, \nu)$ and $\mathcal{R}$ on $(Z, \mathcal{D}, \rho)$. Then any 3-joining of $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{R}$ which is pairwise independent must be the product measure $\mu \otimes \nu \otimes \rho$.*

As a consequence of Lemma 3, we obtain the following.

Theorem 14. *Let $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ be a weakly mixing flow on a standard Borel space $(X, \mathcal{B}, \mu)$. Suppose that there exists a sequence of real numbers (t_n) such that $t_n \to +\infty$ and*

$$T_{t_n} \to \alpha \int_{C(\mathcal{T})} S \, dP(S) + (1 - \alpha) J,$$

where $\alpha > 0$, P is a probability Borel measure on the centralizer $C(\mathcal{T})$ and $J \in J(\mathcal{T})$. Then T_t and T_s are disjoint for almost every pair $(t, s) \in \mathbb{R}^2$. In particular, $\mathcal{T}$ and $\mathcal{T}_s$ are disjoint for almost every $s \in \mathbb{R}$.

Proof. Since $\mathcal{T}$ is ergodic, we can apply Lemma 3 for the unitary representation

$$T_t(f) = f \circ T_t$$

on $L^2(X, \mu)$. Since $T_{t_n} \to K$ weakly where $K = \alpha \int_{C(\mathcal{T})} S \, dP(S) + (1 - \alpha) J$, there exists a measurable subset $E \subset \mathbb{R}$ with $\lambda(E^c) = 0$ such that for every $s \neq 0$ and $r \in E$ if $A_r : L^2(X, \mu) \to L^2(X, \mu)$ is a continuous linear operator such that $A_r(L^2_0(X, \mu)) \subset L^2_0(X, \mu)$ and $A_r T_s = T_{rs} A_r$ then

$$A_r K f = 0 \text{ for every } f \in L^2_0(X, \mu). \tag{14}$$

Let us consider the set

$$E' := \{(t, s) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}) : \frac{t}{s} \in E\}.$$

Since E' is Lebesgue measurable, by Fubini's theorem, the complement of E' has zero Lebesgue measure in $\mathbb{R}^2$. Suppose that $(t, s) \in E'$. Then $t = rs$ for some $r \in E$. Let $J_r : L^2(X, \mu) \to L^2(X, \mu)$ be a Markov operator intertwining T_s and T_t, i.e. $J_r T_s = T_{rs} J_r$. In view of (14),

$$\alpha \int_{C(\mathcal{T})} J_r \circ S \, dP(S) + (1 - \alpha) J_r \circ J = 0 \text{ on } L^2_0(X, \mu),$$

and hence

$$\alpha\int_{C(\mathcal{T})} J_r\circ S\,dP(S)+(1-\alpha)J_r\circ J=\int \quad\text{on } L^2(X,\mu).$$

By the weak mixing of $\mathcal{T}$, the operator $\int$ is indecomposable in $J(T_s,T_{rs})$. Therefore $J_r\circ S=\int$ for P–almost every S, and hence $J_r=\int$. Consequently, T_s and $T_t=T_{rs}$ are disjoint. □

5. Special flow

Let T be an automorphism of a probability standard Borel space $(X,\mathcal{B},\mu)$. If $f:X\to\mathbb{R}$ is a strictly positive integrable function, then by $\mathcal{T}^f=(T^f_t)_{t\in\mathbb{R}}$ we will mean the corresponding special flow under f (see e.g. [3], Chapter 11) acting on $(X^f,\mathcal{B}^f,\mu^f)$, where $X^f=\{(x,s)\in X\times\mathbb{R}:\ 0\le s<f(x)\}$ and $\mathcal{B}^f$ (μ^f) is the restriction of $\mathcal{B}\otimes\mathcal{B}(\mathbb{R})$ ($\mu\otimes\lambda$) to X^f. Under the action of the flow $\mathcal{T}^f$ each point in X^f moves vertically at unit speed, and we identify the point $(x,f(x))$ with $(Tx,0)$. Given $m\in\mathbb{Z}$ we put

$$f^{(m)}(x)=\begin{cases} f(x)+f(Tx)+\ldots+f(T^{m-1}x) & \text{if } m>0\\ 0 & \text{if } m=0\\ -\big(f(T^mx)+\ldots+f(T^{-1}x)\big) & \text{if } m<0.\end{cases}$$

Then for every $(x,s)\in X^f$ we have

$$T^f_t(x,s)=(T^nx,s+t-f^{(n)}(x)),$$

where $n\in\mathbb{Z}$ is a unique number such that $f^{(n)}(x)\le s+t<f^{(n+1)}(x)$.

Remark 2. Note that for every positive s the flow $\mathcal{T}^f_s$ is isomorphic to $\mathcal{T}^{f/s}$. Moreover, $\mathcal{T}^f_{-1}$ is isomorphic to the special flow built over T^{-1} and under $-f^{(-1)}=f\circ T^{-1}$. If T is a rotation on the circle then T^{-1} is isomorphic to T by the symmetry $\zeta:\mathbb{T}\to\mathbb{T}$, $\zeta(x)=1-x$. Therefore the map

$$\mathbb{T}^f\ni(x,t)\mapsto(\zeta x,t)\in\mathbb{T}^{f\circ\zeta}$$

establishes an isomorphism of $\mathcal{T}^f_{-1}$ and $\mathcal{T}^{f\circ\zeta}$.

Assume that $f\in L^2(X,\mu)$. Suppose that there exist an increasing sequence of natural numbers $\{q_n\}$, a sequence $\{a_n\}$ of real numbers and a sequence of Borel sets $\{C_n\}$ such that

$$\mu(C_n)\to\alpha>0,\ \ \mu(C_n\triangle T^{-1}C_n)\to 0\quad\text{and}\quad \sup_{x\in C_n}d(x,T^{q_n}x)\to 0 \tag{15}$$

and the sequence $\{\int_{C_n}|f_n(x)|^2d\mu(x)\}$ is bounded, where $f_n:=f^{(q_n)}-a_n$ for $n\in\mathbb{N}$. As the distributions

$$\left\{\frac{1}{\mu(C_n)}(f_n|_{C_n})_*(\mu|_{C_n}),n\in\mathbb{N}\right\}$$

are uniformly tight, by passing to a further subsequence if necessary we can assume that

$$\frac{1}{\mu(C_n)}(f_n|_{C_n})_*(\mu|_{C_n})\to P$$

weakly in $\mathcal{P}(\mathbb{R})$ the set of probability Borel measures on $\mathbb{R}$.

Proposition 15 (see Theorem 6 in [9])**.** *The sequence $\{(T^f)_{a_n}\}$ converges weakly to the operator*

$$\alpha \int_{\mathbb{R}} (T^f)_{-t}\, dP(t) + (1-\alpha)J,$$

where $J \in J(T^f)$.

Remark 3. Suppose that $T : \mathbb{T} \to \mathbb{T}$ is an ergodic rotation by α on the circle and let $f : \mathbb{T} \to \mathbb{R}$ be a function of bounded variation. By the Denjoy-Koksma inequality, $|f^{(q_n)}(x) - q_n c| \leq \operatorname{Var} f$ for every $x \in \mathbb{T}$, where (q_n) is the sequence of denominators of α and $c = \int f(x)\, dx$. Taking $C_n = \mathbb{T}$ and $a_n = q_n c$, in view of Proposition 15 we obtain that $T^f_{q_n c} \to \int_{\mathbb{R}} (T^f)_{-t}\, dP(t)$ for some $P \in \mathcal{P}(\mathbb{R})$.

Let $T_{\lambda,\pi}$ be an interval exchange transformation on $I = [0,1)$ corresponding to a probability vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ $(m > 1)$ and a permutation π of $\{1, 2, \ldots, m\}$, i.e. T acts on every $I_k^\lambda = [\sum_{i=1}^{k-1} \lambda_i, \sum_{i=1}^{k} \lambda_i)$, $k = 1, \ldots, m$, by a translation in such a way that the intervals I_k^λ are rearranged according to the permutation π.

Suppose that $T = T_{\lambda,\pi}$ is ergodic. Let $f : [0,1] \to \mathbb{R}$ be a positive function of bounded variation. As it was shown in [16] (see also [9]) f satisfies a Koksma-Denjoy type inequality, i.e. there exist an increasing sequence of natural numbers $\{q_n\}$, a sequence $\{a_n\}$ of real numbers and a sequence of towers $\{C_n\}$ satisfying (15), with $\alpha \geq 1/(m+1)^2$, and such that $|f^{q_n}(x) - a_n| \leq \operatorname{Var} f$ for all $x \in C_n$. Now an application of Proposition 15 together with Theorem 14 gives the following.

Corollary 16. *Let T be an ergodic interval exchange transformation and let $f : [0,1] \to \mathbb{R}$ be a positive function of bounded variation. Suppose that the special flow $\mathcal{T}^f = (T_t^f)_{t\in\mathbb{R}}$ is weakly mixing. Then $\mathcal{T}^f$ and $\mathcal{T}_s^f$ are disjoint for almost every $s \in \mathbb{R}$. Moreover, T_t^f and T_s^f are disjoint for almost every pair $(t,s) \in \mathbb{R}^2$.*

Let M be a compact orientable C^∞–surface of genus ≥ 1. A *translation structure* on M consists of a finite set (the singularity set) $\Sigma \subset M$ and an atlas (U_α, ϕ_α) of $M \setminus \Sigma$ such that for all α, β with $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\alpha \circ \phi_\beta^{-1}(v) = v + c$. The surface M endowed with a translation structure is called a *translation surface*. Since transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ preserve constant vector fields, there is a well defined vector field of unit length on $M \setminus \Sigma$ in each direction θ. The corresponding flow $\mathcal{F}^\theta = (F_t^\theta)_{t\in\mathbb{R}}$ is called a *translation flow* in the direction θ. Note that $\mathcal{F}^\theta$ preserves the Liouville measure μ, i.e. the finite measure on M which is determined by images by ϕ_α^{-1} of the Lebesgue measure on $\mathbb{R}^2$.

Theorem 17 (see Veech [27])**.** *If a translation flow $\mathcal{F}^\theta$ has no saddle connection, then it has a special representation under an interval exchange transformation $T_{\lambda,\pi}$ and under a function which is constant over each interval I_k^λ.*

Corollary 18. *If a translation flow $\mathcal{F}^\theta$ is weakly mixing with respect to μ, then $\mathcal{F}^\theta$ and $\mathcal{F}_s^\theta$ are disjoint for almost every s. Moreover, F_t^θ and F_s^θ are disjoint for almost every pair $(s,t) \in \mathbb{R}^2$.*

Recall that recently Avila and Forni [1] proved that given stratum of the moduli space of translation surfaces of genus ≥ 2 for almost every translation surface from the stratum the translation flow $\mathcal{F}^\theta$ is weakly mixing for almost every $\theta \in S^1$.

Although for any weakly mixing translation flow $\mathcal{F}^\theta$ the flows $\mathcal{F}^\theta$ and $\mathcal{F}_s^\theta$ are disjoint for almost every $s \in \mathbb{R}$, $\mathcal{F}^\theta$ can be self-similar.

Example 1. Consider an example of weakly mixing translation flow constructed in [7] which has a special representation $\mathcal{T}^f$ where $T = T_{\lambda,\pi}$ is a 4 interval exchange transformation (π is the symmetric permutation (14)(23)) and $f : [0, 1) \to (0, +\infty)$ is constant and equal to h_k over each interval I_k^λ, $k = 1, 2, 3, 4$. More precisely, the vectors $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $h = (h_1, h_2, h_3, h_4)$ are a right and a left Perron-Frobenius eigenvectors respectively of the primitive matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 3 & 2 & 2 \end{bmatrix}.$$

Let $\theta > 1$ stand for the Perron-Frobenius eigenvalue. Let $J' = [0, 1/\theta)$ and let $T' : J' \to J'$ stand for the induced transformation of T on J'. As it was shown in [7] T' is a 4–interval exchange transformation which is isomorphic to T by the map $[0, 1/\theta) \ni x \mapsto \theta x \in [0, 1)$. Let us consider the interval J' as another cross section for the flow $\mathcal{T}^f$. The corresponding special representation of $\mathcal{T}^f$ is built over $T' : J' \to J'$ and under a piecewise constant function $f' : J' \to (0, +\infty)$ which is equal to h'_k over the k-th interval of the interval exchange transformation T' for $k = 1, 2, 3, 4$. Moreover,

$$(h'_1, h'_2, h'_3, h'_4)^T = A(h_1, h_2, h_3, h_4)^T = \theta(h_1, h_2, h_3, h_4)^T,$$

and hence $f'(\theta^{-1}x) = \theta f(x)$ for all $x \in [0, 1)$. It follows that the map $J'^{f'} \ni (x, s) \mapsto (\theta x, s) \in [0, 1)^{\theta f}$ establishes an isomorphism of $\mathcal{T}'^{f'}$ and $\mathcal{T}^{\theta f}$. In view of Remark 2, $\mathcal{T}^{\theta f} \simeq \mathcal{T}^f_{1/\theta}$, and hence $\mathcal{T}^f \simeq \mathcal{T}^f_{1/\theta}$. Consequently, $\theta^k \in I(\mathcal{T}^f)$ for every $k \in \mathbb{Z}$.

6. Absence of self-similarity for special flows

In this section we present a joining method of proving that a flow has no self-similarities. Let us denote by $M(L^2(X, \mu))$ the simplex of Markov operators $V : L^2(X, \mu) \to L^2(X, \mu)$, i.e. V is a positive operator such that $V(1) = 1$ and $V^*(1) = 1$. Notice that $M(L^2(X, \mu))$ is a compact subset of $\mathcal{L}(L^2(X, \mu))$ endowed with the weak operator topology. Let $\mathcal{V} = (V_t)_{t\in\mathbb{R}}$ be a continuous representation of $\mathbb{R}$ in $M(L^2(X, \mu))$. Given $s \in \mathbb{R} \setminus \{0\}$ by $\mathcal{V}_s$ denote the representation $\mathbb{R} \ni t \mapsto V_{st} \in M(L^2(X, \mu))$. We will say two representation $\mathcal{V} = (V_t)_{t\in\mathbb{R}}$ and $\mathcal{V}' = (V'_t)_{t\in\mathbb{R}}$ are Markov isomorphic if there exists a measure preserving automorphism $S : (X, \mu) \to (X, \mu)$ such that $S \circ V'_t = V_t \circ S$ for all $t \in \mathbb{R}$. Let

$$I(\mathcal{V}) = \{s \in \mathbb{R} \setminus \{0\} : \mathcal{V} \text{ and } \mathcal{V}_s \text{ are Markov isomorphic}\}.$$

Let $R_s : \mathbb{R} \to \mathbb{R}$ stand for the rescaling map $R_s t = st$.

Lemma 19. *For every $P \in \mathcal{P}(\mathbb{R})$ and $s_n \to 0$,*

$$(R_{s_n})_*(P) \to \delta_0 \textit{ weakly}.$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. Then

$$\begin{aligned} &\left|\int_{\mathbb{R}} f(x)\, d(R_{s_n})_*(P)(x) - \int_{\mathbb{R}} f(x)\, d\delta_0(x)\right| = \left|\int_{\mathbb{R}} f(s_n x)\, dP(x) - f(0)\right| \\ \leq\ & \left|\int_{\mathbb{R}} (f(s_n x) - f(0))\, dP(x)\right| \leq \int_{\mathbb{R}} |f(s_n x) - f(0)|\, dP(x). \end{aligned}$$

As $s_n \to 0$, $f(s_n x) \to f(0)$ for every $x \in \mathbb{R}$. Moreover, since $|f(s_n x) - f(0)| \leq 2\|f\|_{\sup}$ for every $x \in \mathbb{R}$, Lebesgue's dominated convergence theorem shows that $\int_{\mathbb{R}} f(x)\, d(R_{s_n})_*(P)(x) \to \int_{\mathbb{R}} f(x)\, d\delta_0(x)$. □

Lemma 20. *If $P_n \to P$ weakly in $\mathcal{P}(\mathbb{R})$ then*

$$\int_{\mathbb{R}} V_t\, dP_n(t) \to \int_{\mathbb{R}} V_t\, dP(t) \text{ weakly in } L^2(X,\mu).$$

Proof. For every $f, g \in L^2(X,\mu)$ the map $t \mapsto \langle V_t f, g\rangle$ is continuous and bounded. It follows that

$$\langle \int_{\mathbb{R}} V_t\, dP_n(t) f, g\rangle = \int_{\mathbb{R}} \langle V_t f, g\rangle\, dP_n(t) \to \int_{\mathbb{R}} \langle V_t f, g\rangle\, dP_n(t) = \langle \int_{\mathbb{R}} V_t\, dP_n(t) f, g\rangle.$$

□

Lemma 21. *Suppose that there exists $s \in I(\mathcal{V}) \setminus \{-1, 1\}$ and there exist $P \in \mathcal{P}(\mathbb{R})$ and $0 < a \leq 1$ such that*

$$a \int_{\mathbb{R}} V_t\, dP(t) + (1-a)J \in \{V_t : t \in \mathbb{R}\}^d$$

for some $J \in M(L^2(X,\mu))$. Then

$$a\,\mathrm{I} + (1-a)K \in \{V_t : t \in \mathbb{R}\}^d$$

for some $K \in M(L^2(X,\mu))$.

Proof. Since $s \in I(\mathcal{V})$, there exists an automorphism $S : (X,\mu) \to (X,\mu)$ such that $S \circ V_{st} = V_t \circ S$ for all $t \in \mathbb{R}$. Therefore,

$$S^m \circ V_{s^m t} = V_t \circ S^m \text{ for every } t \in \mathbb{R} \text{ and } m \in \mathbb{Z}.$$

By the assumption, there exists a sequence (t_n) such that $|t_n| \to +\infty$ and

$$V_{t_n} \to a \int_{\mathbb{R}} V_t\, dP(t) + (1-a)J \text{ weakly.}$$

It follows that

$$\begin{aligned} V_{s^m t_n} &= S^{-m} \circ V_{t_n} \circ S^m \to a \int_{\mathbb{R}} S^{-m} \circ V_t \circ S^m\, dP(t) + (1-a)J_m \\ &= a \int_{\mathbb{R}} V_{s^m t}\, dP(t) + (1-a)J_m = a \int_{\mathbb{R}} V_t\, d(R_{s^m})_*(P)(t) + (1-a)J_m, \end{aligned}$$

and hence

$$a \int_{\mathbb{R}} V_t\, d(R_{s^m})_*(P)(t) + (1-a)J_m \in \{V_t : t \in \mathbb{R}\}^d,$$

where $J_m = S^{-m} \circ J \circ S^m$.

Assume that $|s| < 1$, in the case $|s| > 1$ the proof follows by the same method by taking the sequence $(s^{-m})_{m=1}^{\infty}$ instead of $(s^m)_{m=1}^{\infty}$. By passing to a subsequence, if necessary, we can assume that that $J_m \to K$ weakly. Since $s^m \to 0$ as $m \to +\infty$, by Lemmas 19 and 20,

$$a \int_{\mathbb{R}} V_t\, d(R_{s^m})_*(P)(t) + (1-a)J_m \to a\,\mathrm{I} + (1-a)K \text{ as } m \to +\infty.$$

Thus

$$a\,\mathrm{I} + (1-a)K \in (\{V_t : t \in \mathbb{R}\}^d)^d = \{V_t : t \in \mathbb{R}\}^d.$$

□

Theorem 22. *Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be a measure-preserving flow on (X, μ) such that $\mathcal{T}$ is isomorphic to $\mathcal{T}_s$ for some $s \neq \pm 1$.*

- *If $\int_{\mathbb{R}} T_t\, dP(t)$ belongs to $\overline{\{T_t : t \in \mathbb{R}\}}^d$ for some $P \in \mathcal{P}(\mathbb{R})$ then $\mathcal{T}$ is rigid.*
- *If $a\int_{\mathbb{R}} T_t\, dP(t) + (1-a)J \in \overline{\{T_t : t \in \mathbb{R}\}}^d$ for some $0 < a \leq 1$, $P \in \mathcal{P}(\mathbb{R})$ and $J \in J(\mathcal{T})$ then $\mathcal{T}$ is partially rigid.*

Corollary 23. *If $\mathcal{T}$ is non-rigid and $\int_{\mathbb{R}} T_t\, dP(t)$ belongs to $\overline{\{T_t : t \in \mathbb{R}\}}^d$ for some $P \in \mathcal{P}(\mathbb{R})$ then $\mathcal{T}$ is not self-similar. If $\mathcal{T}$ is not partially rigid and $\alpha\int_{\mathbb{R}} T_t\, dP(t) + (1-\alpha)J$ belongs to $\overline{\{T_t : t \in \mathbb{R}\}}^d$ for some $P \in \mathcal{P}(\mathbb{R})$, $0 < \alpha \leq 1$ and $J \in J(\mathcal{T})$ then $\mathcal{T}$ is not self-similar.*

Example 2. Let us consider a special flow $\mathcal{T}^f$ built over an ergodic interval exchange transformation $T : [0,1) \to [0,1)$ and under piecewise absolutely continuous function $f : [0,1) \to \mathbb{R}$. By Proposition 15, there exist $P \in \mathcal{P}(\mathbb{R})$, $0 < \alpha \leq 1$ and $J \in J(\mathcal{T}^f)$ such that $\alpha\int_{\mathbb{R}} T_t^f\, dP(t) + (1-\alpha)J \in \overline{\{T_t^f : t \in \mathbb{R}\}}^d$. Suppose that the sum of jumps $S(f) = \int_0^1 f'(x)\, dx$ of f is not zero. Then, by Theorem 36 in Appendix A (this is a more general version of Theorem 7.1 in [10]), $\mathcal{T}^f$ is not partially rigid, and hence $\mathcal{T}^f$ is not self-similar.

Example 3. The absence of self-similarity we can observe also for special flows built over ergodic rotations T on the circle by α satisfying a Diophantine condition and under some piecewise constant roof functions $f : \mathbb{T} \to \mathbb{R}$. More precisely, we will deal with rotations with bounded partial quotients and roof functions satisfying conditions (P1) and (P2) from [12]. Such special flows are partially rigid. However, as was noted in Remark 3, $\int_{\mathbb{R}} T_t^f\, dP(t) \in \overline{\{T_t^f : t \in \mathbb{R}\}}^d$. Moreover, as was shown in [12], considered flows are mildly mixing, hence not rigid. Consequently, $\mathcal{T}^f$ is not self-similar.

Let us consider a particular case where $f = a + b\chi_{[0,1/2)}$ and $a, b > 0$, $a, b \notin \mathbb{Q} + \alpha\mathbb{Q}$. Since f verifies (P1) and (P2), $\mathcal{T}_s^f$ is not isomorphic to $\mathcal{T}^f$ for all $s \neq \pm 1$. Observe that $\mathcal{T}_{-1}^f$ and $\mathcal{T}^f$ are isomorphic. Indeed, by Remark 2, $\mathcal{T}_{-1}^f$ is isomorphic to $\mathcal{T}^{f\circ\zeta}$. Putting $R : \mathbb{T} \to \mathbb{T}$, $R(x) = x + 1/2$, we have $f \circ \zeta = f \circ R$. On the other side, the map

$$\mathbb{T}^{f\circ R} \ni (x, s) \mapsto (Rx, s) \in \mathbb{T}^f$$

establishes an isomorphism of $\mathcal{T}^{f\circ R}$ and $\mathcal{T}^f$, and hence $\mathcal{T}_{-1}^f$ and $\mathcal{T}^f$ are isomorphic. Therefore $I(\mathcal{T}^f) = \{-1, 1\}$.

Theorem 24. *Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be a weakly mixing MSJ flow. Suppose that there exists non-zero $s \neq \pm 1$ such that T_s and T_1 are isomorphic. Then $\mathcal{T}$ is either mixing or partially rigid (in fact, α-weakly mixing).*

Proof. By Corollary 6.4 in [15], the flows $\mathcal{T}_s$ and $\mathcal{T}$ are isomorphic, and hence $s \in I(\mathcal{T})$.

Suppose that $\mathcal{T}$ is not mixing. Then there exists a sequence (t_n) with $|t_n| \to +\infty$ such that

$$T_{t_n} \to a\int_{\mathbb{R}} T_t\, dP(t) + (1-a)\int$$

for some $0 < a \leq 1$ and $P \in \mathcal{P}(\mathbb{R})$. An application of Lemma 21 shows that

$$T_{t'_n} \to a\,\mathrm{I} + (1-a)\int$$

for a sequence (t'_n) with $|t'_n| \to +\infty$. $\square$

Theorem 25. *Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be a weakly mixing MSJ flow. Suppose that $\int_{\mathbb{R}} T_t\, dP(t)$ belongs to $\{T_t : t \in \mathbb{R}\}^d$ for some $P \in \mathcal{P}(\mathbb{R})$. Then for every nonzero real s_1, s_2 with $|s_1| \neq |s_2|$ the flows $\mathcal{T}_{s_1}$ and $\mathcal{T}_{s_2}$ are disjoint and the automorphisms T_{s_1} and T_{s_2} are disjoint.*

Proof. Suppose that T_{s_1} and T_{s_2} are not disjoint, or $\mathcal{T}_{s_1}$ and $\mathcal{T}_{s_2}$ are not disjoint for some $s_1, s_2 \in \mathbb{R} \setminus \{0\}$ with $|s_1| \neq |s_2|$. By Corollary 6.4 in [15], $\mathcal{T}_{s_1}$ and $\mathcal{T}_{s_2}$ are isomorphic, and hence $s_1/s_2 \in I(\mathcal{T}) \setminus \{-1, 1\}$. Now an application of Theorem 22 gives the rigidity of $\mathcal{T}$, which is impossible. □

Example 4. Let us consider the special flow $\mathcal{T}$ built over a rotation by α with bounded partial quotients and under a function $f(x) = \{x\} + c$. As it was proved in [11], the flow $\mathcal{T}$ has MSJ. As we noted earlier $\mathcal{T}$ is not self-similar. Furthermore, the absence of self-similarity here has stronger consequence: $\mathcal{T}$ is disjoint from $\mathcal{T}_s$ for every $s \neq \pm 1$. Moreover, $\mathcal{T}$ and $\mathcal{T}_{-1}$ are also disjoint. This is an immediate consequence of Remark 2 and Theorem 40 in Appendix B. Finally, $\mathcal{T}$ is disjoint from $\mathcal{T}_s$ for every $s \neq 1$.

Recall that the same property was observed in [14] for some special flows over Chacon transformation.

Remark 4. Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be an ergodic flow for which $0 < s < 1$ is a scale of self-similarity. It follows that there exists an automorphism $S : (X, \mu) \to (X, \mu)$ such that $S \circ T_t = T_{st} \circ S$ for all $t \in \mathbb{R}$. Then S have to be mixing. Indeed, suppose that S is not mixing. Then there exists an increasing sequence (k_n) of natural numbers and $J \in J(\mathcal{T})$ such that $S^{k_n} \to J$ and $J \neq \int$. Since $S^{k_n} \circ T_t = T_{s^{k_n} t} \circ S^{k_n}$ and $T_{s^{k_n} t} \to \mathrm{I}$, we obtain $T_t \circ J = J$ for every $t \in \mathbb{R}$. By the ergodicity of $\mathcal{T}$, $J = \int$.

7. Self-similar flows

Let G be a countable multiplicative subgroup of $\mathbb{R} \setminus \{0\}$. In this section given we will construct a flow $\mathcal{T}$ such that $I(\mathcal{T}) = G$.

Let $\mathcal{S}$ be a weakly mixing flow on a standard probability Borel space $(Y, \mathcal{C}, \nu)$ which has MSJ property and $\mathcal{S}_s$ if disjoint from $\mathcal{S}$ for every $s \neq 1$. Recall that the flows presented in Example 4 possess such a property. Denote by $(Y_s, \mathcal{C}_s, \nu_s) = (Y, \mathcal{C}, \nu)$ the space of the flow $\mathcal{S}_s$ for $s \neq 0$.

Let us consider the product flow $\mathcal{T} = \prod_{g\in G} \mathcal{S}_g$ which acts on the product space $(X, \mathcal{B}, \mu) = \left(\prod_{g\in G} Y_g, \bigotimes_{g\in G} \mathcal{C}_g, \bigotimes_{g\in G} \nu_g\right)$ by

$$\mathcal{T}_t((y_g)_{g\in G}) = (S_{gt} y_g)_{g\in G}.$$

Assume that $s \in G$. Then the flows $\mathcal{T}$ and $\mathcal{T}_s$ are isomorphic, and the isomorphism is given by

$$\pi : \prod_{g\in G} Y_g \to \prod_{g\in G} Y_g, \quad [\pi((y_g)_{g\in G})]_{g'} = y_{sg'} \text{ for all } g' \in G.$$

Assume that $s \notin G$. We will show that $\mathcal{T}$ and $\mathcal{T}_s$ are disjoint. To prove this we will use PID property of the flow $\mathcal{S}$. Suppose that η is an ergodic joining of $\mathcal{T}$ and $\mathcal{T}_s$. Since $G \cap sG = \emptyset$ and the flow $\mathcal{T}_s = \prod_{g\in G} \mathcal{S}_{sg}$ on $\prod_{g\in G} Y_g$ is isomorphic to the flow $\prod_{g\in sG} \mathcal{S}_g$ on $\prod_{g\in sG} Y_g$, η can be treated as a probability measure on $\prod_{g\in G\cup sG} Y_g$ invariant under the action of the product flow $\prod_{g\in G\cup sG} \mathcal{S}_g$. On the other hand, for any collection $s_1, \ldots, s_k$ of distinct non-zero real numbers every ergodic joining of $\mathcal{S}_{s_1},\ldots,\mathcal{S}_{s_k}$ is the product measure $\rho_{s_1}\otimes\ldots\otimes\rho_{s_k}$. This is a consequence of Proposition 13 and the disjointness of $\mathcal{S}_s$ form $\mathcal{S}_{s'}$ for $s\neq s'$. It follows that the projection of η on any finite product $\prod_{g\in F}Y_g$ ($F\subset G\cup sG$ and finite) is the product measure $\bigotimes_{g\in F}\rho_g$. Therefore $\eta=\bigotimes_{g\in G\cup sG}\rho_g=\mu\otimes\mu$, and hence $\mathcal{T}$ and $\mathcal{T}_s$ are disjoint. Consequently, $I(\mathcal{T})=G$.

8. Spectral theory

Let $\mathbb{A}$ be a locally compact second countable Abelian group. In this paper we will deal only with two cases where $\mathbb{A}=\mathbb{R}$ or $\mathbb{Z}$. Let $\mathcal{T}=(T_a)_{a\in\mathbb{A}}$ be measurable action on a probability Borel space $(X,\mathcal{B},\mu)$. The action $\mathcal{T}$ determined the Koopman representation $U^{\mathcal{T}}$ of $\mathbb{A}$ in $L^2_0(X,\mathcal{B},\mu)$ given by $U^{\mathcal{T}}_a(f)=f\circ T_a$. For any $f\in L^2_0(X,\mathcal{B},\mu)$ we define the *cyclic space* $\mathbb{A}(f)=\overline{\operatorname{span}\{U^{\mathcal{T}}_a f;a\in\mathbb{A}\}}$. By the *spectral measure* $\sigma_{\mathcal{T},f}$ of f we mean a Borel measure on the dual group $\widehat{\mathbb{A}}$ determined by $\int_{\widehat{\mathbb{A}}}\gamma(a)d\sigma_{f,\mathcal{T}}(\gamma)=\langle U^{\mathcal{T}}_a f,f\rangle$ for all $a\in\mathbb{A}$.

By the spectral theorem there exists a spectral decomposition of $L^2_0(X,\mathcal{B},\mu)$, i.e.

$$L^2_0(X,\mathcal{B},\mu)=\bigoplus_{n=1}^{\infty}\mathbb{A}(f_n)\quad \textit{and}\quad \sigma_{f_1,\mathcal{T}}\gg\sigma_{f_2,\mathcal{T}}\ldots\ . \tag{16}$$

Moreover, a spectral sequence $(\sigma_{f_n,\mathcal{T}})_{n\in\mathbb{N}}$ is unique up to equivalence of measures.

The spectral type of $\sigma_{f_1,\mathcal{T}}$ (the equivalence class of measures), denoted by $\sigma_{\mathcal{T}}$, will be called the *maximal spectral type* of $\mathcal{T}$. $\mathcal{T}$ is said to have *Lebesgue spectrum* if $\sigma_{f_1,\mathcal{T}}\equiv\lambda$, where λ is a Haar measure on $\widehat{\mathbb{A}}$. It is said that $\mathcal{T}$ has *simple spectrum* if $L^2(X,\mathcal{B},\mu)=\mathbb{A}(f)$ for some $f\in L^2(X,\mathcal{B},\mu)$.

For any real s let

$$\begin{aligned}
\theta_s&:\mathbb{R}\to\mathbb{R}, &\quad \theta_s(t)&=t+s,\\
R_s&:\mathbb{R}\to\mathbb{R}, &\quad R_s(t)&=st,\\
\chi_s&:\mathbb{R}\to\mathbb{T}, &\quad \chi_s(t)&=\exp 2\pi i st.
\end{aligned}$$

Let $\mathcal{T}=(T_t)_{t\in\mathbb{R}}$ be a measurable flow on $(X,\mathcal{B},\mu)$. Fix $f\in L^2_0(X,\mathcal{B},\mu)$. Then for any $s\neq 0$,

$$\int_{\mathbb{R}}e^{2\pi irt}\,d\sigma_{f,\mathcal{T}_s}(t)=\langle f\circ T_{rs},f\rangle=\int_{\mathbb{R}}e^{2\pi irst}\,d\sigma_{f,\mathcal{T}}(t)=\int_{\mathbb{R}}e^{2\pi irt}\,d(R_s)_*\sigma_{f,\mathcal{T}}(t)$$

and

$$\int_{\mathbb{T}}z^n\,d\sigma_{f,T_s}(z)=\langle f\circ T_{sn},f\rangle=\int_{\mathbb{R}}e^{2\pi inst}\,d\sigma_{f,\mathcal{T}}(t)=\int_{\mathbb{T}}z^n\,d(\chi_s)_*\sigma_{f,\mathcal{T}}(z).$$

It follows that $\sigma_{f,\mathcal{T}_s}=(R_s)_*\sigma_{f,\mathcal{T}}$ and $\sigma_{f,T_s}=(\chi_s)_*\sigma_{f,\mathcal{T}}$, and hence $\sigma_{\mathcal{T}_s}=(R_s)_*\sigma_{\mathcal{T}}$ and $\sigma_{T_s}=(\chi_s)_*\sigma_{\mathcal{T}}$.

Suppose that μ and ν are probability singular Borel measures on $\mathbb{R}$. The following two lemmas are well-known; we give proofs for completeness.

Lemma 26. *For almost every $s\in\mathbb{R}$ measures $(\theta_s)_*\mu$ and ν are orthogonal. If $\mu(\{0\})=\nu(\{0\})=0$ then $(R_s)_*\mu$ and ν are orthogonal for almost every $s\in\mathbb{R}$.*

Proof. Since $\mu*\lambda=\lambda$, there exists a measurable set $E\subset\mathbb{R}$ such that $\nu(E)=1$ and

$$0=\mu*\lambda(E)=\int_{\mathbb{R}}\mu(E-s)\,ds=\int_{\mathbb{R}}(\theta_s)_*\mu(E)\,ds.$$

It follows that $(\theta_s)_*\mu(E)=0$ for almost every $s\in\mathbb{R}$.

Set $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $g(x) = \log|x|$ and $abs : \mathbb{R} \to \mathbb{R}$, $abs(x) = |x|$. Notice that $g_*(\mu)$ and $g_*(\nu)$ are also singular. Let G stand for the set of real numbers s such that $g_*(\nu) \perp (\theta_s)_* g_*(\mu)$. Using once more the non–singularity of g we obtain $\lambda(g^{-1}(G)^c) = 0$. Suppose that $s \in g^{-1}(G)$. Since $g(s) \in G$,

$$g_*(\nu) \perp (\theta_{g(s)})_* g_*(\mu).$$

It follows that (by the non–singularity of the function exp)

$$\exp_* g_*(\nu) \perp \exp_*(\theta_{g(s)})_* g_*(\mu) \text{ and } (-\exp)_* g_*(\nu) \perp (-\exp)_*(\theta_{g(s)})_* g_*(\mu)$$

Since $\exp \circ \theta_{g(s)} \circ g = abs \circ R_s$ on $\mathbb{R} \setminus \{0\}$,

$$\nu|_{(0,+\infty)} \ll abs_*\nu \perp abs_*(R_s)_*\mu \gg (R_s)_*\mu|_{(0,+\infty)}$$

and

$$\nu|_{(-\infty,0)} \ll (-abs)_*\nu \perp (-abs)_*(R_s)_*\mu \gg (R_s)_*\mu|_{(-\infty,0)}.$$

Consequently, $\nu \perp (R_s)_*\mu$. □

Lemma 27. *Let μ and ν be probability singular Borel measures on $\mathbb{R}$. If $\mu \perp \nu$ then $(\chi_s)_*\mu \perp (\chi_s)_*\nu$ for almost every $s \in \mathbb{R}$.*

Proof. By the first part of Lemma 26, there exists a measurable set $E \subset \mathbb{R}$ whose complement has zero Lebesgue measure such that $(\theta_{ks})_*\mu \perp \nu$ for every $s \in E$ and $k \neq \mathbb{Z} \setminus \{0\}$. Therefore $(\theta_{ms})_*\mu \perp (\theta_{ns})_*\nu$ for all $s \in E$ and $m \neq n$. By assumption, $(\theta_{ns})_*\mu \perp (\theta_{ns})_*\nu$ for all $s \in \mathbb{R}$ and $n \in \mathbb{Z}$. Thus $(\theta_{ms})_*\mu \perp (\theta_{ns})_*\nu$ for all $s \in E$ and $m, n \in \mathbb{Z}$. It follows that $(\chi_s)_*\mu \perp (\chi_s)_*\nu$ for every $s \in E$. □

Proposition 28. *Let σ be a probability singular Borel measure on $\mathbb{R}$ which has no atom at zero. Then $(\chi_s)_*\sigma \perp (\chi_t)_*\sigma$ for almost all $(s,t) \in \mathbb{R}^2$.*

Proof. Denote by $\mathcal{P}(\mathbb{T})$ the space of all probability Borel measures on $\mathbb{T}$ provided with the weak topology. As it was shown in [2], the set

$$\{(\mu, \nu) \in \mathcal{P}(\mathbb{T}) \times \mathcal{P}(\mathbb{T}) : \mu \perp \nu\}$$

is a G_δ subset of $\mathcal{P}(\mathbb{T}) \times \mathcal{P}(\mathbb{T})$. Since the map

$$\mathbb{R} \ni t \mapsto (\chi_t)_*\sigma \in \mathcal{P}(\mathbb{T})$$

is continuous,

$$G := \{(s,t) \in \mathbb{R}^2 : (\chi_s)_*\sigma \perp (\chi_t)_*\sigma\}$$

is a G_δ subset of $\mathbb{R}^2$. Let $G' = \{(s,t) \in \mathbb{R}^2 : (s, st) \in G\}$. Since the diffeomorphism

$$(\mathbb{R} \setminus \{0\}) \times \mathbb{R} \ni (s,t) \mapsto (s, st) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$$

is a non-singular automorphism with respect to the Lebesgue measure on $\mathbb{R}^2$, it suffices to prove that the complement of G' has zero Lebesgue measure on $\mathbb{R}^2$. By the second Lemma 26, there exists a set $E \subset \mathbb{R}$ such that $\lambda(E^c) = 0$ and $\sigma \perp (R_t)_*\sigma$ for all $t \in E$. Fix $t \in E$. By Lemma 27,

$$(\chi_s)_*\sigma \perp (\chi_s)_*(R_t)_*\sigma = (\chi_{ts})_*\sigma$$

for almost every $s \in \mathbb{R}$, and hence $(s,t) \in G'$ for almost every $s \in \mathbb{R}$. An application of Fubini's theorem for G' gives that the complement of G' (and hence of G) has zero Lebesgue measure on $\mathbb{R}^2$. □

Theorem 29. *Let $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ be an ergodic flow on a probability standard Borel space $(X, \mathcal{B}, \mu)$. If the spectrum of $\mathcal{T}$ is singular then T_s and T_t are spectrally disjoint for almost every pair $(s,t) \in \mathbb{R}^2$.*

Proof. It suffices to note that if σ is the maximal spectral typ of $\mathcal{T}$ then $(\chi_t)_*\sigma$ is the maximal spectral typ of T_t. □

Corollary 30. *Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be an ergodic flow on a standard Borel space $(X,\mathcal{B},\mu)$. Suppose that there exists a measurable set $E \subset \mathbb{R}$ of positive Lebesgue measure such that T_t and T_s are not spectrally disjoint for all $s,t \in E$. Then the flow $\mathcal{T}$ has an absolutely continuous component in its spectrum.*

Proposition 31. *Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ be a weakly mixing flow on a standard Borel space $(X,\mathcal{B},\mu)$. Suppose that there exists a measurable set $E \subset \mathbb{R}$ of positive Lebesgue measure such that $\mathcal{T}$ and $\mathcal{T}_s$ are spectrally equivalent for all $s \in E$. Then the flow $\mathcal{T}$ has a Lebesgue spectrum.*

Proof. If σ denotes the maximal (reduced) spectral type of $\mathcal{T}$ then $(R_s)_*\sigma$ is the maximal (reduced) spectral type of $\mathcal{T}_s$. By considering $\sigma' = \log_*(\sigma|_{(0,+\infty)})$ we obtain a measure on $\mathbb{R}$ for which the set $H(\sigma')$ of $t \in \mathbb{R}$ such that $\sigma' \equiv (\theta_t)_*\sigma'$ is of positive Lebesgue measure. But $H(\sigma')$ is a Borel subgroup of $\mathbb{R}$ (see [6]), hence $H(\sigma') = \mathbb{R}$ and therefore σ' is equivalent to the Lebesgue measure. It follows that $\sigma|_{(0,+\infty)}$ is also equivalent to the Lebesgue measure restricted to $(0,+\infty)$. Since σ is symmetric, we conclude that σ is also equivalent to the Lebesgue measure. □

9. Gaussian flows

The aim of this section is to show a construction of simple spectrum Gaussian flows with minimal set of self-similarities (Gaussian flows are always reversible) as well as with infinite set of self-similarities.

Let $\mathbb{A}$ be a locally compact second countable Abelian group. A measurable $\mathbb{A}$-action $(S_a)_{a\in\mathbb{A}}$ on a probability Borel space $(X,\mathcal{B},\mu)$ is called a Gaussian action if there exists an infinite dimensional real space $H \subset L^2_0(X,\mathcal{B},\mu)$ which generates $\mathcal{B}$, which is invariant under all S_a, $a \in \mathbb{A}$ and for which all nonzero elements are Gaussian variables. A classical result (see e.g. [3], Ch. 8 for the case of $\mathbb{Z}$-actions) is that a Gaussian $(S_a)_{a\in\mathbb{A}}$ is ergodic iff the spectral type σ of $(S_a)_{a\in\mathbb{A}}$ on the Gaussian space H is continuous. Moreover, the maximal spectral type of $(S_a)_{a\in\mathbb{A}}$ on $L^2_0(X,\mathcal{B},\mu)$ is given by $\exp'\sigma = \sum_{n=1}^{\infty}\frac{1}{n!}\sigma^{(n)}$, where $\sigma^{(n)}$ stands for the n-th convolution power of σ.

Let σ be a finite Borel measure on $\mathbb{R}$. Put $X = \mathbb{R}^{\mathbb{R}}$ and let $\xi_s : X \to \mathbb{R}$ stand for the projection on the s-th coordinate for $s \in \mathbb{R}$, i.e. $\xi_s((x_t)_{t\in\mathbb{R}}) = x_s$. By $\mathcal{B}$ denote the smallest σ-algebra of subsets X for which ξ_s is a measurable map for every real s. Given $s \in \mathbb{R}$ let $T_s : X \to X$ be the shift $T_s((x_t)_{t\in\mathbb{R}}) = (x_{t+s})_{t\in\mathbb{R}}$.

A probability measure μ on $(X,\mathcal{B})$ is called a Gaussian measure if the process $(\xi_s)_{s\in\mathbb{R}}$ on $(X,\mathcal{B},\mu)$ is a stationary centered Gaussian process. The Gaussian measure μ determines the spectral measure σ of the Gaussian process by

$$\widehat{\sigma}(s) = \int_{\mathbb{R}} \xi_s(x)\cdot\xi_0(x)\,d\mu(x) \text{ for } s \in \mathbb{R}.$$

Since the Fourier transform $\widehat{\sigma}$ is real, the measure σ is symmetric. Conversely, every symmetric finite Borel measure on $\mathbb{R}$ is the spectral measure a Gaussian process corresponding to a Gaussian measure μ_σ. Let $\mathcal{T}^\sigma = (T^\sigma_t)_{t\in\mathbb{R}}$ stand for the flow on $(X,\mathcal{B},\mu_\sigma)$ given by

$$T^\sigma_t((x_s)_{s\in\mathbb{R}}) = (x_{s+t})_{s\in\mathbb{R}}.$$

Let $H\subset L^2_0(X,\mu_\sigma)$ be the closed real linear subspace generated by ξ_s, $s\in\mathbb{R}$. Since every non-zero element of H has a Gaussian distribution, the flow $\mathcal{T}^\sigma$ is a Gaussian flow with H as its Gaussian space. Moreover, σ is the spectral measure of $\mathcal{T}^\sigma$ on the Gaussian space H. If σ_1 and σ_2 are equivalent continuous measures on $\mathbb{R}$ then the corresponding flows $\mathcal{T}^{\sigma_1}$ and $\mathcal{T}^{\sigma_2}$ are isomorphic.

Fix $s\neq 0$. The automorphism $T_s=T^\sigma_s:(X,\mu_\sigma)\to(X,\mu_\sigma)$ is a Gaussian automorphism with H as its Gaussian space. Moreover, since

$$\langle \xi_r\circ T_s^n,\xi_r\rangle=\langle\xi_{ns},\xi_0\rangle=\int_{\mathbb{R}}e^{2\pi inst}\,d\sigma(t)=\int_{\mathbb{T}}z^n\,d((\chi_s)_*\sigma)(z),$$

for every $r\in\mathbb{R}$ and $n\in\mathbb{Z}$, the spectral measure of ξ_r with respect to T_s is equal to $(\chi_s)_*\sigma$, and hence $(\chi_s)_*\sigma$ is the spectral measure of T_s on H.

A Borel subset $K\subset\widehat{\mathbb{A}}$ is called independent if for any collection of distinct elements $\chi_1,\ldots,\chi_k\in K$ and any $a_1,\ldots,a_k\in\mathbb{A}$ the condition $\chi_1(a_1)\cdot\ldots\cdot\chi_k(a_k)=1$ implies $a_1=\ldots=a_k=e$, where e denotes the neutral element in $\mathbb{A}$.

By the classical theory of Gaussian systems (see [3] Ch. 8 for the case of $\mathbb{Z}$-actions), if σ is a Borel finite measure on $\mathbb{R}$ concentrated on $K\cup(-K)$, where $K\subset\mathbb{R}$ is an independent Borel set then the flow $\mathcal{T}^\sigma$ has simple spectrum.

Proposition 32 (Corollary 1 in [19])**.** *Let σ and τ be finite positive, symmetric continuous Borel measures on $\mathbb{T}$. Assume that σ is concentrated on $K\cup\overline{K}$, where $K\subset\mathbb{T}$ is an independent Borel set and that τ is concentrated on a countable union of independent Borel sets. Then either $\sigma^{(n)}\perp\tau^{(m)}$ for all $m,n\in\mathbb{N}$, or σ and $\tau*\delta_c$ are not mutually singular for some $c\in\mathbb{T}$.*

Let $\mathcal{P}$ stand for the set of all polynomials in variables $x_1,x_2,\ldots$, i.e. $\mathcal{P}=\bigcup_{k\geq1}\mathbb{R}[x_1,\ldots,x_k]$. In other words every polynomial $P\in\mathbb{R}[x_1,\ldots,x_k]$ we will treat it as a polynomial in variables $x_1,x_2,\ldots$ given by

$$P(x_1,x_2,\ldots)=P(x_1,\ldots,x_k).$$

Let us consider two operators $\mathbf{z},\mathbf{s}:\mathcal{P}\to\mathcal{P}$ given by

$$\begin{aligned}\mathbf{z}(P)(x_1,x_2,x_3,\ldots)&=P(0,x_2,x_3,\ldots)\\ \mathbf{s}(P)(x_1,x_2,x_3,\ldots)&=P(x_2,x_2,x_3,\ldots)\end{aligned}$$

for every $P\in\mathcal{P}$.

For each finite subset $Q\subset\mathcal{P}$ denote by $F_n(Q)$ the smallest subset of $\mathcal{P}$ containing Q and closed under taking all permutations of the first 2^n variables and under the action of the operator $\mathbf{z}$. Of course, $F_n(Q)$ is still finite.

Let $(P_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}\setminus\{0\}$. We now define a sequence $(Q_n)_{n\in\mathbb{N}}$ of finite subsets in $\mathcal{P}$ by putting

$$Q_1=F_1(\{P_1\})\setminus\{0\},\ Q_{m+1}=F_{m+1}(Q_m\cup\{P_{m+1}\})\setminus\{0\}.$$

Lemma 33. *For any sequence $(P_n)_{n\in\mathbb{N}}$ in $\mathcal{P}\setminus\{0\}$ there exists a perfect compact subset $K\subset\mathbb{R}$ such that for every $m\in\mathbb{N}$ and $s\in\mathbb{N}$ if there exists a collection of distinct numbers $y_1,\ldots,y_s\in K$ with*

$$P_m(y_1,\ldots,y_s,0,\ldots)=0$$

then the polynomial $P_m(x_1,\ldots,x_s,0,\ldots)$ is the zero polynomial.

Proof. (The construction of K below is a modification of the construction of a Kronecker set from [21]). The set K is given as $K = \bigcap_{n=0}^{\infty} K_n$, where $K_n = \bigcup_{i=1}^{2^n} F_{n,i}$, $F_{n,i}$ is a closed (non-trivial) interval and $F_{n,i} < F_{n,i+1}$. The notation $A < B$ will mean that $a < b$ for all $a \in A$ and $b \in B$. Our construction goes by induction.

1) $K_0 = F_{0,1}$ is an arbitrary closed non-trivial interval.

2) Suppose we have already constructed $K_n = \bigcup_{i=1}^{2^n} F_{n,i}$. In each interval $F_{n,i}$ we find two open intervals $W_{2i-1} < W_{2i}$. Let Q_{n+1}^0 stand for the subset of polynomials $P \in Q_{n+1}$ such that $P(x_1, \ldots, x_{2^{n+1}}, 0, \ldots)$ is a nonzero polynomial. Since the set

$$\bigcup_{P \in Q_{n+1}^0} \{(x_1, \ldots, x_{2^{n+1}}) \in \mathbb{R}^{2^{n+1}}, \;\; P(x_1, \ldots, x_{2^{n+1}}, 0, \ldots) = 0\}$$

has zero Lebesgue measure, there exists $(\theta_1, \ldots, \theta_{2^{n+1}}) \in W_1 \times \ldots \times W_{2^{n+1}}$ such that

$$P(\theta_1, \ldots, \theta_{2^{n+1}}, 0, \ldots) \neq 0 \;\text{ for all }\; P \in Q_{n+1}^0.$$

Next choose $F_{n+1,i} \subset W_i$, $i = 1, \ldots, 2^{n+1}$ such that

$$\theta_i \in \operatorname{Int} F_{n+1,i}, \; |F_{n+1,i}| \leq \frac{1}{2^{n+1}}$$

and

$$(z_1, \ldots, z_{2^{n+1}}) \in F_{n+1,1} \times \ldots \times F_{n+1,2^{n+1}} \Rightarrow P(z_1, \ldots, z_{2^{n+1}}, 0, \ldots) \neq 0 \tag{17}$$

for every $P \in Q_{n+1}^0$.

We will now show that for every $m, s \in \mathbb{N}$ if the polynomial $P_m(x_1, \ldots, x_s, 0, \ldots)$ is non-zero then for any collection of distinct numbers $y_1, \ldots, y_s$ in K we have

$$P_m(y_1, \ldots, y_s, 0, \ldots) \neq 0.$$

Indeed, fix $m \geq 1$ and $s \in \mathbb{N}$ and suppose that $P_m(x_1, \ldots, x_s, 0, \ldots)$ is non-zero. Take $y_1, \ldots, y_s$ in K such that $y_i \neq y_j$ for $i \neq j$. Let $n \in \mathbb{N}$ be so large that $m \leq n$, $s \leq 2^n$ and $\max_i |F_{n,i}| < \min_{i \neq j} |y_i - y_j|$. We can find a permutation σ of $\{1, \ldots, s\}$ and $1 \leq j(1) < j(2) < \ldots < j(s) \leq 2^n$ such that $y_{\sigma(i)} \in F_{n,j(i)}$ for $i = 1, \ldots, s$. By the definition of Q_n, the polynomial

$$W(x_1, x_2, \ldots) = P_m(x_{j(\sigma^{-1}(1))}, x_{j(\sigma^{-1}(2))}, \ldots, x_{j(\sigma^{-1}(s))}, 0, \ldots)$$

belongs to Q_n^0. Choose $(z_1, \ldots, z_{2^n}) \in F_{n,1} \times \ldots \times F_{n,2^n}$ such that $z_{j(i)} = y_{\sigma(i)}$ for $i = 1, \ldots, s$. From (17),

$$\begin{aligned} P_m(y_1, \ldots, y_s, 0, \ldots) &= P_m(z_{j(\sigma^{-1}(1))}, \ldots, z_{j(\sigma^{-1}(s))}, 0, \ldots) \\ &= W(z_1, \ldots, z_{2^n}, 0, \ldots) \neq 0. \end{aligned}$$

□

Let $A \subset (0, +\infty)$ be an at most countable subset of positive numbers such that for every polynomial $P \in \mathbb{Q}[x_1, \ldots, x_k]$ if there exists a collection $a_1, \ldots, a_k$ of distinct elements of A with $P(a_1, \ldots, a_k) = 0$ then $P \equiv 0$. Note that A can be also empty. Let $G(A)$ stand for the multiplicative subgroup generated by the elements of A. In the case of $A = \emptyset$ we will adhere to the convention that $G(A) = \{1\}$.

Lemma 34. *There exists a perfect compact subset $K \subset \mathbb{R}$ such that the set*

$$\widehat{K} = \bigcup_{g \in G(A)} gK$$

is independent and its symmetrization $\widetilde{K} = \widehat{K} \cup (-\widehat{K})$ satisfies:

$$|r| \notin G(A) \Longrightarrow \#((r\widetilde{K} + t) \cap \widetilde{K}) \leq \aleph_0 \text{ for any real } t.$$

Proof. Let

$$P(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (x_1 - x_3)(x_6 - x_8) - (x_2 - x_4)(x_5 - x_7).$$

Let $\widehat{P}$ denote the smallest subset of $\mathbb{R}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$ containing P, and closed under taking all permutations of coordinates and under the action of the operator $\mathbf{s}$. Let $\mathbb{Q}(A)$ stand for the field which is the extension of $\mathbb{Q}$ by the elements of A. Let Q^* stand for the set all nonzero polynomials of the form

$$q_1 x_1 + \ldots + q_k x_k,$$

where $q_j \in \mathbb{Q}(A)$ for $j = 1, \ldots, k$, and $k \geq 1$.

Let $(P_m)_{m=1}^{\infty}$ be a sequence containing all elements from $(\widehat{P} \cup Q^*) \setminus \{0\}$. Let K satisfy the assertion of Lemma 33. Put

$$\widehat{K} = \bigcup_{g \in G(A)} gK.$$

First note that $\widehat{K}$ is independent. Indeed, suppose that $y_1, \ldots, y_k$ is a collection of distinct elements of $\widehat{K}$ such that

$$q_1 y_1 + \ldots + q_k y_k = 0 \text{ for some rational } q_1, \ldots, q_k.$$

We can find a finite collection $a_1, \ldots, a_m$ of distinct elements of A, $z_1, \ldots, z_k$ in K and an integer matrix $[\beta_{lj}]_{1 \leq l \leq m,\, 1 \leq j \leq k}$ such that

$$y_j = \prod_{l=1}^{m} a_l^{\beta_{lj}} z_j \text{ for every } j = 1, \ldots, k$$

Without loss of generality we can assume that $z_1, \ldots, z_s$ are distinct for some $1 \leq s \leq k$ and $z_i \in \{z_1, \ldots, z_s\}$ for every $1 \leq i \leq k$. Moreover,

$$0 = \sum_{j=1}^{k} q_j y_j = \sum_{j=1}^{s} \prod_{l=1}^{m} a_l^{\beta_{lj}} P_j(a_1, \ldots, a_m) z_j,$$

where the set of coefficients of the polynomials $P_j \in \mathbb{Q}[x_1, \ldots, x_m]$, $j = 1, \ldots, s$ is equal to $\{q_1, \ldots, q_k\}$. Indeed, even if $z_i = z_j$ for $i \neq j$ we still have $\prod_{l=1}^{m} a_l^{\beta_{li}} \neq \prod_{l=1}^{m} a_l^{\beta_{lj}}$, otherwise $y_i = y_j$. Since $\prod_{l=1}^{m} a_l^{\beta_{lj}} P_j(a_1, \ldots, a_m) \in \mathbb{Q}(A)$ for $j = 1, \ldots, s$ and $z_1, \ldots, z_s$ are distinct elements of K, $P_j(a_1, \ldots, a_m) = 0$, and hence $P_j \equiv 0$ for all $j = 1, \ldots, s$. It follows that $q_1 = \ldots = q_k = 0$.

Assume that $|r| \notin G(A)$. It is enough to consider $r \neq 0$. Let $(G_n(A))_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets of $G(A)$ such that $\bigcup_{n \in \mathbb{N}} G_n(A) = G(A)$. Let

$$\widetilde{K}_n = \bigcup_{g \in G_n(A) \cup (-G_n(A))} gK.$$

Since $\widetilde{K}_n \nearrow \widetilde{K}$, it suffices to prove that for each $t \in \mathbb{R}$

$$\#((r\widetilde{K}_n + t) \cap \widetilde{K}_n) < 16(\#G_n(A))^2 \text{ for all } n \geq 1.$$

Suppose, contrary to our claim, that for some $t \in \mathbb{R}$ the set $(r\widetilde{K}_n + t) \cap \widetilde{K}_n$ contains at least $l = 16(\#G_n(A))^2$ distinct elements. It follows that there are $y_1, \ldots, y_{2l} \in K$ and $s_1, \ldots, s_{2l} \in G_n(A) \cup (-G_n(A))$ such that

$$rs_{2k-1}y_{2k-1} + t = s_{2k}y_{2k} \text{ for } k = 1, \ldots, l \tag{18}$$

and $s_{2k}y_{2k}$, $k = 1, \ldots, l$ are distinct numbers. Since $l \geq 4(\#(G_n(A) \cup (-G_n(A))))^2$, the sequence $\{(s_{2k-1}, s_{2k})\}_{k=1}^{l}$ contains at least four identical elements. By renaming points, if necessary, we can assume that $(s_{2k-1}, s_{2k}) = (s_1, s_2)$ for $k = 1, 2, 3, 4$. It follows that

$$r(s_1y_1 - s_1y_3) = s_2y_2 - s_2y_4 \text{ and } r(s_1y_5 - s_1y_7) = s_2y_6 - s_2y_8,$$

and hence

$$(y_1 - y_3)(y_6 - y_8) = (y_2 - y_4)(y_5 - y_7).$$

Moreover, y_2, y_4, y_6, y_8 are distinct, and hence y_1, y_3, y_5, y_7 are distinct. Suppose that $\{z_1, \ldots, z_s\} = \{y_1, \ldots, y_8\}$, where $z_1, \ldots, z_s$ are distinct numbers. Then $s \geq 4$. Let us consider the function $\vartheta : \{1, \ldots, 8\} \to \{1, \ldots, s\}$ determined by $y_i = z_{\vartheta(i)}$ for $i = 1, \ldots, 8$. Note that $\vartheta(1), \vartheta(3), \vartheta(5), \vartheta(7)$ and $\vartheta(2), \vartheta(4), \vartheta(6), \vartheta(8)$ are two collections of distinct numbers. Let

$$W(x_1, \ldots, x_s) = (x_{\vartheta(1)} - x_{\vartheta(3)})(x_{\vartheta(6)} - x_{\vartheta(8)}) - (x_{\vartheta(2)} - x_{\vartheta(4)})(x_{\vartheta(5)} - x_{\vartheta(7)}).$$

Then $W \in \widehat{P}$ and $W(z_1, \ldots, z_s) = 0$. Since $z_1, \ldots, z_s$ belong to K and are distinct, W is the zero polynomial. Observe that $\vartheta(1) \neq \vartheta(6), \vartheta(8)$. Otherwise W contains the monomial $x^2_{\vartheta(1)}$ with a nonzero coefficient, contrary to $W \equiv 0$. It follows that $\vartheta(1) \in \{\vartheta(2), \vartheta(4)\}$. Similar arguments show that $\{\vartheta(1), \vartheta(3)\} = \{\vartheta(2), \vartheta(4)\}$ and $\{\vartheta(5), \vartheta(7)\} = \{\vartheta(6), \vartheta(8)\}$. Since $W \equiv 0$, it follows that $\vartheta(1) = \vartheta(2)$, $\vartheta(3) = \vartheta(4)$, $\vartheta(5) = \vartheta(6)$, $\vartheta(7) = \vartheta(8)$, or $\vartheta(1) = \vartheta(4)$, $\vartheta(3) = \vartheta(2)$, $\vartheta(5) = \vartheta(8)$, $\vartheta(7) = \vartheta(8)$. Thus, by (18),

$$rs_1y_2 + t = s_2y_2, \; rs_1y_4 + t = s_2y_4, \; rs_1y_6 + t = s_2y_6, \; rs_1y_8 + t = s_2y_8.$$

or

$$rs_1y_4 + t = s_2y_2, \; rs_1y_2 + t = s_2y_4, \; rs_1y_8 + t = s_2y_6, \; rs_1y_6 + t = s_2y_8.$$

Since $r \notin G(A) \cup (-G(A))$, we conclude that $y_2 = y_4 = y_6 = y_8$, contrary to our claim. $\square$

Theorem 35. *If ρ is a continuous measure supported on K then the Gaussian flow $\mathcal{T} = (T_t)_{t \in \mathbb{R}}$ given by the measure*

$$\widetilde{\rho} = \sum_{g \in G(A) \cup (-G(A))} p_g (R_g)_*(\rho)$$

($p_g > 0$ and $\sum p_g < +\infty$) has a simple singular spectrum and

- *if $|\tau_1/\tau_2| \in G(A)$ then $\mathcal{T}_{\tau_1}$ is isomorphic to $\mathcal{T}_{\tau_2}$;*
- *if $|\tau_1/\tau_2| \notin G(A)$ then T_{τ_1} is spectrally disjoint from T_{τ_2}, and hence the flows $\mathcal{T}_{\tau_1}$ and $\mathcal{T}_{\tau_2}$ are spectrally disjoint.*

Proof. The simplicity of the spectrum of $\mathcal{T}$ follows directly from the independence of the set $\widehat{K} \subset \mathbb{R}$. The second assertion follows from the fact that

$$(R_\tau)_*(\widetilde{\rho}) \equiv \widetilde{\rho} \text{ for every } \tau \in G(A) \cup (-G(A)).$$

If $|\tau_1/\tau_2| \notin G(A)$ then the measures $(R_{\tau_1})_*(\widetilde{\rho}) * \delta_t$ and $(R_{\tau_2})_*(\widetilde{\rho})$ are orthogonal for every $t \in \mathbb{R}$. It follows that $(\chi_{\tau_1})_*(\widetilde{\rho}) * \delta_c \perp (\chi_{\tau_2})_*(\widetilde{\rho})$ for every $c \in \mathbb{T}$. The measure

$(\chi_{\tau_i})_*(\widetilde{\rho})$ is the spectral measure of T_{τ_i} on its Gaussian space for $i = 1, 2$. Let $D_i \subset \widehat{K}$ be the set of numbers $x \in \widehat{K}$ for which there exist $n_0, n_1, \ldots, n_r, n \in \mathbb{Z} \setminus \{0\}$ and a collection of distinct elements $x_1, \ldots, x_r$ of $\widehat{K}$ different from x such that

$$n_0 x + n_1 x_1 + \ldots + n_s x_s = n/\tau_i.$$

Since $\widehat{K}$ is independent, the set D_i is at most countable. By the definition of D_i, the set $\chi_{\tau_i}(\widehat{K} \setminus D_i) \subset \mathbb{T}$ is an independent Borel set. Moreover, since $\widetilde{\rho}$ is concentrated on $\widehat{K} \cup (-\widehat{K})$ and it is continuous, the measure $(\chi_{\tau_i})_*(\widetilde{\rho})$ is concentrated on $\chi_{\tau_i}(\widehat{K} \setminus D_i) \cup \overline{\chi_{\tau_i}(\widehat{K} \setminus D_i)}$. An application of Proposition 32 for $(\chi_{\tau_1})_*(\widetilde{\rho})$ and $(\chi_{\tau_2})_*(\widetilde{\rho})$ gives the mutual singularity of $(\chi_{\tau_1})_*(\widetilde{\rho})^{(m)}$ and $(\chi_{\tau_2})_*(\widetilde{\rho})^{(n)}$ for all $m, n \in \mathbb{N}$. It follows that $\exp'(\chi_{\tau_1})_*(\widetilde{\rho}) \perp \exp'(\chi_{\tau_2})_*(\widetilde{\rho})$, and hence T_{τ_1} and T_{τ_2} are spectrally disjoint. □

Remark 5. In this case of $A = \emptyset$, Theorem 35 yields an example of weakly mixing flow $\mathcal{T}$ with simple spectrum which has no spectral self-similarity, i.e. $SI(\mathcal{T}) = \{-1, 1\}$ and additionally $\mathcal{T}_s$ is spectrally disjoint from $\mathcal{T}$ for all $s \neq \pm 1$.

10. Open problems

Problem 1. Is $I(\mathcal{T})$ a Borel group for any measurable flow $\mathcal{T}$?

Problem 2. Find a flow $\mathcal{T}$ for which the group $I(\mathcal{T})$ is not countable and has zero Lebesgue measure. Give a classification of multiplicative subgroups of $\mathbb{R}$ that can be obtained as $I(\mathcal{T})$.

The same type of questions can be formulated for smooth system. The existence of smooth flow which is not self-similar was announced by J. Kułaga. She uses a smooth flow on the closed orientable two dimensional surface with genus two isomorphic to the special flow built over an irrational rotation on the circle and under a roof function which is of symmetric logarithmic type.

Problem 3. Solve the self-similarity problem for roof functions of non-symmetric logarithmic type.

In the non-symmetric logarithmic case the special flow built over any irrational rotation is mixing (see [18]), and hence the method of proving the absence of self-similarity presented in Section 6 falls.

Problem 4. Find a self-similar smooth flow for which $I(\mathcal{T})$ has zero Lebesgue measure.

In [22] de la Rue and de Sam Lazaro have shown that a typical automorphisms of a standard Borel space is embeddable in a measurable flow; i.e. a typical automorphism T is isomorphic to the time-1 map T_1 of a measurable flow $(T_t)_{t\in\mathbb{R}}$.

Problem 5. Can we embed a typical automorphism in a self-similar flow?

Let $\mathrm{Flow}(X, \mathcal{B}, \mu)$ stand for the set of measure-preserving flows of a standard probability space $(X, \mathcal{B}, \mu)$. Let $\{A_n : n \in \mathbb{N}\}$ a countable family in $\mathcal{B}$ which is dense in $\mathcal{B}$ for the (pseudo–)metrics $d_\mu(A, B) = \mu(A \triangle B)$. Let us consider the metric d on the group $\mathrm{Aut}(X, \mathcal{B}, \mu)$ of measure preserving automorphisms defined by

$$d(T, S) = \sum_{n=1}^{\infty} \frac{1}{2^n} (\mu(TA_n \triangle SA_n) + \mu(T^{-1}A_n \triangle S^{-1}A_n)).$$

The group $\mathrm{Aut}(X,\mathcal{B},\mu)$ endowed with d is a topological Polish group. Then the distance $d_{\mathcal{F}}$ on $\mathrm{Flow}(X,\mathcal{B},\mu)$ is given by

$$d_{\mathcal{F}}((S_t)_{t\in\mathbb{R}},(T_t)_{t\in\mathbb{R}})=\sup_{0\leq t\leq 1}d(S_t,T_t).$$

Problem 6. Is the absence of self-similarity generic in the set of measure preserving flows $\mathrm{Flow}(X,\mathcal{B},\mu)$?

Appendix A. Absence of partial rigidity

Let $T=T_{\lambda,\pi}:[0,1)\to[0,1)$ be an ergodic m-interval exchange transformation and $f:[0,1]\to\mathbb{R}$ be a piecewise absolutely continuous positive function such that $f\geq c>0$. Let $0<\beta_1<\ldots<\beta_k<1$ stand for all discontinuities of $f:[0,1)\to\mathbb{R}$. Let Ξ_j stand for the set of all discontinuities of T^j, $j>0$. Then $\#\Xi_j=(m-1)j$ and the set of all discontinuities of $f^{(j)}$ is a subset of

$$\bigcup_{l=0}^{j-1}T^{-l}\{\beta_1,\ldots,\beta_k\}\cup\Xi_{j-1}.$$

It follows that $f^{(j)}$ has at most $kj+(m-1)(j-1)\leq(k+m)j-1$ discontinuities.

Theorem 36. *If $S(f)\neq 0$ then the special flow T^f is not partially rigid.*

Proof. Let $C:=\max_{x\in[0,1]}f(x)$. Then $0<c\leq f(x)\leq C$ for every $x\in[0,1]$. Let μ stand for Lebesgue measure on $[0,1]$. Assume, contrary to our claim, that (t_n), $t_n\to+\infty$, is a partial rigidity time for T^f. By Lemma 7.1 in [10], there exists $0<u\leq 1$ such that for every $0<\varepsilon<c$ we have

$$\liminf_{n\to\infty}\mu\{x\in[0,1):\exists_{j\in\mathbb{N}}\ |f^{(j)}(x)-t_n|<\varepsilon\}\geq u. \tag{19}$$

Without loss of generality we can assume that $S:=S(f)>0$, in the case $S<0$ the proof goes along the same lines. Fix

$$0<\varepsilon<\min\left(\frac{Sc^2}{32(k+m)C^2(1+\mathrm{Var}\,f)+Sc^2}u,\frac{c}{4}\right). \tag{20}$$

Since $f'\in L^1([0,1),\mu)$, there exists $0<\delta<\varepsilon$ such that $\mu(A)<\delta$ implies $\int_A|f'|\,d\mu<\varepsilon$. Moreover, by the ergodicity of T (and recalling that $S=\int_0^1 f'\,d\mu$) there exist $A_\varepsilon\subset[0,1)$ with $\mu(A_\varepsilon)>1-\delta$ and $m_0\in\mathbb{N}$ such that

$$\frac{S}{2}\leq\frac{1}{m}f'^{(m)}(x)\text{ for all }m\geq m_0\text{ and }x\in A_\varepsilon. \tag{21}$$

Then take any $n\in\mathbb{N}$ such that $t_n/(2C)\geq m_0$ and $t_n>2\varepsilon$. Now let us consider the set $J_{n,\varepsilon}$ of all natural j such that $|f^{(j)}(x)-t_n|<\varepsilon$ for some $x\in[0,1)$. Then for such j and x we have

$$t_n+\varepsilon>f^{(j)}(x)\geq cj\quad\text{and } t_n-\varepsilon<f^{(j)}(x)\leq Cj,$$

whence

$$t_n/(2C)\leq(t_n-\varepsilon)/C<j<(t_n+\varepsilon)/c\leq 2t_n/c \tag{22}$$

for any $j\in J_{n,\varepsilon}$; in particular, $j\in J_{n,\varepsilon}$ implies $j\geq m_0$.

Let $j^{(n)}=\max J_{n,\varepsilon}$. The points of discontinuity of $f^{(j^{(n)})}$ divide $[0,1)$ into subintervals $I_1^{(n)},\ldots,I_{u_n}^{(n)}$. By the remark preceding the theorem,

$$u_n\leq(k+m)j^{(n)}. \tag{23}$$

Notice that for every $j \in J_{n,\varepsilon}$ the function $f^{(j)}$ is absolutely continuous in the interior of any interval $I_i^{(n)}$, $i = 1, \ldots, u_n$. Moreover, since $(T^i)'(x) = 1$ for all points x of continuity of T^i, $(f^{(j)})'(x) = (f')^{(j)}(x)$ for almost all $x \in [0,1)$ and all natural j.

Fix $1 \leq i \leq u_n$. For every $j \in J_{n,\varepsilon}$ let

$$I_{i,j}^{(n)} = \{x \in I_i^{(n)} : |f^{(j)}(x) - t_n| < \varepsilon\}.$$

Of course, $I_{i,j}^{(n)}$ may be empty. Note that $\overline{I_{i,j}^{(n)}}$, $j \in J_{n,\varepsilon}$ are pairwise disjoint. Indeed suppose that $x \in I_{i,j}^{(n)}$ and $y \in I_{i,j'}^{(n)}$, where $j \neq j'$. In view of (20), $\varepsilon < c/4$, and hence

$$\begin{aligned}
\int_x^y |f'|^{(j^{(n)})}\, d\mu &\geq \left|\int_x^y f'^{(j)}\, d\mu\right| = \left|\int_x^y (f^{(j)})'\, d\mu\right| = |f^{(j)}(y) - f^{(j)}(x)| \\
&\geq |f^{(j)}(y) - f^{(j')}(y)| - |f^{(j')}(y) - t_n| - |f^{(j)}(x) - t_n| \\
&\geq |f^{(j-j')}(T^{j'}y)| - 2\varepsilon \geq c - 2\varepsilon \geq \frac{c}{2}.
\end{aligned}$$

It follows that

$$\int_x^y |f'|^{(j^{(n)})}\, d\mu \geq \frac{c}{2} \text{ whenever } x \in \overline{I_{i,j}^{(n)}},\ y \in \overline{I_{i,j'}^{(n)}} \text{ and } j \neq j', \tag{24}$$

which excludes the possibility that x and y are arbitrarily close, and therefore $\overline{I_{i,j}^{(n)}} \cap \overline{I_{i,j'}^{(n)}} = \emptyset$.

Furthermore, if $x, y \in I_{i,j}^{(n)}$ then

$$\left|\int_x^y f'^{(j)}\, d\mu\right| = \left|\int_x^y (f^{(j)})'\, d\mu\right| = |f^{(j)}(y) - f^{(j)}(x)| \leq 2\varepsilon,$$

therefore

$$\left|\int_x^y f'^{(j)}\, d\mu\right| \leq 2\varepsilon \text{ wherever } x, y \in \overline{I_{i,j}^{(n)}}. \tag{25}$$

Let us consider a sequence of points

$$\inf I_i^{(n)} \leq a_1^i \leq b_1^i \leq a_2^i \leq b_2^i \leq \ldots \leq a_{s_i}^i \leq b_{s_i}^i \leq \sup I_i^{(n)}$$

and a sequence $(j_l)_{l=1}^{s}$ of distinct numbers from $J_{n,\varepsilon}$ determined by the following inductive procedure:

$$a_1^i = \inf\left(\bigcup_{j \in J_{n,\varepsilon}} I_{i,j}^{(n)}\right) = \inf I_{i,j_1}^{(n)}, \quad b_1^i = \sup I_{i,j_1}^{(n)},$$

$$a_{l+1}^i = \inf\left(\bigcup_{j \in J_{n,\varepsilon}} I_{i,j}^{(n)} \cap (b_l^i, 1)\right) = \inf I_{i,j_{l+1}}^{(n)} \cap (b_l^i, 1), \quad b_{l+1}^i = \sup I_{i,j_{l+1}}^{(n)},$$

if $\bigcup_{j \in J_{n,\varepsilon}} I_{i,j}^{(n)} \cap (b_l^i, 1) = \emptyset$, the procedure stops. Since $\bigcup_{j \in J_{n,\varepsilon}} I_{i,j}^{(n)} \cap (b_l^i, a_{l+1}^i) = \emptyset$ for $1 \leq l < s_i$, we have

$$\bigcup_{j \in J_{n,\varepsilon}} I_{i,j}^{(n)} \subset \bigcup_{l=1}^{s_i} [a_l^i, b_l^i].$$

From (25) and (22) for every $1 \le l \le s_i$ we have

$$\left|\int_{a_l^i}^{b_l^i} \frac{f'^{(j_l)}}{j_l}\, d\mu\right| \le \frac{2\varepsilon}{j_l} \le \frac{4C\varepsilon}{t_n}.$$

Moreover, by (24),

$$\int_{b_l^i}^{a_{l+1}^i} |f'|^{(j^{(n)})}\, d\mu \ge \frac{c}{2} \text{ for all } 1 \le l < s_i.$$

It follows that

$$\begin{aligned}
\left|\sum_{l=1}^{s_i}\int_{a_l^i}^{b_l^i} \frac{f'^{(j_l)}}{j_l}\, d\mu\right| &\le s_i\frac{4C\varepsilon}{t_n} = \frac{4C\varepsilon}{t_n} + \frac{8C\varepsilon}{ct_n}(s_i-1)\frac{c}{2}\\
&\le \frac{4C\varepsilon}{t_n} + \frac{8C\varepsilon}{ct_n}\sum_{l=1}^{s_i-1}\int_{b_l^i}^{a_{l+1}^i} |f'|^{(j^{(n)})}\, d\mu \qquad (26)\\
&\le \frac{4C\varepsilon}{t_n} + \frac{8C\varepsilon}{ct_n}\int_{I_i^{(n)}} |f'|^{(j^{(n)})}\, d\mu.
\end{aligned}$$

Since $\mu(A_\varepsilon^c) < \delta$, by (22), we have

$$\begin{aligned}
&\left|\sum_{i=1}^{u_n}\sum_{l=1}^{s_i}\int_{[a_l^i,b_l^i]\cap A_\varepsilon^c} \frac{f'^{(j_l)}}{j_l}\, d\mu\right|\\
&\le \frac{2C}{t_n}\sum_{i=1}^{u_n}\sum_{l=1}^{s_i}\int_{[a_l^i,b_l^i]\cap A_\varepsilon^c} |f'|^{(j^{(n)})}\, d\mu \qquad (27)\\
&\le \frac{2C}{t_n}\int_{A_\varepsilon^c} |f'|^{(j^{(n)})}\, d\mu \le \frac{2C}{t_n} j^{(n)}\varepsilon \le \frac{4C}{c}\varepsilon.
\end{aligned}$$

As

$$B_n := \{x \in [0,1) : \exists_{j\in\mathbb{N}}\, |f^{(j)}(x) - t_n| < \varepsilon\} = \bigcup_{i=1}^{u_n}\bigcup_{j\in J_{n,\varepsilon}} I_{i,j}^{(n)} \subset \bigcup_{i=1}^{u_n}\bigcup_{l=1}^{s_i}[a_l^i, b_l^i],$$

by (21), (26), (27), (23) and (22) we have

$$\begin{aligned}
\frac{S}{2}\mu(B_n\cap A_\varepsilon) &\le \sum_{i=1}^{u_n}\sum_{l=1}^{s_i}\int_{[a_l^i,b_l^i]\cap A_\varepsilon} \frac{f'^{(j_l)}}{j_l}\, d\mu\\
&\le \left|\sum_{i=1}^{u_n}\sum_{l=1}^{s_i}\int_{a_l^i}^{b_l^i} \frac{f'^{(j_l)}}{j_l}\, d\mu\right| + \left|\sum_{i=1}^{u_n}\sum_{l=1}^{s_i}\int_{[a_l^i,b_l^i]\cap A_\varepsilon^c} \frac{f'^{(j_l)}}{j_l}\, d\mu\right|\\
&\le u_n\frac{4C\varepsilon}{t_n} + \frac{8C\varepsilon}{t_n c}\int_{[0,1)} |f'|^{(j^{(n)})}\, d\mu + \frac{4C}{c}\varepsilon\\
&\le \frac{4(k+m)j^{(n)}C\varepsilon}{t_n} + \frac{4C\varepsilon}{c} + \frac{8C\varepsilon j^{(n)}}{t_n c}\|f'\|_{L^1}\\
&\le \frac{8(k+m)C\varepsilon}{c} + \frac{4C\varepsilon}{c} + \frac{16C\varepsilon}{c^2}\|f'\|_{L^1}\\
&\le \frac{16(k+m)C^2}{c^2}(1+\mathrm{Var}\, f)\varepsilon.
\end{aligned}$$

Finally, from (20) we obtain

$$\mu(B_n) \le \mu(B_n \cap A_\varepsilon) + \mu(A_\varepsilon^c) < \frac{32(k+m)C^2}{Sc^2}(1 + \operatorname{Var} f)\varepsilon + \varepsilon < u,$$

contrary to (19). □

Appendix B. Disjointness of special flows under piecewise absolutely continuous roof functions

Let (X, d) and $(Y, \bar{d})$ be σ–compact metric spaces and let $\mathcal{B}$ and $\mathcal{C}$ be the σ–algebras of Borel subsets of X and Y respectively. Let μ and ν be Borel probability measures on (X, d) and $(Y, \bar{d})$. Suppose that $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ and $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ are weakly mixing flows on $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ respectively. Let $P \subset \mathbb{R} \setminus \{0\}$ be a finite subset and $t_0 \in \mathbb{R} \setminus \{0\}$.

Suppose that the pair $(\mathcal{T}, \mathcal{S})$ verifies the following $R(t_0, P)$ ($t_0 \in \mathbb{R}$) property: for every $\varepsilon > 0$ there exist $\kappa = \kappa(\varepsilon) > 0$ and $X(\varepsilon) \in \mathcal{B}$, $Y(\varepsilon) \in \mathcal{C}$ with $\mu(X(\varepsilon)^c) < \varepsilon$, $\nu(Y(\varepsilon)^c) < \varepsilon$ such that for every $x \in X(\varepsilon)$, $y \in Y(\varepsilon)$ and $N \in \mathbb{N}$ there are $L = L(x, y) \ge N$, $M = M(x, y) > 0$, $Q = Q(x, y) \ge 0$ such that $L/(M + Q) \ge \kappa$ and there exists $p = p(x, y) \in P$ such that

$$\begin{aligned} 1 - \varepsilon \quad < \quad & \frac{1}{L}\#\{n \in \mathbb{Z} \cap [M, M + L] : \\ & d(T_{(Q+n)t_0}(x), T_{nt_0}(x)) < \varepsilon,\ \bar{d}(S_{(Q+n)t_0}(y), S_{nt_0+p}(y)) < \varepsilon\}. \end{aligned}$$

Remark 6. Suppose that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic automorphism. Fix $A \in \mathcal{B}$. Notice that if

$$\left| \frac{1}{M} \sum_{n=0}^{M-1} \chi_A(T^n x) - \mu(A) \right| < \eta_1 \text{ and } \left| \frac{1}{M + L + 1} \sum_{n=0}^{M+L} \chi_A(T^n x) - \mu(A) \right| < \eta_2$$

then

$$\left| \frac{1}{L} \sum_{n=M}^{M+L} \chi_A(T^n x) - \mu(A) \right| < \eta_1 \frac{M}{L} + \eta_2 \left(2 + \frac{M}{L}\right).$$

It follows that for every $\varepsilon > 0$, $\delta > 0$ and $\kappa > 0$ there exist $N = N(\varepsilon, \delta, \kappa) \in \mathbb{N}$ and $X(\varepsilon, \delta, \kappa) \in \mathcal{B}$ with $\mu(X(\varepsilon, \delta, \kappa)) > 1 - \delta$ such that for every $M, L \in \mathbb{N}$ with $L \ge N$ and $L/M \ge \kappa$ we have

$$\left| \frac{1}{L} \sum_{n=M}^{M+L} \chi_A(T^n x) - \mu(A) \right| < \varepsilon \text{ for all } x \in X(\varepsilon, \delta, \kappa).$$

Theorem 37. *Suppose that $(\mathcal{T}, \mathcal{S})$ has the $R(s, P)$–property for uncountably many $s \in \mathbb{R}$. Then $\mathcal{T}$ is disjoint from $\mathcal{S}$.*

Proof. Suppose, contrary to our claim, that there exists $\rho \in J^e(\mathcal{T}, \mathcal{S})$ such that $\rho \ne \mu \otimes \nu$. Since the flow $(T_t \times S_t)_{t\in\mathbb{R}}$ is ergodic on $(X \times Y, \rho)$, we can find $t_0 \ne 0$ such that the automorphism $T_{t_0} \times S_{t_0} : (X \times Y, \rho) \to (X \times Y, \rho)$ is ergodic and the pair $(\mathcal{T}, \mathcal{S})$ has the $R(t_0, P)$–property. To simplify notation we assume that $t_0 = 1$.

Since the ergodicity of S_p implies disjointness of S_p from the identity, for every $p \in P$ there exist closed subsets $A_p \subset X$, $B_p \subset Y$ such that

$$\rho(A_p \times S_{-p}B_p) \ne \rho(A_p \times B_p).$$

Let

$$0 < \varepsilon := \min\{|\rho(A_p \times S_{-p}B_p) - \rho(A_p \times B_p)| : p \in P\}. \tag{28}$$

Next choose $0 < \varepsilon_1 < \varepsilon/8$ such that $\mu(A_p^{\varepsilon_1} \setminus A_p) < \varepsilon/4$, $\nu(B_p^{\varepsilon_1} \setminus B_p) < \varepsilon/4$ for $p \in P$, where $A_p^{\varepsilon_1} = \{z \in X : d(z, A) < \varepsilon_1\}$ and $B_p^{\varepsilon_1} = \{\bar{z} \in Y : \bar{d}(\bar{z}, B) < \varepsilon_1\}$. Then

$$|\rho(A_p \times B_p) - \rho(A_p^{\varepsilon_1} \times B_p^{\varepsilon_1})| < \varepsilon/2.$$

Indeed,

$$\begin{aligned} |\rho(A_p \times B_p) - \rho(A_p^{\varepsilon_1} \times B_p^{\varepsilon_1})| &= \rho(A_p^{\varepsilon_1} \times B_p^{\varepsilon_1} \setminus A_p \times B_p) \\ &= \rho(A_p^{\varepsilon_1} \times B_p^{\varepsilon_1} \setminus A_p^{\varepsilon_1} \times B_p) + \rho(A_p^{\varepsilon_1} \times B_p \setminus A_p \times B_p) \\ &\leq \rho(X \times (B_p^{\varepsilon_1} \setminus B_p)) + \rho((A_p^{\varepsilon_1} \setminus A_p) \times Y) \\ &= \nu(B_p^{\varepsilon_1} \setminus B_p) + \mu(A_p^{\varepsilon_1} \setminus A_p) < \varepsilon/2 \end{aligned}$$

and similarly

$$|\rho(A_p \times S_{-p}B_p) - \rho(A_p^{\varepsilon_1} \times S_{-p}(B_p^{\varepsilon_1}))| < \varepsilon/2 \tag{29}$$

for any $p \in P$.

Let $\kappa := \kappa(\varepsilon_1)(> 0)$. Since $T_1 \times S_1$ on $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \rho)$ is an ergodic automorphism, by Remark 6, there exist a measurable set $U \subset X \times Y$ with $\rho(U) > 3/4$ and $N \in \mathbb{N}$ such that if $(x, y) \in U$, $p \in P$, $l \geq N$ and $l/m \geq \kappa$ then

$$\left| \frac{1}{l} \sum_{k=m}^{m+l} \chi_{A_p \times B_p}(T_k x, S_k y) - \rho(A_p \times B_p) \right| < \frac{\varepsilon}{8}, \tag{30}$$

$$\left| \frac{1}{l} \sum_{k=m}^{m+l} \chi_{A_p^{\varepsilon_1} \times S_{-p}(B_p^{\varepsilon_1})}(T_k x, S_k y) - \rho(A_p^{\varepsilon_1} \times S_{-p}(B_p^{\varepsilon_1})) \right| < \frac{\varepsilon}{8} \tag{31}$$

and similar inequalities hold for $A_p^{\varepsilon_1} \times B_p^{\varepsilon_1}$ and $A_p \times S_{-p}B_p$.

By the property $R(1, P)$ applied for ε_1 and N, for every $x \in X(\varepsilon_1)$ and $y \in Y(\varepsilon_1)$ there exist $L = L(x, y) \geq N$, $M = M(x, y) > 0$, $Q = Q(x, y) \geq 0$ with $L/(M + Q) \geq \kappa$ and $p = p(x, y) \in P$ such that $(\#K_p)/L > 1 - \varepsilon_1$, where K_p is equal to

$$\{n \in \mathbb{Z} \cap [M, M + L] : d(T_{Q+n}(x), T_n(x)) < \varepsilon_1,\ \bar{d}(S_{Q+n}(y), S_{n+p}(y)) < \varepsilon_1\}.$$

Since $\mu(X(\varepsilon_1)^c) < \varepsilon_1$, $\nu(Y(\varepsilon_1)^c) < \varepsilon_1$ and $\varepsilon_1 < \varepsilon/8 \leq 1/8$,

$$\begin{aligned} \rho(X(\varepsilon_1) \times Y(\varepsilon_1)) &\geq 1 - \rho((X(\varepsilon_1)^c) \times Y) - \rho(X \times (Y(\varepsilon_1)^c)) \\ &= 1 - \mu(X(\varepsilon_1)^c) - \nu(Y(\varepsilon_1)^c) > 1 - 2\varepsilon_1 \geq 3/4. \end{aligned}$$

Thus we can take $(x, y) \in U \cap (X(\varepsilon_1) \times Y(\varepsilon_1))$. If $k \in K_p$ then $T_{Q+k}x \in A_p$ implies $T_k x \in A_p^{\varepsilon_1}$ and $S_{Q+k}y \in B_p$ implies $S_{k+p}y \in B_p^{\varepsilon_1}$. Hence

$$\begin{aligned} &\frac{1}{L} \sum_{k=Q+M}^{Q+M+L} \chi_{A_p \times B_p}(T_k x, S_k y) = \frac{1}{L} \sum_{k=M}^{M+L} \chi_{A_p \times B_p}(T_{Q+k}x, S_{Q+k}y) \\ &\leq \frac{\#(\mathbb{Z} \cap [M, M + L] \setminus K_p)}{L} + \frac{1}{L} \sum_{k \in K_p} \chi_{A_p \times B_p}(T_{Q+k}x, S_{Q+k}y) \\ &\leq \varepsilon/8 + \frac{1}{L} \sum_{k=M}^{M+L} \chi_{A_p^{\varepsilon_1} \times B_p^{\varepsilon_1}}(T_k x, S_{k+p}y). \end{aligned} \tag{32}$$

Now from (30) (applied to $m = M + Q$ and $l = L$), (32), (31) (applied to $m = M$ and $l = L$) and (29) it follows that

$$\begin{aligned}\rho(A_p \times B_p) &\leq \frac{1}{L}\sum_{k=Q+M}^{Q+M+L} \chi_{A_p\times B_p}(T_k x, S_k y) + \varepsilon/8\\ &\leq \varepsilon/4 + \frac{1}{L}\sum_{k=M}^{M+L} \chi_{A_p^{\varepsilon_1}\times S_{-p}(B_p^{\varepsilon_1})}(T_k x, S_k y)\\ &< \varepsilon/2 + \rho(A_p^{\varepsilon_1} \times S_{-p}(B_p^{\varepsilon_1})) \leq \varepsilon + \rho(A_p \times S_{-p}B_p).\end{aligned}$$

Applying similar arguments we get

$$\rho(A_p \times S_{-p}B_p) < \varepsilon + \rho(A_p \times B_p).$$

Consequently,

$$|\rho(A_p \times B_p) - \rho(A_p \times S_{-p}B_p)| < \varepsilon,$$

contrary to (28). □

While dealing with special flows over irrational rotations on $\mathbb{T}^f$ we will always consider the induced metric from the metric defined on $\mathbb{T} \times \mathbb{R}$ by $d((x, s), (y, t)) = \|x - y\| + |s - t|$.

Lemma 38. *Let $P \subset \mathbb{R} \setminus \{0\}$ be a nonempty finite subset and let $A > 0$. Let $Tx = x + \alpha$ be an ergodic rotation on the circle and let $f, g : \mathbb{T} \to \mathbb{R}$ be positive Riemann integrable functions which are bounded away from zero and $\int f(x)\,dx = \int g(x)\,dx$. Suppose that the special flows T^f and T^g are weakly mixing. Assume that for every $\varepsilon > 0$ there exists $\bar\kappa = \bar\kappa(\varepsilon) > 0$ such that for every $x, y \in \mathbb{T}$ and $N \in \mathbb{N}$ there are natural numbers $\bar L = \bar L(x, y) \geq N$, $\bar M = \bar M(x, y) > 0$, $\bar Q = \bar Q(x, y) \geq 0$ such that $\bar L/(\bar M + \bar Q) \geq \bar\kappa$, $\|\bar Q\alpha\| < \varepsilon$ and there exist $p = p(x, y) \in P$ and $a = a(x, y) \in [-A, A]$ such that*

$$|f^{(\bar Q)}(x) - f^{(\bar Q)}(T^n x) - a| < \varepsilon \text{ and } |f^{(\bar Q)}(x) - g^{(\bar Q)}(T^n y) - a - p| < \varepsilon$$

for all $\bar M \leq n \leq \bar M + \bar L$. Then the pair of special flows (T^f, T^g) has the $R(\gamma, P)$–property for every $\gamma > 0$.

Proof. Let c, C and K be positive numbers such that $0 < c \leq f(x), g(x) \leq C$ for every $x \in \mathbb{T}$ and $P \subset [-K, K]$. Let μ stand for Lebesgue measure on $\mathbb{T}$. Let γ be an arbitrary positive number. We will show that (T^f, T^g) has the $R(\gamma, P)$–property.

Fix $0 < \varepsilon < \frac{c}{4(1+C+K)}$. Put $\varepsilon_1 = \varepsilon/8$. Take $\bar\kappa = \bar\kappa(\varepsilon_1)$ and let $\kappa := \frac{c/2-2\varepsilon}{C+A+K}\bar\kappa$. Let

$$\begin{aligned}\bar X(\varepsilon) &:= \left\{(x, s) \in \mathbb{T}^f : \frac{\varepsilon}{8} < s < f(x) - \frac{\varepsilon}{8}\right\},\\ \bar Y(\varepsilon) &:= \left\{(x, s) \in \mathbb{T}^g : \frac{\varepsilon}{8} < s < g(x) - \frac{\varepsilon}{8}\right\}.\end{aligned}$$

Since $\mu^f(\bar X(\varepsilon)^c) = \mu^g(\bar Y(\varepsilon)^c) = \varepsilon/4$ and T^f_γ and T^g_γ are ergodic, by Remark 6 (applied to $\varepsilon/4$ and $A = \bar X(\varepsilon)^c$ and $\bar Y(\varepsilon)^c$), there exists $N(\varepsilon) \in \mathbb{N}$ and Borel sets $X(\varepsilon) \subset \mathbb{T}^f$, $Y(\varepsilon) \subset \mathbb{T}^g$ with $\mu^f(X(\varepsilon)^c) < \varepsilon$ and $\mu^g(Y(\varepsilon)^c) < \varepsilon$ such that if $l \geq N(\varepsilon)$ and $l/m \geq \kappa$ then

$$\frac{1}{l}\#\{m \leq k < m + l : T^f_{k\gamma}(x, s) \notin \bar X(\varepsilon)\} < \frac{\varepsilon}{2} \text{ for all } (x, s) \in X(\varepsilon) \tag{33}$$

and

$$\frac{1}{l}\#\{m \le k < m+l : T^g_{k\gamma}(y,s') \notin \bar{Y}(\varepsilon)\} < \frac{\varepsilon}{2} \text{ for all } (y,s') \in Y(\varepsilon). \tag{34}$$

Moreover, since T is uniquely ergodic and $\int f(x)\,dx = \int g(x)\,dx$, we can assume that

$$\max(f^{(l)}(x), g^{(l)}(y)) - \min(f^{(l)}(x), g^{(l)}(y)) < \varepsilon \min(\bar{\kappa}, 1) l$$

for all $l \ge N(\varepsilon)$ and $x, y \in \mathbb{T}$.

Let us consider a pair of points $(x,s) \in X(\varepsilon), (y,s') \in Y(\varepsilon)$ and an arbitrary $N \in \mathbb{N}$. By assumption, there are natural numbers $\bar{M} = \bar{M}(x,y)$, $\bar{L} = \bar{L}(x,y) \ge 2\max(1,\gamma/c)\max(1/\varepsilon, N(\varepsilon), N, (C+2K)/c)$ and $\bar{Q} = \bar{Q}(x,y)$ such that $\bar{L}/(\bar{M} + \bar{Q}) \ge \bar{\kappa}$, $\|\bar{Q}\alpha\| < \varepsilon_1$ and there exist $p = p(x,y) \in P$ and $a = a(x,y) \in [-A, A]$ such that

$$|f^{(\bar{Q})}(x) - f^{(\bar{Q})}(T^n x) - a| < \varepsilon_1 \text{ and } |f^{(\bar{Q})}(x) - g^{(\bar{Q})}(T^n y) - a - p| < \varepsilon_1$$

for all $\bar{M} \le n \le \bar{M} + \bar{L}$. Put

$$Q := \frac{f^{(\bar{Q})}(x) - a}{\gamma}, \quad M := \frac{\max(f^{(\bar{M})}(x), g^{(\bar{M})}(y)) + K}{\gamma},$$
$$L := \frac{\min(f^{(\bar{L}+\bar{M})}(x), g^{(\bar{L}+\bar{M})}(y)) - C - K}{\gamma} - M.$$

Then

$$\begin{aligned}
\frac{L}{Q+M} &= \frac{\min(f^{(\bar{L}+\bar{M})}(x), g^{(\bar{L}+\bar{M})}(y)) - \max(f^{(\bar{M})}(x), g^{(\bar{M})}(y)) - C - 2K}{f^{(\bar{Q})}(x) - a + \max(f^{(\bar{M})}(x), g^{(\bar{M})}(y)) + K} \\
&\ge \frac{\min(f^{(\bar{L})}(T^{\bar{M}}x), g^{(\bar{L})}(T^{\bar{M}}y)) - \varepsilon\min(\bar{\kappa},1)(\bar{L}+\bar{M}) - C - 2K}{f^{(\bar{Q})}(x) + \max(f^{(\bar{M})}(x), g^{(\bar{M})}(y)) + A + K} \\
&\ge \frac{(c-\varepsilon)\bar{L} - \varepsilon\bar{\kappa}\bar{M} - C - 2K}{C(\bar{M}+\bar{Q}) + A + K} \ge \frac{(c-\varepsilon)\bar{L} - \varepsilon\bar{\kappa}(\bar{M}+\bar{Q}) - c\bar{L}/2}{(C+A+K)(\bar{M}+\bar{Q})} \\
&\ge \frac{c/2 - 2\varepsilon}{C+A}\bar{\kappa} = \kappa.
\end{aligned}$$

Moreover

$$L \ge \frac{(c-\varepsilon)\bar{L} - \varepsilon\bar{\kappa}\bar{M} - C - 2K}{\gamma} \ge \frac{(c - 2(1+C+K)\varepsilon)\bar{L}}{\gamma} \ge \frac{c\bar{L}}{2\gamma} > \max(N, N(\varepsilon)). \tag{35}$$

Since $L \ge N(\varepsilon)$, $L/M \ge \kappa$, $(x,s) \in X(\varepsilon)$ and $(y,s') \in Y(\varepsilon)$, by (33) and (34),

$$\frac{1}{L}\#\{M \le k < M+L : T^f_{k\gamma}(x,s) \notin \bar{X}(\varepsilon) \text{ or } T^g_{k\gamma}(y,s') \notin \bar{Y}(\varepsilon)\} < \varepsilon. \tag{36}$$

Suppose that $M \le k < M + L$. Then $k\gamma + s \in [f^{(\bar{M})}(x), f^{(\bar{M}+\bar{L})}(x))$, $k\gamma + p + s' \in [g^{(\bar{M})}(y), g^{(\bar{M}+\bar{L})}(y))$ and there exist unique $\bar{M} \le m_k, n_k < \bar{M} + \bar{L}$ such that $k\gamma + s \in [f^{(m_k)}(x), f^{(m_k+1)}(x))$ and $k\gamma + p + s' \in [g^{(n_k)}(y), g^{(n_k+1)}(y))$. Suppose additionally that

$$k \in B := \{M \le j < M+L : T^f_{j\gamma}(x,s) \in \bar{X}(\varepsilon),\ T^g_{j\gamma}(y,s') \in \bar{Y}(\varepsilon)\}.$$

Then

$$f^{(m_k)}(x) + \varepsilon/8 < s + k\gamma < f^{(m_k+1)}(x) - \varepsilon/8.$$
$$g^{(n_k)}(y) + \varepsilon/8 < s' + p + k\gamma < g^{(n_k+1)}(y) - \varepsilon/8.$$

We have

$$\begin{aligned} s+(Q+k)\gamma &= (s+k\gamma)+f^{(\bar{Q})}(x)-a \\ &< f^{(m_k+1)}(x)-\varepsilon/8+f^{(\bar{Q})}(T^{m_k+1}x)+\varepsilon_1 = f^{(m_k+\bar{Q}+1)}(x) \end{aligned}$$

and

$$\begin{aligned} s+(Q+k)\gamma &= (s+k\gamma)+f^{(\bar{Q})}(x)-a \\ &> f^{(m_k)}(x)+\varepsilon/8+f^{(\bar{Q})}(T^{m_k}x)-\varepsilon_1 = f^{(m_k+\bar{Q})}(x) \end{aligned}$$

and

$$\begin{aligned} s'+(Q+k)\gamma &= (s'+p+k\gamma)+f^{(\bar{Q})}(x)-a-p \\ &< g^{(n_k+1)}(y)-\varepsilon/8+g^{(\bar{Q})}(T^{n_k+1}y)+\varepsilon_1 = g^{(n_k+\bar{Q}+1)}(y) \end{aligned}$$

and

$$\begin{aligned} s'+(Q+k)\gamma &= (s'+p+k\gamma)+f^{(\bar{Q})}(x)-a-p \\ &> g^{(n_k)}(y)+\varepsilon/8+g^{(\bar{Q})}(T^{n_k}y)-\varepsilon_1 = g^{(n_k+\bar{Q})}(y). \end{aligned}$$

Thus

$$T^f_{k\gamma}(x,s) = (T^{m_k}x, s+k\gamma-f^{(m_k)}(x)),$$

$$T^f_{(k+Q)\gamma}(x,s) = (T^{m_k+\bar{Q}}x, s+(k+Q)\gamma-f^{(m_k+\bar{Q})}(x))$$

and

$$T^g_{k\gamma+p}(y,s') = (T^{n_k}y, s'+k\gamma+p-g^{(n_k)}(y)),$$

$$T^g_{(k+Q)\gamma}(y,s') = (T^{n_k+\bar{Q}}y, s'+(k+Q)\gamma-g^{(n_k+\bar{Q})}(y)).$$

Hence

$$\begin{aligned} & d(T^f_{k\gamma}(x,s), T^f_{(k+Q)\gamma}(x,s)) = \|\bar{Q}\alpha\|+|Q\gamma-f^{(\bar{Q})}(T^{m_k}x)| \\ &= \|\bar{Q}\alpha\|+|f^{(\bar{Q})}(x)-f^{(\bar{Q})}(T^{m_k}x)-a| < 2\varepsilon_1 < \varepsilon \end{aligned}$$

and

$$\begin{aligned} & d(T^g_{k\gamma+p}(y,s'), T^g_{(k+Q)\gamma}(y,s')) = \|\bar{Q}\alpha\|+|Q\gamma-p-g^{(\bar{Q})}(T^{n_k}y)| \\ &= \|\bar{Q}\alpha\|+|f^{(\bar{Q})}(x)-g^{(\bar{Q})}(T^{n_k}y)-a-p| < 2\varepsilon_1 < \varepsilon \end{aligned}$$

for every $k \in B$.

By (36), $(\#B)/L > 1-\varepsilon$, and the proof complete. □

Let $T:\mathbb{T}\to\mathbb{T}$ be a rotation by an irrational α with bounded partial quotients. Let $(q_n)_{n\in\mathbb{N}}$ stand for the sequence of denominators of α.

Proposition 39. *Let $f:\mathbb{T}\to\mathbb{R}$ be a piecewise absolutely continuous function. Then there exist a finite set $D_f\subset\mathbb{R}$ and $0<\theta_f\leq 1$ satisfying the following property: for every $\varepsilon>0$ there exists $n_f(\varepsilon)\in\mathbb{N}$ such that for every $n\geq n_f(\varepsilon)$, $x,y\in\mathbb{T}$ with $\|y-x\|<1/q_{n+1}$ and any integer interval $I\subset[0,q_{n+1})\cap\mathbb{Z}$ there exist an integer interval $J\subset I$ and $d\in D_f$ such that $\#J\geq\theta_f\#I$ and*

$$|f^{(k)}(y)-f^{(k)}(x)-kS(f)(y-x)-d|<\varepsilon \text{ for all } k\in J.$$

Proof. The proof of this proposition is contained in the proof of Theorem 6.1 in [10]. □

Theorem 40. *Let $T:\mathbb{T}\to\mathbb{T}$ be a rotation by an irrational α with bounded partial quotients. Assume that $f,g:\mathbb{T}\to\mathbb{R}$ be positive piecewise absolutely continuous functions such that $\int f(x)\,dx=\int g(x)\,dx$, $S(f)\neq S(g)$ and T^f, T^g are weakly mixing. Then the special flows T^f and T^g are disjoint.*

Proof. Put

$$P:=\left\{\frac{(S(f)-S(g))\theta_f\theta_g}{8}k:k\in\mathbb{Z}\setminus\{0\}\right\}$$
$$\cap\{x\in\mathbb{R}:|x|\leq|S(f)-S(g)|+\sup|D_f|+\sup|D_g|+\operatorname{Var}f+\operatorname{Var}g+1\}$$

and $A:=2\operatorname{Var}f+1$. Fix $0<\varepsilon<\frac{|S(f)-S(g)|\theta_f\theta_g}{4}$, $N\in\mathbb{N}$ and $x,y\in\mathbb{T}$. Take

$$\bar{\kappa}(\varepsilon)=\frac{\varepsilon\theta_f\theta_g}{32(|S(f)|+|S(g)|)}.$$

Choose $n\geq\max(n_f(\varepsilon/8),n_g(\varepsilon/8))$ (see Proposition 39) such that

$$\{q_n\alpha\}=\|q_n\alpha\|<\frac{\varepsilon}{N}\frac{\theta_f\theta_g}{32(|S(f)|+|S(g)|)}$$

Starting from the interval $[0,q_{n+1})\cap\mathbb{Z}$ and using Proposition 39 twice (first for $I=[0,q_{n+1})\cap\mathbb{Z}$, the function f and the pair $x,T^{q_n}x$ obtaining J and for $I=J$, the function g and the pair $y,T^{q_n}y$) we obtain an integer interval $I\subset[0,q_{n+1})\cap\mathbb{Z}$ and $d_1\in D_f$, $d_2\in D_g$ such that

$$\#I\geq\theta_f\theta_g q_{n+1} \tag{37}$$

$$|f^{(k)}(T^{q_n}x)-f^{(k)}(x)-S(f)k\|q_n\alpha\|-d_1| < \varepsilon/8, \tag{38}$$
$$|g^{(k)}(T^{q_n}y)-g^{(k)}(y)-S(g)k\|q_n\alpha\|-d_2| < \varepsilon/8 \tag{39}$$

for all $k\in I$. Let us consider two sequences $(a_k)_{k\in I}$, $(p_k)_{k\in I}$,

$$a_k=-S(f)k\|q_n\alpha\|-d_1,$$
$$p_k=(S(f)-S(g))(k\|q_n\alpha\|)-d_2+d_1-g^{(q_n)}(y)+f^{(q_n)}(x).$$

Since

$$f^{(k)}(T^{q_n}x)-f^{(k)}(x)=f^{(q_n)}(T^kx)-f^{(q_n)}(x),$$
$$g^{(k)}(T^{q_n}y)-g^{(k)}(y)=g^{(q_n)}(T^ky)-g^{(q_n)}(y),$$

by (38) and (39), for every $k\in I$,

$$|f^{(q_n)}(x)-f^{(q_n)}(T^kx)-a_k|=|f^{(k)}(T^{q_n}x)-f^{(k)}(x)-a_k|<\varepsilon/8 \tag{40}$$

and

$$\begin{aligned}|f^{(q_n)}(x)-g^{(q_n)}(T^ky)-a_k-p_k|&\leq|f^{(q_n)}(x)-f^{(q_n)}(T^kx)-a_k|\\&+|g^{(q_n)}(y)-g^{(q_n)}(T^ky)-(f^{(q_n)}(x)-f^{(q_n)}(T^kx))\\&+f^{(q_n)}(x)-g^{(q_n)}(y)-p_k|\leq3\varepsilon/8.\end{aligned} \tag{41}$$

Moreover

$$|a_k|\leq|f^{(q_n)}(x)-f^{(q_n)}(T^kx)-a_k|+|f^{(q_n)}(x)-f^{(q_n)}(T^kx)|\leq2\operatorname{Var}f+1,$$

for all $k\in I$. Furthermore

$$|p_k|\leq|S(f)-S(g)|+\sup|D_f|+\sup|D_g|+\operatorname{Var}f+\operatorname{Var}g, \tag{42}$$
$$p_{k+1}-p_k=(S(f)-S(g))\|q_n\alpha\|$$

$$\max(p_k)_{k\in I} - \min(p_k)_{k\in I} = |S(f)-S(g)|(\#I-1)\|q_n\alpha\|$$

and

$$|S(f)-S(g)|\theta_f\theta_g/2 < |S(f)-S(g)|\#I\|q_n\alpha\| < |S(f)-S(g)|.$$

It follows that there exist an integer interval $J \subset I$ with

$$\frac{\varepsilon}{8(|S(f)|+|S(g)|)}\#I \le \#J \le \frac{\varepsilon}{8(|S(f)|+|S(g)|)}\#I+1 \tag{43}$$

and an element $p = \frac{(S(f)-S(g))\theta_f\theta_g}{8}m \in P$, with $m \in \mathbb{Z}\setminus\{0\}$, such that

$$|p_k - p| < \varepsilon/8 \text{ for all } k \in J. \tag{44}$$

Fix $k_0 \in J$ and put $a = a_{k_0}$. Then using (43) and the definition of n, for every $k \in J$,

$$\begin{aligned} |a_k - a| = {} & |S(f)||k-k_0|\|q_n\alpha\| \le |S(f)|\#J\|q_n\alpha\| \\ \le {} & \varepsilon\#I\|q_n\alpha\|/8 + |S(f)|\|q_n\alpha\| < \varepsilon/4. \end{aligned} \tag{45}$$

Now from (40), (45), (41), (44),

$$|f^{(q_n)}(x) - f^{(q_n)}(T^kx) - a| \le |f^{(q_n)}(x) - f^{(q_n)}(T^kx) - a_k| + |a_k - a| < \varepsilon/2$$

and

$$\begin{aligned} & |f^{(q_n)}(x) - g^{(q_n)}(T^ky) - a - p| \\ \le {} & |f^{(q_n)}(x) - g^{(q_n)}(T^ky) - a_k - p_k| + |a_k - a| + |p_k - p| < \varepsilon \end{aligned}$$

for every $k \in J$.

Now let $\bar{M}, \bar{L}$ be natural numbers such that $J = [\bar{M}, \bar{M}+\bar{L}]\cap\mathbb{Z}$. Putting $\bar{Q} = q_n$, we have

$$|f^{(\bar{Q})}(x) - f^{(\bar{Q})}(T^kx) - a| < \varepsilon,\ |f^{(\bar{Q})}(x) - g^{(\bar{Q})}(T^ky) - a - p| < \varepsilon$$

for all $\bar{M} \le k \le \bar{M}+\bar{L}$. Moreover, by (43) and (37),

$$\begin{aligned} \frac{\bar{L}}{\bar{M}+\bar{Q}} & \ge \frac{\#J-1}{2q_{n+1}} \ge \frac{\#J}{4q_{n+1}} \ge \frac{\varepsilon\#I}{32(|S(f)|+|S(g)|)q_{n+1}} \\ & \ge \frac{\varepsilon\theta_f\theta_g}{32(|S(f)|+|S(g)|)} = \bar{\kappa}(\varepsilon), \end{aligned}$$

$$\bar{L} = \#J - 1 \ge \frac{\varepsilon q_{n+1}\theta_f\theta_g}{16(|S(f)|+|S(g)|)} > \frac{\varepsilon}{\|q_n\alpha\|}\frac{\theta_f\theta_g}{32(|S(f)|+|S(g)|)} > N.$$

Since the special flow T^f is weakly mixing, the automorphism T^f_γ is ergodic (weakly mixing) for all $\gamma \ne 0$, and hence an application of Lemma 38 and Theorem 37 completes the proof. □

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