NICHOLAS COPERNICUS UNIVERSITY
FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

Conference on Ergodic Theory and Dynamical Systems Toruń 2000

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Introduction

The conference was held between September 1 and 7, 2000 at the Faculty of Mathematics and Computer Science of the Nicholas Copernicus University at Gagarina 12/18 in Toruń, Poland. It was a continuation of a series of international (originally German-Polish and later French-German-Israeli-Polish) workshops: Kazimierz Dolny (Poland) 1992, Aachen 1993, Toruń 1994, Warsaw 1995, Berlin 1996, Szklarska Poręba (Poland) 1997. At the present conference colleagues from Austria, France, Germany, Great Britain, Holland, India, Israel, Japan, Korea, Poland, Russia, Sweden, Ukraine, and USA, contributed, through their lectures and discussions, to a profitable exchange of scientific results and ideas. The main topics of the conference were: measure-theoretic, topological, and smooth dynamics, which includes spectral, combinatorial, and entropy-theoretic aspects of dynamical systems, topological and measurable actions of amenable groups, holomorphic dynamics, Gaussian automorphisms, and applications of ergodic theory to statistical mechanics, thermodynamics, number theory, probability theory and differential equations. This volume contains materials of the conference: full list of participants, full list of lectures, extended abstracts of most of the lectures, and some open questions. The organizers from Toruń would like to express their gratitude to all participants for contributing to the success of the conference; in particular we like to thank:

- all 52 speakers,
- professors Manfred Denker and Feliks Przytycki for their help in obtaining funds of the European Scientific Foundation,
- professor Adam Jakubowski, the dean of the Faculty of Mathematics and Computer Science of the Nicholas Copernicus University,
- sponsors of the Conference.

Toruń, March 2001

Brunon Kamiński
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3

Extended abstracts

3.1 Jon Aaronson[1] *Invariant measures and asymptotics for some skew products*

Let $(X, \mathcal{B})$ be a standard measurable space, and let $\tau : X \to X$ be an invertible measurable map. Let $\mathbb{G}$ be a LCAP topological group and let $\phi : X \to \mathbb{G}$ be measurable.

The *skew product* transformation $\tau_\phi : X \times \mathbb{G} \to X \times \mathbb{G}$ is defined by $\tau_\phi(x, y) := (\tau x, y + \phi(x))$.

For certain examples, we identify all τ_ϕ-invariant locally finite measures (e.g. $p \times m_{\mathbb{G}}$ where $p \circ \tau = p$) and study their asymptotic behaviour.

If τ is a uniquely ergodic homeomorphism of a compact metric space (with invariant probability p), $\mathbb{G}$ is compact (with Haar probability measure $m_{\mathbb{G}}$) and $\phi : X \to \mathbb{G}$ is continuous, then ergodicity of τ_ϕ with respect to the product $p \times m_{\mathbb{G}}$ is equivalent to its unique ergodicity. For non-compact $\mathbb{G}$, if τ is uniquely ergodic (with invariant probability p), and τ_ϕ is ergodic with respect to $p \times m_{\mathbb{G}}$, then (by a version of the so-called coboundary theorem) there is no τ_ϕ-invariant probability on $X \times \mathbb{G}$.

The Maharam measures form a natural class of τ_ϕ-invariant, locally finite measures.

Given $\phi : X \to \mathbb{G}$ be measurable, a continuous homomorphism $\alpha : \mathbb{G} \to \mathbb{R}$ and $\mu = \mu_\alpha \in \mathcal{P}(X, \mathcal{B})$ $(e^{\alpha \circ \phi}, \tau)$-conformal (i.e. $\mu \circ \tau \sim \mu$ and $\frac{d\mu \circ \tau}{d\mu} = e^{\alpha \circ \phi}$ μ-a.e.), the associated *Maharam measure* is

$$dm_\alpha(x, y) := e^{-\alpha(y)} d\mu(x) dy.$$

A Maharam measure is τ_ϕ-invariant, the dilation from the first coordinate being cancelled by the translation in the second.

The transformations τ_ϕ considered here have the following properties:

[1]Joint work with H. Nakada, O. Sarig, R. Solomyak

1) For each continuous homomorphism $\alpha : \mathbb{G} \to \mathbb{R}$, there is a unique $(e^{\alpha\circ\phi}, \tau)$-conformal probability $\mu = \mu_\alpha$ on $(X, \mathcal{B})$;

2) For each continuous homomorphism $\alpha : \mathbb{G} \to \mathbb{R}$, the Maharam measure m_α is ergodic (for τ_ϕ);

3) The only ergodic τ_ϕ-invariant locally finite measures are Maharam measures.

Now let S be a finite set, let $\Sigma \subset S^{\mathbb{N}}$ be a mixing SFT over S, and consider the *tail relation* of the shift $T : \Sigma \to \Sigma$:

$$\mathfrak{T}(T) := \{(x, y) \in \Sigma^2 : \exists\, n \geq 0,\ T^n x = T^n y\}.$$

This is generated by the *adic transformation* $\tau : \Sigma \to \Sigma$, $\tau = V_\Sigma$, the transformation induced by the odometer $V : S^{\mathbb{N}} \to S^{\mathbb{N}}$ on Σ :

$$\tau(x) := V^{\min\{n\geq 1:\ V^n(x)\in\Sigma\}}(x).$$

These adic transformations are uniquely ergodic.

Let $f : \Sigma \to \mathbb{G}$ and consider $T_f : \Sigma \times \mathbb{G} \to \Sigma \times \mathbb{G}$. It turns out that T_f's tail relation is

$$\mathfrak{T}(T_f) := \{((x, y), (x', y')) \in (\Sigma \times \mathbb{G})^2 : (x, x') \in \mathfrak{T}(T),\ y' - y = \psi_f(x, x')\},$$

where $\psi_f : \mathfrak{T} \to \mathbb{G}$ is defined by $\psi_f(x, x') := \sum_{n=0}^{\infty}(f(T^n x) - f(T^n x'))$. Evidently $\mathfrak{T}(T_f)$ is generated by a skew product $\tau_{\phi_f} : \Sigma\times\mathbb{G} \to \Sigma\times\mathbb{G}$ where $\phi_f(x) = \psi_f(x, \tau x) = \sum_0^\infty (f(T^i x) - f(T^i(\tau x)))$.

A measurable function $f : \Sigma \to \mathbb{G}$ is *periodic* if $\exists\, \gamma \in \widehat{\mathbb{G}}$, $z \in \mathbb{S}^1$ and $g : \Sigma \to \mathbb{S}^1$ measurable, not constant, s.t. $\gamma \circ f = z \cdot \overline{g} \cdot g \circ T$; and is called *aperiodic* if it is not periodic.

Suppose that Σ is topologically mixing, and that $f : \Sigma \to \mathbb{G}$ is Hölder continuous and aperiodic. It is known that for every continuous homomorphism $\alpha : \mathbb{G} \to \mathbb{R}$:
1) (see [4]) there is a unique $(e^{-\alpha(\phi_f)}, \tau)$-conformal prob. $\mu_\alpha \in \mathcal{P}(\Sigma_0)$;
2) (see [4]) μ_α is non-atomic;
3) τ_{ϕ_f} is ergodic with respect to the Maharam measure on $\Sigma_0 \times \mathbb{G}$ defined by $dm_\alpha(x, y) = e^{-\alpha\circ y} d\mu_\alpha(x) dy$ ($\iff$ T_f is exact, which is proved in [5]).

Theorem 3.1.1 ([1]). *Suppose that $f : \Sigma \to \mathbb{G}$ is aperiodic and has finite memory.*

If m is an ergodic, τ_{ϕ_f}-invariant locally finite measure on $\Sigma_0 \times \mathbb{G}$, then $m \sim m_\alpha$ for some continuous homomorphism $\alpha : \mathbb{G} \to \mathbb{R}$.

Related results for suspension semiflows appear in [2], and these establish the result that for a horocycle foliation of a $\mathbb{Z}^d$-cover of a compact manifold of constant negative curvature, the only locally finite measures which are ergodic and invariant for the horocycle foliation, and quasi-invariant under the geodesic flow are the Maharam-type measures introduced in [3].

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3.2 Vitaly Bergelson
IP sets and polynomial multiple recurrence

General background

Let us call a set $S \subseteq \mathbb{Z}$ an *IP set* if it consists of an infinite sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{Z}$ together with all finite sums of the form $x_{n_1} + x_{n_2} + \ldots + x_{n_k}$, where $k \in \mathbb{N}$ and $0 < n_1 < n_2 < \ldots < n_k$. It follows that the elements of the IP set generated by $(x_n)_{n=1}^{\infty}$ are naturally indexed by the elements of the set $\mathcal{F}$ of finite nonempty subsets of $\mathbb{N}$. Indeed, for any $\alpha \in \mathcal{F}$ define $x_\alpha := \sum_{n\in\alpha} x_n$. Note that if $(x_\alpha)_{\alpha\in\mathcal{F}}$ is an IP set in $\mathbb{Z}$ then for any disjoint $\alpha, \beta \in \mathcal{F}$ one has $x_\alpha + x_\beta = x_{\alpha\cup\beta}$. Thus IP sets can be viewed as approximate semigroups. It turns out that many results of the ergodic theory of semigroup actions, especially those pertaining to recurrence and multiple recurrence, admit far reaching extensions and refinements in the framework of IP sets.

Theorem 3.2.1 (Hindman, [14]). *If $E \subseteq \mathbb{Z}$ is an IP set, then for any finite partition $E = \bigcup_{i=1}^r C_i$, at least one of the cells C_i contains an IP set.*

Remark 3.2.1. If a set A contains a subsemigroup of $(\mathbb{N}, +)$, it is not hard to construct a partition $A = \bigcup_{i=1}^r C_i$ with the property that no C_i contains a semigroup. Hindman's theorem shows that any similar attempt to destroy the property of containing an IP set is doomed to failure.

IP sets arise naturally in ergodic theory and topological dynamics. First of all, Hindman's theorem itself may be seen as a sort of "iterated" Poincaré recurrence theorem (see [1], Section 3, and [2], Section 3). Second, IP sets provide a natural concept of largeness (namely IP*-ness, see below) which is stronger and, on many occasions, more appropriate than the more familiar and traditional notion of syndeticity ($\equiv$ relative denseness). A set $E \subseteq \mathbb{Z}$ is called IP* if for any IP set S one has $E \cap S \neq \emptyset$. It is not hard to show that if a set $E \subseteq \mathbb{Z}$ is IP* then it is syndetic. On the other hand, not every syndetic set is IP*. (Consider, for example, the odd integers.) It follows from Hindman's theorem that the intersection of any finite collection of IP* sets is also an IP* set.

Let now $(X, \mathcal{B}, \mu)$ be a probability space and T an invertible measure preserving transformation of X. *Khintchine's recurrence theorem* states that for any $A \in \mathcal{B}$ with $\mu(A) > 0$ and any $\lambda \in (0,1)$ the set

$$R_{A,\lambda} = \{n \in \mathbb{Z} : \mu(A \cap T^n A > \lambda\mu(A)^2\}$$

is syndetic. Khintchine's original proof was via the (uniform version of) von Neumann's ergodic theorem. One can however show more: the set $R_{A,\lambda}$ is always an IP* set, and moreover this remains true for measure preserving actions of arbitrary semigroups (with appropriately defined notion of IP*). See the discussion in [1], pp.49-50. Indeed, it was shown in [3] that the IP* version of Khintchine's recurrence theorem also holds for recurrence along polynomials. For example, if T_1, T_2 are commuting invertible measure preserving transformations of a probability space $(X, \mathcal{B}, \mu)$ then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ and any $\lambda \in (0,1)$ the set $\{n \in \mathbb{Z} : \mu(A \cap T_1^{n^2} T_2^{n^3} A) > \lambda\mu(A)^2\}$ is IP*. (The results proved in [3] are much more general.)

We conclude the introductory discussion with a remark about one classical definition. A function $f \colon \mathbb{Z} \longrightarrow \mathbb{R}$ is called *almost periodic* if for any $\varepsilon > 0$ the set of "ε-periods",

$$P_\varepsilon = \{h : \sup_{x \in \mathbb{Z}} |f(x+h) - f(x)| < \varepsilon\}$$

is syndetic. This important notion (introduced by H. Bohr in 1924 for continuous functions on $\mathbb{R}$) has one unpleasant glitch: it is not obvious from the definition that if f and g are almost periodic then $f + g$ also is. However, if one modifies the definition by demanding that for any $\varepsilon > 0$ the set P_ε is IP* then one gets an equivalent definition which is free from this flaw. See the discussion in [1], p. 9.

Polynomial multiple recurrence along IP sets

Traditionally, ergodic theory deals with various Cesàro averages. After all, the very inception of ergodic theory is tightly related to the question about the equality of time and space averages. Also, such basic properties of dynamical systems as ergodicity, weak mixing and strong mixing are naturally expressible in terms of Cesàro averages (for the case of strong mixing see, for example, [4]).

In the IP framework Cesàro convergence is replaced by IP convergence, which we presently define (cf. [10], Chapter 8.) An *$\mathcal{F}$-sequence* is a sequence in an arbitrary space which is indexed by $\mathcal{F}$. For $\alpha, \beta \in \mathcal{F}$ we shall write $\alpha > \beta$ iff $\min \alpha > \max \beta$. Let $(x_\alpha)_{\alpha \in \mathcal{F}}$ be an $\mathcal{F}$-sequence in topological space X and let $x \in X$. One says that IP-lim $x_\alpha = x$ if for every neighborhood U of x there exists $\alpha_0 \in \mathcal{F}$ such that $x_\alpha \in U$ for all $\alpha > \alpha_0$. To define the notion of $\mathcal{F}$-subsequence, one invokes the concept of $\mathcal{F}$-homomorphism. An *$\mathcal{F}$-homomorphism* is a mapping $\varphi \colon \mathcal{F} \longrightarrow \mathcal{F}$ which has the property that if $\alpha, \beta \in \mathcal{F}$ with $\alpha \cap \beta = \emptyset$, then $\varphi(\alpha) \cap \varphi(\beta) = \emptyset$

and $\varphi(\alpha \cup \beta) = \varphi(\alpha) \cup \varphi(\beta)$. An *$\mathcal{F}$-subsequence* of $(x_\alpha)_{\alpha\in\mathcal{F}}$ is defined as any $\mathcal{F}$-sequence of the form $(x_{\varphi(\alpha)})_{\alpha\in\mathcal{F}}$ where φ is an $\mathcal{F}$-homomorphism. It follows now from Hindman's theorem that any $\mathcal{F}$-sequence in a compact metric space has a convergent $\mathcal{F}$-subsequence (see [10], Theorem 8.14, p. 165).

We are now in position to formulate IP analogues of some "Cesàro" facts. Assume, for example, that $(U_i)_{i=1}^{\infty}$ are commuting unitary operators of a Hilbert space $\mathcal{H}$ and let $U_\alpha := \prod_{i\in\alpha} U_i$, $\alpha \in \mathcal{F}$. One immediately observes that $(U_\alpha)_{\alpha\in\mathcal{F}}$ is nothing but a multiplicative IP set of commuting unitary operators. It is, however, its structure as an $\mathcal{F}$-sequence that interests us now. The following result may be viewed as IP analogue of von Neumann's ergodic theorem.

Theorem 3.2.2. (Cf. [12], Theorem 1.7, p. 124). *Let $(U_\alpha)_{\alpha\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in a commutative group of unitary operators acting on a separable Hilbert space $\mathcal{H}$. Then one can find an $\mathcal{F}$-subsequence $(U_{\varphi(\alpha)})_{\alpha\in\mathcal{F}}$ such that for any $f \in \mathcal{H}$*

$$\text{IP-lim}\, U_{\varphi(\alpha)} f = Pf$$

exists in weak topology. In addition, P is an orthogonal projection onto a subspace of $\mathcal{H}$.

We are moving now to some examples of IP results pertaining to multiple recurrence. The theory of multiple recurrence started with the publication of [9], where Furstenberg gave an ergodic proof of the celebrated Szemerédi's theorem on arithmetic progressions by proving the following extension of the Poincaré recurrence theorem and showing that Szemerédi's theorem is a consequence of it.

Theorem 3.2.3 ([9]). *For any invertible probability measure preserving system $(X, \mathcal{B}, \mu, T)$, any $A \in \mathcal{B}$ with $\mu(A) > 0$ and any $k \in \mathbb{N}$ the set*

$$\{n \in \mathbb{Z} : \mu(A \cap T^n A \cap T^{2n} A \cap \ldots \cap T^{kn} A) > 0\}$$

is syndetic.

In [11] Furstenberg and Katznelson obtained a *multidimensional* Szemerédi theorem by deriving it from the following multiple recurrence theorem.

Theorem 3.2.4 ([11]). *Let $(X, \mathcal{B}, \mu)$ be a probability measure space and $T_1, \ldots, T_k$ commuting invertible measure preserving transformations of X. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ one has*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T_1^n A \cap T_2^n A \cap \ldots \cap T_k^n A) > 0.$$

The natural question arises as to whether the set of simultaneous returns,

$$\{n \in \mathbb{Z} : \mu(A \cap T_1^n A \cap T_2^n A \cap \ldots \cap T_k^n A) > 0\}$$

is IP*. The answer is *yes* and is contained in the following IP-Szemerédi theorem proved in [12]:

Theorem 3.2.5 ([12]). *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $k \in \mathbb{N}$. For any $\mathcal{F}$-sequences $(T_\alpha^{(i)})_{\alpha\in\mathcal{F}}$, $i = 1, \ldots, k$, in a commutative group of invertible measure preserving transformations of X and for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists an $\mathcal{F}$-homomorphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{F}$ such that*

$$\text{IP-lim}\, \mu\Big(\bigcap_{i=1}^{k} T_{\varphi(\alpha)}^{(i)} A\Big) > 0.$$

A polynomial extension of Furstenberg-Katznelson's multidimensional Szemerédi theorem was obtained in [5] where, in particular, the following result was proved.

Theorem 3.2.6 ([5]). *Let $(X, \mathcal{B}, \mu)$ be a probability measure space, $T_1, \ldots, T_k$ commuting invertible measure preserving transformations of X, $p_1, \ldots, p_k$ polynomials with rational coefficients taking integer values on the integers and satisfying $p_i(0) = 0$, $i = 1, \ldots, k$. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ one has*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T_1^{p_1(n)} A \cap T_2^{n} A \cap \ldots \cap T_k^{p_k(n)} A) > 0.$$

Again, it was natural to inquire whether sets of the form

$$\{n \in \mathbb{Z} : \mu(A \cap T_1^{p_1(n)} A \cap T_2^{n} A \cap \ldots \cap T_k^{p_k(n)} A) > 0\}$$

are IP*. The following IP polynomial Szemerédi theorem proved in [8] shows that the answer is in the affirmative.

Theorem 3.2.7 ([8]). *Let $T_1, \ldots, T_k$ be commuting invertible measure preserving transformations of a probability measure space $(X, \mathcal{B}, \mu)$, let $d, r \in \mathbb{N}$ and suppose we are given polynomials $p_{i,j} \in \mathbb{Z}[n_1, \ldots, n_d]$, $i = 1, \ldots, r$, $j = 1, \ldots, k$, satisfying $p_{i,j}(0) = 0$. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ the set*

$$\Big\{(n_1, \ldots, n_d) \in \mathbb{Z}^d : \mu\Big(\bigcap_{i=1}^{r} (\textstyle\prod_{j=1}^{k} T_j^{p_{i,j}(n_1,\ldots,n_d)}) A\Big) > 0\Big\}$$

is an IP set in $\mathbb{Z}^d$. (IP sets and IP* sets in $\mathbb{Z}^k$ are defined in the same way as in $\mathbb{Z}$.)*

We collect some of the corollaries of this theorem in the following list. For more discussion and details see [1] and [8].
(i) Taking $d = 1$ we see that the set

$$\{n \in \mathbb{Z} :\ \mu(A \cap T_1^{p_{1,1}(n)} T_2^{p_{1,2}(n)} \ldots T_k^{p_{1,k}(n)} A \cap \ldots \cap T_1^{p_{r,1}(n)} T_2^{p_{r,2}(n)} \ldots T_k^{p_{r,k}(n)} A) > 0\}$$

is IP*.
(ii) Since for any IP sets $(n_\alpha^{(i)})_{\alpha\in\mathcal{F}}$, $i = 1, \ldots, k$, any measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $\alpha \in \mathcal{F}$ such that

$$\mu(A \cap T^{n_\alpha^{(1)}} A \cap T^{n_\alpha^{(1)} n_\alpha^{(2)}} A \cap \ldots \cap T^{n_\alpha^{(1)} n_\alpha^{(2)} \ldots n_\alpha^{(k)}} A) > 0,$$

one obtains (via Furstenberg's correspondence principle) the fact that for any $E \subseteq \mathbb{Z}$ having positive upper density there exist $x \in \mathbb{Z}$ and $\alpha \in \mathcal{F}$ such that

$$\{x, x + n_\alpha^{(1)}, x + n_\alpha^{(1)} n_\alpha^{(2)}, \ldots, x + n_\alpha^{(1)} n_\alpha^{(2)} \ldots n_\alpha^{(k)}\} \subset E.$$

(iii) Let $P\colon \mathbb{Z}^r \longrightarrow \mathbb{Z}^l$ be a polynomial mapping satisfying $P(0) = 0$, let $F \subset \mathbb{Z}^r$ be a finite set, let $S \subseteq \mathbb{Z}^l$ be a set of *positive Banach density* and let $(n_\alpha^{(i)})_{\alpha \in \mathcal{F}}$, $i = 1, 2, \ldots, r$ be arbitrary IP sets. Then for some $u \in \mathbb{Z}^l$ and $\alpha \in \mathcal{F}$ one has

$$\{u + P(n_\alpha^{(1)} x_1, n_\alpha^{(2)} x_2, \ldots, n_\alpha^{(r)} x_r) : (x_1, x_2, \ldots, x_r) \in F\} \subset S.$$

(The *Banach upper density* of a set $S \subseteq \mathbb{Z}^k$ is defined to be

$$d^*(S) = \sup_{\{\Pi_n\}_{n \in \mathbb{N}}} \limsup_{n \to \infty} \frac{|S \cap \Pi_n|}{|\Pi_n|},$$

where the supremum goes over all sequences of parallelepipeds

$$\Pi_n = [a_n^{(1)}, b_n^{(1)}] \times \cdots \times [a_n^{(k)}, b_n^{(k)}] \subset \mathbb{Z}^k, n \in \mathbb{N},$$

with $b_n^{(i)} - a_n^{(i)} \to \infty$, $1 \leq i \leq k$.)

We now turn our attention to polynomial IP theorems in the setup of topological dynamics. We remark in passing that while measure preserving dynamics is applicable to *density* Ramsey theory, theorems in topological dynamics of the kind that we introduce below yield applications in *partition* Ramsey theory. See [1] and [16] for more discussion.

The following theorem proved in [5] gives, as a corollary, the polynomial version of van der Waerden's theorem on arithmetic progressions (and is, in its turn, an important tool in proving the polynomial Szemerédi theorem appearing in [5]).

Theorem 3.2.8 ([5]). *Let (X, ρ) be a compact metric space, let $T_1, \ldots, T_l$ be commuting self-homeomorphisms of X and let $p_{i,j} : \mathbb{Z} \longrightarrow \mathbb{Z}$, $i = 1, \ldots, k$, $j = 1, \ldots, l$, be polynomials satisfying $p_{i,j}(0) = 0$. Then for any $\varepsilon > 0$ there exist $x \in X$ and $n \in \mathbb{N}$ such that $\rho(T_1^{p_{i,1}(n)} \ldots T_l^{p_{i,l}(n)} x, x) < \varepsilon$ for all $i = 1, \ldots, k$.*

Corollary 3.2.1. *Let $d, k \in \mathbb{N}$ and let $P\colon \mathbb{Z}^k \longrightarrow \mathbb{Z}^l$ be a polynomial satisfying $P(0) = 0$. Then for any finite coloring of $\mathbb{Z}^k$ and any finite set $E \subset \mathbb{Z}^k$ there exist $v \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ such that the set $v + P(nE)$ is monochromatic.*

We formulate now a general Abelian polynomial IP multiple recurrence theorem, which is a corollary of the polynomial Hales-Jewett theorem obtained in [6]. It contains the previous theorem as a special case.

Theorem 3.2.9 ([6]). *Let (X, ρ) be a compact metric space, let $k, d \in \mathbb{N}$ and let $T_{j_1,\ldots,j_d}^{(i)}$, $i = 1, \ldots, k$, $j_1, \ldots, j_d \in \mathbb{N}$, be commuting homeomorphisms of X. For any $\varepsilon > 0$ there exist $x \in X$ and a finite nonempty set $\alpha \subseteq \{1, \ldots, N\}$ such that $\rho(\prod_{j_1,\ldots,j_d \in \alpha} T_{j_1,\ldots,j_d}^{(i)} x, x) < \varepsilon$ for all $i = 1, \ldots, k$.*

Corollary 3.2.2. *Let G be an Abelian group and let $g^{(i)}_{j_1,\dots,j_d} \in G$, $i = 1,\dots,k$, $j_1,\dots,j_d \in \mathbb{N}$. For any finite coloring of G there exist $h \in G$ and $\alpha \in \mathcal{F}$ such that the elements $h\prod_{j_1,\dots,j_d\in\alpha} g^{(i)}_{j_1,\dots,j_d}$ all have the same color.*

In the recent paper [7], a nil-IP topological multiple recurrence theorem is established, extending the *Abelian* results mentioned above to a nilpotent setup. To formulate the main result we will need to introduce some definitions and notation. Some care and precision are needed here due to the fact that we are dealing with non-commutativity. We start with extending (and somewhat modifying) the definition of IP set to a non-commutative situation. Let $\prec$ be (any) linear order on $\mathbb{N}$. (In particular, it may be the standard order $<$ on $\mathbb{N}$.) Let G be a (not necessarily commutative) semigroup. Given a sequence $\{g_j\}_{j\in\mathbb{N}}$ in G and $\alpha \in \mathcal{F}$, let $g_\alpha = \prod^{\prec}_{j\in\alpha} g_j$ denote the product of g_j, $j \in \alpha$, in the order which $\prec$ induces on α. Let $\mathrm{FP}(\{g_j\}_{j\in\mathbb{N}},\prec) = \{g_\alpha\}_{\alpha\in\mathcal{F}}$. The elements of the set of $\prec$-ordered finite products, $\mathrm{FP}(\{g_j\}_{j\in\mathbb{N}},\prec)$, satisfy the relation $g_{\alpha\cup\beta} = g_\alpha g_\beta$ whenever $\alpha \prec \beta$ (which means that $k \prec l$ for all $k \in \alpha$, $l \in \beta$). The objects of the form $\mathrm{FP}(\{g_j\}_{j\in\mathbb{N}},\prec)$ are non-commutative IP sets, which, alternatively, may be defined as follows: Given a semigroup G, an IP set in G is a mapping $\mathcal{F} \longrightarrow G$, $\alpha \mapsto g_\alpha$, such that for some linear order $\prec$ on $\mathbb{N}$ one has $g_{\alpha\cup\beta} = g_\alpha g_\beta$ whenever $\alpha \prec \beta$.

To give the reader a flavor of what nil-IP topological multiple recurrence theorem is about we will formulate first its special, "linear" case.

Theorem 3.2.10. *Let G be a nilpotent group of self-homeomorphisms of a compact metric space (X,ρ). For any $\varepsilon > 0$ and any IP-systems $\{g^{(1)}_\alpha\}_{\alpha\in\mathcal{F}},\dots,\{g^{(k)}_\alpha\}_{\alpha\in\mathcal{F}}$ in G there exist $x \in X$ and $\alpha \in \mathcal{F}$ such that $\rho(g^{(i)}_\alpha x, x) < \varepsilon$ for all $i = 1,\dots,k$.*

We are moving now toward a formulation of the polynomial nil-IP theorem. (It is worth mentioning that the only way known to us of proving the "linear" case above is to derive it as a corollary from this much more general fact. This situation is quite different in the abelian case where one can get the proof of the "linear" result in a self-contained way.) Before introducing general *nil-IP polynomials*, let us summarize the pertinent definitions and facts about IP polynomials with values in Abelian groups. Call a mapping P from $\mathcal{F}$ into a commutative (semi)group G *an IP polynomial of degree 0* if P is constant, and, inductively, define an IP-polynomial of degree $\leq d$ if for any $\beta \in \mathcal{F}$ there exists a polynomial mapping $D_\beta P\colon \mathcal{F}(\mathbb{N}\setminus\beta) \longrightarrow G$ of degree $\leq d-1$ where $\mathcal{F}(\mathbb{N}\setminus\beta)$ is the set of finite subsets of $\mathbb{N}\setminus\beta$ such that $P(\alpha\cup\beta) = P(\alpha) + (D_\beta P)(\alpha)$ for every $\alpha \in \mathcal{F}$ with $\alpha\cap\beta = \emptyset$.

If G is an Abelian group, it follows from [6], Theorem 8.3, that for any IP-polynomial $P\colon \mathcal{F} \longrightarrow G$ there exist $d \in \mathbb{N}$ and a family $\{g_{(j_1,\dots,j_d)}\}_{(j_1,\dots,j_d)\in\mathbb{N}^d}$ of elements of G such that for any $\alpha \in \mathcal{F}$ one has $P(\alpha) = \prod_{(j_1,\dots,j_d)\in\alpha^d} g_{(j_1,\dots,j_d)}$. It is this latter characterization of commutative IP-polynomials which makes sense in the nilpotent setup as well. Namely, let now G be a nilpotent group. We will call a mapping $P\colon \mathcal{F} \longrightarrow G$ *an IP-polynomial* if for some $d \in \mathbb{N}$ there exist a family $\{g_{(j_1,\dots,j_d)}\}_{(j_1,\dots,j_d)\in\mathbb{N}^d}$ of elements of G and a linear order $\prec$ on $\mathbb{N}^d$ such that for any $\alpha \in \mathcal{F}$ one has $P(\alpha) = \prod^{\prec}_{(j_1,\dots,j_d)\in\alpha^d} g_{(j_1,\dots,j_d)}$.

Theorem 3.2.11 ([7]). *Let G be a nilpotent group of self-homeomorphisms of a compact metric space (X,ρ) and let $P_1,\dots,P_k\colon \mathcal{F} \longrightarrow G$ be polynomial mappings.*

For any $\varepsilon > 0$, there exist $x \in X$ and $\alpha \in \mathcal{F}$ such that $\rho\big(P_i(\alpha)x, x\big) < \varepsilon$ for all $i = 1, \ldots, k$.

Corollary 3.2.3. *Let G be an infinite nilpotent group, let F be the free group generated by a (finite) set $\{z_1, \ldots, z_m\}$, let $E \subset F$ be finite, let $\prec_1, \ldots, \prec_m$ be linear orders on $\mathbb{N}$ and let $r \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that for any r-coloring $G = \bigcup_{m=1}^{r} C_m$ of G and any $g_j^{(i)} \in G$, $1 \leq i \leq m$, $1 \leq j \leq N$, there exist $m \in \{1, \ldots, r\}$ and a nonempty $\alpha \subseteq \{1, \ldots, N\}$ such that if $\varphi \colon F \longrightarrow G$ is the homomorphism defined by $\varphi(z_i) = \prod_{j\in\alpha}^{\prec_i} g_j^{(i)}$, $i = 1, \ldots, m$, then the set $\{h \in G : h\varphi(E) \subseteq C_m\}$ is infinite.*

For example, taking E to be $\{z_1 z_2^2 z_1^{-3}, z_2^{-1} z_1^2 z_2\}$, one can find N such that for any r-coloring of G and any $g_1^{(1)}, \ldots, g_N^{(1)}, g_1^{(2)}, \ldots, g_N^{(2)}$ there exist $1 \leq j_1 < \ldots < j_l \leq N$ and $1 \leq m \leq r$ such that for $h_1 = g_{j_1}^{(1)} \ldots g_{j_l}^{(1)}$ and $h_2 = g_{j_1}^{(2)} \ldots g_{j_l}^{(2)}$, the products $hh_1h_2^2h_1^{-3}$ and $hh_2^{-1}h_1^2h_2$ have the same color for infinitely many $h \in G$.

Let G be a nilpotent group with uniformly bounded torsion, that is, for some $d \in \mathbb{N}$, $g^d = 1_G$ for all $g \in G$. (Examples: (i) the group of upper triangular matrices with unit main diagonal over a field of finite characteristic; (ii) any finite p-group, where p is a prime number.) It is easy to see that any finitely generated subgroup of G is finite, and, moreover, one can estimate its cardinality in terms of the number of generators and the nilpotency class of G.

For convenience of discussion let us temporarily assume that G is infinite. Let a finite coloring of G be given. In accordance with the principles of Ramsey theory, one should be able to find in one color arbitrarily large "highly organized" configurations. In the case of our group G, which has uniformly bounded torsion, it is natural to look for monochromatic cosets of arbitrarily large subgroups. While getting such monochromatic cosets is itself a nontrivial task, an even better result would be not only to get monochromatic cosets of arbitrarily large subgroups, but to have these subgroups be as "noncommutative" as G is.

We will now formalize these considerations. Given $B \subseteq G$, let $\langle B\rangle$ denote the subgroup of G generated by B, and let $\gamma_l\langle B\rangle$ be the l-th term of the lower central series of this subgroup. Let G be a (finite or infinite) nilpotent group of class q and let $N \in \mathbb{N}$; we call G *N-large* if there are N elements $g_1, \ldots, g_N \in G$ and $K \in \mathbb{N}$, $K \leq N$, such that for all $k = K, \ldots, N$ and every nonempty $B \subseteq \{g_1, \ldots, g_{k-1}\}$ one has $\gamma_q\langle B \cup \{g_k\}\rangle \neq \gamma_q\langle B\rangle$. It then follows that for any $1 \leq l_1 < l_2 < \ldots < l_K \leq N$ and $1 \leq j_1 < \ldots < j_{l_K} \leq N$, the group generated by $h_1 = g_{j_1} \cdots g_{j_{l_1}}$, $h_2 = g_{j_{l_1+1}} \cdots g_{j_{l_2}}$, $\ldots$, $h_m = g_{j_{l_{K-1}+1}} \cdots g_{j_{l_K}}$ has nilpotency class q.

The following theorem is a consequence of the nil-IP topological recurrence result.

Theorem 3.2.12 ([7]). *For any $m, d, q, r \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that if G is an N-large nilpotent group of class q with $g^d = 1_G$ for all $g \in G$, then for any r-coloring of G there is a subgroup H of G of nilpotency class q and of cardinality $\geq m$ such that for some $h \in G$ the coset hH is monochromatic.*

Discussion of open problems

In this section we shall formulate some conjectures which naturally arise in the IP context. (For more conjectures and discussion see [1], section 5, and [8], Chapter 8.)

We start with an alternative definition of an IP polynomial in a multiplicative commutative group (cf. [3], Section 3 and [6], Section 8). Let us say that an $\mathcal{F}$-sequence $\{V_\alpha\}_{\alpha\in\mathcal{F}}$ in a multiplicative commutative group G with identity I is a *VIP*-system if for some d (called *the degree* of the system if it is the least such) we have

$$\prod_{0\le i_1<\ldots<i_k\le d} V^{(-1)^k}_{\alpha_{i_1}\cup\ldots\cup\alpha_{i_k}} = I, \text{ if } \alpha_i\cap\alpha_j=\emptyset,\ 0\le i<j\le d.$$

For $d=1$, this reduces to a characterization of IP sets.

In general, VIP-systems may be infinitely generated. For example, let $\{S_{i,j}\}_{\substack{i,j\in\mathbb{N},\\ i\le j}}$ be elements of a commutative group and put $T_\alpha = \prod_{i,j\in\alpha, i\le j} S_{i,j}$, $\alpha\in\mathcal{F}$. Then $\{T_\alpha\}_{\alpha\in\mathcal{F}}$ is a VIP-system of degree (at most) 2.

We remark that the topological version of the following conjecture is known to be true ([6]).

Conjecture 1. If $k\in\mathbb{N}$, $(X,\mathcal{B},\mu)$ is a probability space and $\{V_\alpha^{(i)}\}_{\alpha\in\mathcal{F}}$, $1\le i\le k$, are commuting VIP-systems of measure preserving transformations of X then for every $A\in\mathcal{B}$ with $\mu(A)>0$ there exists $\alpha\in\mathcal{F}$ with

$$\mu\Big(\bigcap_{i=1}^{k}(V_\alpha^{(i)})^{-1}A\Big) > 0.$$

Conjecture 2. Let $(V_\alpha)_{\alpha\in\mathcal{F}}$ be a VIP-system of unitary operators on a Hilbert space $\mathcal{H}$ such that IP-lim $V_\alpha f = Qf$ exists weakly for all $f\in\mathcal{H}$. Then $\langle f, Qf\rangle \ge 0$ for all $f\in\mathcal{H}$.

We remark that Conjecture 2, if true, would settle in affirmative the $k=1$ case of Conjecture 1; that is, the case of single recurrence. (See also [3], Section 3.)

The following conjecture (from [1], p. 56) asserts that a "density polynomial Hales-Jewett theorem" holds. Such a result would extend both the partition results from [6] and the density version of the ("linear") Hales-Jewett theorem proved in [13]. For $q,d,N\in\mathbb{N}$ let $\mathcal{M}_{q,d,N}$ be the set of q-tuples of subsets of $\{1,2,\cdots,N\}^d$:

$$\mathcal{M}_{q,d,N} = \big\{(\alpha_1,\cdots,\alpha_q) : \alpha_i\subset\{1,2,\cdots,N\}^d,\ i=1,2,\cdots,q\big\}.$$

Conjecture 3. For any $q,d\in\mathbb{N}$ and $\epsilon>0$ there exists $C=C(q,d,\epsilon)$ such that if $N>C$ and a set $S\subset\mathcal{M}_{q,d,N}$ satisfies $\frac{|S|}{|\mathcal{M}_{q,d,N}|}>\epsilon$ then S contains a "simplex" of the form:

$$\big\{(\alpha_1,\alpha_2,\cdots,\alpha_q),(\alpha_1\cup\gamma^d,\alpha_2,\cdots,\alpha_q),(\alpha_1,\alpha_2\cup\gamma^d,\cdots,\alpha_q), \cdots,(\alpha_1,\alpha_2,\cdots,\alpha_q\cup\gamma^d)\big\},$$

where $\gamma\subset\mathbb{N}$ is a finite non-empty set and $\alpha_i\cap\gamma^d=\emptyset$ for all $i=1,2,\cdots,q$.

The discussion of topological nil-IP polynomial recurrence in Section 2 above leads one to believe that nil-IP theory should produce corresponding results in the measure preserving framework as well. A. Leibman proved in [15] that the polynomial Szemerédi theorem obtained in [5] holds when the transformations T_i generate a nilpotent group. The following conjecture, if true, would contain both the IP polynomial Szemerédi Theorem from [8] and the nilpotent polynomial Szemerédi Theorem from [15] as special cases.

Conjecture 4. Let G be a nilpotent group of invertible measure preserving transformations of a probability measure space $(X, \mathcal{B}, \mu)$ and let $P_1, \dots, P_k : \mathcal{F} \longrightarrow G$ be polynomial mappings. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $\alpha \in \mathcal{F}$ with

$$\mu\Big(\bigcap_{i=1}^{k} P_i(\alpha)A\Big) > 0.$$

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3.3 Julien Brémont
On some random walks on $\mathbb{Z}$ in random medium

Model

Let $(\Omega, \mathcal{F}, \mu, T)$ be an invertible and ergodic dynamical system. The space Ω is interpreted as the space of environments. Fix an integer L and set $\Lambda = \{-L, \ldots, 0, 1\}$. Let $(p_z)_{z\in\Lambda}$ be positive random variables such that $\sum_{z\in\Lambda} p_z(\omega) = 1$, μ-a.e. We assume the existence of some $\varepsilon > 0$ such that $p_z \geq \varepsilon$, μ-a.e., for $z \neq 0$, $z \in \Lambda$.

For fixed ω, let $(\xi_n(\omega))_{n\geq 0}$ be a Markov chain on $\mathbb{Z}$, defined by $\xi_0(\omega) = 0$ and the transition probabilities in the medium ω:

$$\forall x, y \in \mathbb{Z},\ p(x, y, \omega) := p_{y-z}(T^x\omega).$$

If P'_ω denotes the measure induced by $(\xi_n(\omega))_{n\geq 0}$ on the space of jumps $\Lambda^{\mathbb{N}}$, we want to study this chain with P'_ω-probability 1 for a given ω, ω-a.e. We shall study first the asymptotic behaviour and second, conditions under which the model exhibit classical behaviour.

Results

Set $M := \begin{pmatrix} a_1 & \cdots & a_{L-1} & a_L \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$, where $a_i = \dfrac{p_{-L} + \cdots + p_{-i}}{p_1}$, for $1 \leq i \leq L$.

From directional contraction properties of M in the positive cone, there exists a unique positive vector $V \in \mathbb{R}^L$ and a unique strictly positive scalar λ such that: $\|V\|_1 = 1$ and $MT^{-1}V = \lambda V$. We set $\gamma(M, T)$ the dominant Lyapunov exponent of M with respect to T. We have:

$$\mu\text{-a.e.}, \forall x \succ 0, \gamma(M, T) = \lim_{n\to+\infty} \frac{1}{n} \log \|T^n M \cdots M x\|$$

and then

$$\gamma(M, T) = \int \log(\lambda)\, d\mu.$$

Theorem 3.3.1 (asymptotic behaviour). *Let $(f_n)_{n\geq 0}$ be the sequence of random variables defined by: $f_0 = \cdots = f_{L-2} = 1$ and by*

$$f_n = a_1 + \frac{a_2}{T^{-1}f_{n-1}} + \cdots + \frac{a_L}{T^{-1}f_{n-1}\cdots T^{-L+1}f_{n-L+1}}, \quad \textit{for } n \geq L-1.$$

Then $(f_n(\omega))$ converges, uniformly in ω an exponential rate, to a function $f(\omega)$ which is unique positive solution of the equation:

$$f = a_1 + \frac{a_2}{T^{-1}f} + \cdots + \frac{a_L}{T^{-1}f\ldots T^{-L+1}f}.$$

We have $\gamma(M,T) = \int \log(f)\, d\mu$. The asymptotic behaviour of the walk is the following one:
(i) *If $\int \log(f)\, d\mu < 0$, then: $\xi_n(\omega) \longrightarrow +\infty$, P'_ω-a.e., μ-a.e.*
(ii) *If $\int \log(f)\, d\mu = 0$, then: $-\infty = \underline{\lim}\xi_n(\omega) < \overline{\lim}\xi_n(\omega) = +\infty$, P'_ω-a.e., μ-a.e.*
(iii) *If $\int \log(f)\, d\mu > 0$, then: $\xi_n(\omega) \longrightarrow -\infty$, P'_ω-a.e., μ-a.e.*

Fixing ω, let $(\omega_n)_{n\geq 0}$, with $\omega_n = T^{\xi_n(\omega)}\omega$, be the sequence of the environments from the point of view of a moving particle. Collecting all the chains $(\xi_n(\omega))$, we observe that (ω_n) is a Markov chain on Ω with generator P. We write (IM) the assumption of the existence of a P-invariant probability equivalent to μ. Under (IM) a law of large numbers is valid.

We also consider the Markov chain on $(\Omega \times \Lambda)$ defined by $x_k = (\omega_k, z_k)$, where $z_k = \xi_{k+1}(\omega) - \xi_k(\omega)$. We write (HC) for the assumption of the existence of "harmonic coordinates" in $L^2(\Omega \times \Lambda)$ for the previous chain (cf [4]). Under (IM) and (HC), a functional CLT is valid.

Theorem 3.3.2 (characterization of both (IM) and (HC)).
(i) *If $\gamma(M,T) = 0$, (IM) + (HC) $\Longleftrightarrow$ $\exists \varphi > 0$ with φ and $\frac{1}{\varphi} \in L^1(\mu)$ such that $\lambda = \dfrac{\varphi}{T^{-1}\varphi}$.*
(ii) *If $\gamma(M,T) < 0$, (IM) + (HC)*

$$\Longleftrightarrow \int \left(\sum_{n=0}^{+\infty} \lambda \cdots T^{-n}\lambda\right)^2 \left(\sum_{n=0}^{+\infty} \lambda \cdots T^{n}\lambda\right) d\mu < +\infty.$$

(iii) *If $\gamma(M,T) > 0$, (IM) + (HC)*

$$\Longleftrightarrow \int \left(\sum_{n=0}^{+\infty} \lambda^{-1} \cdots T^{n}\lambda^{-1}\right)^2 \left(\sum_{n=0}^{+\infty} \lambda^{-1} \cdots T^{-n}\lambda^{-1}\right) d\mu < +\infty.$$

One can also characterize (IM) alone in a similar way. The next result shows the importance of the notions of (IM) and (HC).

Theorem 3.3.3. *Assume that $\gamma(M,T) = 0$. Then (IM) $\Longleftrightarrow$ "μ-a.e., the normalized paths converge to a non-degenerated Brownian motion".*

We now give an example:

Theorem 3.3.4. *Assume that Ω is the circle S^1, $T = T_\alpha$ is an irrational rotation and μ is Lebesgue measure.*
(i) *If $\gamma(M,T) \neq 0$, then $\xi_n(\omega) \longrightarrow +\infty$, P'_ω-a.e., μ-a.e.*
If M is continuous, a drift>0 and a functional CLT occur, μ-a.e.
(ii) *If $\gamma(M,T) = 0$, then $(\xi_n(\omega))$ is recurrent, μ-a.e.*
If M belongs to $C^{m+\delta}(S^1, \mathbb{R})$ and if α is of finite type η such that $m + \delta > \eta$, then there is a law of large numbers, with a drift equal to 0, and a functional CLT, μ-a.e.

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3.4 Georgi Chakvetadze *Measurable dynamics and one-dimensional model of drilling*

We consider a family of interval maps serving as a model for the process of rock destruction by drilling bit. These maps depend on two parameters - a scalar and a function - and are defined as follows. Let $p : \mathbb{R} \circlearrowleft$ be a 1-periodic function, twice continuously differentiable on $[0,1]$, $p''(r) < 0$, $p(1-r) = p(r)$, $r \in \mathbb{R}$. This function is determined by the geometry of the drilling bit. Set $m = \inf_{r \in [0,1]} |p''(r)|$, $M = \sup_{r\in[0,1]} |p''(r)|$, $a = \frac{M}{m}$ and assume that $0 < b < 1$. Given $s \in \mathbb{R}$ the graph of the function

$$h_s(r) = p(s) + p'(s+0)(r-s) - \frac{bm}{2}(r-s)^2, \quad r \geq s, \tag{1}$$

intersects the graph of p in a finite number of points with abscissas $s < r_1(s) < r_2(s) < \cdots < r_{q(s)}(s)$. Set $F(s) = F_{b,p}(s) = r_1(s) \bmod 1$, $s \in [0,1)$. The transformation F is piecewise monotonic.

We study the dynamical properties of F with respect to Lebesgue measure on the unit interval. The next lemma describes the occurrence of trivial dynamics in the system.

Lemma 3.4.1. *Assume that $b > \frac{a}{2}$. Then all the phase space is attracted to the unique fixed point of the transformation F.*

The next theorem provides the sufficient condition for the system to behave chaotically on some large set on the unit interval.

Theorem 3.4.1. *Assume that p'' has a bounded variation on $[0, 1]$ and the values of the parameters a and b satisfy the inequality $(1-b)^3/b > (a-b)^2 + \frac{1}{8}(a-1)(1-b)$. Then the transformation F has an absolutely continuous invariant probability (acip).*

Note, that the measure $v = v_{b,p}$ constructed in Theorem 1 has a property that any acip for F is absolutely continuous with respect to v.

Once having obtained the acip v we study the endomorphism $< F, v >$.

Theorem 3.4.2. *Assume that p'' has a bounded variation on $[0, 1]$ and the inequality $(1-b)^3/b > 2(a-b)^2$ holds. Then the system $< F, v >$ is exact, isomorphic to some Bernoulli shift and the central limit theorem holds for the functions of bounded p–variation ($p \geq 1$).*

Next we discuss the stability property of the measure v with respect to stochastic perturbation Y_e, $e > 0$, introduced as follows. Given $e > 0$ and $s \in [0, 1)$ we replace $p(s)$ ($p'(s+0)$) in the equality (1) by the stochastic variable distributed with some density on the interval $[p(s) - e, p(s) + e]$ ($[p'(s+0) - e, p'(s+0) + e]$). Then $F(s)$ is replaced by the stochastic variable distributed on some set including $F(s)$ with the density $u_e(s, \cdot)$. The Markov chain Y_e has $u_e(s, \cdot)$ as the densities of transition probabilities. In the proof of the next theorem additionally some technical conditions are used which we omit.

Theorem 3.4.3. *Assume that $p \in C^3([0, 1])$ and the transformation F has no periodic turning points. Then the densities of stationary distributions of the chains Y_e tend to the density of v in L_1-metric as $e \to 0$.*

The dynamical model of drilling was suggested by Lasota and Rusek. The problem of existence of acip for one dimensional maps was stated by Ulam. The stochastic perturbations of dynamical systems were studied by Blank, Keller, Kifer, Sinai et al.

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3.5 Maurice Courbage
Stochastic process determined or not determined by their 2-marginals laws

Let $\omega = (\omega_i)_{i\in\mathbb{Z}}$ be a stationary stochastic process on a finite space $\mathbb{K} = \{0, 1, 2, .., k-1\}$. Let $\Omega = \mathbb{Z}^{\mathbb{Z}}$, σ be the shift on Ω, $(\sigma\omega)_i = \omega_{i+1}$ and μ be the associated σ-invariant probability measure on Ω. We say that a σ-invariant measure ν on Ω has the same 2-marginal laws as μ if $\nu(\{\omega_0 = i, \omega_n = j\}) = \mu(\{\omega_0 = i, \omega_n = j\})$ for all $n \in \mathbb{Z}$, $i, j, \in \mathbb{K}$.

We say that μ is determined by its 2-dimensional marginals if for any σ-invariant measure ν on Ω, having the 2-marginals as μ, $\nu = \mu$.

We consider here ergodic stochastic processes determined or not determined by their 2-marginals in the class of ergodic measures.

Our motivation in this problem comes from the problem of entropy in non-equilibrium thermodynamics. The problem can be formulated in measure-theoretical language as follows.

Let (X, T, μ) be a dynamical system and $P = (P_i)$, $i = 0, \ldots, k-1$, be a finite measurable partition. Let Π_n be the family of stochastic matrices:

$$(\Pi_n)_{ij} = \mu(T^{-n}P_j|P_i), \quad n \geq 0.$$

For any probability ν on the σ-algebra generated by P, we define the probability measure at time n by:

$$\nu_n(P_j) = \sum_i \nu(P_i)(\Pi_n)_{ij}$$

and the non-equilibrium entropy by

$$S(\nu_n) = \sum_j \nu_n(P_j)\frac{\log \nu_n(P_j)}{\mu_n(P_j)}.$$

If Π_n is a semi-group of irreducible and aperiodic stochastic matrices, then $\nu_n(P_j) \to \mu(P_j)$ as $n \to \infty$ and $S(\nu_n)$ is monotonically decreasing to zero as $n \to \infty$. The above properties of Π_n define what we call a Chapmann-Kolmogorov partition. If, for a σ-invariant measure μ on Ω, the zero coordinate partition is a Chapmann-Kolmogorov partition, μ is called a Chapman-Kolmogorov measure.

Our first result states that any ergodic Markov chain is not determined by its 2-marginal laws. The proof is based on a construction described in the two following theorems.

Theorem 3.5.1. *Let μ be a probability measure on $\mathbb{K}^{n+1}$, $n \geq 2$, invariant with respect to σ.*

We say that $\mu \in M_n$ if

$$\sum_y \mu(\omega_0 = y, \omega_1 = x_0, \ldots, \omega_n = x_{n-1}) = \mu(\omega_0 = x_0, \ldots, \omega_{n-1} = x_{n-1}).$$

Let $\nu_0(\mu)$ be a measure on $\mathbb{K}^{\mathbb{N}}$,

$$\begin{aligned}&\nu_0(\omega_0 = x_0, \ldots, \omega_{pn} = x_{pn}) \\ &= \mu(x_0, \ldots, x_n)\mu(x_{n+1}, \ldots, x_{2n}|x_n) \ldots \mu(x_{(p-1)n+1}, \ldots, x_{pn}|x_{(p-1)n}).\end{aligned}$$

Let

$$\Phi_n(\mu) := \frac{1}{n}\sum_{j=0}^{n-1}\sigma^j\nu_0(\mu).$$

Then $\Phi_n : M_n \to M(\Omega, \sigma)$ has the properties:

i) *Φ_n is one-to-one,*

ii) *Φ_n preserves the 2-marginals,*

iii) *$\Phi_n\mu = \mu \Longleftrightarrow \mu$ is a Markov chain.*

Theorem 3.5.2. *If the stochastic matrix $A_n(\mu)$ defined by*

$$(A_n(\mu))_{ij} = \mu(\omega_n = j|\omega_0 = i)$$

is irreducible then

i) *$\Phi_n(\mu)$ is ergodic,*

ii) *$\Phi_n(\mu)$ has infinite memory.*

It follows from i)-iii) of Theorem 1 that if $\nu \in M_n$ is not Markovian, but having the same 2-marginals of a Markov chain μ_Π, then $\Phi_n(\nu)$ is not Markovian but having the same 2-marginals as μ_Π. These results are contained in [1,3]. Theorem 2 can be generalized to p-marginal laws.

The above processes have positive entropy. As to zero entropy systems, we found pairwise independent partitions in the Anzai skew product of translations [2]. A necessary condition for the existence of a Chapman-Kolmogorov partition is that the dynamical system contains a Lebesgue spectral type.

The following open problems can be formulated:

Problem 3.5.1. Find Chapman-Kolmogorov partitions that are pairwise independent in the skew product.

Problem 3.5.2. Find a class of dynamical systems having a Chapman-Kolmogorov partition.

Ergodic measures on $\{0, 1\}^{\mathbb{Z}}$ determined by their 2-marginals

(1). Let $\frac{\alpha}{2\pi}$ be irrational, $\mathbb{T} = [0, 2\pi[$, $Rx = x + \alpha\,(\mathrm{mod}\, 2\pi)$ and $d\lambda = \frac{dx}{2\pi}$. Let P be the partition of $\mathbb{T}$ given by two arcs:

$$P_0 = [0, \beta[, \quad p_1 = [\beta, 2\pi[,$$

where $\frac{\beta}{2\pi}$ is irrational. Let μ be the measure on $\{0,1\}^{\mathbb{Z}}$ defined by

$$\mu(\omega_0 = i_0, \dots, \omega_n = i_n) = \lambda(P_{i_0} \cap R^{-1}P_{i_1} \cap \dots \cap R^{-n}P_{i_n}).$$

Then for any ergodic measure ν on $\{0,1\}^{\mathbb{Z}}$ s.t.

$$\nu(\omega_0 = i, \omega_n = j) = \mu(\omega_0 = i, \omega_n = j)$$

we have $\nu = \mu$.

(2). Let $(\mathbb{R}^{\mathbb{Z}}, T, \mu)$ be a Kronecker-Gauss dynamical system with $\int X_0 d\mu = 0$. Let

$$P_0 = \{X_0 > 0\}, \quad P_1 = \{X_0 < 0\},$$
$$\nu(\omega_0 = i_0, \dots, \omega_n = i_n) = \mu(P_{i_0} \cap \dots \cap T^{-n}P_{i_n}).$$

Then (σ, ν) is determined by its 2-marginals.

These results are contained in [4].

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3.6 Aleksandre I. Danilenko *Entropy of cocycles of measurable equivalence relations and applications*

Let $\mathcal{R}$ be a countable measure preserving equivalence relation on a standard probability space $(X, \mathfrak{B}_X, \mu)$. Given a Borel cocycle α of $\mathcal{R}$ with values in the automorphism group of another standard probability space $(Y, \mathfrak{B}_Y, \nu)$, a type I subrelation $\mathcal{S}$ of $\mathcal{R}$ and a finite partition P of $X \times Y$, we set

$$h(\mathcal{S}, \alpha, P) := \int_X \frac{1}{\#\mathcal{S}(x)} H\Big(\bigvee_{x' \in \mathcal{S}(x)} \alpha(x, x')P_{x'}\Big) d\mu(x),$$

where $P_{x'}$ is the "restriction" of P to the fiber $\{x'\} \times Y$. Recall that $\#\mathcal{S}(x) < \infty$ since $\mathcal{S}$ is of type I. Now we define

$$\begin{aligned} h(\alpha, P) &= \inf\{h(\mathcal{S}, \alpha, P) \mid \mathcal{S} \text{ is a type } I \text{ subrelation of } \mathcal{R}\} \\ h(\alpha) &= \sup\{h(\alpha, P) | P \text{ is a finite partition of } X \times Y\}, \\ \Pi(\alpha) &= \vee\{P \mid h(\alpha, P) = 0\}. \end{aligned}$$

We say that α is CPE if $\Pi(\alpha)$ is the least possible, i.e. $\Pi(\alpha) = \mathfrak{B}_X \otimes \mathfrak{N}_Y$, where $\mathfrak{N}_Y$ stands for the trivial sub-σ-algebra of $\mathfrak{B}_Y$.

Theorem 3.6.1. 1. *If two cocycles $\alpha, \beta : \mathcal{R} \to Aut(Y, \nu)$ are cohomologouus (or even weakly equivalent) then $h(\alpha) = h(\beta)$. If α is CPE then β so is.*

2. *If the group $\{\gamma \in Aut(X, \mu) \mid (\gamma \times \gamma)\mathcal{R} = \mathcal{R}$ and $\alpha \circ \gamma = \alpha\}$ is ergodic then there exists a sub-σ-algebra $\mathfrak{F} \subset \mathfrak{B}_Y$ such that $\Pi(\alpha) = \mathfrak{B}_X \otimes \mathfrak{F}$.*

3. *If α is recurrent then $h(\alpha) = 0$, i.e. $\Pi(\alpha) = \mathfrak{B}_{X\times Y}$.*

4. *If $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$ for an increasing sequence $(\mathcal{R}_n)$ of type I $\mathcal{R}$-subrelations then $h(\alpha, P) = \lim_n h(\mathcal{R}_n, \alpha, P)$.*

Our purpose now is to provide a new definition for the entropy of a process. Let G be a countable amenable group, R a free action of G on X generating $\mathcal{R}$, $T = (T_g)_{g\in G}$ a ν-preserving action of G on Y and Q a finite Y-partition. We set

$$\widehat{h}(T, Q) := h(\beta_T, \mathfrak{N}_X \otimes Q),$$

where $\beta_T : \mathcal{R} \to \mathrm{Aut}(X, \nu)$ is a cocycle given by $\beta_T(R_g x, x) = T_g$.

Theorem 3.6.2. 1. *$\widehat{h}(T, Q)$ is well defined.*

2. *$\widehat{h}(T, Q) = h(T, Q)$, where $h(T, Q)$ is the classical entropy of the process (T, Q)* [2], [4].

3. *$h(\alpha, P) = h(R^\alpha, P \mid \mathfrak{B}_X \otimes \mathfrak{N}_Y)$, where $R^\alpha = (R_g^\alpha)_{g\in G}$ stands for the α-skew-product extension of R.*

As applications we obtain short proofs of the following recent results.

Theorem 3.6.3. 1. *If $\mathfrak{F}$ is a factor of T then $h(T) = h(T \restriction \mathfrak{F}) + h(T \mid \mathfrak{F})$ (the entropy of the $\mathfrak{F}$-quotient action + the $\mathfrak{F}$-relative entropy),* [6].

2. *If T is $\mathfrak{F}$-relatively CPE. Then for each $\epsilon > 0$ there exists a finite G-subset K such that $\left| \frac{1}{\#F} H\left(\bigvee_{g\in F} T_g^{-1} Q \mid \mathfrak{F}\right) - H(Q \mid \mathfrak{F}) \right| < \epsilon$ for any finite subset F with $g_1 g_2^{-1} \notin K$ for all $g_1 \neq g_2 \in K$ (see* [5] *for $\mathfrak{F}$ trivial).*

3. (see [1]) *Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{E}$ are three factors of T with $\mathfrak{E} \subset \mathfrak{A}_1 \cap \mathfrak{A}_2$.*

If $T \restriction \mathfrak{A}_1$ is $\mathfrak{E}$-relatively CPE and $h(T \restriction \mathfrak{A}_2 \mid \mathfrak{E}) = 0$ then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $\mathfrak{E}$-relatively independent

If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $\mathfrak{E}$-relatively independent then $\Pi(T \restriction (\mathfrak{A}_1 \vee \mathfrak{A}_2) \mid \mathfrak{E}) = \Pi(T \restriction \mathfrak{A}_1 \mid \mathfrak{E}_1) \vee \Pi(T \restriction \mathfrak{A}_2 \mid \mathfrak{E})$.

$\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $\mathfrak{E}$-relatively independent iff $\Pi(T \restriction \mathfrak{A}_1 \mid \mathfrak{E})$ and $\Pi(T \restriction \mathfrak{A}_2 \mid \mathfrak{E})$ are $\mathfrak{F}$-relatively independent and $h(T \restriction \mathfrak{A}_1 \vee \mathfrak{A}_2 \mid \mathfrak{E}) = h(T \restriction \mathfrak{A}_1 \mid \mathfrak{E}) + h(T \restriction \mathfrak{A}_2 \mid \mathfrak{E})$.

4. (see [3] for the case $G = \mathbb{Z}$) *If $\mathfrak{F}$ is a class-bijective factor of T and $h(T \mid \mathfrak{F}) < \log n$ for some integer n then there exists a finite partition Q of Y such that $\#Q = n$ and $\mathfrak{F} \vee \bigvee_{g\in G} T_g Q = \mathfrak{B}_Y$.*

This research was inspired by [5], where the orbit theory was used to prove the absolute version of Theorem 3(2). Unlike [5], [1], [6] we do not use Rokhlin lemma, Sannon-MaMillan theorem, castle analysis, joining techniques. Our approach is independent of [2], [4] where the classical entropic concepts for amenable actions were elaborated.

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3.7 Manfred Denker[2] *Relative thermodynamic formalism*

Introduction

Let T be a continuous map of a compact metric space Y onto itself and $\varphi \in C(Y)$. The well known variational principle states that

$$\sup_{\mu \in M(T)} \left\{ h_\mu(T) + \int_Y \varphi(y)\mu(dy) \right\} = P_T(\varphi), \tag{3.1}$$

where $M(T)$ denotes the set of T-invariant probability measures. The supremum is attained (if at all) for equilibrium measures. $h_\mu(T) + \int_Y \varphi(y)\mu(dy)$ is called the *free energy* of the measure μ. Under suitable additional assumptions on T like expansiveness the existence of an equilibrium measure μ is well known. Uniqueness and the Gibbs property of equilibrium measures require much stronger assumptions on T and φ. These topics constitute a part of the theory of thermodynamic formalism.

For certain non-invertible locally expanding maps the *transfer operator*

$$(V_\varphi g)(x) = \sum_{y:T(y)=x} g(y)\exp(\varphi(y)) \tag{3.2}$$

has an eigenvalue $\lambda = \exp P(T, \phi)$ and an eigenmeasure μ for V^* (called a *Gibbs measure*) which is an equilibrium measure.

[2]The talk is based on joint research with M. Gordin and St.-M. Heinemann.

Fibred Systems and Transfer Operators

We consider a fibred system where the map T is foliated over a continuous map $S : X \to X$ on some compact space X, with a continuous *factorisation map* $\pi : Y \to X$ which semi-conjugates T to S. We always suppose that T, S and π are surjective.

Ledrappier and Walters (1976) extended the variational principle to the fibred case:

$$\sup_{\mu \in M(T,\nu)} \left\{ h_\mu(T|S) + \int_Y \varphi \, d\mu \right\} = \int_X P_T(\varphi, Y_x)\nu(dx), \tag{3.3}$$

where $\nu \in M(S)$, $M(T,\nu) = \{\mu : \mu \in M(S), \mu \circ \pi^{-1} = \nu\}$, $Y_x = \pi^{-1}(\{x\})$ for $x \in X$, $h_\mu(T|S)$ denotes the mean relative entropy of T with respect to S, and $P_T(\psi, Z)$ denotes the relative pressure of a function $\psi : Z \to \mathbb{R}$ with respect to T and a closed set $Z \subset Y$. In analogy to the non-fibred case, the quantity $h_\mu(T|S) + \int_Y \varphi \, d\mu$ is called the *relative free energy* (of μ). Furthermore, a probability measure μ which maximizes this expression is called a *relative equilibrium measure*. The problems of determining the maximum (or supremum) of free energy, existence, uniqueness and other properties of equilibrium measures are of obvious interest.

The relative transfer operator is given by a family of operators between spaces of all bounded Borel measurable functions B_x on the fibres Y_x. For $x \in X$ and $k \geq 1$ the operators $V_x^{(k)} : B_x \to B_{S^k(x)}$ are defined by

$$(V_x^{(k)} g)(y) = \sum_{\substack{T^k(y')=y \\ \pi(y')=x}} g(y') \exp\left(\sum_{j=0}^{k-1} \varphi(T^j(y')) \right), \tag{3.4}$$

where $y \in Y_{S^k(x)}$. Under suitable continuity assumptions $V_x^{(1)}$ preserves continuous functions and $V_x^{(1)*}$ maps measures of bounded total variation on $Y_{S(x)}$ to measures of bounded total variation on Y_x. Unlike for $V_x^{(1)*}$, there is no natural way to derive a global operator on bounded measurable functions from the family $\{V_x^{(1)} : x \in X\}$, unless S is invertible. Clearly, for an invertible base map S the relative transfer operator coincides with the usual one.

Main Results

We assume that T is bounded-to-one on fibres. A fibred system is called *fibre expanding*, if the fibre maps $T_x : Y_x = \pi^{-1}(\{x\}) \to Y_{S(x)}$ are uniformly expanding in Ruelle's sense and *topologically exact along fibres* if, for every $\varepsilon > 0$, balls of radius ϵ centered at any $y \in Y$ are mapped under T^n onto the fibre of $T^n(y)$ for all sufficiently large n.

Let $\varphi : Y \to \mathbb{R}$ be a Borel measurable function. A system $\{\mu_x : x \in X\}$ of conditional probabilities for $\mathcal{Y}$ is called a *Gibbs family* for φ, if there exists a measurable function $A_\varphi : X \to \mathbb{R}$ such that for $x \in X$ and $f \in L_1(\mu_x)$ we have that

$$\int V_x f(y) \mu_{S(x)}(dy) = A_\varphi(x) \int f(y) \mu_x(dy). \tag{3.5}$$

Theorem 3.7.1. ([1]) *Let $\mathcal{Y} = (Y, T, X, S, \pi)$ be a fibred system. Assume that $\mathcal{Y}$ is fibrewise expanding and topologically exact along fibres. Then, for every Hölder continuous function $\varphi : Y \to \mathbb{R}$, there exists a unique Gibbs family $\{\mu_x : x \in X\}$ for φ. The function A_φ is also unique and one has that $supp\{\mu_x\} = Y_x$.*

In addition, if $i_1(\cdot) = (\pi(\cdot), T(\cdot))$ is a local homeomorphism and S and π are open maps, then the Gibbs family $\{\mu_x : x \in X\}$ and the function $A_\varphi : X \to \mathbb{R}$ are continuous.

Theorem 3.7.2. ([2]) *If the assumptions of the last theorem are satisfied (including the addition), then, for any S-invariant measure ν on X, we have that*

$$\sup_{\mu \in M(T,\nu)} \left(h_\mu(T|S) + \int_Y \varphi d\mu \right) = \int_X \log A_\varphi d\nu. \tag{3.6}$$

Moreover, the supremum is finite and is attained by a unique probability measure.

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3.8 Jérôme Depauw
Résistivité du réseau cubique de résistances aléatoires stationnaires

1. — Considérons un conducteur de forme parallélépipédique rectangle, de section horizontale S et de hauteur h, composé dans un matériau homogène et isotrope. Lorsque ses faces latérales sont isolées, et une différence de potentiel U imposée entre ses faces horizontales, il est traversé par un courant I. Ce dernier est, pour un "bon" matériau, proportionnel à la différence de potentiel U. Le coefficient de proportionnalité $R = \frac{U}{I}$ est la résistance du conducteur. Celle-ci dépend naturellement des dimensions de ce dernier: elle est inversement proportionnelle à sa section, et proportionnelle à sa hauteur. Cette proportionnalité fait apparaître un coefficient $\rho = \frac{S}{h}R$ qui ne dépend que du matériau. C'est la résistivité de ce dernier. Nous étudions ici la notion analogue pour un réseau aléatoire.

2. — Soit le réseau $\mathcal{Z} = \mathbf{Z}^3$ et $(o; \vec{x}_1, \vec{x}_2, \vec{x}_3)$ son repère affine canonique, le premier vecteur correspondant à la direction verticale. Deux nœuds voisins e et $e + \vec{x}_j$, $e \in \mathcal{Z}$, $j = 1, 2, 3$ sont reliés par une résistance de valeur $\mathcal{R}j(e)$. Pour N fixé, soit $\mathcal{Z}_N$ le sous-réseau de $\mathcal{Z}$ correspondant aux nœuds de la boite $[0, N]^3$. Lorsque les quatre faces verticales sont isolées, et qu'une différence de potentiel U est imposée entre les deux faces horizontales, ce réseau est traversé par un courant I_N. Sa résistance est le coefficient de proportionnalité entre la différence de potentiel U imposée au réseau et le courant I_N: $R_N = \frac{U}{I_N}$. Suivant l'analogie avec la notion de résistivité définie ci-dessus, la résistivité de $\mathcal{Z}_N$ est la quantité $\frac{N^2}{N} R_N = N R_N$.

Supposons que les résistances $\mathcal{R}_j(e)$ soient des variables aléatoires indépendantes, prenant deux valeurs r, r', $0 < r < r'$, avec probabilité $\frac{1}{2}$, et soit $(\Omega, \mathcal{T}, P)$ l'espace de probabilité associé à cette expérience. Kesten a posé la question suivante ([4], pb.12, p.385): quel type de convergence a-t-on pour la suite de variables aléatoires NR_N, et quelle est la limite?

Nous avons montré la convergence ponctuelle vers une constante:

Théorème 3.8.1. ([1]) *Sous les hypothèses décrites ci-dessus, NR_N converge presque sûrement vers une constante, quand N tend vers l'infini.*

La valeur de la limite reste un problème ouvert (le seul résultat connu concerne la dimension 2; Marchant et Gabillard [5] ont montré que dans cette dimension, la seule valeur possible est $\sqrt{rr'}$). Nous nous proposons ici de donner une expression de la limite et d'indiquer pourquoi sa valeur est inconue.

3. — Considérons le système dynamique constitué par les réalisations de l'expérience aléatoire décrite dans l'introduction. L'espace des réalisations est l'ensemble Ω des suites $\omega = (\mathcal{R}_j(e))_{j,e}$ indexée sur $\{1, 2, 3\} \times \mathcal{Z}$, à valeurs dans $\{r, r'\}$. Il est muni de la tribu usuelle, et de la probabilité P faisant des coordonnées $\mathcal{R}_j(e)$ des variables aléatoires indépendantes équidistribués et de loi uniforme. L'action de $\mathbf{Z}^3$ est le décalage, défini avec des notations évidentes par $T_{\vec{x}}(\omega) = (\mathcal{R}_j(e + \vec{x}))_{j,e}$. En notant $R_j(\omega)$, $j = 1, 2, 3$, les applications de premiere coordonnée, $R_j(\omega) = \mathcal{R}_j(o)$, la résistance de l'arête $[e, e+\vec{x}_j]$ pour la réalisation ω prend donc l'expression $R_j(T_{\vec{oe}}\omega)$. Considérons des fonctions $(A_j)_{j=1,2,3}$ et $(B_j)_{j=1,2,3} \in L^2(\Omega)$ telles que $A_j(T_{\vec{oe}}\omega)$ et $B_j(T_{\vec{oe}}\omega)$ représentent respectivement le champ électrique et le courant dans l'arête $[e, e + \vec{x}_j]$. Les lois de l'électricité s'écrivent:

$$\begin{aligned} \sum_{j=1}^{3} (T_j^{-1} B_j - B_j) &= 0 && \text{Loi des nœuds,} \\ T_j A_k - A_k &= T_k A_j - A_j && \text{Lois de Mailles,} \\ A_j &= R_j B_j && \text{Loi de Ohm ;} \end{aligned}$$

où T_j est l'opérateur de L^2 défini par $T_j f = f \circ T_{\vec{x}_j}$. Les conditions aux bords n'ayant pas de sens pour un courant spatialement stationnaire, elles sont remplacées par des conditions en moyenne:

$$\int_\Omega A_1 = U, \qquad \int_\Omega A_2 = \int_\Omega A_3 = 0.$$

L'existence et l'unicité d'un tel courant sont donnés par un théorème d'analyse hilbertienne reposant sur le théorème de Lax Milgram.

Théorème 3.8.2. ([2],[7]) *Le système constitué des quatre équations ci-dessus admet une unique solution* $\vec{A} = (A_j)_j \in (L^2(\Omega))^3$.

L'idée de la preuve de la convergence ponctuelle de NR_N peut alors se résumer grossièrement de la manière suivante. La conductivité NR_N est de l'ordre de

$$\frac{\sum_{e \in Z_N} A_1(T_{\vec{oe}}\omega)}{\sum_{e \in Z_N} B_1(T_{\vec{oe}}\omega)}.$$

D'aprés le théorème ergodique ponctuel, cette variable aléatoire converge presque sûrement vers $\frac{\int_\Omega A_1}{\int_\Omega B_1}$, ce qui est encore égal à $\frac{U}{\int_\Omega R_1^{-1} A_1}$. C'est la valeur du dénominateur qui est inconue, puisque le théorème de Lax Milgram ne donne pas explicitement la solution A.

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3.9 Yves Derriennic *On Hopf's decomposition and the ratio ergodic theorem for a* $\mathbb{Z}^d$*-action in infinite measure*

Let S and T be two commuting, measure preserving transformations of a measure space $(\Omega, \mathcal{F}, m)$. According to Wiener's ergodic theorem

$$\lim_n \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} f(S^k T^l \omega)$$

exists m-a.e. when $f \in L^1$; the limit is 0 when the system is ergodic and the measure m is infinite. Thus in this situation it is natural to look for a version of the ergodic theorem similar to Hopf's ratio ergodic theorem: given f and $g \in L^1$ with $g > 0$ m-a.e., do the ratios

$$\frac{\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f(S^kT^l\omega)}{\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} g(S^kT^l\omega)}$$

converge m-a.e., the measure m being infinite? It is known since a long time that the answer is negative. A counter-example is given in Krengel's book (p.217). It is worthwhile to recall the basic idea of this counter-example. The measure space is $\mathbb{Z}$ with m the counting measure; the transformations are both equal to the unit translation: $Tj = j+1$, $S = T$. Then

$$\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f(S^kT^l 0) = \sum_{j=0}^{n-1}(j+1)f(j) + \sum_{j=n}^{2n-2}(2n-j-1)f(j).$$

It is clear that for $f \in l^1$ and $f > 0$, these sums $\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f(S^kT^l 0)$ may converge or diverge, therefore in general the ergodic ratios do not converge to a finite limit. Moreover this example shows that the set where $\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f(S^kT^l) = +\infty$ may depend on f, hence Hopf's maximal ergodic inequality cannot be true for the double sums $\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f(S^kT^l)$. It is emphasized by Krengel that, using stacking arguments, it is possible to build such a counter-example with S and T being individually conservative, ergodic and the action of (S,T) being free.

It is remarkable that the replacement of the one-sided sums $\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} f(S^kT^l)$ by the symmetric sums $\Xi_n(f) = \sum_{k=1-n}^{n-1}\sum_{l=1-n}^{n-1} f(S^kT^l)$ produces an essential change of the problem (from now on we assume the invertibility of the transformations).

Maximal ergodic inequality for symmetric sums. *There is a universal constant c (depending only on the dimension) such that for every f and $g \in L^1_+$, with $g > 0$ m-a.e., and for every $t > 0$ we have*

$$\int_{E(t)} g\,dm \le \frac{c}{t}\int f\,dm$$

where $E(t) = \left\{\omega \in \Omega; \sup_n \dfrac{\Xi_n(f)}{\Xi_n(g)} > t\right\}$.

The proof follows the classical pattern of Wiener's proof of the maximal ergodic inequality. But instead of the Vitali type covering lemma we use the Besicovitch covering lemma (we refer to the book of Wheeden and Zygmund "Measure and Integral", Marcel Dekker ed.). At the time of his talk at the Toruń conference the author ignored that the preceding inequality, with the same proof based on the Besicovitch covering lemma, was the main content of a 1983 paper by Maria Becker. This reference was kindly provided by Jon Aaronson to whom the author expresses his thanks.

From this maximal inequality it is easy to deduce a version of Hopf's decomposition for symmetric sums.

Hopf's decomposition for symmetric sums. *The space Ω splits into two disjoint measurable parts C, the conservative part, and D, the dissipative part, such that for every $f \in L^1_+$, $\{\lim_n \Xi_n(f) = +\infty\} = C \cap \{\lim_n \Xi_n(f) > 0\}$*

In Aaronson's book there is another approach to Hopf's decomposition for a group action.

To get a ratio ergodic theorem for the symmetric sums it would then suffice to prove for $f \in L^1_+$

$$\lim_n \frac{\sum_{k=1-n}^{n-1} f(S^k T^n)}{\Xi_n(f)} = 0 \quad m\text{-a.e.} \quad (?)$$

One might think, at first glance, that this should be easy. But, as far as the author knows, it is an unsolved problem. So, following the invitation of the organizers of the Toruń conference, we propose:

Problem 3.9.1. Prove or disprove the convergence (?) stated above. (S and T are two commuting measure preserving invertible transformations of an infinite measure space).

If we return to Krengel's example that we recalled at the beginning, we observe that the convergence (?) holds, therefore in that case the ratio ergodic theorem for symmetric sums holds.

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3.10 Tomasz Downarowicz[3] *Relative variational principle*

This note is a survey of results which are being prepared for publication as [3] in a mathematical journal, where they will appear with complete proofs, examples, and historical comments.

At the beginning, Abramov and Rokhlin's definition of measure-theoretic fiber entropy is extended, using disintegration. A strong connection with measure-theoretic conditional entropy is proved.

Largest part of the work is devoted to the case of a pair of topological dynamical systems on compact Hausdorff (not necessarily metrizable) spaces, one being a factor of another. Topological notions of fiber entropy and conditional entropy are defined

[3] Joint work with J. Serafin

and studied. We prove three variational principles of conditional nature, some of them generalizing results known before in metric spaces.

A tail entropy of a measure is introduced in totally disconnected spaces. As an application of our variational principles it is proved that the tail entropy estimates from below the "defect of upper semi-continuity" of the entropy function.

Let (X, Σ, μ) be a probability space and α a finite measurable partition of X. Throughout this note we shall write:

$$\begin{aligned} \mathrm{H}(\mu, \alpha) &:= -\textstyle\sum_{A \in \alpha} \mu(A) \log(\mu(A)); \\ \mathrm{H}(\mu, \alpha|\beta) &:= \mathrm{H}(\mu, \alpha \vee \beta) - \mathrm{H}(\mu, \beta). \end{aligned}$$

We start with recalling the definition of disintegration of a measure, and then by defining the measure theoretic conditional and fiber entropies:

Definition 3.10.1 (classical). Let $\pi : (X, \Sigma, \mu) \to (Y, \Theta, \nu)$ be a homomorphism of measure spaces. By a *disintegration of* μ, *with respect to* ν we shall mean a family of measures μ_y defined for ν-almost every $y \in Y$, each supported by $\pi^{-1}y$, such that for every bounded measurable function f on X the function $\overline{f}(y) := \int f(x) d\mu_y(x)$ is measurable, and

$$\int \overline{f}(y) d\nu(y) = \int f(x) d\mu(x).$$

Assume now, that π is a factor map between measure preserving dynamical systems (X, μ, T) and (Y, ν, S).

Definition 3.10.2 (classical). For a partition α of X we set

$$\mathrm{h}(\mu, \alpha|\nu) := \inf_{\beta} \inf_{n} \frac{1}{n} \mathrm{H}(\mu, \alpha^n|\beta),$$

where β ranges over all Y-measurable partitions of X. Furthermore, we set

$$\mathrm{h}(\mu|\nu) := \sup_{\alpha} \mathrm{h}(\mu, \alpha|\nu),$$

with α ranging over all partitions of X. The latter is called *the conditional entropy of the system* (X, μ, T) *given the factor* (Y, ν, S)

Definition 3.10.3. Let α be a finite partition of X. By the *fiber entropy of* α we shall mean the function $\mathrm{h}(\mu, \alpha|\cdot)$ defined ν-almost everywhere on Y by the formula

$$\mathrm{h}(\mu, \alpha|y) := \lim_{n} \mathrm{H}(\mu_y, \alpha|T^{-1}\alpha^n).$$

We shall say that the disintegration is *invariant* if $\mu_{Sy} = T\mu_y$ ν-a.e. It is known that such disintegration exists in skew products (with respect to the base), or in the case where S is invertible and the measure space (X, μ) is compact in the sense of Marczewski (see [6]), for example, when it is a compact Hausdorff space with a Radon measure μ. In such case we have the following result (a generalization of Abramov-Rokhlin's theorem for skew products [1]):

Theorem 3.10.1. *If (Y, ν, S) is a factor of (X, μ, T) via a map π such that an invariant disintegration of μ with respect to ν exists then, for every partition α of X holds*

$$\int h(\mu, \alpha|y)d\nu(y) = h(\mu, \alpha|\nu).$$

From now on we will assume that (Y, S) is a topological factor of (X, T) (both systems are actions of a continuous map on a compact Hausdorff space). By a *measure* we shall always mean an invariant Radon probability measure on the corresponding space. The letter $\mathcal{A}$ will denote a finite open cover of X.

We introduce the following notions (compare [5]):

Definition 3.10.4. For $y \in Y$ set

$$\mathbf{H}(\mathcal{A}|y) := \log \min\{\#\mathcal{F} : \mathcal{F} \subset \mathcal{A}, \bigcup \mathcal{F} \supset \pi^{-1}y\}.$$

If ν is a measure on Y then we set

$$\mathbf{H}(\mathcal{A}|\nu) := \int \mathbf{H}(\mathcal{A}|y)d\nu.$$

Definition 3.10.5. The *topological fiber entropy of the cover $\mathcal{A}$ given y* equals

$$\mathbf{h}(\mathcal{A}|y) := \limsup_{n\to\infty} \frac{1}{n}\mathbf{H}(\mathcal{A}^n|y).$$

If ν is a measure on Y then we denote

$$\mathbf{h}(\mathcal{A}|\nu) := \inf_n \frac{1}{n}\mathbf{H}(\mathcal{A}^n|\nu).$$

Definition 3.10.6. We set

$$\begin{aligned} \mathbf{h}(X|y) &:= \sup_{\mathcal{A}} \mathbf{h}(\mathcal{A}|y), \\ \mathbf{h}(X|\nu) &:= \sup_{\mathcal{A}} \mathbf{h}(\mathcal{A}|\nu), \end{aligned}$$

where $\mathcal{A}$ ranges over all finite covers, and ν is a measure on Y. The above quantities will be called *the topological fiber entropy of X given y*, and *given ν*, respectively. Of course, infinity is admitted in both cases.

The last quantity to be defined has purely topological nature and it will be called the *topological conditional entropy of (X, T) given the factor (Y, S)*.

Definition 3.10.7. Let

$$\begin{aligned} \mathbf{H}(\mathcal{A}|Y) &:= \sup_{y\in Y} \mathbf{H}(\mathcal{A}|y) \\ \mathbf{h}(\mathcal{A}|Y) &:= \inf_n \tfrac{1}{n}\mathbf{H}(\mathcal{A}^n|Y) \\ \mathbf{h}(X|Y) &:= \sup_{\mathcal{A}} \mathbf{h}(\mathcal{A}|Y). \end{aligned}$$

Let $\mathcal{P}_T(X)$ and $\mathcal{P}_S(Y)$ denote the sets of invariant measures on X and Y, respectively. In the theorems below we connect all previously defined types of entropies.

Theorem 3.10.2 (outer variational principle). *Let (Y, S) be a topological factor of (X, T). Then*

$$\mathbf{h}(X|Y) = \sup_{y\in Y} \mathbf{h}(X|y) = \sup_{\nu} \mathbf{h}(X|\nu) = \sup_{\tilde{\nu}} \mathbf{h}(X|\tilde{\nu}),$$

where the last two suprema run over all invariant and all ergodic measures on Y, respectively.

Theorem 3.10.3 (inner variational principle). (compare Theorem 2.1 in [4]). *Let $\pi : X \to Y$ be a topological factor map between topological dynamical systems (X, T) and (Y, S). If ν is an invariant measure on the factor (Y, S) then*

$$\mathbf{h}(X|\nu) = \sup\{h(\mu|\nu) : \ \mu \in \mathcal{P}_T(X), \pi\mu = \nu\}.$$

Combining the above two results, we obtain:

Theorem 3.10.4 (conditional variational principle). (Compare [2]).

Let $\pi : X \to Y$ be a topological factor map between topological dynamical systems (X, T) and (Y, S). Then

$$\mathbf{h}(X|Y) = \sup_{\mu\in\mathcal{P}_T(X)} h(\mu|\pi\mu).$$

M. Misiurewicz introduced in [5] the notion of the so-called "topological conditional entropy" of a dynamical system. We would rather call it the *tail entropy*, because "topological conditional entropy" has in our paper a different meaning. Assume that the space X is totally disconnected. Then, as it is well known, the dynamical system (X, T) is isomorphic to an inverse limit of a net of symbolic systems

$$(X, T) = \overleftarrow{\lim_{\iota}} (Y_\iota, \sigma),$$

where for each index ι, (Y_ι, σ) is a subshift over a finite alphabet. In this case, applying the outer variational principle, the tail entropy can be defined in terms of the topological fiber entropy:

$$\mathbf{h}^*(X) = \inf_{\iota} \sup_{\mu\in\mathcal{P}_T(X)} \mathbf{h}(X|\mu_\iota),$$

where μ_ι denotes the projection of μ onto the factor space Y_ι. The following definition sounds naturally in this context:

Definition 3.10.8. Let $\mu \in \mathcal{P}_T(X)$. The *tail entropy* of μ is defined as

$$\mathbf{h}^*(\mu) := \inf_{\iota} \mathbf{h}(X|\mu_\iota).$$

Our next result states that the "defect of upper semi-continuity" of the entropy function $h(\cdot)$ defined on $\mathcal{P}_T(X)$ is estimated by the tail entropy. Let

$$(\mathrm{h}^+ - \mathrm{h})(\mu) := \limsup_{\mu'\to\mu} \mathrm{h}(\mu') - \mathrm{h}(\mu), \quad \mu, \mu' \in \mathcal{P}_T(X).$$

Theorem 3.10.5. (see [5] for the first inequality) *For every $\mu \in \mathcal{P}_T(X)$ with $h(\mu) < \infty$,*

$$\mathbf{h}^*(X) \geq (h^+ - h)(\mu) \geq \mathbf{h}^*(\mu).$$

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3.11 Sébastien Ferenczi[4] *Structure of three-interval exchanges*

In this lecture we give a detailed analysis of the spectral and ergodic properties of a *symmetric three-interval exchange transformation* i.e., a three-interval exchange transformation T with probability vector $(\alpha, \beta, 1-(\alpha+\beta))$, $\alpha, \beta > 0$, and permutation $(3,2,1)$ [5] defined by

$$Tx = x + 1 - \alpha \quad \text{if} \quad x \in [0, \alpha[,$$

$$Tx = x + 1 - 2\alpha - \beta \quad \text{if} \quad x \in [\alpha, \alpha+\beta[,$$

$$Tx = x - \alpha - \beta \quad \text{if} \quad x \in [\alpha+\beta, 1[.$$

Our approach is mostly combinatorial and relies on arithmetic results and a combinatorial description of return words (with respect to the natural coding) to a special family of intervals. The aim of this study was to develop a theory for three-interval exchange transformations analogous to that developed by Morse-Hedlund, Coven-Hedlund, and Arnoux-Rauzy which links together the diophantine properties of an irrational number α, the ergodic dynamical properties of a circle rotation by angle α, and the combinatorial symbolic properties of a class of binary sequences known as the *Sturmian infinite words*: we define a new vectorial algorithm of simultaneous approximation for two real numbers and study its arithmetic properties. Then we use this algorithm to combinatorially characterize the symbolic sub-shifts canonically associated to three-interval exchanges. Finally we apply this description to solve long-standing problems on the spectral properties of three-interval exchanges.

[4] Joint work with Charles Holton (Berkeley) and Luca Zamboni (North Texas)

[5] All other permutations on three letters reduce the transformation to an exchange of two intervals.

It is well known that every three-interval exchange transformation is induced by a rotation on the circle, and some properties of three-interval exchange transformations are readily traced back to the underlying rotation. For instance, under the assumption that T satisfies the *infinite distinct orbit condition* of Keane, the system is known to be both minimal and uniquely ergodic. Also, in the case of three intervals, the associated surface (obtained by suspending an interval exchange transformation via so-called 'zippered rectangles' is nothing more than a torus devoid of singularities. Finally, our 2-dimensional vectorial algorithm, as it underlies the dynamics of a three-interval exchange, verifies only a *quadratic* form of Lagrange's Theorem: the algorithm is eventually periodic if and only if the parameters α and β lie in the same quadratic extension of $\mathbb{Q}$.

On the other hand, other more subtle spectral properties of three-interval exchange transformations appear *not* to be directly linked to the underlying rotation. These include for instance the existence and characterization of the eigenvalues of the associated unitary operator (in particular the weak mixing) and joinings (minimal self-joining and simplicity). Katok and Stepin have proved that almost all three-interval exchange transformations are weakly mixing.

In this lecture we obtain necessary and sufficient conditions on α and β for T to be weak mixing. These conditions unify all previously known examples, and show that the weak mixing comes from the presence of either a spacer above a column of positive measure (like for Chacon's map; this property was also the basis of the famed Ratner's R-property), or of an isolated spacer above a column of small measure (like for del Junco-Rudolph's map). In addition, we exhibit new interesting examples of weak mixing three-interval exchanges. The conditions stem from a combinatorial recursive construction for generating three sequences of nested Rokhlin stacks which describe the system from a measure-theoretic point of view, and which combined with a result of Choksi and Nadkarni in the class of *rank one* systems, provide an explicit computation of the eigenvalues.

While it is known that all three-interval exchange transformations are topologically weakly mixing, Veech proved the surprising existence of three-interval exchange transformations with eigenvalue $\lambda = -1$. This was later extended by Stewart who showed that for all rational numbers $\frac{p}{q}$ there exists a three-interval exchange transformation with eigenvalue $e^{\frac{2\pi pi}{q}}$. In this lecture we give a simple combinatorial process for constructing the three-interval exchange transformations of Veech and Stewart. In addition we exhibit examples of three-interval exchange transformations having a p-adic odometer as factor.

However the question concerning the existence of irrational eigenvalues remained unsolved, in spite of some partial results due to Merrill and Parreau. We are now able to give affirmative answers to two questions asked by Veech in 1984:

Theorem 3.11.1. *Let γ be an irrational number, $[0; y_1, y_2 \ldots]$ its usual continued fraction expansion, and q_k, $k \geq 1$ the denominators of its convergents, given by $q_{k+1} = y_{k+1}q_k + q_{k-1}$. If*

$$\sum_{k=1}^{+\infty} \frac{q_k}{y_{k+1}} < +\infty,$$

then there exists a symmetric three-interval exchange transformation, satisfying the i.d.o.c. condition, which is measure-theoretically isomorphic to the rotation of angle γ, and hence has discrete (pure point) spectrum.

We also show

Theorem 3.11.2. *For every quadratic irrational number γ there exists a symmetric three-interval exchange transformation, satisfying the i.d.o.c. condition, with eigenvalue $e^{2\pi i\gamma}$.*

Theorems 1 and 2 are extremes of one another in that in one case the partial quotients tend to infinity very quickly, while in the other they are eventually periodic.

Theorems 1 and 2 suggest that not all properties of a three-interval exchange transformation can be traced back to the underlying rotation: the irrational rotation by angle γ of Theorem 1 has no connection with the underlying rotation inducing the interval exchange, and in the case of Theorem 2 the three-interval exchange transformation has as factor a rotation with a quadratic angle, while the angle of the inducing rotation is a Liouville number.

Among the remaining open questions, we formulate the following:

Problem 3.11.1. We do not know whether every complex number of modulus 1 is an eigenvalue of some nondegenerate three-interval exchange transformation. The question to know whether a nondegenerate three-interval exchange transformation may have k rationally independent irrational eigenvalues was recently solved by Guenais and Parreau for $k = 2$ but remains open for $k \geq 3$.

3.12 Doris Fiebig *Factor theorems for Markov shifts*

We study the question when is there a factor map from a locally compact transitive Markov shift S onto a locally compact transitive Markov shift T. Here by a factor map we mean a continuous shift commuting onto map. There is a trivial necessary periodic point condition for the existence of a factor map. So we consider only Markov shifts S, T which satisfy this trivial periodic point condition. In case that S and T are both compact then a necessary condition for the existence of a factor map is that the topological entropy does not increase, $h_{\text{top}}(S) \geq h_{\text{top}}(T)$. And $h_{\text{top}}(S) > h_{\text{top}}(T)$ is sufficient, [1]. If one relaxes the compactness assumption on S then surprisingly the entropy condition vanishes completely: If S is a non-compact (but locally compact transitive) Markov shift and T is compact transitive then there is always a factor map from S onto T, [2]. The case that S or T is compact is thus satisfactorily solved.

We consider here the case that both, S and T are non-compact (but locally compact transitive). In this setting there are various necessary conditions for the existence of a factor map. Assume there is a factor map from S onto T. If S has a rome, that means there is a compact open set K in S such that for every point in S visits the set K at least once, then also T has a rome. Similarly, we say S is periodic at infinity, if there is a compact open set K such that the points in S which never visit K are contained in the periodic points of S. Again, if S is periodic at infinity, then so is T. A property which lifts under factor map is the following. We say that S is big at infinity if there is an uncountable set A of points in S such that for every compact open set K in S there is some integer N such that $S^n A \cap K = \emptyset$ for all n with $|n| \geq N$.

Having these necessary conditions for the existence of a factor map from S onto T we restrict first to the case that S is big at infinity and that T is mixing with a rome. We prove that then a factor map from S onto T exists iff the trivial periodic point condition is satisfied. We obtain two interesting applications: There are two mixing locally compact Markov shifts being a factor of each other but having distinct Gurevic entropy (the "loop counting entropy" defined in [5]), and using a result from [3] we can show that there is a mixing locally compact Markov shift having an endomorphism which is uncountable-to-1.

Then we extend the class of Markov shifts T by considering now T which are periodic at infinity. Then a new periodic point condition at infinity arises, and we give a complete description when a factor map from S onto T exists, in case that S is big at infinity and T is mixing and periodic at infinity.

Finally we consider the other extremal case that S has a rome. We show that in this situation an entropy condition restores, and also return time constraints arise.

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3.13 Ulf Fiebig[6]

Pressure and equilibrum states for countable state Markov shifts

We consider continuous real valued functions f on transitive two-sided countable state Markov shifts (X, S) given in the graph presentation. We introduce topological pressure, prove a variational principle and study the existence of equilibrium states.

[6]Joint work with Doris Fiebig and Michiko Yuri

Our methods avoid the use of the Perron-operator, this enables us to study the thermodynamic formalism in situations where this operator is not well defined.

We give a general definition of the topological pressure $P_{\text{top}}(f, S)$ which extends the notion from the compact setting. For functions f which satisfy a mild distortion property, namely $D_n(f) = O(1/n)$, the pressure $P_{\text{top}}(f, S)$ is determined by the values of f on periodic points which visit a fixed finite path in the defining graph of the Markov shift. Functions where the values depend only on the zero-coordinate or more generally Hölder continuous functions always have bounded distortions, $\sup_n D_n(f) < \infty$, in particular they satisfy the mild distortion property. This implies that our notion of topological pressure also extends those introduced by Sarig [3] for Hölder continuous functions and by Gurevic and Savchenko [2] for functions where the values depend only on the zero-coordinate.

We prove a variational principle for functions satisfying the mild distortion property. To study the existence of equilibrium state we introduce a new notion of positive recurrence for functions with bounded distortions, $\sup_n D_n(f) < \infty$. The function $f = 0$ is positive recurrent in our sense iff the Markov shift (X, S) is positive recurrent in the classical sense.

To construct equilibrium states we rather follow the classical approach in using point masses on periodic points than the ideas of Sarig, which use the Ruelle-Perron-Frobenius-operator, or of Gurevic and Savchenko, which use matrix techniques. For positive recurrent functions we construct a sequence (μ_n) of probability measures, each concentrated on finitely many periodic points. Our notion of positive recurrence is designed to make this sequence of measures (μ_n) tight. Thus any subsequence has a weak limit μ. The next essential point is that the positive recurrence implies that a certain set of closed open pairwise disjoint sets built from first return loops is actually an almost everywhere partition of the space X with respect to every weak limit μ. We then show that a weak limit μ is an equilibrium state for f if and only if $\int f^- \, d\mu < \infty$. In particular, any such function which is bounded from below has an equilibrium state. If f is bounded from above, a modified version of this construction makes to easier to check conditions for the existence of equilibrium states for a large class of Markov shifts, which include the Bernoulli shift. Finally we extend our results to continuous functions on one-sided Markov shifts, give an application from number theory and give an example how our results extend the work of Sarig and that of Gurevic and Savchenko.

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3.14 Alan Forrest
Dynamics on ordered Cantor sets

We introduce ordered dynamics with the following elementary definition and a theorem that admits many generalizations:

Definition 3.14.1. Let I be the unit interval with unit Lebesgue measure and the usual total ordering, $<$. An aperiodic point, $x \in I$, defines a total order, $<_x$, on $\mathbb{N}$: $n <_x m$ if and only if $T^n x < T^m x$ (here $\mathbb{N}$, the natural numbers, include 0).

Theorem 3.14.1. *If (I, T) is ergodic, Lebesgue measure preserving, then there is a subset B of I of full measure such that $\phi : x \mapsto <_x$ is 1-1 when restricted to B.*

Thus we encode almost every point in I with an order on $\mathbb{N}$. Coding by order opens an interesting alternative to classical symbolic coding. Note that the issues of expansiveness, generation and finite entropy make no appearance. The remainder of this note, which summarizes the work in [3], takes this approach to the topological category.

Definition 3.14.2. Suppose that X is a Cantor subset of $\mathbb{R}$ inheriting an order $<$ from the usual order on $\mathbb{R}$. We call such an order *standard*; this structure can be defined intrinsically.

Say that a point $x \in X$ is a *left end* if $[x, y) \cap X$ is a neighbourhood of x for any $y > x$. Write X^- for the set of left ends; it is countable and infinite. The unique minimal element, x^-, is a left end.

Examples include certain Cantor extensions or subsystems of classical 1-dimensional maps such as rotations on the circle, the unimodal map or more general maps. The classical Feigenbaum classification of transition to Chaos is expressed in terms of the order $<_x$ where x is the critical point. Recently Blokh and Misiurewicz [1,2] have considered interval maps using the canonical ordering, generalizing the idea of rotation number and analyzing sequences of runs in $<_x$.

Lexicographical order on $A^{\mathbb{N}}$ or $A^{\mathbb{Z}}$ (using some well-ordering on $\mathbb{Z}$), where A is an ordered finite set, is standard. Thus every Cantor subshift, one-sided or two-sided, supports a natural standard order.

We consider Cantor systems (X, T) here in which T is invertible and aperiodic. This is not the full generality possible [3].

Definition 3.14.3. Let Ω be the set of all orders on $\mathbb{Z}$. Define $<^{(n)}$ to be the restriction of $< \in \Omega$ to $I_n = \mathbb{Z} \cap [-(n-1)/2, n/2]$ (an interval of n integers).

Ω is naturally a Cantor set with basis of clopen sets $U(<', n) = \{ < \in \Omega : <^{(n)} = <'^{(n)} \}$. For each $n \geq 1$, we write $\mathcal{Q}_n$ for the partition of Ω defined by clopen sets $U(<, n)$. Ω also supports a homeomorphism shift map, S: $n(S <)m$ if $(n+1) < (m+1)$.

In an invertible aperiodic Cantor system (X, T), we can extend Def 1 to make $<_x$ an order on $\mathbb{Z}$, and redefine $\phi : X \longrightarrow \Omega$ as $\phi(x) = <_x$.

Theorem 3.14.2. *Suppose that $(X, T, <)$ is a standard ordered Cantor system then ϕ is continuous and equivariant with respect to the dynamical actions of T and S.*

If $\{T^n x^- : n \in \mathbb{Z}\} = X^-$, then ϕ is 1-1 on the set of transitive points. In particular, if (X, T) is also minimal then ϕ embeds (X, T) as a subsystem of (Ω, S).

Simple counter-examples show that a condition on X^- such as the one above is necessary. It is not hard to show that every transitive Cantor system has a standard order which has this condition. When $\phi : X \longrightarrow \Omega$ is an injection, we call such a set-up *generating*.

If $<$ is a standard order on (X, T), then we can define, for each $n \geq 1$, a clopen partition $\mathcal{P}_n$ of X according to the value of the function, $x \mapsto <_x^{(n)}$, i.e. $\mathcal{P}_n = \phi^{-1}(\mathcal{Q}_n)$.

In our analogue with symbolic coding, the sequence of partitions $\mathcal{P}_n$ corresponds to $\bigvee_{i \in I_n} T^{-i}\mathcal{P}$ where $\mathcal{P}$ is a generating partition. The growth of $|\mathcal{P}_n|$, the number of atoms in $\mathcal{P}_n$, is an alternative measure of the complexity or entropy of the system. The fact that $1 \leq |\mathcal{P}_n| \leq n! = \exp(n \log n + O(n))$ allows room for super-exponential growth and the classification of infinite entropy systems.

Zero entropy example *If $(X, T, <)$ is the Morse-Thue in its usual representation in $\{0, 1\}^{\mathbb{Z}}$, then the lexicographical order is generating. In this case $|\mathcal{P}_n|$ is bounded linearly in n, above and below.*

Theorem 3.14.3 (Finite entropy cases). *Suppose that (X, T) is an aperiodic transitive Cantor system. If $<$ is a generating standard order, then $h_{top}(X, T) = \lim_n(\log|\mathcal{P}_n|)/n$.*

Theorem 3.14.4. *If (Y, S) is a transitive Cantor subsystem of (Ω, S), then there is an aperiodic generating standard ordered Cantor system $(X, T, <)$ for which $\log|\mathcal{P}_n| = \log c_n + o(n)$, where c_n is the number of atoms of $\mathcal{Q}_n$ intersected by Y.*

Infinite order entropy examples *For $\alpha = 1$ or $1/2$, there are aperiodic standard ordered Cantor systems $(X, T, <)$ for which $\lim_n(\log|\mathcal{P}_n|)/n \log n = \alpha$.*

The example for $\alpha = 1/2$ is given by Theorem 7 applied to the S-invariant subset, Y, of Ω which excludes all those orders, $<$, that have some n such that $n < (n+1) < (n+2)$: a natural ordered dynamical generalization of subshift of finite type. I am grateful to Bernard Host for suggesting this example.

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3.15 Krzysztof Frączek *Classification of diffeomorphisms on the torus*

Let M be a compact Riemannian smooth manifold and μ its probability Lebesgue measure. Let $f : (M, \mu) \to (M, \mu)$ be a smooth measure-preserving ergodic diffeomorphism. An important question of smooth ergodic theory is: what is the relation between asymptotic properties of the sequence $\{Df^n\}_{n\in\mathbb{N}}$ and dynamical or spectral properties of the dynamical system $f : (M, \mu) \to (M, \mu)$. There are results well describing this relation in the case where M is the torus. For example, if a diffeomorphism f is homotopic to the identity and the sequence $\{Df^n\}_{n\in\mathbb{N}}$ is uniformly bounded, then f is C^0-conjugate to an ergodic rotation (see [3] p. 181). Hence f has purely discrete spectrum. Moreover, if $\{Df^n\}_{n\in\mathbb{N}}$ is bounded in the C^r-norm ($r \in \mathbb{N} \cup \{\infty\}$), then f and the ergodic rotation are C^r-conjugated (see [3] p. 182). On the other hand, if $\{Df^n\}_{n\in\mathbb{N}}$ has "exponential growth", precisely if f is an Anosov diffeomorphism, then it is metrically isomorphic to a Bernoulli shift (see [6]). Hence f has countable Lebesgue spectrum. Moreover, f is C^0-conjugate to an algebraic automorphism of the torus (see [5]).

A natural question is: what can happen between the above extreme cases? We would like to propose the following definition, which was introduced in [1].

Definition 3.15.1. We say that the derivative of a smooth diffeomorphism $f : M \to M$ has *β-strong polynomial growth* ($\beta > 0$) if the sequence $\{\frac{1}{n^\beta} Df^n\}_{n\in\mathbb{N}}$ converges μ-a.e. to a measurable μ-nonzero function. We will use the word *linear* instead of *1-polynomial.*

Let us fix our attention on diffeomorphisms of 2-dimensional torus $\mathbb{T}^2$. One of the examples of ergodic measure-preserving diffeomorphisms with linear growth of the derivative is any skew product of any irrational rotation on the circle and any circle smooth cocycle with nonzero degree. Let $\alpha \in \mathbb{T}$ be an irrational number and let $\varphi : \mathbb{T} \to \mathbb{T}$ be a C^1-cocycle. We denote by $d(\varphi)$ the topological degree of φ. Consider the skew product $T_{\alpha,\varphi} : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ defined by

$$T_{\alpha,\varphi}(x_1, x_2) = (x_1 + \alpha, x_2 + \varphi(x_1)).$$

Observe that

$$\frac{1}{n} DT^n_{\alpha,\varphi}(x_1, x_2) = \begin{bmatrix} \frac{1}{n} & 0 \\ \frac{1}{n}\sum_{k=0}^{n-1} D\varphi(x_1 + k\alpha) & \frac{1}{n} \end{bmatrix}.$$

By the Ergodic Theorem, the sequence $\frac{1}{n}\sum_{k=0}^{n-1} D\varphi(\,\cdot\, + k\alpha)$ converges uniformly to the number $\int_{\mathbb{T}} D\varphi(x)dx = d(\varphi)$. Therefore the sequence $\frac{1}{n} DT^n_{\alpha,\varphi}$ converges uniformly to the nonzero matrix $\begin{bmatrix} 0 & 0 \\ d(\varphi) & 0 \end{bmatrix}$. The following result is proved in [1].

Theorem 3.15.1. *Every ergodic measure-preserving C^1-diffeomorphism of $\mathbb{T}^2$ with strong linear growth of the derivative is algebraically conjugated (i.e. by a group automorphism) to a skew product of an irrational rotation on $\mathbb{T}$ and a circle C^1-cocycle with nonzero degree. Moreover for no positive real $\beta \neq 1$ does an ergodic measure-preserving C^2-diffeomorphism of polynomial strong growth of the derivative with degree β exist.*

Moreover, every skew product of an irrational rotation on the circle and a circle C^2–cocycle with nonzero degree has countable Lebesgue spectrum on the orthocomplement of the space of functions depending only on the first variable (see [4]). It follows that every measure–preserving, ergodic diffeomorphism with the above–mentioned linear growth of the derivative has countable Lebesgue spectrum on the orthocomplement of its eigenfunctions.

Now I would like to propose a seemingly weaker and more natural definition of the linear growth of the derivative.

Definition 3.15.2. We say that the derivative of a smooth diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ has *linear growth* if there exist positive constants c, C such that

$$0 < c \leq \frac{1}{n}\|Df^n(\bar{x})\| \leq C \tag{3.1}$$

for every $\bar{x} \in \mathbb{T}^2$ and $n \in \mathbb{N}$.

This definition has a nice property, because the linear growth of the derivative is invariant under the relation of smooth conjugation. Of course, if $d(\varphi) \neq 0$, then $T_{\alpha,\varphi}$ has linear growth of the derivative. Moreover, it is easy to check that if φ is of class C^3, then the sequence $\frac{1}{n}DT^n_{\alpha,\varphi}$ is bounded in $C^2(\mathbb{T}^2, M_2(\mathbb{R}))$. The following result is proved in [2]

Theorem 3.15.2 (Main Theorem). *Let $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ be a measure-preserving C^3-diffeomorphism. Suppose that*

- *f is ergodic,*
- *f has linear growth of the derivative,*
- *the sequence $\{\frac{1}{n}Df^n\}_{n\in\mathbb{N}}$ is bounded in $C^2(\mathbb{T}^2, M_2(\mathbb{R}))$.*

Then f is algebraically conjugate to a skew product of an irrational rotation on the circle and a circle C^3-cocycle with nonzero degree.

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3.16 Sergey Gefter *On dense embeddings of discrete groups into locally compact groups and on associated with them equivalence relations*[7]

Let G be a continuous locally compact second countable group with left Haar measure μ, and Γ be a countable dense subgroup of G. Then (G, μ) is a standard measure space and Γ acts on (G, μ) by left translations: $g \overset{\gamma}{\longmapsto} \gamma g,\ \gamma \in \Gamma,\ g \in G$. The action of Γ on (G, μ) is ergodic because Γ is dense in G. Let us denote the Γ-orbital equivalence relation by R_Γ. If G is amenable as a discrete group (for example, if G is Abelian or is solvable), the equivalence relation R_Γ is amenable too. That is why this case is not especially interesting from the orbital theory point of view. The opposite case in partially described with the following important Zimmer's theorem.

Theorem 3.16.1. (R. Zimmer, 1987) *If G is a connected non-solvable Lie group, then R_Γ is non-amenable.*

A lot of interesting questions on the equivalence relation R_Γ arise in a non-amenable case. In the present talk we would like to consider only one of them, namely the calculation of the fundamental group of R_Γ in some particular cases when Γ acts by translations on a non-compact group G.

Now let G be a non-compact locally compact group. Denote by $\mathrm{Aut}(R_\Gamma)$ the automorphism group of R_Γ. It is well known that if $\theta \in \mathrm{Aut}(R_\Gamma)$ then there exists $\lambda = \mathrm{mod}\ \theta > 0$, such that $\mu \circ \theta = \lambda\mu$.

Definition 3.16.1. The fundamental group of the equivalence relation R_Γ is the following subgroup in $\mathbb{R}^*_+$:

$$F(R_\Gamma) = \{\mathrm{mod}\,\theta \mid \theta \in \mathrm{Aut}(R_\Gamma)\}.$$

The definition and properties of the fundamental group of type II_1 equivalence relations see in [1] and [2, §2].

If the equivalence relation R_Γ is amenable, then $F(R_\Gamma) = \mathbb{R}^*_+$.

1. Actions of the irreducible lattices

[7] The work is partially supported by INTAS grant 97-1843

For each prime $p \in \mathbb{Z}$ let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $\mathbb{Q}_\infty = \mathbb{R}$. Let $V = \{primes\ in\ \mathbb{Z}\} \cup \{\infty\}$, and $p_1, p_2, \dots, p_m \in V$. Suppose that for each p_i, H_{p_i} is a connected almost $\mathbb{Q}$-simple linear algebraic $\mathbb{Q}$-group such that the group $H_{p_i}(\mathbb{Q}_{p_i})$ is not compact. Let $B = \prod_{i=1}^{m} H_{p_i}(\mathbb{Q}_{p_i})$, so that B is a locally compact non-compact group. Now let us suppose that Λ is an irreducible lattice and we fix some non-trivial subset $I_0 \subset \{1, \dots, m\}$. We let $G = \prod_{i \in I_0} H_{p_i}(\mathbb{Q}_{p_i})$ and $\Gamma = \pi_{I_0}(\Lambda)$, where $\pi_{I_0} : B \to G$ is a projection onto G. Then G is a locally compact non-compact group and Γ is a dense subgroup in G.

Theorem 3.16.2. *Suppose* $\sum_{i \notin I_0} \mathbb{Q}_{p_i} - \mathrm{rank}(H_{p_i}) \geq 2$. *Then all automorphisms of equivalence relation* R_Γ *preserve Haar measure, i.e.* $F(R_\Gamma) = \{1\}$.

The proof of this theorem uses Zimmer's rigidity theorem for ergodic actions of semisimple groups and the notion of the fundamental group for an ergodic action of a continuous locally compact group [3].

Example 1.1. We let $\mathbb{Z}[\sqrt{2}] = \{m + n\sqrt{2} \mid m, n \in \mathbb{Z}\}$ and $\Gamma = \mathrm{SL}_n(\mathbb{Z}[\sqrt{2}])$. Then Γ is isomorphic to the irreducible lattice in $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$. Consider the equivalence relation R_Γ generated by left translations of Γ on $G = \mathrm{SL}_n(\mathbb{R})$. If $n \geq 3$, then all automorphisms of R_Γ preserve Haar measure on G.

Example 1.2. Let $S = \{p_1, p_2, \dots, p_m\}$ be a finite set of primes. We denote by $\mathbb{Z}[S]$ the subring of $\mathbb{Q}$ generated by the elements $\frac{1}{p_1}, \dots, \frac{1}{p_m}$. Let $\Gamma = \mathrm{SL}_n(\mathbb{Z}[S])$, $n \geq 3$. Identify Γ with its image under the diagonal embeddings in $B = \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_{p_1}) \times \dots \times \mathrm{SL}_n(\mathbb{Q}_{p_m})$. Then Γ is an irreducible lattice in B. Consider the equivalence relation R_Γ generated by translations of Γ on $G = \mathrm{SL}_n(\mathbb{R})$. Then all automorphisms of R_Γ preserve Haar measure on G.

2. Actions of the groups $\mathbb{Q}$-rational points

Let H be a connected semisimple linear algebraic $\mathbb{Q}$-group, and let p be a prime. We let $G = H(\mathbb{Q}_p)$ and $\Gamma = H(\mathbb{Q})$. Assume that H is a simply connected, almost $\mathbb{Q}$-simple and the groups $H(\mathbb{R})$ and $H(\mathbb{Q}_p)$ are non-compact. Then G is a locally compact group and Γ is a dense subgroup in G.

Theorem 3.16.3. *Suppose that* $H(\mathbb{R})$ *has Kazhdan's property (T). Then all automorphisms of equivalence relation* R_Γ *preserve Haar measure on* G.

Example. Let $G = \mathrm{SL}_n(\mathbb{Q}_p)$, $\Gamma = SL_n(\mathbb{Q})$, $n \geq 3$. Then $F(R_\Gamma) = \{1\}$.

3. Countable groups which do not have ergodic actions by translations on non-compact locally compact groups

It is well known that the group of integers cannot be densely embedded into a non-discrete non-compact locally compact group (in other words, a monothetic locally compact group is either compact or isomorphic to $\mathbb{Z}$).

Definition 3.16.2. We say that a discrete group Γ has Z-property if Γ cannot embedded densely into a continuous locally compact non-compact group.

The following theorem provides a complete description of the Abelian groups with Z-property.

For a prime p we let

$$\mathbb{C}_{p^\infty} = \{z \in \mathbb{T} \mid z^{p^n} = 1 \quad \text{for some } n\}.$$

Theorem 3.16.4 (Kulagin, Gefter, 1999). *1. The group $\mathbb{C}_{p^\infty}$ has Z-property. 2. Let Γ be an infinite discrete Abelian group with Z-property. Then either $\Gamma \cong \mathbb{Z} \times F$ or $\Gamma \cong \mathbb{C}_{p^\infty} \times F$, where F is a finite Abelian group.*

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3.17 Eli Glasner[8]
The topological Rokhlin property and topological entropy

Abstract

For a compact metric space X let $G = H(X)$ denote the group of self homeomorphisms with the topology of uniform convergence. The group G acts on itself by conjugation and we say that X satisfies the topological Rokhlin property if this action has dense orbits. We show that the Hilbert cube, the Cantor set and, with a slight modification, also even dimensional spheres, satisfy this property. We also show that zero entropy is generic for homeomorphisms of the Cantor set, whereas it is infinite entropy which is generic for homeomorphisms of cubes of dimension $d \geq 2$ and the Hilbert cube.

Introduction

Let $(X, \mathcal{X}, \mu, T)$ be an aperiodic probability measure preserving system with μ non-atomic. Given $\epsilon > 0$ and a positive integer n, Rokhlin's lemma tells us that there is a measurable subset $A \subset X$ such that $A, TA, T^2A, \ldots, T^{n-1}A$ are disjoint and cover X up to a set of measure less than ϵ. This simple lemma is an essential tool in ergodic theory. It is used in one way or another in most aspects of this theory. One well known consequence of it is the following.

[8]Joint work with B. Weiss

Theorem 3.17.1. *For a non-atomic probability space $(X, \mathcal{X}, \mu)$ let G be the Polish group of measure preserving transformations with a measurable inverse, equipped with the weak topology. Then the action of the group G on itself by conjugation is topologically transitive; i.e. there exists a transformation $T \in G$ such that the set $\{STS^{-1} : S \in G\}$ is dense in G.*

One can consider a more general situation where a (say countable discrete) group Γ acts by measure preserving transformations on a probability space $(X, \mathcal{X}, \mu)$. Again the space $\mathbb{A} = \mathbb{A}_\Gamma$ of all such Γ-actions can be endowed with the weak topology, making it a Polish space, and the group G of all bi-measure-preserving-transformations of $(X, \mathcal{X}, \mu)$, acts on $\mathbb{A}$ by conjugation. In [2] the following definition was introduced. Say that the group Γ has the *Rokhlin property* if the action of G on $\mathbb{A}_\Gamma$ is topologically transitive. It is observed there that every amenable Γ has the Rokhlin property, and the question which groups have the Rokhlin property is raised. (See [2] for more details).

In the present work we are dealing with an analogous question in the topological context. For a compact metric space X, denote the group of self homeomorphisms of X by $G = H(X)$. With the topology of uniform convergence, G is a Polish topological group.

We say that a Polish topological group G has the *topological Rokhlin property* (or just the Rokhlin property) when it acts transitively on itself by conjugation. We say that the space X has the *Rokhlin property* when $G = H(X)$ has the Rokhlin property; i.e. $H(X)$ is the closure of a single conjugacy class. Which compact metric spaces poses the Rokhlin property? We show that the Hilbert cube and the Cantor set have it. For some connected spaces like spheres the existence of orientation of a homeomorphism, which is clearly preserved under conjugation, means that $H(S^d)$ can not have the Rokhlin property; therefore we say that a sphere satisfies the Rokhlin property when the group $H_0(S^d)$ — the connected component of the identity in $H(S^d)$ — has the Rokhlin property. With this definition we show that even dimensional spheres have the Rokhlin property. On the other hand it appears that for general compact manifolds of positive finite dimension the answer is rather different. For circle homeomorphisms, Poincaré's rotation number, $\tau : H^+(S^1) \to \mathbb{R}/\mathbb{Z}$, $h \mapsto \tau(h)$, where $H^+(S^1) = H_0(S^1)$ is the subgroup of index 2 of orientation preserving homeomorphisms, is a continuous conjugation invariant and thus there are at least a continuum of different closed disjoint conjugation invariant subsets.

We refer the reader to the recent paper [1], by E. Akin, M. Hurley and J. Kennedy, for a detailed discussion of circle homeomorphisms. Their main result (on the circle) can be briefly formulated by saying that the circle has the local Rokhlin property, where a space X has the *local Rokhlin property* if $H(X)$ contains an open dense subset which is the union of interior of conjugacy class closures. More precisely they show that for a rational number c, the set $\tau^{-1}(c)$ has a nonempty interior in $H^+(S^1)$ and that in each such set $\tau^{-1}(c)$ there is a — necessarily unique — residual $H^+(S^1)$ conjugacy class. On the other hand for irrational rotation numbers we have the following information. Denote by $\mathcal{T}$ the set of topologically transitive homeomorphisms of S^1, then $\mathcal{T}$ is a G_δ subset of $H^+(S^1)$ on which τ takes irrational values and for an irrational number c the set $\mathcal{T} \cap \tau^{-1}(c)$ *is* the conjugacy class of the "rigid" rotation h_c. It is also easy to see that no odd dimensional sphere has the Rokhlin property[9].

The motivation for the definition of the Rokhlin property came from the work [2]. The Hilbert cube case was done during the special year in ergodic theory at the Institute for Advanced Studies of the Hebrew University in Jerusalem, 1996-7. The question regarding the Cantor set was raised recently by J. King and was answered independently by E. Akin, ([1]).

A related problem is the question: what is the topological entropy of the typical homeomorphism in $H(X)$? The machinery we develop for dealing with the Rokhlin property, enables us to answer the entropy problem as follows. For the Hilbert cube and spheres S^d, $d \geq 2$, the set of homeomorphisms with infinite entropy is residual while for the Cantor set it is the set of zero entropy which is a dense G_δ subset of $H(X)$.

The Hilbert cube is dealt with in section 1, the Cantor set in section 2 and in section 3 we consider finite dimensional cubes and spheres. We wish to thank E. Akin for his helpful comments and for supplying information concerning the Annulus conjecture.

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3.18 Gernot Greschonig[10]
Invariant cocycles have Abelian ranges

Let $(X, \mathcal{S})$ be a standard Borel space and $\mathrm{Aut}(X, \mathcal{S})$ the group of Borel automorphisms of X. A Borel set $R \subset X \times X$ is a *discrete Borel equivalence relation* on X if R is an equivalence relation whose *equivalence classes* $R(x) = \{y \in X : (x, y) \in R\}$, $x \in X$, are all countable.

Now let R be a discrete Borel equivalence relation on X. The *full group* $[R]$ of R is the group of all $W \in \mathrm{Aut}(X, \mathcal{S})$ with $Wx \in R(x)$ for every $x \in X$. For any countable group $\Gamma \subset \mathrm{Aut}(X, \mathcal{S})$ we denote by $R[\Gamma]$ the discrete Borel equivalence relation $\{(x, \gamma x) : x \in X, \gamma \in \Gamma\}$.

We denote by

$$\mathrm{Aut}(R) = \{V \in \mathrm{Aut}(X, \mathcal{S}) : (Vx, Vy) \in R \text{ if and only if } (x, y) \in R\} \qquad (3.1)$$

[9]For the proof it suffices to note that there are orientation preserving homeomorphisms with an attracting fixed point — hence all small perturbations have a fixed point, while there are orientation preserving homeomorphisms with no fixed points — and any small perturbation will not have one either.

[10]Joint work with Klaus Schmidt

the *automorphism group* of R. If μ is a probability measure on $\mathcal{S}$ which is quasi-invariant under R we set

$$\operatorname{Aut}(R,\mu) = \{V \in \operatorname{Aut}(R) : \mu \text{ is quasi-invariant under } V\}. \tag{3.2}$$

Definition 3.18.1. Let R be a discrete Borel equivalence relation on a standard Borel space $(X,\mathcal{S})$ and μ a probability measure on $\mathcal{S}$ which is quasi-invariant under R. An element $V \in \operatorname{Aut}(R,\mu)$ is *weakly asymptotically central* if it preserves μ and

$$\lim_{|n|\to\infty} \mu(B \triangle V^n W V^{-n} B) = \lim_{|n|\to\infty} \mu(V^{-n} B \triangle W V^{-n} B) = 0 \tag{3.3}$$

for every $W \in [R]$ and $B \in \mathcal{S}$. The automorphism V is *strongly asymptotically central* if it preserves μ and

$$\lim_{|n|\to\infty} \mu(\{x \in X : WV^{-n}W'V^n x = V^{-n}W'V^n W x\}) = 1 \tag{3.4}$$

for all $W, W' \in [R]$.

Remark 3.18.1. The terminology chosen here is consistent with the weak and strong topology on the set of ergodic transformations: if V is a weakly asymptotically central automorphism of (R,μ), then W and $V^{-n}W'V^n$ commute asymptotically in the weak topology. If V is strongly asymptotically central, then W and $V^{-n}W'V^n$ commute asymptotically in the strong topology.

Proposition 3.18.1. *Let R be a discrete Borel equivalence relation on a standard Borel space $(X,\mathcal{S})$ and μ a probability measure on $\mathcal{S}$ which is quasi-invariant and conservative under R. Then every strongly asymptotically central automorphism V of (R,μ) is weakly asymptotically central.*

There exists a simple counterexample which shows that Proposition 3.18.1 fails without the hypothesis of conservativity. The following theorem is useful to construct examples of strongly asymptotically central automorphisms.

Theorem 3.18.1. *Let Γ be a countable abelian group of Borel automorphisms of a standard Borel space $(X,\mathcal{S})$, and let $V \in \operatorname{Aut}(X,\mathcal{S})$ with $V^{-1}\Gamma V \subset \Gamma$.*

Suppose that $R \subset R[\Gamma]$ is a V-invariant subrelation (i.e. $V \in \operatorname{Aut}(R)$) and μ a V-invariant probability measure on $\mathcal{S}$ which is quasi-invariant under R. If $V \in \operatorname{Aut}(R,\mu)$ is weakly asymptotically central and mixing then it is strongly asymptotically central.

Remark 3.18.2. If the equivalence relation R in Theorem 3.18.1 is ergodic then every weakly asymptotically central automorphism V of (R,μ) is mixing (Theorem 2.3 in [2]).

Examples of strongly asymptotically central automorphisms are shift spaces with shift-invariant mixing probability measures and the *Gibbs relation*. Here we have to restrict the Gibbs relation to a shift invariant Borel set of full measure where the measure is quasi-invariant. Such a set exists according to Lemma 2.3 in [1], and we use the fact that the Gibbs relation on the full shift over $\mathbb{Z}/k\mathbb{Z}$) is generated by the

Abelian group $\sum_{\mathbb{Z}} \mathbb{Z}/k\mathbb{Z}$). Another example is the homoclinic equivalence relation on the k-torus with a hyperbolic automorphism. Furthermore there exist examples of weakly but not strongly asymptotically central automorphisms.

Now we want to turn to the ranges of invariant cocycles.

Definition 3.18.2. Let G be a Polish (i.e. complete separable metric) group with identity element $1 = 1_G$ and Borel field $\mathcal{B}_G$, R a discrete nonsingular equivalence relation on a standard probability space $(X, \mathcal{S}, \mu)$ and $V \in \mathrm{Aut}(X)$. A Borel map $c\colon R \longrightarrow G$ is a *cocycle* on R if

$$c(x, x')c(x', x'') = c(x, x'') \tag{3.5}$$

for every $(x, x'), (x, x'') \in R$. A cocycle $c\colon R \longrightarrow G$ is *V-invariant* if

$$c(Vx, Vy) = c(x, y) \tag{3.6}$$

for every $(x, y) \in R$.

Theorem 3.18.2. *Let R be a discrete nonsingular equivalence relation on a standard probability space $(X, \mathcal{S}, \mu)$, V an ergodic automorphism of (R, μ) which is both weakly and strongly asymptotically central, and $c : R \longrightarrow G$ a V-invariant cocycle with values in a Polish group G. Then there there exist a μ-null set $N \in \mathcal{S}$ and a closed Abelian subgroup $G_0 \subset G$ such that $c(x, y) \in G_0$ for every $(x, y) \in R_{X \smallsetminus N}$.*

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3.19 Stefan-M. Heinemann[11] *Geometric exponents for hyperbolic Julia sets*

Abstract

For hyperbolic rational maps we show that the Hausdorff dimension of the associated Julia set is bounded away from 2, where the bounds depend exclusively on certain intrinsic geometric exponents. This result is derived via lower estimates for the iterate-counting-function and for the dynamical Poincaré series. Subsequently, we deduce some interesting consequences, such as upper bounds for the decay of the area of parallel-neighbourhoods of the Julia set, and lower bounds for the Lyapunov exponents with respect to the measure of maximal entropy.

[11] Joint work with B. O. Stratmann

Structure of the talk

We consider Julia sets $J(T) \subsetneq \overline{\mathbb{C}}$ of hyperbolic holomorphic endomorphisms $T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Riemann sphere. It is well-known that these maps form a large open set inside the set of all holomorphic endomorphisms, and it is conjectured that they are in fact dense ('Fatou-conjecture'). Furthermore, for hyperbolic maps there already exists an affirmative answer to the 'analogue of the Ahlfors-conjecture for Kleinian groups', which asserts that the Julia set has always vanishing 2-dimensional Lebesgue measure. That this conjecture is true for hyperbolic rational maps is of course an immediate consequence of the well-known fact that in this case the Julia set is porous, and hence its Hausdorff dimension h is strictly less than 2 (see e.g. [2]).

We add to the latter result an estimate for h from above in terms of certain intrinsic parameters. Our estimate clearly reveals the geometric obstacles which prevent a hyperbolic Julia set from being of Hausdorff dimension 2. In order to state the estimate, let T have *critical distance* c (the distance of $J(T)$ to the forward orbit of the critical points of T), *core exponent* κ (the inverse of the maximal distortion of T on $J(T)$), and *inner lacunarity exponent* λ (that is roughly, the area of $U(J(T))/\langle T_*^{-1}\rangle$, for $U(J(T))$ denoting a suitable neighbourhood of $J(T)$ and $\langle T_*^{-1}\rangle$ the semigroup generated by the holomorphic inverse branches of T). Our main result relates these three geometric exponents to the Hausdorff dimension of $J(T)$. Namely with $U_c(J(T))$ denoting the c-neighbourhood of $J(T)$ we derive the formula

$$\boxed{h < 2 - \frac{2\,\lambda\,\kappa^{10}}{\operatorname{area}(U_c(J(T)))}.}$$

Originally, this type of formula arose from attempts to clarify the relationship between the spectrum of the Laplacian and certain intrinsic geometric quantities such as volume and length-spectrum for hyperbolic manifolds. For geometrically finite, infinite-volume hyperbolic manifolds a lower bound for the bottom of the Laplace-spectrum was obtained in [1] in terms of the convex core of the manifold (namely, its volume and the area of its boundary), which in certain cases leads to upper bounds for the Hausdorff dimension of the associated Kleinian limit sets. Recently, we derived by purely geometric means a similar type of formula for all convex cocompact Kleinian groups ([2]).

We adopt the geometric method of [2] and show how to adjust it to the setting of hyperbolic rational maps. For this we require the existence of geometrically well-behaved coverings of $J(T)$ and packings of the Fatou set $F(T) := \overline{\mathbb{C}} \setminus J(T)$. This allows us to introduce the concepts 'iterate-counting-function' and 'dynamical Poincaré series' (representing the natural analogues of the 'orbital counting function' and 'Poincaré series' for Kleinian groups). We derive a finer estimate for the h-conformal measure, and apply a folklore-lemma in elementary number theory, which in turn leads to some lower estimates for the iterate-counting-function and the dynamical Poincaré series. Subsequently, we interpret these estimates in terms of the fractality of $J(T)$ and the lacunarity of $F(T)$, which then gives our main result, the afore-stated formula for the Hausdorff dimension of $J(T)$.

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3.20 Teturo Kamae

Stochastic analysis based on deterministic Brownian motion

Deterministic Brownian motions are stochastic processes with noncorrelated, stationary and strictly ergodic increments having 0-entropy and 0-expectation. The self-similarity of order 1/2 follows from these properties. Such processes have a lot of variety and have different properties. It is not the case of the Brownian motion where the process is characterized as a process with stationary and independent increments with 0-expectation and standard variance.

Among the deterministic Brownian motions, the simplest one is the N-process which is introduced in [2] and studied in [1]. It comes from a piece-wise linear function consisting of 3 pieces with shape of letter "N" and is called an N-process and denoted by $\{\mathrm{N}_t;\ t \in \mathbf{R}\}$. It is a deterministic Brownian motion in the above sense together with the time reversibility.

We consider a process $Y_t = H(\mathrm{N}_t, t)$, where the function $H(x, s)$ is twice continuously differentiable in x and once continuously differentiable in s and $H_x(x, s) > 0$. The function H is considered completely unknown except for these properties. We want to predict the value Y_c from the observation $Y_J := \{Y_t;\ t \in J\}$, where $J = [a, b]$ and $a < b < c$. It is proved in [1] that there exists an estimator $\hat{Y}_c$ such that

$$\mathrm{E}[(\hat{Y}_c - Y_c)^2] = o((c-b)^2) + O(\frac{(c-b)^2}{b-a}). \tag{3.1}$$

One of the motivations of the research is given by Benoit B. Mandelbrot [3], who mentioned that the simulation of stock market by the Brownian motion contains too much randomness. Actual market has a strong negative correlation between the fluctuations of price on a day and the next day. He is suggesting to use the N-shaped function as the base of the simulation.

Our model has a lot of similarities to the Itô process. For example, we get a formula corresponding to the Itô formula. Nevertheless, there is a big difference between them. Our process has 0-entropy while the Itô process has ∞-entropy. Therefore, we have much better possibility of predicting the future. Theoretically, if we have complete information of the function H, and have complete data of Y_t in the past, we should be able to predict the future without error. But the actual setting is with the unknown function H and the limited observation Y_t for a bounded interval J. The best we can do is the order $O((c-b)^2)$ in the above estimate (1), while $O(c-b)$ in the case of Itô process.

The sample path of N-process repeats the N-function in various scales. The main idea for the prediction called *synchronization* is to find out the positions and the scales of the appearances of an N-function in the sample path. An N-function in the sample path is a part of bigger N-functions while containing smaller ones. Along the 3 line segments in an N-function, the sample path either increases at the first part, then decreases and increases, or decreases at the first part, then increases and decreases. Thus, it has a strong correlation along the synchronized intervals, while the process itself has noncorrelated increments.

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3.21 Gerhard Keller *Spectral perturbation theory for transfer operators of Anosov diffeomorphisms*

Outline

In this talk I reported about two related results:

I) G. Keller, C. Liverani, *Stability of the Spectrum for Transfer Operators*, Annali della Scuola Normale Superiore di Pisa, Scienze Fisiche e Matematiche, (4) **XXVIII**, (1999), pp. 141–152.

II) M. Blank, G. Keller, C. Liverani, *Perron-Frobenius spectrum for Anosov maps*, Preprint (2000), http://www.mi.uni-erlangen.de/~keller/publications/bkl2000.ps.gz

Motivation for I): Spectral theory of piecewise expanding interval maps

The statistical properties of a piecewise C^2 expanding map $T : [0,1] \to [0,1]$ are most effectively studied in terms of spectral properties of its associated transfer operator

$$P : L^1_{dx} \to L^1_{dx}, \quad \int Pf \cdot g\, dx := \int f \cdot (g \circ T)\, dx.$$

As an operator on L^1_{dx} its spectrum is always the full complex unit disk and carries no useful information. However it is known since long that it has the following nice property: For each $\sigma > \overline{\lim}_{n\to\infty} \left\| \frac{1}{|(T^n)'|} \right\|_\infty^{1/n}$ there are constants $C_1, C_2 > 0$ such that

$$\mathrm{var}(P^n f) \leq C_1\, \sigma^n\, \mathrm{var}(f) + C_2 \int |f|\, dx$$

where $\mathrm{var}(f)$ denotes the minimal variation of a function in the Lebesgue equivalence class of f. Let $V := \{f : \mathrm{var}(f) < \infty\}$. In terms of the two norms

$$|f| := \int |f|\, dx, \quad \|f\| := \mathrm{var}(f) + \int |f|\, dx$$

one thus has the following abstract setting:

$$|P^n f| \leq C_0\, |f|, \quad \|P^n f\| \leq C_1\, \sigma^n\, \|f\| + C_2\, |f|, \quad \{f \in V : \|f\| \leq 1\} \text{ is } |\,.\,| \text{ - compact} \tag{3.1}$$

where C_2 is not the same constant as before. In this abstract setting

– $(V, \|\,.\,\|)$ is a Banach space with a second norm $|\,.\,|$, $P : V \to V$ is linear, and (3.1) is satisfied – it is well known that

P is quasicompact and its essential spectral radius is bounded by σ.

Now let P_ε be a Markov operator describing a dynamically interesting perturbation of T with $P_0 = P$. This could be a small deterministic perturbation of T, a small stochastic perturbation of T or also an Ulam discretization of T. In all of these cases $\|P - P_\varepsilon\|, |P - P_\varepsilon| \gg 0$, so classical perturbation theory is not applicable to the isolated eigenvalues of P. But for this kind of perturbations we have $|||P - P_\varepsilon||| \leq \mathrm{const}\, \varepsilon$, where

$$|||P - P_\varepsilon||| := \sup \{|Pf - P_\varepsilon f| : f \in V, \|f\| \leq 1\}\ . \tag{3.2}$$

The main result of I): Stability of the spectrum

For $\lambda \in \mathbb{C}$ and $\delta > 0$ denote the spectral projections

$$\Pi_\varepsilon^{\lambda,\delta} := \frac{1}{2\pi i} \int_{|z-\lambda|=\delta} (z - P_\varepsilon)^{-1}\, dz\ .$$

Theorem 3.21.1. *The eigenvalues λ of $P = P_0$ with $|\lambda| > \sigma$ and their associated eigenprojections are stable under perturbations in the following sense: Suppose that (3.1) holds uniformly for a family $(P_\varepsilon)_{\varepsilon \geq 0}$ (and not only for $P = P_0$). Let $r \in (\sigma, 1)$ and let $\delta > 0$ be sufficiently small. Then there are $\varepsilon_0 > 0$ and $K > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ and for all $\lambda \in \mathbb{C}$ with $|\lambda| > r$*

$$|||\Pi_\varepsilon^{\lambda,\delta} - \Pi_0^{\lambda,\delta}||| \leq K \cdot |||P_\varepsilon - P_0|||^\eta$$

where $\eta := \log \frac{r}{\sigma} \sigma \log \frac{1}{\sigma} \in (0, 1)$.

The main result of II): Spectral and perturbation theory for Anosov diffeomorphisms

Now we consider C^2-Anosov diffeomorphisms $T : \mathbb{T}^d \to \mathbb{T}^d$. In order to fix the notation denote by $E^s(x)$ and $E^u(x)$ the stable and unstable distributions which have the property that there are constants $K > 0$ and $0 < \lambda_s < 1 < \lambda_u$ such that $\|(d_xT^n)_{|E^s(x)}\| \leq K\,\lambda_s^n$ and $\|(d_xT^{-n})_{|E^u(x)}\| \leq K\,\lambda_u^{-n}$ for all $x \in \mathbb{T}^d$ and all $n \geq 0$.

Recall that each individual stable (unstable) manifold $W^s(x)$ ($W^u(x)$) is an immersed C^2 submanifold of $\mathbb{T}^d$, but that the foliations $(W^s(x))_x$ and $(W^u(x))_x$ are only Hölder continuous in general. However, in case $d = 2$ they are always of class $C^{1+\alpha}$ for some $\alpha > 0$.

Given the constants λ_s, λ_u and α we fix two further auxiliary constants $\delta > 0$ (small) and $b > 0$ (large). For $0 < \beta \leq 1$ and $\varphi : \mathbb{T}^d \to \mathbb{R}$ we define a local β-Hölder constant in stable direction

$$H^s_\beta(\varphi) := \sup\left\{\frac{|\varphi(y) - \varphi(x)|}{d^s(y,x)^\beta} : y \in W^s(x), d^s(y,x) \leq \delta\right\} \tag{3.3}$$

($d^s(x,y)$ denotes the distance within $W^s(x)$) and a class of test functions

$$\mathcal{D}_\beta := \left\{\varphi : \mathbb{T}^d \to \mathbb{R} \text{ measurable} : |\varphi|_\infty \leq 1, H^s_\beta(\varphi) \leq 1\right\} .$$

In order to control f in the unstable direction we consider test vector fields $v : \mathbb{T}^d \to \mathcal{T}\mathbb{T}^d$, define $H^s_\beta(v)$ just as in (3.3) and denote

$$\mathcal{V}_\beta := \left\{v : \mathbb{T}^d \to \mathcal{T}\mathbb{T}^d : |v|_\infty \leq 1, H^s_\beta(v) \leq 1, v(x) \in E^u(x)\,\forall x\right\}$$

Fix now $0 < \beta < \gamma \leq \alpha$. For $f \in C^1(\mathbb{T}^d)$ this gives rise to the norms

$$\begin{aligned}
\|f\|_s &:= \sup\left\{\int_{\mathbb{T}^d} f\,\varphi\,dx : \varphi \in \mathcal{D}_\beta\right\}\\
\|f\|_u &:= \sup\left\{\int_{\mathbb{T}^d} d_xf(v(x))\,dx : v \in \mathcal{V}_\beta\right\}\\
\|f\| &:= \|f\|_u + b\,\|f\|_s\\
|f| &:= \sup\left\{\int_{\mathbb{T}^d} f\,\varphi\,dx : \varphi \in \mathcal{D}_\gamma\right\}
\end{aligned}$$

Denote by V the completion of $C^1(\mathbb{T}^d)$ w.r.t. the norm $\|\,.\,\|$.

Theorem 3.21.2. *Suppose that $d = 2$. For any $\sigma > \max\left\{\lambda_u^{-1}, \lambda_s^\beta\right\}$ there are constants $C_0, C_1, C_2 > 0$ such that*

1) $|P^nf| \leq C_0\,|f|$,

2) $\|P^nf\| \leq C_1\,\sigma^n\,\|f\| + C_2\,|f|$,

3) $\{f \in V : \|f\| \leq 1\}$ *is* $|\,.\,|$ *- compact.*

4) *In particular, $P : V \to V$ is quasicompact.*

5) *Let* $Q_\varepsilon f(x) := \int f(y) q_\varepsilon(x,y)\,dy$ *with some reasonable assumptions on the kernels* q_ε, *e.g.* $q_\varepsilon(x,y) = \varepsilon^{-d} q(\frac{y-x}{\varepsilon})$ *and consider* $P_\varepsilon := Q_\varepsilon \circ P$. *Then 1) and 2) hold for* P_ε *and* $|||P - P_\varepsilon||| \le \text{const}\ \varepsilon^{\gamma-\beta}$. *In particular, Theorem 1 applies.*

For $d > 2$ assertions 1) - 3) remain valid whith a different constant α. The results also carry over to other smooth compact manifolds than the torus.

3.22 Yuri Kifer[12]

Generating partitions for random transformations

The Kolmogorov–Sinai theorem says that the entropy of a measure preserving transformation is equal to its entropy with respect to any generating partition. Rokhlin showed in [11] that any ergodic transformation T of finite entropy has a countable generating partition whose entropy is arbitrarily close to the entropy of T. Krieger [7] proved that there exist finite generators with such properties and if the entropy of T is less than $\log \ell$ than there exists a generator of cardinality ℓ.

In the work [6] we study generating partitions in the relative ergodic theory setup. Here T is an invertible skew product transformation acting by $T(\omega, x) = (\vartheta\omega, T_\omega x)$ where ϑ is an invertible ergodic map of a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and T_ω, called often a random transformation, maps a Polish space X into itself. The relative measure entropy of T was first studied in [1] under the name "mixed entropy of the fiber of a skew product" where it was obtained as a difference of entropies of T and ϑ which makes sense when these entropies are finite but in many interesting cases both these entropies are infinite though the relative entropy is still finite. Later, a general setup was considered and other notions of the relativized ergodic theory such as the relative topological entropy and pressure were introduced and studied (see [9], [4], [2], [5]).

A relative version of the Kolmogorov–Sinai theorem is not difficult to prove (see, for instance, [4] and [2]) but the construction of relative generating partitions has not appeared in the literature so far. Of course, if the usual entropy of T is finite then Rokhlin's countable and Krieger's finite generators will serve as relativized generators, as well. However, the more precise analogues of Krieger's result which we obtain in this work, namely, that if T is ergodic and ϑ is aperiodic then there is a relative generating partition with ℓ elements provided that the relative entropy is less than $\log \ell$, cannot be obtained in this way whenever the usual entropy of ϑ is larger than $\log 2$. Moreover, if the usual entropy of T is infinite though the relative one is finite the known results cannot be employed, at all. Our relative result complements Krieger's theorem so that both together yield the general theorem on skew products. We stress that our relative results do not require extra assumptions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which would undermine the probabilistic philosophy behind random transformations. On the other hand, Rokhlin's countable and Krieger's finite generators theorems rely on separability and they are valid only for

[12] Joint work with Benjamin Weiss

Lebesgue spaces. Following the talk by the first author about this work at Toruń's conference in September 2000 A. Danilenko suggested another proof of our result above based on orbit equivalence considerations from [3] and [12] and valid also only for Lebesgue spaces.

The finite generator of the relative Krieger theorem provides a coding of the system for one ergodic invariant measure. In this work we are interested also in relative topological generators $\mathcal{Q}$ which are not only generating partitions for all $T-$invariant measures with the marginal $\mathbb{P}$ on Ω but also that $\mathcal{Q}$ separate orbits of T_ω in the sense that $\mathbb{P}-$almost surely (a.s.) for any $x, y \in X$, $x \neq y$ there exists $n \in \mathbb{Z}$ such that $T^n(\omega, x)$ and $T^n(\omega, y)$ belong to different elements of $\mathcal{Q}$. We introduce the notion of asymptotically entropy expansive random transformations (cf. [10]) which include as very particular cases expansive random transformations and those which have zero relative topological entropy. Assuming that ϑ is aperiodic, X is compact, T_ω are homeomorphisms and that the relative topological entropy of T is less than $\log \ell$ for an integer $\ell > 1$ we construct for asymptotically entropy expansive random transformations relative topological generators $\mathcal{Q}$ with ℓ elements. Actually, we construct such partitions in a more general situation when T is defined on a $T-$invariant set $\mathcal{E} \subset \Omega \times X$ with compact fibers $\mathcal{E}_\omega$ which are mapped by T_ω homeomorphically. This result does not have a counterpart in the usual deterministic theory without strong additional assumptions since, for instance, the only generator for the identity transformation is the partition into points. The relative generating partitions described above enable us to represent the system as a random symbolic shift. We construct both measure and topological generators relying on the strong Rokhlin lemma from [8].

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3.23 Renaud Leplaideur *Existence of SRB-measures for some topologically hyperbolic diffeomorphisms*

Let us consider a smooth dynamical system (M, f), where M is a compact smooth riemannian manifold, f a C^2 diffeomorphism acting on M and μ some invariant ergodic probability measure on M. By the Ergodic Theorem, the set G_μ of generic points carries μ-all the dynamic, and the question is then to ask for if we can observe it.

For instance, may we have $Leb_M(G_\mu) > 0$, where Leb_M denotes Lebesgue measure on M ?

If this property holds we say that *μ is a physical measure.* In the case of hyperbolic dynamical systems we look for special physical measures, the so-called *Sinai-Ruelle-Bowen measures.*

It is well-known that *uniformly hyperbolic dynamical systems* admit some SRB-measure, but the existence is not clear for *non-uniformly hyperbolic dynamical systems.* Actually, hyperbolicity may fail in multiple ways, and so we are far away to get some general theory. Nevertheless, several works (see e.g. [1], [5], [2], [3] or [4]) prove existence or non-existence of SRB-measures for some systems such that hyperbolicity fails in a topological way.

Our work is in keeping with this topological point of view; to be more precise we set

Definition 3.23.1. f is said to be Almost-Axiom-A (3a) if there exist an open set U and some f-invariant compact set $\Omega \subset U$ such that :
(i) $\forall x \in U$ there is a df-invariant (where it makes sense) splitting of the tangent space $T_xM = E^u(x) \oplus E^s(x)$ with $x \mapsto E^u(x)$ and $x \mapsto E^s(x)$ two Lipschitz maps (with uniformly bounded Lipschitz constant).
(ii) There exist two continuous and positive functions $x \mapsto k^u(x)$ and $x \mapsto k^s(x)$ such that

$$\forall x \in U, \qquad \begin{array}{ll} \forall v \in E^s(x) & \|df(x).v\|_{f(x)} \leq e^{-k^s(x)}\|v\|_x \\ \forall v \in E^u(x) & \|df(x).v\|_{f(x)} \geq e^{k^u(x)}\|v\|_x, \end{array}$$

(iii) For p in U, $k^u(p) = 0 \iff k^s(p) = 0$. If S denotes the set $(k^s)^{-1}(0)$, then $f(S) = S$.

Let $\lambda \in]0, +\infty[$. A point x in Ω is said to be *λ-hyperbolic* if $\forall v \in E^u(x) \setminus \{0\}$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|df^{-n}(x).v\| < -\lambda$$

and $\forall v \in E^s(x) \setminus \{0\}$,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|df^{n}(x).v\| < -\lambda.$$

If x is some λ-hyperbolic point, there exist two immersed manifolds $W^u(x)$ and $W^s(x)$ such that

$$T_xW^i(x) = E^i(x) \qquad \text{for } i = u, s.$$

Hence it makes sense to define the *Leb^u-measure*: a set A has positive Leb^u-measure if there exists some λ-hyperbolic point x such that $Leb^u_x(A \cap W^u(x)) > 0$, where Leb^u_x denotes the Riemannian measure on $W^u(x)$.

If λ and ε_0 are fixed, a point x is said to be *(ε_0, λ)-regular* if it is some λ-hyperbolic point such that $d^i(x, \partial W^i(x)) > \varepsilon_0$, where $i = u, s$ and d^i denote the Riemannian distance on $W^i(x)$. A set Λ is said to be a *(ε_0, λ)-regular set* if it is some f-invariant set of (ε_0, λ)-regular points. Then the result is the following:

Theorem 3.23.1. *If there exists some (ε_0, λ)-regular set Λ such that $Leb^u(\Lambda) > 0$, then f admits either a σ-finite SRB measure or a probability SRB measure.*

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3.24 E. Lesigne[13]
Weak disjointness in ergodic theory

Let $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ be two probability measure-preserving dynamical systems. We say that these two systems are *weakly disjoint* if, given any $f \in L^2(\mu)$ and $g \in L^2(\nu)$,

> there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\mu(A) = \nu(B) = 1$, and, for **all** $(x, y) \in A \times B$, the sequence
> $$\left(\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \cdot g(S^n y)\right)_{N>0} \tag{3.1}$$
> converges.

The study of this weak disjointness property is in its early stages. In this abstract we make some basic remarks, give some examples and ask some questions. A detailed version of this note is in preparation. Let us make two remarks on the vocabulary.

- When no confusion is possible, we refer to the dynamical system by the transformation alone and we speak of the weak disjointness of the transformations T and S.
- Until the Conference on Ergodic Theory and Dynamical Systems, Toruń 2000, the weak disjointness property was called the "strong product ergodic property", as in [9].

We begin the study of weak disjointness with the following two remarks.

- The weak disjointness property is an invariant of metric isomorphism of dynamical systems. If the transformations T' and S' are metric factors of T and S respectively, and if these latter transformations are weakly disjoint, then T' and S' are weakly disjoint.
- By a careful but standard use of an ergodic maximal inequality, it can be shown that T and S are weakly disjoint as soon as Property (3.1) is satisfied by all f and g belonging to dense subsets of $L^2(\mu)$ and $L^2(\nu)$ respectively.

Using the notion of generic points in topological measure-preserving dynamical systems, we deduce from these remarks the following proposition.

Proposition 3.24.1. *If two measure-preserving dynamical systems are disjoint (in the sense of [3]), then they are weakly disjoint.*

The next proposition shows that the converse is not true. A measure-preserving dynamical system is said to have quasi-discrete spectrum if the generalized proper functions in the sense of [5] generate a dense linear subspace of L^2. This notion is studied in [1]. The following proposition can be deduced from the generalized Wiener-Wintner ergodic theorem proved in [8].

[13] Joint work with A. Quas, T. de la Rue and B. Rittaud

Proposition 3.24.2. *Any quasi-discrete spectrum transformation is weakly disjoint from any measure preserving transformation, and in particular any quasi-discrete spectrum transformation is weakly disjoint from itself.*

This is not the only known example of transformation weakly disjoint from itself. A direct consequence of the study developed in [6] is that Chacón's transformation is disjoint from itself. Thanks to deep Ratner's theorem (see [4]), we see that if both T and S are translations by unipotent elements on finite volume homogeneous spaces of connected Lie groups, then T and S are weakly disjoint. Another example, whose study needs some more effort, is the Morse dynamical system: the subshift associated to the Morse sequence (see [7]), which is known to be strictly ergodic, is weakly disjoint from itself.

What about negative results?

Proposition 3.24.3. *Two transformations with positive entropy are never weakly disjoint.*

We know also that negative results can be obtained in zero entropy: there exists a zero entropy measure preserving dynamical system which is not weakly disjoint from itself. Such an example can be constructed by a cutting and stacking procedure.

We conclude by a list of open questions.

Problem 3.24.1. Let us say that a dynamical system (or a transformation) is *universal* if it is weakly disjoint from any measure-preserving transformation. Is Chacón's transformation universal? Is the Morse dynamical system universal?

Problem 3.24.2. (V. Bergelson) If T is invertible and weakly disjoint from itself, is it weakly disjoint from its inverse?

Problem 3.24.3. Inspired by the ideas of the proof of the ergodic return time theorem ([2] and [10]), we ask the following question, which contains the two previous ones. If T is weakly disjoint from itself, is it necessarily universal?

Problem 3.24.4. (B. Kamiński) Is the weak disjointness property generic?

We think that the weak disjointness property could be studied in the class of distal dynamical systems, beginning with the case of Anzai skew products. It is also sure that the links between weak disjointness and joinings have to be better understood. Lastly, D. Rudolph suggested studying the property in the class of rank one systems, and V. Bergelson suggested looking at rigid weakly mixing systems.

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3.25 Michael Lin[14] *Ergodic characterizations of reflexivity of Banach spaces*

I. Using the spectral theorem, von-Neumann [6] proved that for every unitary operator T in a complex Hilbert space,

$$Px := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T^k x \quad \text{exists } \forall x. \tag{$*$}$$

A linear operator T on a (real or complex) Banach space X is called *mean ergodic* if $(*)$ is satisfied, and *uniformly ergodic* if the convergence in $(*)$ is uniform on the unit ball, i.e., $\lim_{n\to\infty} \|\frac{1}{n}\sum_{k=1}^{n} T^k - P\| = 0$. A power-bounded T in a Banach space is mean ergodic if and only if X has the following *ergodic decomposition*

$$X = \{y \in X : \, Ty = y\} \oplus \overline{(I-T)X} \tag{$**$}$$

We denote by $F(T)$ the set of fixed points of the linear operator T.

A Banach space X will be called *mean ergodic* if every power-bounded operator $T \in B(X)$ satisfies $(*)$. Lorch [5] proved that *all reflexive Banach spaces are mean ergodic.* We refer the reader to [4] for additional references (to which [8] should be added). Sucheston [7] posed the following question, concerning the converse of Lorch's result: *If every contraction in a Banach space X is mean ergodic, is X*

[14] Joint work with V. Fonf and P. Wojtaszczyk

reflexive? Even under the stronger assumption, that all power-bounded operators are mean ergodic, i.e., X is mean ergodic, the problem is still unsolved.

Main Theorems. *Let X be a Banach space with a Schauder basis.*
(i) *X is finite-dimensional if and only if every power-bounded operator is uniformly ergodic.*
(ii) *X is reflexive if and only if every power-bounded operator is mean ergodic.*
(iii) *X is quasi-reflexive of order one (i.e., $\dim X^{**}/X = 1$) if and only if for every power-bounded operator T, T or T^* is mean ergodic.*

The basis is used in the paper for the constructions of operators in the proofs of the "if" parts.

Corollary 3.25.1. *A Banach space X is reflexive if and only if every closed subspace of X is mean ergodic (i.e., each power bounded operator defined on a closed subspace is mean ergodic).*

Proof. Suppose that X is non-reflexive. By a result of Pelczynski [2, p. 54], X has a non-reflexive (separable) closed subspace with a basis, which yields a contradiction to Main Theorem (ii). The converse follows from Lorch's Theorem. □

Butzer and Westphal [1] proved that every power-bounded operator S in a reflexive Banach space Y satisfies

$$\left\{ y : \sup_n \|\sum_{k=0}^{n} S^k y\| < \infty \right\} = (I - S)Y \qquad (***)$$

This equality was further studied in [3], where additional references are given.

Theorem. *A Banach space X is reflexive if and only if every power-bounded operator S defined on a closed subspace $Y \subset X$ satisfies $(***)$.*

Proof. If X is reflexive, the result follows from [1].

For the converse, we show that every power-bounded T on a closed subspace Z is mean ergodic, and apply the previous corollary. Assume T is power-bounded on Z, and let S be the restriction of T to the invariant subspace $Y = \overline{(I-T)Z}$. For $z \in Z$, the vector $y_0 = (I-T)z$ obviously satisfies $\sup_n \|\sum_{k=0}^n S^k y_0\| < \infty$, so by $(***)$ there is $y_1 \in Y$ with $(I-S)y_1 = y_0$. Hence $(I-T)(z-y_1) = 0$, so $z - y_1 \in F(T)$, which yields $z \in F(T) \oplus \overline{(I-T)Z}$. Hence $Z = F(T) \oplus \overline{(I-T)Z}$, and T is mean ergodic. □

The previous proof can be used to show the following

Proposition 3.25.1. *Let T be power-bounded on X. If $\overline{(I-T)X}$ is reflexive, then T is mean ergodic.*

An equivalent formulation of the proposition is:

Proposition 3.25.2. *Let A be a bounded operator from a Banach space X into a reflexive subspace. If $T = I + A$ is power-bounded on X, then T is mean ergodic.*

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II. In this paper, we obtain a positive solution to this last problem for Banach spaces with bases (throughout this paper, a basis means a *Schauder basis*). From this result we conclude that *a Banach space X is reflexive if and only if every closed subspace is mean ergodic.* Our construction also yields that a Banach space with basis is finite-dimensional if and only if every power-bounded operator is uniformly ergodic. We show that a non-reflexive Banach space with basis is 1-quasi-reflexive if and only if for every power-bounded T, T or T^* is mean ergodic, and such a space is *not* mean ergodic.

Recently, Emel'yanov and Wolff [8] have proved that on any (not necessarily separable) Banach space X which contains c_0 there is a power-bounded operator which is not mean ergodic. Our methods yield a different proof of this result.

We mention that Eeml'yanov [7] proved that if every power-bounded operator on a Banach lattice E is mean ergodic, then E is reflexive. For a dual Banach lattice, Zaharopol [22] proved that if all power-bounded *positive* operators are mean ergodic, then the Banach lattice is reflexive.

2. Ergodic characterizations of reflexivity and 1-quasi-reflexivity

Definition 3.25.1. A *Schauder decomposition* of a Banach space X is an infinite sequence $\{E_k\}_{k=1}^{\infty}$ of closed subspaces $\{0\} \neq E_k \subset X$ such that each $x \in X$ has a unique representation $x = \sum_{k=1}^{\infty} x_k$, with $x_k \in E_k$, $(k = 1, 2, \ldots)$. We denote it by $X = \sum_k E_k$.

The corresponding "coordinate" projectors $Q_k : X \to E_k$ are defined for $x = \sum x_j$ $(x_j \in E_j)$ by $Q_k x = x_k$ $(k = 1, 2, ...,)$. The "partial sum" operators $P_n = \sum_{k=1}^{n} Q_k$ $(n = 1, 2, ...)$ satisfy $\lim_n P_n x = x$ for every x. An adaptation of the proof given in [20, vol. I, pp. 18-20] for bases shows that the partial sums operators are continuous and uniformly bounded (see [20, vol. II, p. 499]). Hence also the coordinate projectors are continuous and uniformly bounded. By introducing the following norm (which is equivalent to the original one)

$$|||x||| = \sup\{\|Q_k x\|,\ \|P_k x\| :\ k = 1, 2, ...\},$$

we get

$$|||Q_k||| = |||P_k||| = 1 \quad \forall k \geq 1. \tag{1}$$

Since power-boundedness of a linear operator is the same in all equivalent norms, whenever necessary we may assume that the original norm $\|\cdot\|$ satisfies (1).

Definition 3.25.2. A Schauder decomposition $X = \sum_{k=1}^{\infty} E_k$ is called *shrinking* if for each $f \in X^*$ we have $\lim_{n\to\infty} \|f_{|\sum_{i=n}^{\infty} E_i}\| = 0$.

Note that if each subspace E_k of a Schauder decomposition is spanned by one vector e_k, then $\{e_k\}$ is a basis of X, and when this decomposition is shrinking we call $\{e_k\}$ a *shrinking basis*.

Lemma 3.25.1. *Let $X = \sum_k X_k$ be a non-shrinking Schauder decomposition of a Banach space X. Then there is a Schauder decomposition $X = \sum_k E_k$ with the following property: there exist a linear functional $h \in X^*$ and a sequence $\{e_k\}$, such that for every $k \geq 1$ we have $e_k \in E_k$, $\|e_k\| \leq 1$, and $h(e_k) = 1$.*

Proof. Since the decomposition $X = \sum_k X_k$ is not shrinking, there is a functional $f \in X^*$ with $\|f\| = 1$ and $\limsup_n \|f_{|\sum_{k\geq n} X_k}\| = \alpha > 0$ (obviously $\alpha \leq 1$). Take a vector y_1 such that

$$y_1 = \sum_{k=n_1+1}^{\infty} a_k^{(1)} x_k^{(1)}, \quad x_k^{(1)} \in X_k, \quad \|y_1\| = 1, \quad |f(y_1)| > \frac{\alpha}{2}.$$

Find $n_2 > n_1$ with $\|\sum_{k=n_2+1}^{\infty} a_k^{(1)} x_k^{(1)}\| < \alpha/4$, and take a vector y_2 such that

$$y_2 = \sum_{k=n_2+1}^{\infty} a_k^{(2)} x_k^{(2)}, \quad x_k^{(2)} \in X_k, \quad \|y_2\| = 1, \quad |f(y_2)| > \frac{\alpha}{2}.$$

We continue inductively and obtain a strictly increasing sequence of integers $\{n_j\}$ and a sequence of vectors $\{y_j\}$, such that for each j,

$$y_j = \sum_{k=n_j+1}^{\infty} a_k^{(j)} x_k^{(j)}, \quad x_k^{(j)} \in X_k, \quad \|y_j\| = 1, \quad |f(y_j)| > \frac{\alpha}{2},$$

and $\|\sum_{k=n_j+1}^{\infty} a_k^{(j-1)} x_k^{(j-1)}\| < \frac{\alpha}{4}$.

Define $E_1 = \sum_{i=1}^{n_2} X_i$, and $E_j = \sum_{i=n_j+1}^{n_{j+1}} X_i$ for $j \geq 2$. Clearly $\{E_j\}$ is a Schauder decomposition. Put $z_j = \sum_{k=n_j+1}^{n_{j+1}} a_k^{(j)} x_k^{(j)}$. Then $z_j \in E_j$, and, by the construction, $1 - \alpha/4 \leq \|z_j\| \leq 1 + \alpha/4$ and $\alpha/4 \leq |f(z_j)| \leq 1 + \alpha/4$. Finally, let $h = \frac{4+\alpha}{\alpha} f$, and define $e_j = \frac{\alpha}{(4+\alpha)f(z_j)} z_j$. Then $\|e_j\| \leq 1$, and $h(e_j) = 1$ for every j. □

Theorem 3.25.1. *If a Banach space X admits a non-shrinking Schauder decomposition, then there exists a power bounded linear operator $T \in B(X)$ which is not mean ergodic.*

Proof. Let $X = \sum_{k=1}^{\infty} E_k$ be the decomposition given by the Lemma, so we have $h \in X^*$ and a sequence $\{e_k\}$ such that $e_k \in E_k$, $h(e_k) = 1$, $\|e_k\| \leq 1$, $k = 1, 2, \ldots$. The change to a norm satisfying (1) yields that $\|e_k\| \leq M$, so we replace e_k by $M^{-1}e_k$ and h by Mh. Thus, we can assume that the norm satisfies (1) (for the projectors defined by $\{E_k\}$).

Take an arbitrary sequence $a = \{a_j\}_{j=1}^{\infty}$ of positive numbers with

$$\sum_{j=1}^{\infty} a_j = 1, \quad a_j > 0, \ j = 1, 2, \ldots, \tag{2}$$

and denote $A_n = \sum_{j=1}^{n} a_j$. For $x \in X$ and $m > n \geq 2$ we then have

$$\sum_{k=n}^{m} A_k Q_k x = \sum_{k=n}^{m} Q_k(\sum_{j=1}^{n-1} a_j + \sum_{j=n}^{k} a_j)x = (\sum_{j=1}^{n-1} a_j)(\sum_{k=n}^{m} Q_k x) + \sum_{j=n}^{m} a_j(\sum_{k=j}^{m} Q_k x).$$

Since $\sum_k Q_k x$ converges, we see that $\{\sum_{k=1}^{m} A_k Q_k x\}_m$ is a Cauchy sequence in the norm, hence converges. Denoting $P_0 = 0$, we obtain by (1) that

$$\|\sum_{k=1}^{m} A_k Q_k x\| = \|\sum_{j=1}^{m} a_j(\sum_{k=j}^{m} Q_k x)\| = \|\sum_{j=1}^{m} a_j(P_m - P_{j-1})x\| \leq 2\|x\|. \tag{3}$$

We now define an operator $T_a : X \to X$ by

$$> T_a x = \sum_{k=1}^{\infty} A_k Q_k x + \sum_{j=2}^{\infty} h(P_{j-1}x) a_j e_j. \tag{4}$$

Since $\|e_j\| \leq 1$ for every j, (1), (2) and (3) yield $\|T_a x\| \leq (2 + \|h\|)\|x\|$, so

$$\|T_a\| \leq 2 + \|h\| \tag{5}$$

The bound $(2 + \|h\|)$ for $\|T_a\|$ does not depend on the choice of the sequence $\{a_j\}$ satisfying (2), so in order to prove that the operator T_a is power-bounded, it is enough to show that for sequences a and b satisfying (2), the composition $T_a T_b$ is of the same type (say T_c). We formulate it precisely:

Claim. *Let the sequences $a = \{a_j\}$ and $b = \{b_j\}$ satisfy (2), and define the operators T_a and T_b by (4) (with $B_0 = 0$ and $B_n = \sum_{j=1}^{n} b_j$). Then the sequence $c = \{c_j\}$, defined by $c_j = A_j b_j + B_{j-1} a_j$, $j = 1, 2, \ldots$, satisfies (2), and the composition satisfies $T_a T_b = T_c$.*

Proof. Clearly $C_1 = A_1 B_1$. We obtain that c satisfies (2), since for $n \geq 2$ we have

$$C_n = \sum_{j=1}^{n} c_j = \sum_{j=1}^{n}(B_{j-1}a_j + b_j A_j) = \sum_{k=1}^{n-1} b_k \sum_{j=k+1}^{n} a_j + \sum_{k=1}^{n} b_k \sum_{j=1}^{k} a_j$$

$$= \sum_{k=1}^{n-1} b_k A_n + b_n A_n = A_n B_n.$$

Now we show that $T_a T_b = T_c$. In view of the decomposition $X = \sum_k E_k$, it is enough to show that $T_a T_b e = T_c e$ for each vector $e \in E_k$, $k = 1, 2, \ldots$. Fix k, and take $x_k \in E_k$. The definition (4) yields

$$> T_a x_k = A_k x_k + \sum_{j=k+1}^{\infty} h(x_k) a_j e_j = A_k x_k + h(x_k) \sum_{j=k+1}^{\infty} a_j e_j \qquad (6)$$

We apply (6) to T_b and to T_a, and obvious computations yield

$$T_a(T_b x_k) = T_a(B_k x_k + h(x_k) \sum_{j=k+1}^{\infty} b_j e_j) = A_k B_k x_k + h(x_k) \sum_{j=k+1}^{\infty} (B_{j-1} a_j + b_j A_j) e_j$$

Since $A_k B_k = C_k$, an application of (6) to c yields $T_a(T_b x_k) = T_c x_k$, and the claim is proved. □

To prove that the power-bounded operator T_a is not mean ergodic, it is enough to show (by the above mentioned Sine's criterion [17]) that the *non-zero* functional h is a fixed point for T_a^*, while zero is the only fixed point for T_a.

Suppose that $T_a x = x$. Using the definition (4), we have

$$\sum_{k=1}^{\infty} Q_k x = x = T_a x = \sum_{k=1}^{\infty} A_k Q_k x + \sum_{k=2}^{\infty} h(P_{k-1} x) a_k e_k.$$

We look at the components in each E_k. For $k = 1$ we have $Q_1 x = A_1 Q_1 x$, so $Q_1 x = 0$ since $A_1 = a_1 < 1$. For $k > 1$ we obtain $(1 - A_k) Q_k x = h(P_{k-1} x) a_k e_k$. Assume now that $Q_j x = 0$ for every $j < k$; then $P_{k-1} x = 0$, and thus $(1 - A_k) Q_k x = 0$, yielding $Q_k x = 0$. Hence by induction we have $Q_k x = 0$ for every $k \geq 1$, so $T_a x = x$ implies $x = 0$.

Fix an arbitrary $k \geq 1$ and take an arbitrary $e \in E_k$. Applying h to (6) and using $h(e_j) = 1$ for every j, we obtain

$$(T_a^* h)(e) = h(T_a e) = h(A_k e + h(e) \sum_{j=k+1}^{\infty} a_j e_j) = A_k h(e) + h(e) \sum_{j=k+1}^{\infty} a_j = h(e).$$

In view of the decomposition $X = \sum_k E_k$, we have $T_a^* h = h$. The Theorem is now proved. □

Remark 3.25.1. Clearly, $\sum_{j=1}^{\infty} a_j P_{j-1} x$ converges in norm for $\{a_j\}$ satisfying (2), and the equality of the vector sums appearing in the first and third terms of (3) yields

$$\sum_{k=1}^{\infty} A_k Q_k x = \sum_{j=1}^{\infty} a_j (I - P_{j-1}) x = x - \sum_{j=1}^{\infty} a_j P_{j-1} x \quad \forall x \in X. \qquad (7)$$

Remark 3.25.2. In fact, the functional h of the previous proof is the *only* fixed point for T_a^* (up to a scalar multiplier). We now prove this fact, though not needed for Theorem 1, since it will be important for Theorem 4.

So, we assume that $T_a^* f = f$, and prove that $f = th$ for some scalar t. With $P_0 = 0$, we can write (4) as

$$T_a x = \sum_{m=1}^{\infty} (A_m Q_m x + h(P_{m-1}x) a_m e_m).$$

Duality yields (with w^*-convergence of the series)

$$T_a^* f = \sum_{m=1}^{\infty} (A_m Q_m^* f + a_m f(e_m) P_{m-1}^* h). \tag{8}$$

Since $f = \sum_{m=1}^{\infty} Q_m^* f$ (again, w^*-convergence of the series), the assumption $T_a^* f = f$ and (8) yield

$$\sum_{m=1}^{\infty} [(1 - A_m) Q_m^* f - a_m f(e_m) P_{m-1}^* h] = 0. \tag{9}$$

Now fix an integer n, and apply the functional of the left side of (9) to a vector $z_n \in E_n$:

$$\sum_{m=1}^{\infty} [(1 - A_m)(Q_m^* f)(z_n) - a_m f(e_m)(P_{m-1}^* h)(z_n)] =$$

$$\sum_{m=1}^{\infty} [(1 - A_m) f(Q_m z_n) - a_m f(e_m) h(P_{m-1} z_n)] =$$

$$(1 - A_n) f(z_n) - \sum_{m=n+1}^{\infty} a_m f(e_m) h(z_n) = 0. \tag{10}$$

Since $h(e_n) = 1$, (10) with $z_n = e_n$ yields the following system of linear equations in the unknowns $t_m = f(e_m)$ $m = 1, 2, \ldots$:

$$(1 - A_n) t_n = \sum_{m=n+1}^{\infty} a_m t_m \qquad n = 1, 2, \ldots \tag{11}$$

Subtraction of equation number n from equation number $(n + 1)$ shows that the only solution of the system (11) is $t_1 = t_2 = \ldots = t$, so

$$f(e_n) = t = th(e_n), \; n = 1, 2, \ldots. \tag{12}$$

In order to prove that $f = th$, we show the equality on each E_n. Fix $x \in E_n$. Then $h(x - h(x)e_n) = h(x) - h(x)h(e_n) = 0$ since $h(e_n) = 1$. Denote $z = x - h(x)e_n$. By (10) with $z_n = z$ we have

$$(1 - A_n) f(z) - \sum_{m=n+1}^{\infty} a_m f(e_m) h(z) = 0.$$

Since $h(z) = 0$, this yields $f(z) = 0$, so $f(x) = h(x) f(e_n) = th(x)$.

Corollary 3.25.2. *Let X be a (separable) Banach space with a basis. Then X is reflexive if and only if every power-bounded operator in X is mean ergodic.*

Proof. Zippin [23] proved that if a non-reflexive Banach space has a basis, then it has a non-shrinking basis. Thus, if X is not reflexive, Theorem 1 yields a power-bounded operator which is not mean ergodic. If X is reflexive, apply Lorch's Theorem. □

Corollary 3.25.3. *For every Banach space X the following assertions are equivalent:*

(i) *X is reflexive.*

(ii) *Every closed subspace of X is mean ergodic (i.e., each power bounded operator defined on a closed subspace is mean ergodic).*

Proof. $(ii) \Rightarrow (i)$: Suppose that X is non-reflexive. By a result of Pelczynski [6, p. 54], X has a non-reflexive (separable) closed subspace with a basis, and Corollary 1 yields a contradiction.

$(i) \Rightarrow (ii)$ follows from Lorch's Theorem, since a closed subspace of a reflexive Banach space is reflexive. □

Theorem 3.25.2. *If an infinite-dimensional Banach space X admits a Schauder decomposition, then there is a mean ergodic power-bounded operator $T \in B(X)$ which is not uniformly ergodic.*

Proof. We may assume that the norm satisfies (1). For a sequence $\{a_j\}$ satisfying (2), let $T_a x = \sum_{k=1}^{\infty} A_k Q_k x$ (T_a is defined as in (4) with $h = 0$). By the proof of Theorem 1, T_a is power-bounded, and has no fixed points except 0 (this part of the proof did not require the special properties of h, which were used only to show that T_a^* had h as fixed point).

Let $f \in X^*$ satisfy $T_a^* f = f$. Then for $z_n \in E_n$ we have

$$f(z_n) = f(T_a z_n) = \sum_{k=1}^{\infty} A_k f(Q_k z_n) = A_n f(z_n).$$

Since $A_n < 1$ for each n, we have $f(z_n) = 0$ for any $z_n \in E_n$. Hence $T_a^* f = f$ implies $f = 0$, which yields (Hahn-Banach) that $\overline{(I - T_a)X} = X$, so T_a is mean ergodic.

Since T_a has no non-zero fixed points, it is uniformly ergodic if and only if $I - T_a$ is invertible on X [19]. By definition,

$$(I - T_a)x = x - \sum_{k=1}^{\infty} A_k Q_k x = \sum_{k=1}^{\infty} (1 - A_k) Q_k x = \sum_{k=1}^{\infty} \Big(\sum_{j=k+1}^{\infty} a_j \Big) Q_k x.$$

We now take $a_j = 2^{-j}$ for $j \geq 1$, and put $T = T_a$. Then $(I - T)x = \sum_{k=1}^{\infty} 2^{-k} Q_k x$. Take a sequence $e_k \in E_k$ with $\|e_k\| = 1$ for every k. Then $\sum_{k=1}^{\infty} \frac{1}{k^2} e_k$ converges, say to y. When we try to solve $(I - T)x = y$, we obtain the equations $Q_k x = \frac{2^k}{k^2} e_k$, which imply $\|Q_k x\| \to \infty$. Since for $x \in X$ we have $Q_k x \to 0$, there is no $x \in X$ with $(I - T)x = y$, so $I - T$ is not invertible, and therefore T is not uniformly ergodic. □

Remark 3.25.3. The existence of T which is not uniformly ergodic in X with an unconditional basis was proved in [12].

Corollary 3.25.4. *Let X be a Banach space with basis. Then the following conditions are equivalent:*

(i) X is finite-dimensional.

(ii) Every power-bounded operator is uniformly ergodic.

(iii) Every mean ergodic power-bounded operator is uniformly ergodic.

Proof. Clearly, if X is finite-dimensional, every power-bounded $T \in B(X)$ is uniformly ergodic. Obviously, $(ii) \Rightarrow (iii)$, and $(iii) \Rightarrow (i)$ follows from Theorem 2. □

The ideas in the proof of Theorem 1 can be used to obtain the following generalization, which applies also to non-separable spaces:

Theorem 3.25.3. *Let a Banach space X admit a sequence of projectors $\{P_n\}$ such that*

(i) $\sup \|P_n\| < \infty$

(ii) $P_n P_m = P_{\min(m,n)}$

(iii) *There exists a functional $h \in X^*$ such that for each $n \geq 1$ there is a vector $e_n \in (P_n - P_{n-1})X$ with $\|e_n\| \leq 1$ and $h(e_n) = 1$.*

Then, for a sequence $\{a_n\}$ which satisfies (2), the operator

$$Sx = x - \sum_{n=2}^{\infty} a_n P_{n-1}x + \sum_{n=2}^{\infty} a_n h(P_{n-1}x)e_n, \tag{13}$$

is power-bounded and not mean ergodic.

Proof. It is immediate from the assumptions that S is well defined. Denote $Y_n = P_n X$. By (ii), $\{Y_n\}$ is an increasing sequence of subspaces, and $Y = \overline{\bigcup_{n\geq 1} Y_n}$ is a S-invariant subspace. By (ii), $\lim_n P_n y = y$ for $y \in Y_k$, so by (i) $\lim_n P_n y = y$ for every $y \in Y$. Let $Q_1 = P_1$, and $Q_k = P_k - P_{k-1}$ for $k \geq 2$. It is easily checked, using (ii), that each Q_k is a projection, and $Q_k Q_j = 0$ for $j \neq k$. Since $\sum_{k=1}^{n} Q_k y = P_n y \to y$ for every $y \in Y$, the sequence $\{E_k\}$ with $E_k = Q_k X = Q_k Y$ is a Schauder decomposition of Y. Assumption (iii) allows us to apply the proof of Theorem 1 to Y – the restriction of S to Y is the operator T_a constructed in that proof, when we substitute (7) into (4). Hence there is a vector $y \in Y$ such that the sequence $\{\frac{1}{n}\sum_{k=1}^{n} T_a^k y\}$ does not converge, which shows that S is not mean ergodic.

To complete the proof, we have to show that S is power-bounded on all of X (this does not follow from the proof of Theorem 1, since Y is not necessarily complemented in X).

Denote S by S_a to indicate the dependence on $\{a_j\}$ (which satisfies (2)). Clearly

$$\|S_a x\| \leq \|x\| + \|x\|(1 + \|h\|) \sup_n \|P_n\|$$

so we have an estimate of the norm of S_a, which is independent of a. As in the proof of Theorem 1, the power-boundedness follows from the following claim. □

Claim. *Let the sequences* $a = \{a_j\}$ *and* $b = \{b_j\}$ *satisfy* (2), *and define the sequence* $c = \{c_j\}$ *by* $c_j = A_j b_j + B_{j-1}a_j$, $j = 1, 2, \ldots,$. *Then* $\{c_j\}$ *satisfies* (2), *and the operators* S_a, S_b *and* S_c *defined by* (13) *satisfy* $S_a S_b = S_c$.

Proof. $\{c_j\}$ satisfies (2) by the claim in the proof of Theorem 1. Apply property (ii) to (13), we obtain

$$P_n(S_b x) = P_n x - \sum_{i=2}^{\infty} b_i P_n P_{i-1} x + \sum_{i=2}^{\infty} b_i h(P_{i-1}x) P_n e_i$$

$$= P_n x - \sum_{i=1}^{n} b_i P_{i-1} x - (1 - B_n) P_n x + \sum_{i=1}^{n} b_i h(P_{i-1}x) e_i.$$

We substitute this into

$$S_a(S_b x) = S_b x - \sum_{n=2}^{\infty} a_n P_{n-1}(S_b x) + \sum_{n=2}^{\infty} a_n h(P_{n-1}(S_b x)) e_n,$$

and some straight forward (tedious) calculations prove the claim. □

Corollary 3.25.5. *Let* X *be a Banach space which contains a closed subspace isomorphic to* c_0. *Then there exists a power-bounded* $T \in B(X)$ *which is not mean ergodic.*

Proof. Let Y be a closed subspace of X isomorphic to c_0, and let $y_n \in Y$ be the image of the of the unit vector $e_n \in c_0$. Then $\{y_n\}$ is a basis of Y, and there is $K > 0$ such that $\|\sum_{j=1}^{\infty} a_j y_j\| \le K \sup_j |a_j|$. Let $\{y_n^*\} \subset Y^*$ be the coefficient functionals, which are uniformly bounded, and take $f_n \in X^*$ a Hahn-Banach extension of y_n^*. We now define $x_n = \sum_{j=1}^{n} y_j$ and $g_n = f_n - f_{n+1}$. Then $g_k(x_n) = \delta_{kn}$, and the operators $P_n x = \sum_{k=1}^{n} g_k(x) x_k$ are commuting projections satisfying assumption (ii) of Theorem 3. The functional $h = f_1$ satisfies assumption (iii) since $h(x_n) = 1$. Finally, the isomorphism of Y and c_0 yields that $\sup_n \|P_n\| \le 2K \sup_n \|f_n\|$, since

$$P_n x = f_1(x) y_1 - f_{n+1}(x) x_n + \sum_{k=2}^{n} f_k(x)(x_k - x_{k-1}) = \sum_{k=1}^{n} [f_k(x) - f_{n+1}(x)] y_k.$$

□

Remark 3.25.4. The Corollary was first proved in [8] using a different method. Note that if X is separable (as any space with a basis is) and contains c_0, then (even without a basis), there is a power-bounded operator $T \in B(X)$ which is not mean ergodic, since c_0 is complemented in X [6, p. 71], and $T_0 \in B(c_0)$ defined by $T_0(a_1, a_2, a_3, \ldots) = (a_1, a_1, a_2, \ldots)$ is power-bounded and not mean ergodic. Thus, the novelty of the result is for non-separable spaces, in which c_0 need not be complemented.

For a basis $\{x_i\}$ of a Banach space X, we denote by $\{x_i^*\}$ the associated coefficient functionals. Recall [20, vol. I p. 268] that a basis $\{x_i\}$ is called *k-shrinking* if codim $[x_i^*]_{i=1}^{\infty} = k$ (where $[y_j]_{j=1}^{\infty}$ denotes the closed linear manifold generated by the sequence $\{y_j\}_{j=1}^{\infty}$). It is well known [4],[20, vol. I p. 272] that a basis is 0-shrinking if and only if it is shrinking in the sense of Definition 2.

Definition 3.25.3. A Banach space X is called *quasi-reflexive of order* k if $\dim X^{**}/X = k < \infty$ (we identify X with its natural embedding in X^{**}). The original construction of the James space [15], valid over the real or complex field, yields an example of a Banach space with basis which is quasi-reflexive of order 1.

Theorem 3.25.4. *Let X be a Banach space with a basis, such that* $\dim X^{**}/X \geq 2$. *Then there exists a power-bounded operator $T \in B(X)$ such that neither T nor T^* are mean ergodic.*

Proof. According to Zippin's result [23] mentioned above, the (non-reflexive) space X has a non-shrinking basis, say $\{u_i\}$; that is, $\{u_i\}$ is a basis which is not 0-shrinking. If $\{u_i\}$ is k-shrinking with $k \geq 2$, we keep it. If $\{u_i\}$ is 1-shrinking, we use Theorem 1 of [5]: *Let X be a Banach space which is not quasi-reflexive of order k* (in our case $\dim X^{**}/X \geq 2$, so X is not quasi-reflexive of order 1). *If X has a k-shrinking basis, then X has a $(k+1)$-shrinking basis.* Thus, we have established that there exists in X a basis $\{x_i\}$ such that

$$\operatorname{codim}\,[x_i^*]_{i=1}^{\infty} \geq 2. \tag{14}$$

Since this basis $\{x_i\}$ is not shrinking, the Lemma (with $X_k = \{tx_k : t \in \mathbf{R}\}$) yields a Schauder decomposition $X = \sum_k E_k$ with the following property: there exist a functional $h \in X^*$ and a sequence $\{e_k\}$, $e_k \in E_k$, $k = 1, 2, \ldots$ such that $h(e_k) = 1$, $\|e_k\| \leq 1, k = 1, 2, \ldots$ By the construction in the proof of the Lemma, each E_k is finite-dimensional, and the decomposition $X = \sum_k E_k$ has the following additional property: the "partial sum" operators P_m are of the form

$$P_m x = \sum_{i=1}^{n_m} x_i^*(x) x_i, \quad x \in X, \quad m = 1, 2, \ldots$$

This yields $P_m^* f = \sum_{i=1}^{n_m} f(x_i) x_i^*$ for $f \in X^*$, and so, $\bigcap_m \ker P_m^{**} = [x_i^*]_{i=1}^{\infty\perp}$. By (14),

$$\dim \bigcap_{m=1}^{\infty} \ker P_m^{**} = \dim(X^*/[x_i^*]_{i=1}^{\infty})^* \geq \operatorname{codim}\,[x_i^*]_{i=1}^{\infty} \geq 2. \tag{15}$$

We now proceed as in the proof of Theorem 1. For a sequence $a = \{a_j\}$ satisfying (2), define the operator T_a according to (4). It was shown that T_a is power-bounded and not mean ergodic, $F(T_a) = \{0\}$, and $F(T_a^*) = \{th\}$. We will choose $\{a_j\}$ satisfying (2) such that $\sum_{n=1}^{\infty}(1 - A_n) < \infty$ (e.g., $a_j = 2^{-j}$). Since $\dim F(T_a^*) = 1$, to prove that the operator T_a^* is not mean ergodic we have to show (by Sine's criterion) that $\dim F(T_a^{**}) \geq 2$. By (15), it is enough to show

$$F(T_a^{**}) = \bigcap_{m=1}^{\infty} \ker P_m^{**}. \tag{16}$$

From (4), (1), and the condition $\sum_n (1 - A_n) < \infty$, it follows that

$$T_a^{**}\psi = \psi + \sum_{m=1}^{\infty}(A_m - 1) Q_m^{**}\psi + \sum_{m=2}^{\infty} a_m P_{m-1}^{**}\psi(h) e_m.$$

Hence $\psi \in F(T_a^{**})$ is equivalent to

$$\sum_{m=1}^{\infty}(1-A_m)Q_m^{**}\psi = \sum_{m=2}^{\infty} a_m P_{m-1}^{**}\psi(h)e_m. \tag{17}$$

If $P_n^{**}\psi = 0$ for every $n \geq 1$, then clearly (17) holds, so $\psi \in F(T_a^{**})$. Suppose now that $\psi \in F(T_a^{**})$; we apply the operators Q_n^{**} to both sides of (17), and obtain the equations $(1-A_n)Q_n^{**}\psi = a_n P_{n-1}^{**}\psi(h)e_n$, $n = 1, 2, \ldots$. Solving successively, we obtain $Q_n^{**}\psi = 0$ for $n \geq 1$, which proves (16) and completes the proof of the theorem. □

Remark 3.25.5. Every operator T on X is the restriction of T^{**} to its invariant subspace X, so if T^{**} is mean ergodic, so is T. Hence, if both operators T and T^* are not mean ergodic, then automatically all the next conjugates (T^{**}, $T^{***}, \ldots$) are not mean ergodic.

Theorem 3.25.5. *Let X be a non-reflexive Banach space with a basis. Then the following assertions are equivalent:*
(i) *X is quasi-reflexive of order one.*
(ii) *For each power-bounded operator $T \in B(X)$, T or T^* is mean ergodic.*

Proof. *(ii)* ⇒ *(i)*: If $\dim X^{**}/X \geq 2$, then Theorem 4 yields a contradiction to (ii).

(i) ⇒ *(ii)* : Let T be a power-bounded operator on X which is not mean ergodic. By Sine's criterion, $F(T)$ does not separate $F(T^*)$, so there is $f_0 \in F(T^*)$ such that $f_0(y) = 0$ for every $y \in F(T)$. To show that T^* is mean ergodic, we will prove that $F(T^*)$ separates $F(T^{**})$. As mentioned in the introduction, $F(T^*)$ always separates $F(T)$. Hence $F(T^{**})$ separates $F(T^*)$, so there is $\psi_0 \in F(T^{**})$ such that $\psi_0(f_0) \neq 0$. By the definition of f_0, ψ_0 is not in $F(T)$, so $\psi_0 \notin X$. Since $\dim X^{**}/X = 1$, every $\phi \in X^{**}$ is of the form $\phi = \alpha\psi_0 + x$ with $x \in X$, so each $\psi \in F(T^{**})$ is of the form $\psi = \alpha\psi_0 + y$ with $y \in F(T)$. We then have $\psi(f_0) = \alpha\psi_0(f_0) \neq 0$ for $\psi \in F(T^{**})$ with $\alpha \neq 0$. If $\alpha = 0$, then ψ is in $F(T)$, and the separation of $F(T)$ by $F(T^*)$ provides an $f \in F(T^*)$ with $\psi(f) = f(\psi) \neq 0$. Hence $F(T^*)$ separates $F(T^{**})$, so T^* is mean ergodic by Sine's criterion. □

Remark 3.25.6. The implication *(i)* ⇒ *(ii)* does not require a basis for X.

If $T \in B(X)$ is power-bounded, then it is easily shown that

$$(I-T)X \subset \{y : \sup_n \|\sum_{k=1}^{n} T^k y\| < \infty\} \subset \overline{(I-T)X}.$$

When T is uniformly ergodic, then [19] $(I-T)X$ is closed, which yields

$$> (I-T)X = \{y : \sup_n \|\sum_{k=1}^{n} T^k y\| < \infty\} \tag{18}$$

If X is a *dual* space and T is a power-bounded *dual* operator, then (18) holds [18]. It now follows from Theorem 2 that in every infinite-dimensional reflexive Banach space X with a basis there is a power-bounded T which is not uniformly ergodic, but satisfies (18). It was shown in [12] that if X is a separable Banach space which does not contain infinite-dimensional dual spaces, then (18) implies uniform ergodicity. This result is true also in complex Banach spaces, since the needed result of [11], stated for real spaces, is valid also in complex spaces, with the same proof.

Proposition 3.25.3. *Let Z be an infinite-dimensional Banach space which is the dual of a separable Banach space (e.g., Z is a separable dual space). Then there exists an infinite-dimensional Banach space E with a basis such that E^* is isomorphic to a closed subspace of Z.*

Proof. This proposition is an immediate consequence of the results of [16]: Let F be separable, with $F^* = Z$. Since the unit ball of Z is compact in the weak-* topology and not in the norm, there is a sequence $\{y_n\}$ in Z which is weak-* convergent to 0, such that $\limsup_n \|y_n\| > 0$. Combining Theorem III.1 and Proposition II.1(a) of [16], we obtain a separable Banach space E with a basis, such that E^* is isometrically isomorphic to the weak-* closed subspace generated in Z by a subsequence $\{y_{n_k}\}$. □

Theorem 3.25.6. *Let X be a Banach space. Then the following conditions are equivalent:*

(i) *X does not contain an infinite-dimensional closed subspace isomorphic to the dual of a separable Banach space.*

(ii) *Every power-bounded operator T defined on a closed subspace Y, which satisfies $F(T) = \{0\}$ and $(I - T)Y = \{y \in Y : \sup_n \|\sum_{k=1}^n T^k y\| < \infty\}$, is uniformly ergodic.*

If X is separable, each of the previous conditions is equivalent to

(iii) *Every power-bounded operator T defined on a closed subspace Y which satisfies $(I - T)Y = \{y \in Y : \sup_n \|\sum_{k=1}^n T^k y\| < \infty\}$ is uniformly ergodic.*

Proof. The proof of (i) ⇒ (ii) is the same as that of Corollary 3.4(ii) of [12], noting that the results of [9], [10] used there yield a dual of a *separable* space. For the complex case, we observe that the proofs of Proposition 7.1 and Corollary 7.2 in [14] are valid also for complex Banach spaces, and these results imply the required result of [9]. (The result of [10], proved for real spaces, is more general than needed here; see also Theorem 3.2 in [13])

We now assume that (ii) holds. If (i) does not hold, then X has an infinite-dimensional closed subspace Z which is isomorphic to the dual F^* of a separable Banach space F. By the Proposition, there is an infinite-dimensional Banach space E with a basis, such that E^* is isomorphic to a subspace of F^*. Hence E^* is isomorphic to a closed subspace of Z, say Y. By Theorem 2, there is a power-bounded $S \in B(E)$ which is not uniformly ergodic, with $F(S^*) = \{0\}$. Let $T \in B(Y)$ correspond to S^*. Then $(I - T)Y = \{y \in Y : \sup_n \|\sum_{k=1}^n T^k y\| < \infty\}$ by [18], but T is not uniformly ergodic since S is not – contradicting (ii). Hence (i) must hold.

When X is separable, *(i)* ⇒ *(iii)* follows from Theorem 3.3 of [12], applied to any closed subspace Y (which also satisfies (i)). Clearly *(iii)* ⇒ *(ii)*. □

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3.26 Wacław Marzantowicz *Homotopy minimal periods for nilmanifold maps and number theory*

A natural number m is called the homotopy minimal period of a map $f : X \to X$ if it is a minimal period for every map g homotopic to f. Of course the set $\mathrm{HPer}(f)$ is a subset of the set of all minimal periods $\mathrm{Per}(f)$. In a work with Jezierski [1] we give a complete description of the set $\mathrm{HPer}(f)$ of homotopy minimal periods of a map of a compact nilmanifold X, $\dim X = d$, which extends a previous result of Jiang and Llibre for the torus ([3]). The main theorem of is the following. Let $A = A_f$ be the integral matrix of linearization of f given by the Fadell-Husseini construction (cf. [1]), $N(f)$ the Nielsen number, $L(f)$ the Lefschetz number, and

$$T_A := \{n \in \mathbb{N} \,|\, \det(\mathrm{I} - A^n) \neq 0\}.$$

Theorem 3.26.1. *Let $f : X \to X$ be a map of a compact nilmanifold X of dimension r, A a matrix associated with f and called the linearization of f* (cf Thm. 2.1 of [1]), *and $T_A \subset \mathbb{N}$ the subset defined above.*

Then $\mathrm{HPer}(f) \subset T_A$ *and it is in one of the following three (mutually exclusive) types, where the letters E, F, and G are chosen to represent "empty", "finite" and "generic" respectively:*

(E) $\mathrm{HPer}(f)$ *is empty if and only if $N(f) = L(f) = 0$;*

(F) $\mathrm{HPer}(f)$ *is nonempty but finite if and only if all eigenvalues of A are either zero or roots of unity;*

(G) $\mathrm{HPer}(f)$ *is infinite and $T_A \setminus \mathrm{HPer}(f)$ is finite.*

Moreover, for every dimension r of X, there are finite sets $P(r)$, $Q(r)$ of integers such that $\mathrm{HPer}(f) \subset P(r)$ *in Type F and* $T_A \setminus \mathrm{HPer}(f) \subset Q(r)$ *in Type G.*

The proof follows the approach of Jiang and Llibre and uses the Nielsen theory. It is based on the motion of the Nielsen number of m-periodic points, denoted by $NP_m(f)$. It is defined as the number of essential Reidemeister classes of fixed points of f^m which are irreducible, i.e. are not in the Reidemeister relation with a fixed point of f^n, $n < m$. Remind that two fixed points x, $y \in X$ of f are in the Reidemeister relation if there exist a path ω, in X such that $\omega(0) = x$, $\omega(1) = y$, and $\omega \sim f(\omega)$ rel $\{x, y\}$. A class $[x]$ of the Reidemeister relation is called essential if the fixed point index of f at $[x]$ is different from 0, and by the definition the Nielsen number $N(f)$ is equal to the number of essential classes. The main geometric ingredient is a theorem on cancelling m-periodic points of a local homeomorphism which leads to the following.

Theorem 3.26.2. *Let $f : X \to X$ be a selfmap of a compact nilmanifold X and $P_n(f)$ denote the of the set of all points with the minimal period equal to n.*

If $NP_n(f) = 0$ then f is homotopic to a map $g : X \to X$ such that $P_n(g) = \emptyset$.

For a map of a nilmanifold the general case reduces to it by a homotopy argument. The remaining arguments of the proof, combinatorial and number theory, are taken from [3]. It is worth of pointing out that the fact that in an infinite essential case (G) we have $N(f^m) > N(f^n)$ for $n < m$, $n|m$ if $m \geq m_0$, m_0 depending on $r = \dim A$ only, is the main and most difficult algebraic step in the proof such a theorem in [3]. It is a theorem which says that powers of an algebraic number of module 1 not being a root of unity can not converge to 1 too fast. We observed also that the above inequality for Nielsen numbers can be also deduced from an earlier result of Schinzel on primitive divisors in algebraic number fields.

Note that the set $\mathrm{HPer}(f)$ of all minimal homotopy periods is an invariant of the dynamics of f which is the same for a small perturbation of f. In the forthcoming paper [2] we give a complete description of the sets of homotopy minimal periods of self-maps of nonabelian three dimensional nilmanifold which is a counterpart of the corresponding characterization for three dimensional torus proved by Jiang and Llibre ([3]). As a corollary we get a Šarkovskii type theorem:

If a self map of a 3-nilmanifold different than 3-torus is such that $3 \in \mathrm{HPer}(f)$ then $\mathbb{N} \setminus 2\mathbb{N} \subset \mathrm{HPer}(f) \subset \mathrm{Per}(f)$. If $2 \in \mathrm{HPer}(f)$ then $\mathbb{N} = \mathrm{HPer}(f) = \mathrm{Per}(f)$. In particular the first assumption is satisfied if $L(f^3) \neq L(f)$ and the second if $L(f^2) \neq L(f)$.

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3.27 Christian Mauduit *Correlations in infinite and finite words*

The aim of this talk is to present a new notion of measure of correlation for finite and infinite sequences introduced by A. Sárközy and myself in a series of papers. More precisely if $E_N = (e_i)_{1 \leqslant i \leqslant N} \in \{-1, 1\}^N$ we consider for any positive integer $k \leqslant N$ the quantity

$$C_k(E_N) = \max_{\substack{M,D \\ M + d_k \leqslant N}} V_k(E_N, M, D),$$

where $D = (d_1, d_2, \ldots, d_k)$ with $0 \leqslant d_1 < d_2 < \cdots < d_k$ and

$$V_k(E_N, M, D) = \sum_{n=1}^{M} e_{n+d_1} e_{n+d_2} \cdots e_{n+d_k}.$$

A first probabilistic result shows that for any positive integer k and any $\varepsilon > 0$, there exist $N_0 \in \mathbb{N}$ and $\delta > 0$ such that if $N \geqslant N_0$, then we have with probability greater than $1 - \varepsilon$

$$\delta\sqrt{N} < C_k(E_N) < 5\sqrt{kN \log N}.$$

Then, we focus our interest on the study of correlations in the case of the following constructions:

— the Rudin-Shapiro sequence $(R_n)_{n\geqslant 0}$ defined by the relations $R_0 = 1$, $R_{2n} = R_n$ and $R_{2n+1} = (-1)^n R_n$ for $n \geqslant 0$, for which we have for any positive integers d and M

$$\sum_{n<M} R_n R_{n+d} < 2d + 4d \log_2 \frac{2M}{d};$$

— a construction suggested by P. Erdös, defined for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ by

$$e_n(\alpha) = \begin{cases} 1 & \text{if} \quad 0 \leqslant \{n^2\alpha\} < \frac{1}{2} \\ -1 & \text{if} \quad \frac{1}{2} \leqslant \{n^2\alpha\} < 1, \end{cases}$$

for which we show that for D fixed and $k = 2$ or odd $V_k(E_N(\alpha), M, D) = o(M)$, whereas

$$\lim \frac{1}{M} \sum_{n=1}^{M} e_n(\alpha) e_{n+1}(\alpha) e_{n+2}(\alpha) e_{n+3}(\alpha) \neq 0$$

for a set of α with positive measure;

— the Liouville sequence $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ denotes the number of prime factors of the positive integer n for which it seems to be hopeless to prove that

$$\underline{\lim} \frac{1}{N} \sum_{n=1}^{N} \lambda(n)\lambda(n+1) < 1,$$

but for which we proved, in collaboration with J. Cassaigne, S. Ferenczi and J. Rivat that

$$\overline{\lim} \frac{1}{N} \sum_{n=1}^{N} \lambda(n)\lambda(n+1)\lambda(n+2) \leqslant \frac{2}{3}.$$

In a recent work, we introduce a new combinatorial function, defined for any positive integer k by

$$M_k(N) = \min_{E_N \in \{-1,1\}^N} C_k(E_N) \quad \text{for } N \geqslant k.$$

If k is odd, then $M_k(N) = 1$ for $N \geqslant k$, and we proved that if k is even, then $M_k(N) \geqslant \dfrac{1}{\log 2} \log \dfrac{N}{k}$ for $N \geqslant 2k$ and we ask the following questions:

Problem 3.27.1. Is it true that there are positive constants c and γ such that $M_2(N) \geqslant cN^\gamma$ for N big enough.

The various examples we studied and computations we made lead us to ask the following questions:

Problem 3.27.2. Are there sequences E_N such that $C_2(E_N) = O(\sqrt{N})$ and $C_3(E_N) = o(N)$ for $N \to +\infty$?

Problem 3.27.3. What are the connections between $M_2(N)$ and $M_4(N)$? Is it true for example that $M_4(N) > M_2(N)$ and even that $(M_4(N) - M_2(N)) \longrightarrow +\infty$ when $N \longrightarrow +\infty$?

3.28 Miłosz M. Michalski *Some problems in statistical mechanics of superadditive lattice systems*

The subject of this report is a multidimensional generalization of so-called subadditive thermodynamic formalism developed by Falconer [3,4] in the context of dimension theory for nonconformal repellers (cf. also [1]). We are mainly interested in the statistical mechanics of $\mathbb{Z}^d$ lattice systems of interacting particles, the potential energy being given by a *superadditive* net of functions on configuration space (self-adjoints of an appropriate C*-algebra in quantum formulation, respectively) — local Hamiltonians of the system in question rather than by an explicit interaction [2,5,6]. We introduce an appropriate Banach space of Hamiltonians and study the properties of free energy density functional P on this space: similarly as in the case of standard "additive" statistical mechanics, the free energy density is shown to be convex and Lipshitz.

Using the technique of coarse graining, we define also the entropy and the mean energy densities for a superadditive system. This allows us to formulate an analogon of variational principle binding the above mentioned quantities. However, unlike in the additive case, in general no equilibrium states can be found: probability measures realizing the variational extremality condition need not obey any translational invariance, thus precluding a sound definition of entropy. Our results conform with those of Falconer.

We conclude this report with a list of problems yet to be addressed in detail.

Basic notions and properties

We consider classical and quantum lattice systems on $\mathbb{Z}^d$. In the classical case, the configuration space is $\mathcal{X} = \mathcal{X}_0^{\mathbb{Z}^d}$ with product topology, where $\mathcal{X}_0 = \{1, 2, \ldots, p\}$ represents particle species in the model (e.g. possible spin orientations).

Classical observables form the commutative real C*-algebra $\mathcal{C}(\mathcal{X})$. Macroscopic states of the system are represented by probability measures on $\mathcal{X}$, identified with respective elements of the dual space $\mathcal{C}^*(\mathcal{X})$. In quantum formulation, $\mathcal{X}_0$ is a finite dimensional Hilbert space, and for each finite $\Lambda \subset \mathbb{Z}^d$ the corresponding local observables are self-adjoint elements of the C*-algebra $\mathfrak{A}_\Lambda$ of linear operators on $\bigotimes_\Lambda \mathcal{X}_0$. The appropriate inductive limit of such C*-algebras is the algebra $\mathfrak{A}$ of quantum observables. Quantum macrostates are represented by density operators in $\mathfrak{A}$.

Definition 3.28.1. A superadditive classical hamiltonian is a net $\mathbf{H} = \{H_\Lambda\}_{\Lambda \in \mathrm{Fin}}$ of functions $H_\Lambda \in \mathcal{C}(\mathcal{X})$ satisfying the following properties:

1° superadditivity: $\quad H_{\Lambda_1 \cup \Lambda_2} \geq H_{\Lambda_1} + H_{\Lambda_2}, \quad$ for $\quad \Lambda_1 \cap \Lambda_2 = \emptyset$;

2° translation invariance: $\quad H_{\Lambda+r} = H_\Lambda \circ \tau^{-r}\;$ for all $\Lambda \in \mathrm{Fin}$, $r \in \mathbb{Z}^d$;

3° uniform bound: $\quad |||\mathbf{H}||| = \sup_\Lambda \dfrac{\|H_\Lambda\|}{|\Lambda|} < \infty.$

Quantum systems are made up of nets of self-adjoint elements H_Λ of local C*-algebras $\mathfrak{A}_\Lambda$. The superadditivity condition reads:

1° $\qquad H_{\Lambda_1 \cup \Lambda_2} \;\geq\; H_{\Lambda_1} \otimes 1\!\!1_{\Lambda_2} \,+\, 1\!\!1_{\Lambda_1} \otimes H_{\Lambda_2}, \qquad$ for $\quad \Lambda_1 \cap \Lambda_2 = \emptyset\,.$

In what follows, we deal with the classical case only, yet the transition to the quantum one is straightforward.

Proposition 3.28.1. *Nets* $\mathbf{H}$ *possessing the properties* 2° *and* 3° *form a Banach space* $\mathcal{B}$. *The collection* $\mathcal{S}$ *of superadditive nets is a closed cone in* $\mathcal{B}$. *Those* $\mathbf{H}$ *for which* $H_\Lambda/|\Lambda|$ *are equicontinuous form a closed subspace* $\mathcal{B}_e$ *of* $\mathcal{B}$.

Given $r \in \mathbb{Z}^d$ with strictly positive components, we use the following notation:

- $\Pi(r) \stackrel{\mathrm{df}}{=} \{q \in \mathbb{Z}^d:\ 0 \leq q_i < r_i,\ \ i = 1, \ldots, d\}$,
- $r|\Lambda$ (r divides Λ) $\;\Leftrightarrow\;$ Λ can be tessellated by $r\mathbb{Z}^d$-translates of $\Pi(r)$,
- $|r| \stackrel{\mathrm{df}}{=} |\Pi(r)| \qquad$ and $\qquad |\Lambda|_r \stackrel{\mathrm{df}}{=} |\Lambda|/|r|\;$ when $\;r|\Lambda$.

Free energy

The free energy density for a superadditive net $\mathbf{H} = \{H_\Lambda\}$ is defined by the limit

$$P(\mathbf{H}) \stackrel{\mathrm{df}}{=} \underset{\Lambda\to\infty}{\text{vH-lim}}\ \frac{1}{|\Lambda|}\,\log \int_{\mathcal{X}} e^{-H_\Lambda}\, d\eta,$$

where vH-lim denotes the van Hove limit, η is a fixed *reference* probability measure representing the state of noninteracting particle system (usually a product measure on $\mathcal{X}$, $\eta = \bigotimes \eta_0$).

For any $r \in \mathbb{Z}^d$ ($r > 0$) we also define the *coarse-grained* free energy

$$P_r(H_{\Pi(r)}) \;=\; \underset{\Lambda \xrightarrow{r} \infty}{\text{vH-lim}}\ \frac{1}{|\Lambda|_r}\,\log \int_{\mathcal{X}} e^{-H_\Lambda^{(r)}}\, d\eta,$$

with $\Lambda \xrightarrow{r} \infty$ meaning that Λ tends to ∞ through regions divisible by r and

$$H_\Lambda^{(r)} \;=\; \sum_{s \in \Lambda \cap r\mathbb{Z}^d} H_{\Pi(r)+s} \;=\; \sum_{s \in \Lambda \cap r\mathbb{Z}^d} H_{\Pi(r)} \circ \tau^{-s}$$

(compare this with local Hamiltonians of the standard "additive" theory,

$$H_\Lambda^{\text{add}} = \sum_{s\in\Lambda} H_{\{s\}} = \sum_{s\in\Lambda} H_0 \circ \tau^{-s},$$

all arising from a single function H_0 which gives the "per head" interaction energy of a particle at site 0 with the rest of configuration).

Proposition 3.28.2. *For a superadditive net* $\mathbf{H} = \{H_\Lambda\}$

$$P(\mathbf{H}) = \lim_{r\to\infty} \frac{1}{|r|} P_r\big(H_{\Pi(r)}\big).$$

Moreover, $P(\,\cdot\,)$ *is convex on* $\mathcal{S}$ *and Lipshitz with constant 1 there, i.e. for any* $\mathbf{G}, \mathbf{H} \in \mathcal{S}$

$$\big|P(\mathbf{G}) - P(\mathbf{H})\big| \leq |||\mathbf{G} - \mathbf{H}|||.$$

Entropy and mean energy

Entropy density is defined for $\mu \in \bigcup_r \Delta_r$, where Δ_r is the collection of $r\mathbb{Z}^d$-invariant probability measures on $\mathcal{X}$. In what follows η is a fixed reference (product) measure. We have

Proposition 3.28.3. *Let* $\mu \in \Delta_r$. *Then the following limit exists:*

$$\underset{\Lambda\to\infty}{\text{vH-lim}} -\frac{1}{|\Lambda|} \int_{\mathcal{X}_\Lambda} \frac{d\mu}{d\eta} \log \frac{d\mu}{d\eta}\, d\eta,$$

defining the entropy density $S(\mu|\eta)$. *Moreover,*

$$S(\mu|\eta) = \frac{1}{|r|} S_r(\mu|\eta),$$

where S_r *is the corresponding coarse-grained limit.*

We also obtain, along similar lines, the existence of superadditive mean energy density E for $\mu \in \bigcup_r \Delta_r$ and its relation with the respective coarse-grained quantity E_r,

$$E(\mathbf{H};\, \mu) \stackrel{\text{df}}{=} \underset{\Lambda\to\infty}{\text{vH-lim}} \frac{1}{|\Lambda|} \int_{\mathcal{X}} H_\Lambda \, d\mu = \frac{1}{|r|} E_r(\mathbf{H};\, \mu).$$

Superadditive variational principle

We have the following

Theorem 3.28.1. *Let* $\mathbf{H} = \{H_\Lambda\} \in \mathcal{S} \cap \mathcal{B}_e$. *Then*

$$P(\mathbf{H}) = \sup_{\mu\in\bigcup\Delta_r} \{\, S(\mu|\eta) - E(\mathbf{H};\, \mu) \,\},$$

yet the supremum need not be attained in the set $\bigcup_r \Delta_r$.

The lack of equilibrium states, i.e. the ones realizing the supremum above, is the consequence of the noncompactness of $\bigcup_r \Delta_r$, even though the entropy is an upper semicontinuous function of μ. The wk-* closure of this set contains states without any translational symmetry, for which the entropy is not defined.

Some further problems

Problem 3.28.1. To reformulate the subadditive variational principle in terms of P-bounded and P-tangent functionals in $\mathcal{B}^*$ using the convexity of P. This would provide a rigorous background for approximation of "pseudo-equilibrium" states by those in $\bigcup_r \Delta_r$.

Problem 3.28.2. To develop a superadditive Gibbs state theory and study relations between Gibbs and "pseudo-equilibrium" states.

Problem 3.28.3. In the one-dimensional setting, to study spectral properties of transfer operator families corresponding to superadditive Hamiltonians. To investigate properties of "superadditive" zeta functions.

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3.29 Mahendra Nadkarni[15]
Sets with doubleton sections, good sets, and ergodic theory

Let X and Y be two non-empty sets. Let S be a non-empty subset of $X \times Y$. We say that S is *good*, if every complex valued function f on S can be expressed in the form

$$f(x, y) = u(x) + v(y), \quad (x, y) \in S,$$

where u and v are functions on X and Y respectively.

If one looks for some qualitative properties of Borel support of a probability measure μ on R^Z such that the co-ordinate functions span $L^2(R^Z, \mu)$, one is led to sets of the above type. Good sets can be characterized as follows: Say that two

[15] Talk based on joint work with A. Kłopotowski, H. Sarbadhikari, and S. M. Srivastava

points $(x, y), (z, w) \in S \subseteq X \times Y$ (S is not necessarily good) are *linked* (and write $(x, y)L(z, w)$), if there exists a finite sequence of points $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ in S such that:

(i) $(x_1, y_1) = (x, y)$, $(x_n, y_n) = (z, w)$;
(ii) for any $1 \leq i \leq n - 1$ *exactly one* of the following equalities holds:

$$x_i = x_{i+1}, \; y_i = y_{i+1};$$

(iii) for any i, $1 \leq i \leq n - 2$, it is not possible to have $x_i = x_{i+1} = x_{i+2}$ or $y_i = y_{i+1} = y_{i+2}$.

An equivalence class of the relation L is called *a linked component* of S. A link L is said to be uniquely linked if any two points $(x, y), (z, w) \in L$ are joined by one and only one link. A set $S \subset X \times Y$ is good if and only if each linked component of S is uniquely linked. A good set can be written as a union if two graphs G and H, G being the graph of a function on a subset of X with values in Y and H being the graph of a subset of Y with values in X. Some natural questions arise:

(1) If $S \subset R \times R$ is a Borel set which is good and if f is a Borel measurable function on S, can one choose the function u, v to be Borel measurable?

(2) If S is good and Borel measurable, can one choose the graphs G and H mentioned above to be Borel measurable?

The answer to both these questions is affirmative if we assume that the partition of S into linked components admits a Borel cross-section. On the other hand if the partition into linked components is not countably generated, the answer to both the questions above is, in general, negative. The counter-examples are provided by the mathematics of subsets of R^2 with doubleton sections. A subset S of R^2 is said to have doubleton sections if each section $S_x = \{(t, s) \in S : t = x\}$ and $S^y = \{(t, s) \in S : s = y\}$ of S is a two point set.

To a set with doubleton sections are associated groups G and G_0 defined as follows: if $(x, y) \in S$ then there is a unique $(x, y') \in S$ with $y \neq y'$ which we denote by $v(x, y)$ and there is a unique $(x', y) \in S$ with $x \neq x'$ which we denote by $h(x, y)$. Clearly $v^2 = h^2 =$ identity map. We write G for the group generated by v and h and G_0 the subgroup of G generated by hv. G/G_0 is a two point set. G and G_0 act on S in a natural fashion. One observes that a set S with doubleton sections is good if and only if all the G orbits (hence G_0 orbits) are infinite.

M. Laczkovich has given an example of a Borel set $S \subset [0, 1] \times [0, 1]$ with doubleton sections such that its partition into G-orbits is not countably generated and that S does not admit a Borel subset which is the graph of a one-one function on $[0, 1]$. (*Closed Sets Without Measurable Matchings*, Proc. Amer. Math. Soc., Vol. 103, No.3, July 1988, pp. 894-896.) One can show that this set can not be written as a union two Borel graphs G and H as above and that there are Borel measurable functions f on this set for which u and v can not be chosen to be Borel.

We raise some question connected with the above theme.

Problem 3.29.1. If S is a good Borel subset of R^2 whose linked components are not countably generated, does S admit a Borel subset with doubleton sections whose linked components are not countably generated?

Problem 3.29.2. Call a probability measure μ on R^2 good if every $f \in L^2(R^2, \mu)$ is of the form $f(x, y) = u(x) + v(y)$, with $u, v \in L^2(R^2, \mu)$. If one tries to characterize such μ's, one is led to the following question. Let $p = (p_1, p_2, \cdots)$ be an infinite probability vector viewed as a probability measure on natural numbers N. For a function $f \in L^2(N, p)$, write $S(f) = (f(1), f(1) + f(2), f(1) + f(2) + f(3), \cdots)$. Describe p for which $f \in L^2(N, p)$ implies that $S(f)$ is in $L^2(N, p)$.

A characterization of good subsets in product $X_1 \times X_2 \times \cdots \times X_n$ of n non-empty sets $X_1, X_2, \cdots X_n$, $n \geq 2$ is given by K. P. S. Bhaskara Rao. Let $S \subset X_1 \times X_2 \cdots \times X_n$ and let $M = \{x(1) = (x_1^1, x_2^1, \cdots x_n^1), \cdots, x(k) = (x_1^k, x_2^k, \cdots, x_n^k)\}$ be a set k points in S. We say that M forms a loop in S if there exist non-zero integers $p_1, \cdots, p_k$ such that $p_1 x(1) + \cdots p_k x(k) = 0$, i.e., for each $j, 1 \leq j \leq n$ the formal sum $n_1 x_j^1 + p_2 x_2 + \cdots + n_k x_j^k$ is zero. A subset S of $X_1 \times X_2 \times \cdots \times X_n$ is good (in the sense that every complex valued function $f(x_1, \cdots, x_n)$ on S can be written as a sum $u_1(x_1) + \cdots + u_n(x_n)$ of n functions of one variable) if and only if S does admit a loop.

The above discussion has a consequence for measures arising from stochastic processes of "multiplicity one".

Let μ be a measure on R^Z such that the *co-ordinate functions span* $L^2(R^Z, \mu)$ (multiplicity one). Let $Z = A_1 \cup A_2 \cup \cdots \cup A_n$, with $A_i \cap A_j = \emptyset$ if $i \neq j$. Write $X_i = R^{A_i}$. Then μ admits a support which is good in $X_1 \times X_2 \times \cdots \times X_n$.

3.30 Hitoshi Nakada *On a transformation associated to mediant convergents of Rosen's continued fractions*

Let $\mathbb{H}$ be the Hecke group of index 4, i.e. $\mathbb{H}$ is the group of linear fractional transformations generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$. We consider the following generalized Diophantine inequality for a fixed $c > 0$:

$$(1) \qquad |x - \frac{p}{q}| < \frac{c}{q^2}, \quad q > 0,$$

where $\frac{p}{q} = g(\infty)$ for some $g \in \mathbb{H}$. Since $\mathbb{H}$ is a discontinuous group, q is uniquely determined for each $g(\infty)$. It is known that there exists the Hurwitz constant c_H: (1) has infinitely many solutions $g(\infty)$ for each $x \in \mathbb{R} \setminus \{g(\infty),\ g \in \mathbb{H}\}$ if $c \geq c_H$ and has at most finitely many solutions for some $x \in \mathbb{R} \setminus \{g(\infty),\ g \in \mathbb{H}\}$ if $c < c_H$.

Each x has the Rosen continued fraction expansion, which is the nearest-integer type continued fraction with $\mathbb{Z} \cdot \sqrt{2}$-valued coefficients. This expansion is given by the following transformation T of $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$Tx = |\frac{1}{x}| - [|\frac{1}{x}|]_*,$$

where $[\cdot]_*$ denotes the nearest point in $\mathbb{Z}\cdot\sqrt{2}$. There exists a Legendre constant c_L: for any x,

$$(2) \qquad |x - \frac{p}{q}| < \frac{c}{q^2}$$

implies $\frac{p}{q}$ is the principal convergent of Rosen continued fraction whenever $c \leq c_L$ and this property does not hold for some x_0 when $c > c_L$.

It is possible to show that $c_L = \sqrt{2}-1$ by the individual ergodic theorem with the natural extension of T. This means that c_L is determined by an almost everywhere property. It is easy to see that $c_L < c_H$, which implies that we can not determine the explicit value of the Hurwitz constant only by the Rosen continued fractions.

Let S be the map of $[-\frac{1}{\sqrt{2}}, \sqrt{2})$ defined by

$$Sx = \begin{cases} -\frac{1}{x} - \sqrt{2}, & x \in [-\frac{1}{\sqrt{2}}, -\frac{\sqrt{2}}{3}) \\ -\frac{x}{\sqrt{2}x+1}, & x \in [-\frac{\sqrt{2}}{3}, 0) \\ 0, & 0 \\ \frac{x}{-\sqrt{2}x+1}, & x \in [0, \frac{\sqrt{2}}{3}) \\ \frac{1}{x} - \sqrt{2}, & x \in [\frac{\sqrt{2}}{3}, \sqrt{2}). \end{cases}$$

It is easy to get the mediant convergents of Rosen continued fractions by this map S. If we take the jump transformation of S with $J = [-\frac{1}{\sqrt{2}}, -\frac{\sqrt{2}}{3}) \cup [\frac{\sqrt{2}}{3}, \frac{1}{\sqrt{2}})$ (one jump after hitting J), we get T. It is possible to construct the natural extension of S on a subset of $\mathbb{R}^2$. With this, we see that the Legendre constant c_L^* associated to the mediant convergents is equal to $\frac{1}{\sqrt{2}}$. Since S is infinite measure preserving, we use the ratio ergodic theorem to get this value.

As an application, we can show that $c_H = \frac{1}{2}$. Moreover this method holds for the Hecke group of index k for any $k \geq 4$. A. Haas and C. Series('86) calculated these Hurwitz constants by consideration of the maximum height of the congruence class of geodesics.

3.31 Gunter Ochs *One more proof of the multiplicative ergodic theorem*

Abstract

The Multiplicative Ergodic Theorem of Oseledets is a fundamental tool in the local analysis of deterministic and stochastic dynamical systems. There are several published proofs. Here we present a new approach which relies on the construction of "random" invariant measures on the projective space (resp. on Grassmannians)

Introduction

Our goal is to give a short proof of the following result due to V. I. Oseledets [3] well known as the Multiplicative Ergodic Theorem (MET):

Theorem 3.31.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\theta : \Omega \to \Omega$ an invertible and ergodic $\mathbb{P}$ preserving transformation, and $A : \Omega \to \mathrm{Gl}(d, \mathbb{R})$ a measurable map, where $\mathrm{Gl}(d, \mathbb{R})$ denotes the space of invertible $d \times d$ matrices endowed with any matrix norm $|\cdot|$ and corresponding Borel σ-algebra. Assume $\int \log^+ |A(\omega)| \mathbb{P}(d\omega) < \infty$, where $\log^+ x = \max\{\log x, 0\}$. Then there exist numbers (independent of ω)*

$$\infty > \lambda_1 > \lambda_2 > ... > \lambda_p \geq -\infty \qquad \textit{called } \text{Lyapunov exponents}$$

and linear subspaces $E_1(\omega), ..., E_p(\omega)$ (called Oseledets spaces, *they depend measurably on ω) with ($\mathbb{P}$ a.s.)*

- $\mathbb{R}^d = E_1(\omega) \oplus ... \oplus E_p(\omega)$,
- *$E_1, ..., E_p$ are invariant in the sense $A(\omega)E_k(\omega) = E_k(\theta\omega)$,*
- *they are dynamically characterized by*

$$x \in E_k(\omega) \setminus \{0\} \Leftrightarrow \lim_{n \to \pm\infty} \frac{1}{n} \log |A(n, \omega)x| = \lambda_k,$$

where $A(n, \omega)$ denotes the cocycle *generated by A, which is defined by*

$$A(n, \omega) = \begin{cases} A(\theta^{n-1}\omega)...A(\omega), & \text{if } n \geq 0, \\ A(\theta^n\omega)^{-1}...A(\theta^{-1}\omega)^{-1}, & \text{if } n < 0. \end{cases}$$

This theorem is a fundamental result in smooth ergodic theory, where the situation Ω a smooth manifold and θ a diffeomorphism preserving any Borel probability measure $\mathbb{P}$ is considered. In this setup the application of Oseledets' theorem to the tangent cocycle $A(n, \omega) = D_\omega \theta^n$ serves as a basis for the construction of invariant (stable, unstable,...) manifolds.

Another important application lies in the analysis of linear stochastic dynamical systems, where the MET provides a substitute of linear algebra. In the case of nonlinear stochastic systems it can be applied to the linearization (for an extensive description of the role of the MET in stochastic dynamics see Arnold [1]).

Before we pass to the main ideas of our proof we make some remarks on the assumptions.

Remark 3.31.1. (i) Here we deal exclusively with discrete time. A continuous time version could be derived by considering the time–one mapping and controlling "what happens in between" by an additional integrability condition.

(ii) Invertibility and ergodicity of θ is not an essential restriction. If θ is not ergodic, one can use an ergodic decomposition and apply the theorem to each ergodic component separately. In this case the Lyapunov exponents in general depend on ω. In the non–invertible case it is possible to consider a canonical extension.

(iii) The case of non-invertible (even rectangular) matrices $A(\omega)$ can be handled by restriction to the invariant subbundle

$$F(\omega) := \bigcap_{n \geq 0} A(n, \theta^{-n}\omega)\mathbb{R}^d.$$

Then $A(\omega) : F(\omega) \to F(\theta\omega)$ is invertible.

There are at least 16 published proofs of the MET (cf. [1], page 112). The original one of Oseledets relies on triangularization of the cocycle. In the today most established way of proving the MET (following Ragunathan [4], Ruelle [5] and others) Kingman's subadditive ergodic theorem is used to derive the existence of the Lyapunov exponents. The construction of the Oseledets spaces is "hard work" consisting of matrix calculations such as singular value decomposition. An alternative approach is due to Walters [7], who uses some properties of invariant measures on the projective space.

The following section is devoted to a sketch of the main ideas of our proof. It consists of a "direct" construction of invariant measures on the projective space respectively on suitable Grassmannians (in contrast to Walter's approach, who uses the existence of some invariant measures without studying there structure explicitly), whose supports will be the Oseledets spaces. The main ingredient of our proof besides standard results in ergodic theory like Poincaré recurrence, Birkhoff's Ergodic Theorem and Kingman's Subadditive Ergodic Theorem (which was not available for Oseledets when he found his original proof, but is an important tool in later proofs of the MET) is a "random version" of the Krylov–Bogolyubov theorem due to H. Crauel.

Sketch of the proof

We start with some notation. For $1 \leq k \leq d$

$$\mathrm{Gr}(k) := \{U \subset \mathbb{R}^d : \dim U = k\}$$

is a compact manifold called *Grassmannian* ($\mathrm{Gr}(1)$ coincides with the projective space $\mathcal{P}^{d-1}$). A *skew product* is well defined by

$$\Theta_k : \Omega \times \mathrm{Gr}(k) \to \Omega \times \mathrm{Gr}(k), \quad (\omega, U) \mapsto (\theta\omega, A(\omega)U).$$

The following statements on random measures can be found in Crauel [2], see also Arnold [1], Chapter 1.5. The space

$$M_k := \{\mu \text{ probability on } \Omega \times \mathrm{Gr}(k) : \pi_\Omega \mu = \mathbb{P}\},$$

where π_Ω denotes the projection onto Ω, is compact in the topology of weak convergence (for a definition see below). Each $\mu \in M_k$ can be identified with a *random measure* $(\mu_\omega)_{\omega \in \Omega}$ ($\mathbb{P}$–a.s. uniquely defined by μ) via the *disintegration* $\mu(d\omega, dU) = \mu_\omega(dU)\mathbb{P}(d\omega)$. The *invariant measures*

$$I_k := \{\mu \in M_k : \Theta_k \mu = \mu\}$$

form a closed (and hence compact) subset of M_k. We have $\mu \in I_k$ if and only if $A(\omega)\mu_\omega = \mu_{\theta\omega}$ $\mathbb{P}$–a.s., which in turn implies $A(\omega)K(\omega) = K(\theta\omega)$, where $K(\omega)$ denotes the support of μ_ω. Finally set

$$\phi_k : \Omega \times \mathrm{Gr}(k) \to \mathbb{R}, \quad (\omega, U) \mapsto \log |\det (A(\omega)|_U)|,$$

where $|\det (A(\omega)|_U)|$ is the growth rate of the k dimensional volume in U under application of $A(\omega)$. Then we have

$$\begin{aligned}
&\log |\det (A(n,\omega)|_U)| \\
&= \log \left|\det (A(\omega)|_U) \cdot \det \left(A(\theta\omega)|_{A(\omega)U}\right) \cdot \det \left(A(\theta^2\omega)|_{A(2,\omega)U}\right)\right| \cdot \ldots \\
&= \sum_{i=0}^{n-1} \log \left|\det \left(A(\theta^i\omega)|_{A(i,\omega)U}\right)\right| = \sum_{i=0}^{n-1} \phi_k \left(\theta^i\omega, A(i,\omega)U\right) \\
&= \sum_{i=0}^{n-1} \phi_k \left(\Theta_k^i(\omega, U)\right) =: S_n\phi_k(\omega, U).
\end{aligned}$$

Now we are prepared to perform our proof in three steps.

Step 1. We start with the construction of an invariant subspace, where the highest exponential growth rate is realized, i.e. the Oseledets space $E_1(\cdot)$. Note that this is in contrast to the usual proofs of the MET, which start "from below", i.e. with λ_p and $E_p(\cdot)$.

For simplicity we assume the additional integrability condition

$$\int \log |A^{-1}(\omega)| \mathbb{P}(d\omega) < \infty,$$

which implies

$$\int \|\phi_k(\omega, \cdot)\|_\infty \mathbb{P}(d\omega) < \infty \Leftrightarrow: \phi_k \in \mathrm{L}^1\left(\mathbb{P}; C(\mathrm{Gr}(k), \mathbb{R})\right)$$

the space of functions from $\Omega \times \mathrm{Gr}(k)$ to the reals which are measurable in the first and continuous in the second coordinate, and whose supremum norm over U is integrable with respect to $\mathbb{P}$

$$\|\phi_k(\omega, \cdot)\|_\infty = \sup_{U \in \mathrm{Gr}(k)} \phi_k(\omega, U).$$

We emphasize that we do not really need this second integrability condition, but it makes the argumentation easier.

For $1 \le k \le d$ define

$$\gamma_k := \lim_{n\to\infty} \frac{1}{n} \log \sup_{U \in \mathrm{Gr}(k)} |\det (A(n,\omega)|_U)|.$$

The limit exists and is $\mathbb{P}$–a.s. independent of ω by the Subadditive Ergodic Theorem. Furthermore, convergence holds also in $\mathrm{L}^1(\mathbb{P})$.

The relation

$$|\det(A(n,\omega)|_{U\oplus V})| \leq |\det(A(n,\omega)|_U)| \cdot |\det(A(n,\omega)|_V)|$$

for orthogonal subspaces U, V implies $\gamma_k \leq k\lambda_1$ with $\lambda_1 := \gamma_1$. Choose now k maximal with $\gamma_k = k\lambda_1$ (such a k exists, since equality holds at least for $k = 1$).

For each $n > 0, \omega \in \Omega$ choose a subspace $U(n,\omega) \in \mathrm{Gr}(k)$, which maximizes $|\det(A(n,\omega)|_U)|$. Define a measure $\nu_n \in M_k$ via its disintegration

$$\nu_n(d\omega, dU) = \delta_{U(n,\omega)}(dU)\mathbb{P}(d\omega)$$

and set

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} \Theta_k^i \nu_n \in M_k.$$

Then

$$\int \phi_k d\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\int \phi_k \circ \Theta_k^i d\nu_n = \frac{1}{n}\int S_n\phi_k d\nu_n$$
$$= \frac{1}{n}\int \log|\det(A(n,\omega)|_U)|\,\nu_n(d\omega, dU)$$
$$= \frac{1}{n}\int \log \sup_{U\in \mathrm{Gr}(k)} |\det(A(n,\omega)|_U)|\,\mathbb{P}(d\omega),$$

which converges by the Subadditive Ergodic Theorem to γ_k as $n \to \infty$.

At this point we have to go a little bit deeper into the theory of random measures. We say $(\mu_n)_{n\in\mathbb{N}} \subset M_k$ *converges weakly* to $\mu \in M_k$, if $\lim_{n\to\infty}\int \psi d\mu_n = \int \psi d\mu$ for every $\psi \in \mathrm{L}^1(\mathbb{P}; C(\mathrm{Gr}(k), \mathbb{R}))$.

Theorem 3.31.2. (Crauel [2], see also [1] Theorem 1.5.8) (i) *Every sequence in M_k has an accumulation point with respect to weak convergence.*

(ii) (*"random" Krylov-Bogolyubov theorem*)
Let $(\nu_n)_{n\geq 0} \subset M_k$ and $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\Theta_k^i\nu_n$.

Then every accumulation point μ of (μ_n) lies in I_k, i.e. it is invariant under the skew product Θ_k.

Of course $\mathrm{Gr}(k)$ could be replaced with any compact metric space X and $A(\omega)$ with any continuous map $f(\omega) : X \to X$ depending measurably on ω.

To continue Step 1 of our proof choose any accumulation point μ of (μ_n). It follows $\int \phi_k d\mu = \lim_{n\to\infty}\int \phi_k d\mu_n = \gamma_k$.

By Birkhoff's ergodic theorem we have for μ-a.e. (ω, U)

$$\lim_{n\to\pm\infty}\frac{1}{n}\log|\det(A(n,\omega)|_U)| = \gamma_k. \qquad (*)$$

If $(*)$ holds for (ω, U) and (ω, V), then one can show that the angle between $A(n,\omega)U$ and $A(n,\omega)V$ converges to 0 as $n \to \infty$ (otherwise it would be possible to construct an $(k+1)$ dimensional subspace with "too large" volume growth, i.e. then $\gamma_{k+1} \geq$

$\gamma_k + \lambda_1 = (k+1)\lambda_1$, which contradicts the choice of k). This in turn implies, that μ_ω must be concentrated in one point, i.e. there exists $E_1(\omega) \in \mathrm{Gr}(k)$ with $\mu_\omega = \delta_{E_1(\omega)}$ $\mathbb{P}$–a.s. The invariance of μ implies the invariance of $E_1(\cdot)$, i.e. $A(\omega)E_1(\omega) = E_1(\theta\omega)$.

Since the asymptotic growth rate of k dimensional volumina in $E_1(\cdot)$ under application of the cocycle $A(n,\omega)$ equals $\gamma_k = k\lambda_1$ and the growth rate of one dimensional volumina is bounded from above by λ_1 (by the definition of $\lambda_1 = \gamma_1$), all one dimensional subspaces within $E_1(\cdot)$ must have asymptotic growth rate exactly λ_1. In addition the growth rate of l dimensional subspaces of $E_1(\cdot)$ (with $1 \le l \le k$) must be $l\lambda_1$.

Step 2. Here we construct a higher dimensional invariant random subspace which will turn out to be the direct sum of the Oseledets spaces $E_1(\cdot)$ and $E_2(\cdot)$. To do this we apply Step 1 to the projection of our system to the orthogonal complement of $E_1(\cdot)$.

Denote by $P(\omega)$ the orthogonal projection onto $E_1(\omega)^\perp$ and set

$$\hat{A}(n,\omega) := P(\theta^n\omega)A(n,\omega) : E_1(\omega)^\perp \to E_1(\theta^n\omega)^\perp.$$

The invariance of $E_1(\cdot)$ implies that $\hat{A}$ defines a cocycle, i.e.

$$\hat{A}(n+m,\omega) = \hat{A}(m,\theta^n\omega)\hat{A}(n,\omega) \quad \text{for } n,m \in \mathbb{Z},\ \omega \in \Omega.$$

Using the arguments of Step 1 we derive the existence of $\lambda_2 \in \mathbb{R}$ and of an l dimensional subspace $\hat{E}(\omega) \subset E_1(\omega)^\perp$ such that all j dimensional volumes within $\hat{E}(\cdot)$ for $1 \le j \le l$ have asymptotic growth rate $j\lambda_2$ under the application of the cocycle $\hat{A}(\cdot)$. This implies that $F(\omega) := E_1(\omega) \oplus \hat{E}(\omega)$ is invariant under $A(\cdot)$. If $E_1(\omega) \subset U \subset F(\omega)$ and $\dim U = k + j$, then the growth rate of $(k+j)$ dimensional volumes within U equals $k\lambda_1 + j\lambda_2$. This in particular implies (by the choice of k) that $\lambda_2 < \lambda_1$.

Step 3. Here we construct an invariant subspace $E_2(\omega) \subset F(\omega)$, which is "complementary" to $E_1(\omega)$.

In order to do this, choose $V(n,\omega) \in \mathrm{Gr}(l)$, $V \subset F(\omega)$, which minimizes $|\det(A(n,\omega)|_U)|$. Define $\nu_n \in M_l$ via

$$\nu_n(d\omega, dU) = \delta_{V(n,\omega)}(dU)\mathbb{P}(d\omega),$$

set

$$\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} \Theta_l^i \nu_n \in M_l,$$

and choose an accumulation point of (μ_n). In a similar way as in Step 1 it is possible to show $\int \phi_l d\mu = l\lambda_2$ and that μ_ω is supported in one point $E_2(\omega)$, which yields the desired invariant subspace.

Further steps. Now repeat Steps 2 and 3 in order to construct λ_3, $E_1(\cdot) \oplus E_2(\cdot) \oplus E_3(\cdot)$, $E_3(\cdot)$, λ_4,...

Concluding remarks

We did not only give another proof of Oseledets' result, but also a slight generalization. Namely, we are able to establish the full Oseledets splitting (in place of a *flag* of invariant subspaces) without integrability condition on the inverse $A(\cdot)^{-1}$, even in the case that the inverse does not exist. We did not point out here how to argue in this case, but the main idea is to restrict our attention to an invariant subbundle, where the cocycle is invertible and the norm of the inverse is log integrable.

However, there are several other (and probably more important) generalizations of the MET, which we did not consider up to now. There exists an infinite dimensional version due to Ruelle [6]. It is not clear, if our approach carries over to this case. The main problem seems to be that the proof presented here crucially depends on compactness of the Grassmannians, which does not hold in infinite dimensions. Other generalizations of the MET deal with more general group actions in place of the action of $\mathrm{Gl}(d,\mathbb{R})$ on $\mathbb{R}^d$. It seems possible, that our proof works in a more general (and more abstract) setup, but we haven't tried this out yet.

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3.32 François Parreau
On the Foiaş and Stratila theorem

Let T be an ergodic measure-preserving automorphism of a standard probability space $(X,\mathcal{B},\mu)$. The Foiaş and Stratila theorem ([2]) asserts that, if the spectral measure σ of a non zero function f in $L^2(\mu)$ is continuous and supported by a Kronecker set, then the process $(f\circ T^n)$ must be Gaussian (recall that a closed subset K of the circle group $\mathbb{T}$ is a Kronecker set if every modulus one continuous function defined on K is a uniform limit of characters).

This result of spectral determination is almost unique in ergodic theory. However the assumption is very restrictive and the original proof does not really allow to understand this phenomenon. So, it is natural to ask about possible extensions, and to try to explain the role of harmonic properties of the spectral support. Another motivation comes from recent common work of the author with M. Lemańczyk and J.-P. Thouvenot ([3], [4]).

Our main results are the following:

Theorem 3.32.1. *Let f be a non-zero function in $\bigcap_{2\le p<+\infty} L^p(\mu)$ such that*

$$(C) \qquad \sum_{p=2}^{\infty} (1/\|f\|_p) = +\infty.$$

If the spectral measure of f is continuous and concentrated on an independent compact set, then $(f \circ T^n)$ is a Gaussian process.

A closed set $K \subset \mathbb{T}$ is said to be a Helson set with constant α $(0 < \alpha \le 1)$, or a Helson–α set, if $\sup_{n\in\mathbf{Z}} |\hat{\sigma}(n)| \ge \alpha \|\sigma\|$ for every complex Borel measure σ on K. A Kronecker set is an independent Helson–1 set ([5], Chap. X).

Theorem 3.32.2. *Let f be a non-zero function in $L^2(\mu)$. If the spectral measure of f is continuous and concentrated on an independent Helson set, then $(f \circ T^n)$ is a Gaussian process.*

Besides, the Foiaş and Stratila Theorem, as well as the results above, extend to all ergodic measure–preserving actions $(T_g)_{g\in G}$ of a locally compact second countable Abelian group G.

The problem remains open if we only assume that the spectral measure is continuous and concentrated on an independent compact set. However, under the weaker hypothesis that the corresponding Gaussian automorphism has simple spectrum, there are counter-examples of measures σ (e.g. the spectral type of the Chacon transformation) which do not satisfy the spectral determination property.

While spectral measures concentrated on Kronecker sets lead to rigid Gaussian automorphisms, a positive answer for all independent sets would yield mixing examples, by a well known result of Rudin.

The Helson hypothesis forbids mixing for the corresponding Gaussian processes but it allows mild mixing, and even a spectral form of partial mixing (usual partial mixing, without strong mixing, never occurs for Gaussian automorphisms). Indeed, by a result of T. Körner (see [5], Chap. X), for each α in $(0,1]$, there exists a Borel probability measure σ, concentrated on an independent Helson–α set K of $\mathbb{T}$, with the property:

$$\textit{for every Borel set } B \subset \mathbb{T}, \quad \limsup |(\sigma|_B)\hat{}\,(n)| = \alpha\, \sigma(B).$$

Then the corresponding Gaussian automorphism T satisfies: for every zero–mean square integrable function h,

$$\limsup |\langle T^n h, h\rangle| = \limsup |\hat{\sigma}_h(n)| \le \alpha \|h\|_2^2.$$

The basic idea of the proofs consists, as in [2], in showing that the spectral process associated to f has independent increments (for the extension to other Abelian group actions, we construct an analogue of the spectral process with the same continuity properties). It is sufficient to show that, for every disjoint open sets U_1 and U_2 in $\operatorname{supp}\sigma$, the factors $\mathcal{B}_i$ generated by the spectral projections $E(U_i)f$ are independent. Under the condition (C), which is known as the Carleman condition for Hamburger's moment problem, each $L^2(\mathcal{B}_i)$ is spanned by polynomials of functions with spectral support in U_i. Then Foiaş' group property (if g, h and $gh \in L^2$, then $\operatorname{supp}\sigma_{gh} \subset \operatorname{supp}\sigma_g \cdot \operatorname{supp}\sigma_h$, [1], Proposition 2) yields that the spectral type of $T|_{\mathcal{B}_i}$ is concentrated on the group generated by U_i. If $\operatorname{supp}\sigma$ is algebraically independent, the factors are spectrally disjoint and Theorem 1 follows.

Now, in order to prove Theorem 2, it is enough to show that, when K is a Helson set, the corresponding spectral subset is spanned by functions with property (C). This follows from a result of independent interest:

Theorem 3.32.3. *Let K be a Helson subset of $\mathbb{T}$. Then, for every $p \geq 2$, the spectral projector $E(K)$ maps $L^p(\mu)$ into itself with, for every $g \in L^p(\mu)$,*

$$\|E(K)g\|_p \leq C\,\|g\|_p$$

where C depends only on the Helson constant of K.

Theorem 3 is proved by approximation of $E(K)$ by operators of the form $\varphi(T)$, where φ approximates the indicator function of K and $\varphi \in A(\mathbb{T})$ (the space of functions with absolutely convergent Fourier series). A lower bound $C(\varepsilon)$ for the $A(\mathbb{T})$-norm of such a φ equal to 1 on K and less than ε on a given closed set F disjoint from K yields a bound for the norm of $\varphi(T)$ on $L^p(\mu)$. The main lemma in the solution by S. Drury and N. Varopoulos [7] of the long-standing union problem for Helson sets gives a bound depending only on the Helson constant of K and a universal construction. Finally, a later work of J.-F. Méla [6] allows us to show $C(\varepsilon) \leq C\,|\log\varepsilon|$, which is the best possible and precisely what we need to conclude, using interpolation between the L^p spaces.

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3.33 Alexandre Prihod'ko *Stochastic constructions of mixing local rank one flows*

Let us consider a measure preserving flow (T^t) on a Lebesgue probability space $(X, \mathcal{B}, \mu)$. A *Rokhlin-Halmos tower* of length h for (T^t) is a map $\phi\colon U \to (0, h)$, $U \subseteq X$, such that for any Borel set $B \subseteq (0, h)$ the pre-image $\phi^{-1}B \in \mathcal{B}$, and if $B, t + B \subseteq (0, h)$ then $T^t\phi^{-1}B = \phi^{-1}(t + B)$. Denote $[\phi_n] = U$.

A flow (T^t) is called a *rank one* flow if there is a sequence of towers ϕ_n of height h_n with two properties: $[\phi_n] \to 1$ and for any measurable A there are sets $A_n = \phi_n^{-1}B_n$ (B_n are Borel subsets of $(0, h_n)$) such that $\mu(A \triangle A_n) \to 0$.

The first example of a mixing rank one transformation was given by Ornstein [2]. We propose a kind of stochastic construction of mixing rank one flows as well as construction (pure) local rank one flows.

Let (T^t) be a rank one flow given by a cutting-and-stacking construction with heights h_n and spacers $s_{n,k}$. We consider a family of constructions for which $s_{n,k}$ are equidistributed random values with the distribution ξ_n. The formal definition is as follows. Let h_n and ξ_n be given and let $\Phi_n(t)$ be the stationary process, $\Phi_n(t) \in (0, h_n) \sqcup \{*\}$, which sample consists of disjoint intervals $\mathcal{I} = (u, u + h_n)$ of length h_n separated by spacers independently distributed as ξ_n, such that $\Phi(t) = t - u$ if $t \in \mathcal{I}$, and $\Phi(t) = *$ if t does not belong to any interval $\mathcal{I}$. We restrict Φ_n to $(0, h_{n+1})$ and say that intervals of length h_n which lies entirely inside $(0, h_{n+1})$ correspond to subcolumns of n th tower and the spacers between them in a sample path of Φ_n correspond to spacers $s_{n,k}$. Thus, we have constructed a random family of flows determined by the parameters h_n and ξ_n.

Suppose that $\xi_n \ll h_n$ and has density p_n which is a function of bounded variation. Let $\sigma_n^2 = D\xi_n$, $\bar{\kappa}_n = h_n^2/\sigma_n^2$ and $\bar{q}_n = h_{n+1}/h_n$.

Theorem 3.33.1. *If for some constants $\epsilon, \gamma, C_1, C_2 > 0$ and $0 < \chi < 1$ the following conditions hold*

(1) $\|(1 + C_1\sigma_n^2x^2)^{1/2}\,\hat{p}_n(x)\|_\infty = 1;$ (2) $\prod_{j<n} \bar{q}_j \le C_2\,\bar{q}_n^{\gamma};$

(3) $\|\bar{q}_{n-1}^{-\chi} h_n p_n\|_\infty \to 0;$ (4) $\bar{\kappa}_n/\bar{q}_n^{1-\chi-\epsilon} \to 0,$

then the flow (T^t) is mixing almost surely.

We say that a flow (T^t) has *β-rank one* if there is a sequence of towers ϕ_n of height h_n satisfying the following conditions: $[\phi_n] \to \beta$ and for any measurable A there are sets $A_n = \phi_n^{-1}B_n$ (B_n are Borel subsets of $(0, h_n)$) and $C_n \subseteq X \setminus [\phi_n]$ such that $\mu(A \triangle (A_n \sqcup C_n)) \to 0$ (see [1]).

Modifying this construction as follows we'll provide a construction of mixing rank one flows. Fix $\beta \in (0,1)$. Letting $I_n = (0, h_n)$, consider a family of independent stationary processes

$$\Phi_n^{(k)} \colon \mathbb{R} \to (I_n \times \{0, 1, \ldots, r_n\}) \sqcup \{*\}, \qquad k = 0, 1, \ldots, r_{n+1},$$

which are defined in the same way as Φ_n, the only difference is that all the points in a non-$*$ interval $\mathcal{I}$ of length h_n in a sample path of $\Phi_n^{(k)}$ have common second coordinate which is distributed on the sequence of intervals $\mathcal{I}$ according to the Bernoulli scheme with probability vector $(\beta, (1-\beta)/r_n, \ldots, (1-\beta)/r_n)$. As before let $\phi_n^{(k)} \colon (0, h_{n+1}) \to (I_n \times \{0, 1, \ldots, r_n\}) \sqcup \{*\}$ be the map (which coincides mainly with $\Phi_n^{(k)}$) such that $\phi_n^{(k)}(t) = \Phi_n^{(k)}(t)$ iff t belongs to an interval $\mathcal{I}$ of length h_n on which $\Phi_n^{(k)}(t) \neq *$ and $\mathcal{I} \subset (0, h_{n+1})$, otherwise $\phi_n^{(k)}(t) = *$. Let

$$\psi_n \colon (I_{n+1} \times \{0, 1, \ldots, r_{n+1}\}) \sqcup \{*\} \to (I_n \times \{0, 1, \ldots, r_n\}) \sqcup \{*\}$$

be the map coinciding with $\phi_n^{(k)}$ on $I_{n+1} \times \{k\}$, and such that $\psi_n(*) = *$.

Theorem 3.33.2. *Assume that $r_n = n$ and h_n and ξ_n satisfy the conditions of Theorem 1. Then almost surely we can define correctly a Lebesgue space $(X, \mathcal{B}, \mu)$ which is the inverse limit of spaces $(I_n \times \{0, 1, \ldots, r_n\}) \sqcup \{*\}$ with respect to maps ψ_n, as well as a natural flow on it. This construction produces a family of pairwise disjoint β-rank one mixing flows (T^t), which are of infinite rank and not of $\tilde{\beta}$-rank one for any $\tilde{\beta} > \beta$. Moreover, the flows (T^t) have almost surely* MSJ(2) *property.*

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3.34 Mary Rees *The resident's view of parameter space*

In holomorphic dynamics, it is natural to study parameter spaces of holomorphic maps for which dynamics is constant on some finite forward-invariant set Z. The set Z could be a periodic orbit, for example, or a subset of the full orbit of a periodic orbit. Indeed, this is a common strategy in dynamics in general. For example, take some hyperbolic periodic orbit $Z(f)$ of a diffeomorphism f. Then $Z(g)$ varies isotopically for sufficiently small C^1 perturbations of f, as do compact subsets of the unstable and stable manifolds. Then one might try to study movement of orbits of other points under g, relative to $Z(g)$, as g moves: for example, of homoclinic intersection points of the stable and unstable manifolds. Such studies have been made. This is one example of what might be described loosely as the *resident's view*, that is , a local piece of parameter space viewed and analysed in terms of f.

Such an approach is particularly attractive in complex dynamics, because the dynamics of a rational map is very much influenced by the behaviour of its critical points, as the classical studies of Fatou and Julia show. The best-known parameter space, that of quadratic polynomials $f_c(z) = z^2 + c$, is well-understood, at least modulo some hard conjectures, because, conjecturally every polynomial in the Mandelbrot set can be viewed in terms of the map (and resident) f_0. Of course, the Julia set moves as the parameter is varied in the Mandelbrot set, but very often (whenever the Julia set is locally connected), the Julia set is a topological and dynamical quotient of $(\{z : |z| = 1\}, f_0)$, and the view of f_0 is of the critical point moving relative to the original circle. This gives rise to a conjectural description of the Mandelbrot set, its topology and full variation of dynamical behaviour within it.

The quadratic family f_c is unusually simple (although far from completely understood analytically) because there is a canonical choice of path between any two polynomials in the Mandelbrot set, at least if one assume that the Mandelbrot set is locally connected. In most parameter spaces, there is no "best" path between two given maps, even up to crude homotopy. This is important, because it is often possible to describe one map in terms of another given map, and a path between the two in parameter space [1], that is, one resident can view another in terms of its own dynamics and a chosen line of vision. The resident views paths in parameter space as paths in its own dynamical plane.

As an example of the parameter spaces under consideration, consider rational maps

$$g_{c,d}(z) = 1 + c/z + d/z^2$$

for which the critical points are 0 and $-2d/c$, and the orbit of 0 starts $0 \mapsto \infty \mapsto 1 \mapsto 1+c+d$. Let $V(n)$ $(n \geq 3)$ denote the set of $g_{c,d}$ for which 0 has period n. For any map $g \in V(n)$ let

$$Z_m(g) = g^{-m}(\{g^i(0) : i \geq 0\}).$$

Let $v_2(g) = g_{c,d}(-2d/c)$, the second critical value. Let

$$P(n,m) = \{g_{c,d} \in V(n) : v_2(g) \in Z_m(g)\},$$

$$V(n,m) = V(n) \setminus P(n,m).$$

Then $V(n,m)$ is a punctured variety whose points are rational maps g such that $Z_m(g) \cup \{v_2(g)\}$ varies isotopically for $g \in V(n,m)$, and the punctures of $V(n)$ are the points $P(n,m)$. Fix some basepoint $h_0 \in V(n,m)$. Consider $\pi_1(V(n,m), P(n,m), h_0)$, the set of paths in $V(n,m)$ from h_0 to $P(n,m)$, modulo homotopy. Consider also the set $\pi_1(\hat{\mathbf{C}} \setminus Z_m(h_0), Z_m(h_0), v_2(h_0))$ of paths in $\hat{\mathbf{C}} \setminus Z_m(h_0)$ from $v_2(h_0)$ to $Z_m(h_0)$ modulo homotopy. There is a natural map (described briefly in the talk and also in [R1])

$$\Phi : \pi_1(V(n,m), P(n,m), h_0) \to \pi_1(\hat{\mathbf{C}} \setminus Z_m(h_0), Z_m(h_0), v_2(h_0))$$

Both $V(n,m)$ and $\hat{\mathbb{C}} \setminus Z_m(h_0)$ are Riemann surfaces covered by the disc D, and the two homotopy sets of paths to punctures above can be considered as subsets of the boundary ∂D. Part of the Theorem we have is [2]

Theorem 3.34.1. *Φ is injective. Regarded as a map on a subset of ∂D, the inverse map is monotone and extends continuously to a monotone map of ∂D to itself which collapses countably many intervals.*

One consequence of this is that the universal cover of $V(n,m)$ can be regarded as a subset of the universal cover of $\hat{\mathbb{C}} \setminus Z_m(h_0)$. So the resident h_0 views the universal cover of the parameter space within the universal cover of its punctured dynamical plane. The rest of the universal cover of the resident's punctured dynamical plane is a disjoint union of regions, each of which corresponds to the universal cover of a set of branched coverings which, up to homotopy, has some geometric structure. For example, if $n = 4$ and $m = 0$, there are regions corresponding to degree two branched coverings which leave invariant some subsurface which is a four-holed surface, and the branched coverings act as a pseudo-Anosov on this. All pseudo-Anosovs occur in this way.

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3.35 Ben-Zion Rubshtein *Classifications of one-sided Markov shifts*

One-sided Markov shifts, corresponding to positive recurrent Markov chains with countable (finite or infinite) state spaces, are considered. A complete system of explicit invariants for the isomorphism classes of the shifts is found. Every one-sided Markov shift can be represented in a canonical form by means of a uniquely defined stochastic graph and by a minimal n-point extension of the graph. It is shown that two one-sided Markov shifts are isomorphic iff their canonical stochastic graphs are isomorphic and the corresponding extensions are equivalent. Our approach to the isomorphic problem applies the theory of decreasing sequences of measurable partitions (cofiltrations) and a classification of the cofiltrations generated by one-sided Markov shifts is obtained.

3.36 Ryszard Rudnicki *Typical properties of fractal dimensions*

We present some recent results concerning box and packing dimensions of typical compact sets [4] and box and correlation dimensions of typical probabilistic measures

[5,6].

Let $\mathcal{X}$ be a complete metric space. Recall that a countable union of nowhere dense sets is said to be of the *first Baire category*. A subset of a complete metric space $\mathcal{X}$ is called *residual* in $\mathcal{X}$ if its complement is of the first Baire category. If the set of all elements of $\mathcal{X}$ satisfying some property P is residual in $\mathcal{X}$, then the property P is called *typical* or *generic*. We also say that the typical element of $\mathcal{X}$ has property P.

The space $\mathcal{C}(X)$ of all non-empty compact subsets of a complete metric space (X, ρ) endowed with the Hausdorff metric

$$d(A, B) = \max\{\max_{x\in A} \rho(x, B), \max_{x\in B} \rho(x, A)\}$$

is a complete metric space. The study of typical properties of compact sets has a long history starting with the works of Kuratowski [3] and Ostaszewski [7]. A survey of many results on this subject is given in Renfro [8]. We recall some of them to show that the typical compact set (t.c.s.) is simultaneusly "small" and "large".

If μ is a continuous Borel measure on X then t.c.s. has μ-measure zero. If $h : [0, \infty) \to [0, \infty)$ is a strictly increasing function and $h(0) = 0$ then t.c.s. has Hausdorff h-measure zero. T.c.s. has also the lower box dimension zero. We recall that the *lower (upper) box dimension* of a compact set A is the lower (upper) limit of the function $\alpha(r) = -\log N(A, r)/\log r$ as $r \to 0$, where $N(A, r)$ is the least number of balls of radius r which cover the set A.

The above results show that t.c.s. is "small". But t.c.s. is also "large". Namely, if the space X has no isolated points, then t.c.s. has no isolated points, which implies that t.c.s. has cardinality $\geq \mathfrak{c}$. Moreover, t.c.s. in the space $\mathbb{R}^d$ has the upper box dimension d. More general result has been proved by Gruber [2]: *if the collection of compact sets in X having lower box dimension at least δ is dense in $\mathcal{C}(X)$, then t.c.s. has upper box dimension at least δ.*

We need two auxiliary notions. The number

$$\text{sl-}\overline{\dim}\, A = \inf\{\overline{\dim}\,(B(x, r) \cap A) : x \in A,\ r > 0\}$$

is called *the smallest local upper box dimension* of the set A. Analogously, we define *the smallest local lower box dimension* $\text{sl-}\underline{\dim}\, A$ replacing in the above definition $\overline{\dim}$ by $\underline{\dim}$. Now we can formulate the Gruber theorem in a different way: *if A is t.c.s. then* $\overline{\dim}\, A \geq \text{sl-}\underline{\dim}\, X$. In [4] we strengthen the Gruber's result.

Theorem 3.36.1. *If A is t.c.s. then* $\overline{\dim}\, A \geq \text{sl-}\overline{\dim}\, A \geq \text{sl-}\overline{\dim}\, X$.

If $\dim_P A$ is the packing dimension of a closed set A, then $\dim_P A \geq \text{sl-}\overline{\dim}\, A$. From this inequality and Theorem 1 follows immediately

Corollary 3.36.1. *If A is t.c.s. then* $\dim_P A \geq \text{sl-}\overline{\dim}\, X$.

Typical properties of measures has been studied in [1,5,6]. We present here some results concerning box dimension [5] and correlation dimension [6] of probabilistic measures. By $\mathcal{M}$ we denote the set of all probability Borel measures on a complete metric space (X, ρ). The space $\mathcal{M}$ is endowed with the *Fortet-Mourier distance:*

$$\operatorname{dist}(\mu, \nu) = \sup\{|\int_X f(x)\,\mu(dx) - \int_X f(x)\,\nu(dx)| : f \in \mathcal{L}\}, \quad \mu,\ \nu \in \mathcal{M},$$

where $\mathcal{L}$ is the subset of $C(X)$ which contains all the functions f such that $|f(x)| \leq 1$ and $|f(x) - f(y)| \leq \rho(x, y)$ for $x, y \in X$. The space $(\mathcal{M}, \operatorname{dist})$ is complete. The sequence (μ_n), $\mu_n \in \mathcal{M}$, is weakly convergent to a measure $\mu \in \mathcal{M}$ if and only if $\lim_{n\to\infty} \operatorname{dist}(\mu_n, \mu) = 0$. The quantity

$$\underline{\dim}\,\mu = \lim_{\kappa\to 0^+} \inf\{\underline{\dim}\, E : \ E \in \mathcal{B}(X),\ \mu(E) \geq 1 - \kappa\}$$

is called *lower box dimension* of μ. Analogously, we define *the upper box dimension* $\overline{\dim}\,\mu$ replacing in the above definition $\underline{\dim}$ by $\overline{\dim}$. The *lower correlation dimension* of μ is defined by

$$\underline{\dim}_c\,\mu = \underline{\lim}_{r\to 0} \frac{1}{\log r} \log \int_X \mu(B(x, r))\, d\mu(x).$$

Analogously, we define *the upper correlation dimension* $\overline{\dim}_c\,\mu$.

Theorem 3.36.2 ([5]). *If μ is the typical probabilistic measure then*

1. $\underline{\dim}\,\mu = 0$,
2. $\overline{\dim}\,\mu \geq \inf\{\overline{\dim}\,E : \mu(E) > 0\} \geq \text{sl-}\underline{\dim}\,X$,
3. *if X is separable then* $\operatorname{supp}\mu = X$.

Theorem 3.36.3 ([6[). *If μ is the typical probabilistic measure then $\underline{\dim}_c\,\mu = 0$ and*

$$\text{sl-}\underline{\dim}\,X \leq \overline{\dim}_c\,\mu \leq \text{sl-}\overline{\dim}\,X.$$

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3.37 Jörg Schmeling[16] *Applications of multifractal analysis to number theory*

We study the Hausdorff dimension of a large class of sets of real numbers satisfying certain number-theoretical properties related to the representation in some base. In particular, our results unify and extend classical work of Borel and Eggleston. Our methods are based on recent results concerning the multifractal analysis of dynamical systems.

Instead of trying to formulate general statements at this point, we want to discuss explicit examples, which illustrate well the nature of our work. Given an integer $m > 1$, for each number $x \in [0,1]$ we shall denote by $x = 0.x_1x_2\cdots$ a base-m representation of x. It is easy to see that this representation is unique except for a countable set of points. We remark that since countable sets have zero Hausdorff dimension, the non-uniqueness of the representation does not interfere with our study. Due to the non-uniqueness we shall always use the representation for which the digits are the smallest possible.

For each $k \in \{0,\ldots,m-1\}$, $x \in [0,1]$, and $n \in \mathbb{N}$ we set

$$\tau_k(x,n) = \operatorname{card}\{i \in \{1,\ldots,n\} : x_i = k\}.$$

Whenever there exists the limit

$$\tau_k(x) = \lim_{n\to\infty} \frac{\tau_k(x,n)}{n},$$

it is called the *frequency* of the number k in the base-m representation of x. A classical result of Borel [4], says that for Lebesgue-almost every $x \in [0,1]$ we have $\tau_k(x) = 1/m$ for every k. Therefore, Lebesgue-almost all numbers are normal in every base.

This remarkable result does not mean that the set of non-normal numbers is empty. In particular, it was established by Eggleston [5] that the set of numbers

$$F_m(\alpha_0,\ldots,\alpha_{m-1}) = \{x \in [0,1] : \tau_k(x) = \alpha_k \text{ for } k = 0, \ldots, m-1\},$$

composed of those numbers having a percentage α_k of digits equal to k in its base-m representation for each k, has Hausdorff dimension

$$\dim_H F_m(\alpha_0,\ldots,\alpha_{m-1}) = -\sum_{k=0}^{m-1} \alpha_k \log_m \alpha_k, \tag{3.1}$$

whenever $\alpha_0 + \cdots + \alpha_{m-1} = 1$ with $\alpha_i \in [0,1]$ for each i. Define

$$L = \{(\alpha_0,\ldots,\alpha_{m-1}) \in [0,1]^m : \alpha_0 + \cdots + \alpha_{m-1} = 1\}$$

[16] The talk is based on joint work with L. Barreira and B. Saussol

and

$$M_k = \left\{ x \in [0,1] : \liminf_{n\to\infty} \frac{\tau_k(x,n)}{n} < \limsup_{n\to\infty} \frac{\tau_k(x,n)}{n} \right\}.$$

We have

$$[0,1] = \bigcup_{(\alpha_0,\ldots,\alpha_{m-1})\in L} F_m(\alpha_0,\ldots,\alpha_{m-1}) \cup \bigcup_{k=0}^{m-1} M_k. \tag{3.2}$$

An easy consequence of the work of Eggleston is that if $0 < \alpha_i < 1$ for some i, then the set $F_m(\alpha_0,\ldots,\alpha_{m-1})$ is non-empty, and is in fact everywhere dense in $[0,1]$, and has positive Hausdorff dimension.

Here we want to provide further detailed information about the decomposition in (3.2). In particular we shall establish the following statement.

Theorem 3.37.1. *For each k the set M_k is everywhere dense in $[0,1]$, and*

$$\dim_H \bigcap_{k=0}^{m-1} M_k = 1.$$

This implies that the union $\bigcup_{k=0}^{m-1} M_k$ in (3.2) also has Hausdorff dimension equal to 1, and thus from the point of view of dimension theory it is as large as the interval $[0,1]$. On the other hand the set $\bigcup_{k=0}^{m-1} M_k$ has zero Lebesgue measure as well as zero measure with respect to any Bernoulli measure $\mu_{\alpha_0,\ldots,\alpha_{m-1}}$ on m symbols. We note that all these measures are mutually singular, and thus the set $\bigcup_{k=0}^{m-1} M_k$ must be considered rather small from the point of view of measure theory.

In order to simplify the exposition let us now consider $m = 3$, and the set

$$F = \{x \in [0,1] : \tau_1(x) = 5\tau_0(x)\}.$$

This is the set of number in $[0,1]$ such that its base-3 expansion has a percentage of 1's which is five times the percentage of 0's. The percentage of 2's is arbitrary. It is easy to see that

$$F \supset \bigcup_{\alpha\in[0,1/6]} F_3(\alpha, 5\alpha, 1-6\alpha). \tag{3.3}$$

We shall show that

$$\dim_H F = \frac{\log(1 + 6/5^{5/6})}{\log 3} \approx 0.85889\cdots. \tag{3.4}$$

We remark that it is easy to show that the number in (3.4) is a lower bound for $\dim_H F$. Namely, it follows from (3.1) and (3.3) that

$$\dim_H F \geq \max_{\alpha\in[0,1/6]} -\frac{\alpha\log\alpha + 5\alpha\log(5\alpha) + (1-6\alpha)\log(1-6\alpha)}{\log 3}. \tag{3.5}$$

The maximum is attained at $\alpha = 1/(5^{5/6} + 6)$, and it is a straightforward computation to show that it is equal to the right-hand side in (3.4). This establishes the lower bound.

The corresponding upper bound is more delicate, since the union in (3.3) is composed of an uncountable number of non-empty mutually disjoint sets. Our approach is based on a *conditional variational principle*, which we now formulate in the particular case considered here.

Theorem 3.37.2. *For each $k \neq \ell$ and $\beta \geq 0$ we have*

$$\dim_H \left\{ x \in [0,1] : \frac{\tau_k(x)}{\tau_\ell(x)} = \beta \right\} = \max \left\{ -\sum_{j=0}^{m-1} \alpha_j \log_m \alpha_j : \frac{\alpha_k}{\alpha_\ell} = \beta \right\}.$$

An easy consequence of Theorem 3.37.2 and (3.1) is that for each $k \neq \ell$ and $\beta \geq 0$, there exists $(\alpha_0, \ldots, \alpha_{m-1}) \in L$ such that

$$F_m(\alpha_0, \ldots, \alpha_{m-1}) \subset \left\{ x \in [0,1] : \frac{\tau_k(x)}{\tau_\ell(x)} = \beta \right\}$$

with

$$\dim_H F_m(\alpha_0, \ldots, \alpha_{m-1}) = \dim_H \left\{ x \in [0,1] : \frac{\tau_k(x)}{\tau_\ell(x)} = \beta \right\}.$$

In particular, this implies that the inequality in (3.5) is in fact an identity, thus establishing the claim in (3.4).

The statements formulated above are consequences of much more general statements established in [1]. Our results are based on recent work concerning the multifractal analysis of dynamical systems, and in particular they require a multidimensional version of the classical multifractal analysis. The multifractal analysis of dynamical systems is in turn strongly based on the thermodynamic formalism. This method allows to consider more advanced problems:

Let A be an $m \times m$ matrix and $v \in \mathbb{R}^n$. Then we are able to compute the dimension of the set of reals having the digit relation $\mathbf{A}\tau + \mathbf{v} = \tau$.

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3.38 Martin Schmoll
Pointwise asymptotic formulas on families of translation surfaces

The aim of the talk is to describe new ways to obtain results on the quadratic growth rate of geodesics on translation surfaces. Translation surfaces are pairs (Σ, ω), where Σ is a compact connected Riemann surface without boundary and ω is a holomorphic differential on Σ. ω induces a flat Riemann metric $|\omega|^2 := \omega \otimes_S \bar{\omega}$ on Σ. $|\omega|$ has conic singularities of total angle $2n\pi$ (where $n \in \mathbb{N}$), in the zeros of ω. Away from the zeros of ω the metric space $(\Sigma, |\omega|)$ is locally isometric to Euclidean plane. Natural coordinates $\zeta := \int_\gamma \omega$ of (Σ, ω) are obtained by integrating ω along a path γ, coordinate changes in this coordinates are obviously translations: $\zeta = \zeta' + c$ with $c \in \mathbb{C}$. This causes the name "translation surface" or "translation structure". Examples for translation surfaces are the invariant surfaces in the phase space of rational polygon billiard. Furthermore any translation structure can be viewed as point in the cotangent space of the Teichmüller space of Σ. There is a family of vector fields X_θ dual to ω parameterized by their "direction" $\theta \in S^1$. The leaves of these vector fields are geodesics and they are isometric to straight lines in $\mathbb{R}^2$. The problem is to count growth rates of certain types of geodesics with finite length. Examples of interesting kinds are closed geodesics or saddle connections (:= geodesics which start and end at singular points). Since the notion of direction makes sense on translation surfaces one can associate to any geodesic of finite length a vector in $\mathbb{R}^2$ defined by the length and the direction of the geodesic. For simplicity from now on we restrict to closed geodesics. If $\mathcal{V}_{cg}(\sigma, \omega)$ denotes the set of image vectors of all closed geodesics on (Σ, ω) in $\mathbb{R}^2$ (with multiplicity if they are not homotopic (modulo singular points)), let

$$N_{cg}((\Sigma, \omega), T) := |\{v \in \mathcal{V}_{cg}(\Sigma, \omega) : \ |v| < T\}|$$

be the associated counting function. The question is, does the limit

$$\lim_{T \to \infty} \frac{N_{cg}((\Sigma, \omega), T)}{T^2}$$

as a function of (Σ, ω) exist? And if it exists, can one compute it? A result of Eskin and Masur [1] shows that the existence question is equivalent to the existence of the limit (where we use the shorthand $u := (\Sigma, \omega)$):

$$\lim_{t \to \infty} \int_0^{2\pi} \sum_{v \in \mathcal{V}_{cg}(u)} f(a_t \exp(i\theta) v)\, d\theta \quad \text{with} \quad a_t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \in SL_2(\mathbb{R}^2), \ f \in C_0^\infty(\mathbb{R}^2_+)$$

and $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$. The $SL_2(\mathbb{R})$ action on translation surfaces which is used here comes from post composition of natural charts with $A \in SL_2(\mathbb{R})$. Moduli spaces of translation surfaces (up to isomorphy) with fixed orders of singularities and normalized volume are smooth manifolds in a natural way. On each connected component $\mathcal{M}$ of such a space there is a natural $SL_2(\mathbb{R})$ invariant probability measure μ which is ergodic with respect to the a_t action. Veech [5] obtains (for more general setup) the following formula: there exists a constant $c_{\mathcal{V},\mu}$ such that for any $f \in C_0^\infty(\mathbb{R}_+^2)$:

$$\int_M \hat{f}(u)\, d\mu(u) = c_{\mathcal{V},\mu} \int_{\mathbb{R}^2} f(x,y)\, dx\, dy \quad \text{with} \quad \hat{f}(u) := \sum_{v \in \mathcal{V}_{cg(u)}} f(v).$$

For the space $(\mathcal{M}, \mu)$ Eskin and Masur [1] proved furthermore:

$$\lim_{t \to \infty} \int_0^{2\pi} \hat{f}(a_t \exp(i\theta)u)\, d\theta = \int_{\mathcal{M}} \hat{f}(u)\, d\mu(u) \quad \mu \text{ a.e.}$$

These results are valid only almost everywhere, but if the parameter space of translation structure is a homogeneous space, then Ratner's classification theorem for ergodic measures on homogeneous spaces be used to obtain pointwise results. Families of torus coverings branched over given points are examples for homogeneous parameter spaces [2]. In the case of saddle connections (sc) on two marked tori, one can use elementary methods to compute the function $x \mapsto c_{sc}(x)$ (see [3]):

$$\begin{aligned} \left[\frac{p_1}{n}, \frac{p_2}{n}\right] &\longmapsto \frac{5}{\pi}\left(1 + \prod_{p|n \text{ prime}} \left(1 - \frac{1}{p^2}\right)^{-1} \sum_{\gcd(i,n)=1} \left(\frac{1}{p^2} - \frac{1}{n^2}\right)\right) \quad \text{on } \mathbb{Q}^2 \\ [x,y] &\longmapsto \frac{6}{\pi} + \pi \quad \text{for } (x,y) \notin \mathbb{Q}^2 \end{aligned}$$

(where we assume $\gcd(p_1, p_2, n) = 1$). This function is continuous at non rational points, thus the above formula approaches $\frac{6}{\pi} + \pi$ as $\left(\frac{p_1}{n}, \frac{p_2}{n}\right)$ converges to a non rational number. This continuity can be proven by a direct estimate, it is not restricted to 2 markings. It also holds for general n markings (with other limiting constants and sets, of course). For any n the limit quadratic growth constants as functions of the marking are maximal exactly at the points of continuity. The constant in the Siegel-Veech formula for the above example is the value $\frac{6}{\pi} + \pi$ at the continuity points. In view of the theory of polygonal billiards one question is:

Problem 3.38.1. Are the Siegel Veech constants $c_{\mathcal{V},\mu}$ (at least in the case of saddle connections) an upper bound for the growth rates in the spaces $(\mathcal{M}, \mu)$, even if there are no pointwise asymptotic growth rates in $\mathcal{M}$?

If the Siegel Veech constants are maximal, the next question is:

Problem 3.38.2. Can one understand their behaviour as function of the genus and singularity pattern of the underlying translation surface? The computation of the Siegel Veech constants is of course a problem in its own right.

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3.39 Jacek Serafin[17]
On the finite generator theorem

This is an abstract of an article, which will be submitted elsewhere, in which we address the means to obtain a finite generating partition for an ergodic finite entropy transformation of a probability space (the theorem of Krieger). We present an elementary reduction of the cardinality of a generating partition from the countable to the finite number, using a simple coding technique due originally to Shannon.

Let X be the set of all bilaterally infinite sequences (x) in which each coordinate x_t is a positive integer, S the (left) shift transformation on X, and let μ be a probability measure μ defined on the σ-algebra $\mathcal{A}$ generated by the coordinate mappings, which is S-invariant and ergodic. We define

$$p_n := \mu(\{x \in X : x_0 = n\}) \quad (n \in \mathbb{N}).$$

and assume that

$$-\sum_{n \in \mathbb{N}} p_n \log p_n < \infty;$$

so the entropy of a countable generator is finite. Further, let A be a finite set and denote by Y the set of all bilaterally infinite sequences (y) with coordinates y_t in A. Denote by T the (left) shift transformation on Y, and by $\mathcal{B}$ the σ-algebra on Y generated by the coordinate mappings. We prove the following statement, which, together with the countable generator theorem implies the existence of a finite generating partition.

Theorem 3.39.1. *If the cardinality of A is sufficiently large, then there exists a mapping ϕ defined for μ-almost every point $x \in X$, taking values in Y, which is an isomorphism between the dynamical systems $(X, \mathcal{A}, \mu, S)$ and $(Y, \mathcal{B}, \mu \circ \phi^{-1}, T)$.*

We fix a positive integer M (which will be specified later). For each $x \in X$ define the sequence

$$\phi_1(x) := z := (\ldots, z_{-1}, z_0, z_1, \ldots)$$

by setting

$$z_t = \begin{cases} x_t & \text{if} \quad x_t \le M \\ \star & \text{if} \quad x_t > M. \end{cases}$$

[17] Joint work with Michael Keane

The map ϕ_1 is called the *marker mapping* and a symbol $m > M$ a *marker*. We now specify the alphabet A for Y. We set

$$A := \{(a, \epsilon) : a \in \{1, \dots, M, \star\}, \epsilon \in \{0, 1\}\}.$$

We also improve on the description of the mapping ϕ of the theorem:

$$\phi(x) = (\phi_1(x), \phi_2(x)),$$

where $\phi_1(x)$ is as above, and where $\phi_2(x)$ will be (for μ-almost every x) a sequence of zeroes and ones, containing the necessary information to determine which symbols $m, (m > M)$ are hidden behind the $\star$'s.

To each symbol (marker) $m > M$ we associate a finite binary prefix code $c(m)$ of length l_m by defining

$$\delta := \sum_{m>M} p_m > 0, q_m := \frac{p_m}{\delta} \qquad (m > M),$$

and setting

$$l_m := \lceil \log(1/q_m) \rceil \qquad (m > M).$$

It is not difficult to prove that a prefix code with the length function l_m as above exists (it is usually called the Shannon code) and for this code we have the following bound

$$\sum_{m>M} l_m q_m \leq - \sum_{m>M} q_m \log q_m + 1 \tag{3.1}$$

(prefix code means that no code word is a prefix of another code word).

As $\delta > 0$, μ-almost every $x \in X$ will contain infinitely many markers occurring at coordinates

$$\dots t_{-1} < 0 \leq t_0 < t_1 < \dots.$$

The marker m_i which occurs at t_i has a prefix code $c(m_i)$ as described in the previous section of length $h_i := l_{m_i}$. Thus we have also a doubly infinite sequence of binary words, the i-th word having length h_i. We now produce, in a shift invariant manner, a sequence

$$\phi_2(x) = (\dots, \epsilon_{-1}, \epsilon_0, \epsilon_1, \dots) = \epsilon$$

of zeroes and ones from the words $c(m_i), i \in \mathbb{Z}$, so that given ϵ and $\phi_1(x)$, we can recover the $c(m_i)$. We use the following algorithm:

1. For each i, start writing the code word $c(m_i)$ at coordinate t_i into ϵ. Stop if either the code word has been exhausted ($t_{i+1} - t_i \geq h_i$) or if coordinate t_{i+1} is reached. Do this for each $i \in \mathbb{Z}$.

2. Now shift all of the unwritten parts of the code words one place to the right (from i to $i+1$), consider these as new code words and the remaining unwritten ϵ-places between t_i and t_{i+1} as new possible receptors, and repeat step 1.

3. If, after an infinite number of iterations of steps 1 and 2 some unwritten ϵ-coordinates remain, write 0 into each such coordinate.

It is easy to see that the above procedure succeeds in writing all of the code word $c(m_i)$ if and only if there exists a coordinate $j > i$ such that

$$t_j - t_i \geq \sum_{k=i}^{j-1} h_k. \tag{3.2}$$

The algorithm to recover the code words $c(m_i)$ and thus the markers m_i from the sequence ϵ together with the knowledge of the t_i, making essential use of the prefix property of the code c, is the following:

1. For each i, start reading ϵ at t_i and stop if either a code word has been recognized or coordinate t_{i+1} is reached. Do this for each i.

2. Shift the symbols unead to the left (from i to $i-1$) and continue reading as in step 1, for each i.

Condition (3.2) is fulfilled for each i for almost every $x \in X$ if M is sufficiently large, by a simple application of the ergodic theorem. Namely, it will hold if the average gap size $t_{i+1} - t_i$ is larger than the average code length h_i. We know that the average length is given by (3.1), and from the Kac lemma, the average gap size is $1/\delta$. Hence a sufficient condition is

$$-\sum_{m>M} q_m \log q_m + 1 < \frac{1}{\delta}$$

or

$$\delta + \delta \cdot (-\sum_{m>M} q_m \log q_m) < 1.$$

A simple calculation shows that the left hand side converges to 0 as M tends to ∞, so our condition is satisfied for large enough M, and our theorem is proved.

3.40 Artur Siemaszko[18] *Relative topological Pinsker factors*

I. The existence of the largest factor of a topological dynamical system with zero entropy (so called topological Pinsker factor) was shown in [3]. The idea was to find the largest factor without entropy pairs (see [1] for the definition of an entropy pair). In [8] another approach is presented. Let (X, T) be a topological dynamical system. So X is assumed to be a σ-compact Hausdorff topological space and $T : X \longrightarrow X$ to be a homeomorphism. In the sequel $\mathcal{B}(X)$ denotes the σ-algebra of Borel subset of a topological space X.

[18]The talk is based on joint work with M. Lemańczyk

Let $\{R_i\}_{i\in I}$ be a family of closed (as subsets of $X \times X$) equivalence relations on X. Put $R = \bigcap_{i\in I} R_i$. Denote by π_i (π) the canonical map $\pi_i : X \to X/R_i$ ($\pi : X \to X/R$), $i \in I$. Fix $K \subset X$ a compact set. Denote $K/R := \pi(K) \subset X/R$ and $K/R_i := \pi_i(K) \subset X/R_i$. Let $p_{i,K} : K/R \to K/R_i$ stand for the corresponding canonical maps.

Let m be a probability measure on $\mathcal{B}(X/R)$. By $m|$ denote its restriction to K/R. Put $\tilde{\mathcal{B}}(K/R_i) = p_{i,K}^{-1}(\mathcal{B}(K/R_i))$ and let the join $\bigvee_{i\in I} \tilde{\mathcal{B}}(K/R_i)$ denote the smallest σ-algebra of $\mathcal{B}(K/R)$ containing all $\tilde{\mathcal{B}}(K/R_i)$, $i \in I$. The main ingredient in the approach of [8] is the following lemma.

Lemma 3.40.1. *Assume that X is a topological Hausdorff space (not necessarily σ-compact). Then $\mathcal{B}(K/R) = \bigvee_{i\in I} \tilde{\mathcal{B}}(K/R_i) \mod m$ for any compact set $K \subset X$.*

If we assume in addition that X is σ-compact we get the following corollary.

Lemma 3.40.2. $\mathcal{B}(X/R) = \bigvee_{i\in I} \tilde{\mathcal{B}}(X/R_i) \mod m$.

Using Lemma 3.40.2 for the family $\{R_i\}$ of all relations with $h(X/R_i) = 0$ and the variational principle (twice) we obtain the following result.

Proposition 3.40.1. *If T is a homeomorphism of a σ-compact Hausdorff space X and the variational principle holds for all factors then the topological Pinsker factor of (X,T) exists.*

II. The first approach to the relativization of the notion of the topological Pinsker factor was presented in [6] as a generalization of results of [3]. Now we assume that the factor $\pi : (X,T) \longrightarrow (Y,S)$ is given. Let R_π denote the invariant closed equivalence relation associated with π and $E_\pi = E_X \cap R_\pi$, where E_X denotes the set off all entropy pairs of (X,T). In [6] a version of relative topological Pinsker factor is defined by $X_{P_1|Y} = X/\langle E_\pi\rangle$, where $\langle A\rangle$ denotes the smallest invariant closed equivalence relation containing A. We call it a relative topological Pinsker$_1$ factor of (X,T) with respect to (Y,S). So $X_{P_1|Y}$ is the greatest factor between (X,T) and (Y,S) with no entropy pairs in fibers. It is shown in [6] that $h(X_{P_1|Y}) = h(Y)$.

The idea described in Section I leads to another notion of a relative topological Pinsker factor. Now consider the family $\{R_i\}$ of relations with $h_m(\mathcal{B}(X/R_i)|Y) = 0$ for every $m \in M(X,T)$ and $R = \bigcap_i R_i$. The similar methods to those used in the proof of Proposition 3.40.1 show that X/R is the greatest factor between (X,T) and (Y,S) with the property that $h_m(\mathcal{B}(X/R)|Y) = 0$ for every $m \in M(X,T)$. We denote the factor X/R by $X_{P_2|Y}$ and call it the *relative topological Pinsker$_2$ factor* of (X,T) with respect to (Y,S). It is shown in [8] that $h(X_{P_2|Y}) = h(Y)$. Unlike in the absolute case, where both notions become equivalent, it turns out that in general $X_{P_1|Y}$ is a proper factor of $X_{P_2|Y}$. In [8] $X_{P_2|Y}$ is described as the greatest factor between (X,T) and (Y,S) with the property that for every $\nu \in M(Y,S)$, $h(X_{P_2|Y}|\pi^{-1}(y)) = 0$ ν-a.e. Using the results of [5] one can easily show that the "ν-a.e" condition can actually be replaced by the "for every $y \in Y$" condition.

One may naturally raise the following question. Why do not we consider the family $\{R_i\}$ of relations with $h(X/R_i) = h(Y)$ instead of the more complicated condition with invariant measures given above? The answer is given by Downarowicz example ([4]), presented in [8] which shows that it is possible to have two relations R_i, $i = 1, 2$ with $h(X/R_i) = h(Y)$ while $h(X/R_1 \cap R_2) > h(Y)$. There is, however, at least one natural case when both ways are equivalent. Namely when we assume that (Y, S) has a constant entropy function property, i.e. $h_\nu(Y) = h_{\nu'}(Y)$ for all ν, $\nu' \in M(Y, S)$ (for example all distal extensions of uniquely ergodic systems have this property) then $X_{P_2|Y}$ is the greatest factor between (X, T) and (Y, S) with $h(X_{P_2|Y}) = h(Y)$.

III. The relative topological Pinsker_1 factor is defined in terms of entropy pairs. Is it possible to describe $X_{P_2|Y}$ in a similar way? The affirmative answer is given in [7].

Given $m \in M(X, T)$ a pair $(x, x') \in X \times X$ is called a *relative m-entropy pair* with respect to the factor Y if for every Borel partition $\mathcal{F} = \{F, F'\}$ of X with $x \in \text{Int}(F)$ and $x' \in \text{Int}(F')$

$$h_m(\mathcal{F}|Y) > 0.$$

See [2] for the definition in the absolute case. The set of all relative m-entropy pairs we denote by $E_{m|Y}$. Note that if Y is the one-point flow then $E_{m|Y} = E_m$, where E_m denotes the set of all m-entropy pairs. It is worth to mention that the usual in topological dynamics way of relativization according to which one should consider $E_{m|Y} = E_m \cap R_\pi$ instead of the above definition does not work in this case. Let us denote $R_m = \langle E_{m|Y} \rangle$. We call $X/R_m = X_{P|Y}(m)$ the *relative topological m-Pinsker factor* of (X, T) with respect to (Y, S). See [2] for the definition in the absolute case. Let $\sigma_m : (X, m) \longrightarrow (\mathcal{P}_{X|Y}m, \nu)$ be the measure theoretical Pinsker factor of X with respect to Y and let P_{σ_m} denote the relation given by $(x, x') \in P_{\sigma_m}$ if $\sigma_m(x) = \sigma_m(x')$. Let R be the relation defining $X_{P_2|Y}$.

Proposition 3.40.2. (1) $E_{m|Y} = E_m \cap \overline{P}_{\sigma_m}$;

(2) $R = \bigvee_{m \in M(X,T)} R_m = \bigvee_{m \in M^e(X,T)} R_m,$

hence $X_{P_2|Y}$ is the greatest common factor of all $X_{P|Y}(m)$'s;

(3) $R = \left\langle \bigcup_{m \in M(X,T)} E_{m|Y} \right\rangle = \left\langle \bigcup_{m \in M^e(X,T)} E_{m|Y} \right\rangle.$

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3.41 Sergey Sinelshchikov[19] *Complete positivity of entropy and non-Bernoullicity for transformation groups*

We present a construction of c.p.e. non-Bernoullian actions for a class of countable amenable groups. The construction itself is quite plausible. Nevertheless, to prove that it possesses the property of completely positive entropy (c.p.e.), we need a reversion of a quite recent result by Rudolph and Weiss, which also makes some independent interest.

The following definition is due to D. Rudolph and B. Weiss. Let G be a countable amenable group and $K \subset G$ a finite set. A finite set $S \subset G$ is said to be *K-spread* if for all $\gamma_1 \neq \gamma_2 \in S$ one has $\gamma_1\gamma_2^{-1} \notin K$.

Recall also that the action of G is said to have a completely positive entropy (c.p.e.) if for any finite partition ξ, the mean entropy $h(\xi, G)$ is positive.

Theorem 3.41.1. *A free action of a countable amenable group G on a Lebesgue space (X, μ) has c.p.e. if and only if for any finite partition ξ and any $\varepsilon > 0$ there exists a finite subset $K \subseteq G$ such that for any finite set S which is K-spread*

$$\left| \frac{1}{\#S} H\left(\bigvee_{g\in S} g\xi\right) - H(\xi)\right| < \varepsilon.$$

Let $H \subset G$ be a subgroup and (X, μ) a free H-space which has c.p.e. and is non-Bernoulli. Denote by $\pi : G \to H\backslash G$ the natural projection and by $s : H\backslash G \to G$ a section, which possesses the property $s(H) = e$.

Form the product space $Y = X^{H\backslash G}$ with the associated product measure ν and introduce an action of G on Y by

$$(gy)_\gamma = s(\gamma)gs(\gamma g)^{-1}y_{\gamma g}, \qquad y \in Y, \quad \gamma \in H\backslash G, \quad g \in G,$$

with the given action of H in each direct multiple of Y. An easy verification shows that this action is well defined (in particular, $s(\gamma)gs(\gamma g) \in H$). In some cases it

[19] This note presents an exposition of the results of the author's joint work with V. Golodets, *Complete positivity of entropy and non-Bernoullicity for transformation groups*, Coll. Math. 84/85 (2000), 412-429.

is possible to prove that this G-action inherits the properties of c.p.e. and non-Bernoullicity. So, starting from a $\mathbb{Z}$-action with these properties produced by D. Ornstein and P. C. Shields, one can arrange certain iteration of the above construction in order to prove the following theorems.

Theorem 3.41.2. *Any countable Abelian group G containing an element of infinite order has a non-Bernoullian c.p.e. action.*

Theorem 3.41.3. *Let G be a countable nilpotent group. There exists a c.p.e. non-Bernoullian G-space.*

Theorem 3.41.4. *Let G be a countable solvable group whose commutant $[G, G]$ is nilpotent. Then G admits a non-Bernoullian action with completely positive entropy. In particular, any countable solvable subgroup of $GL(n, \mathbb{R})$ admits a non-Bernoullian action with c.p.e.*

3.42 Meir Smorodinsky *Necessary and sufficient conditions for a process to admit independent extension*[20]

Let $X_n, n \leq 0$ be a stochastic process defined on a Lebesgue probability space. Such a process defines a *filtration* $\mathcal{F}_n = \sigma\{X_i, i \leq -n\}$. Our goal is to classify processes up to their filtration. We assume that $\cap_{n\geq 0}\mathcal{F}_n$ is trivial. Among such processes there is a distinguish class, the *independent* processes. An independent process process has the property that the conditional distributions of $\mathcal{F}_{n-1}$ given $\mathcal{F}_n$ are a.e. equivalent. We say that 2 distributions (say on the real line) are equivalent if there is a measure preserving 1-1 map from one to the other. Among the distributions we single out *homogeneous* which are either of finite number of atoms, all of equal probability, or non atomic (continuous). We call a process *conditionally homogeneous* if for each n the conditional distributions of $\mathcal{F}_{n-1}$ given $\mathcal{F}_n$ are a.e. equivalent, and are homogeneous (the distribution can depend on n). It was discovered by Vershik [4] that there are homogeneous processes which their filtration is not equivalent to one generated by independent process. Let U_n be an independent process. A *parameterization* of X_n by U_n is a joining of U_n and X_n with the following properties

(1) $U_{-1}, \cdots, U_{-n}$ are independent of $\mathcal{F}_{n-1}$.
(2) The complete σ-field generated by U_{-n} and $\mathcal{F}_n$ contains $\mathcal{F}_{n-1}$.
If also
(3) $\sigma[u_{-n}, u_{-n-1}, \cdots]$ contains $\mathcal{F}_{n-1}$ for each positive n,
then we say the parameterization is *generating*. In such case we say that X_n admits an independent extension.

Let X_n be a finitely homogeneous process i.e. for each n, the conditional distributions consist of finitely many atoms say p_n. Denote by C_n the set $\{1, \ldots p_n\}$. Let $\mathcal{Q}$ be a finite measurable partition. Let Ω_n be the space of $n-1$ tuples

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$\Omega_n = \{(\xi_{n-1}, ..., \xi_1) : \xi_i \in C_i\}$. The set Ω_n has a natural tree structure. Denote by A_n, the set of the tree automorphisms of Ω_n. Now, given a point x, there is an atom $x_n \in \mathcal{F}_n(x)$. A tree Ω_n can be attached to x_n in such a way so that for each $\omega_n \in \Omega_n$ the pair (x_n, ω_n) defines a point $x(\omega_n)$.

We define an n-distance $n \geq 0$, between any 2 points x, x' as follows. Begin by putting $d_0(x, x') = 0$ if x, x' belong to the same atom of $\mathcal{Q}$ and $d_0(x, x') = 1$ otherwise.

Definition 3.42.1.

$$d_n(x, x') = (\prod_{i<n} p_i)^{-1} \# \min_{a \in A_n} \#\{\omega | \omega \in \Omega_n, d_0(x(\omega), x'(a(\omega)) = 1\}.$$

Theorem 3.42.1. *A finitely homogeneous process X_n admits independent extension if and only if for every finite partition $\mathcal{Q}$, $d_n(x, x')$ tends weakly to 0 as n tends to infinity.*

Using Theorem 1, with appropriate approximation we get,

Theorem 3.42.2. ([2]) *Homogeneous process X_n admits independent extension if and only if its filtration is equivalent to independent filtration.*

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3.43 Örjan Stenflo *Non-uniqueness of invariant probability measures for contractive iterated function systems with place-dependent probabilities*

Let f_0 and f_1 be two maps from $[0, 1]$ into itself defined by

$$f_0(x) = \alpha x \quad \text{and} \quad f_1(x) = \alpha + (1 - \alpha)x,$$

where σ and α are fixed parameters satisfying $0 < \sigma, \alpha < 1$. Let p be a real valued continuous function on $[0,1]$ with $0 < p(x) < 1$.

Suppose a point $x \in [0,1]$ moves randomly with probability $p(x)$ to $f_0(x)$ and with probability $1 - p(x)$ to $f_1(x)$. This procedure generates a Markov chain on $[0,1]$.

It is a natural question to ask whether a Markov chain generated in this way necessarily has a unique stationary probability measure. We give an example, based on a result by M. Bramson and S. Kalikow, *Israel J. Math.* **84** (1993), 153-160, showing that this is not necessarily the case. This constitutes a counterexample to a conjecture raised by an incomplete proof by S. Karlin from 1953.

Further details can be found in the paper "A note on a theorem of Karlin", electronically available at: http://www.math.umu.se/~stenflo/research.html

3.44 Anatole Stepin
A remark on $\mathbb{Z}^2$-actions of finite type with phase transitions

Let A be a finite alphabet, G be a countable group and $T : G \times A^G \longrightarrow A^G$ be the natural G-action. Given G-invariant closed (with respect to the product topology on A^G) subset $\Omega \subset A^G$ consider the restriction of T to Ω. This symbolic transformation group is of finite type if configuration from Ω are defined by universal local rules. In the case $G = \mathbb{Z}$ these actions are also called subshifts of finite type or topological Markov chains. It is well known that invariant probability measure of maximal entropy for any transitive subshift of finite type is unique. This is not the case for the class of transitive $\mathbb{Z}^H$-actions of finite type as R. Burten and J. Steiff showed in [1]. In particular, they proposed an example (described below) of $\mathbb{Z}^2$-action of finite type with $\#A = 8526$ possessing two measures of maximal entropy (phase transition). Our remark (joint work with S. Shepovalov) is that the number of states can be reduced in this context.

Let n be a positive integer. ,

$$\begin{aligned} A &= \{-n, -n+1, \ldots, -1, 1, \ldots, n-1, n\} \\ G &= \{(i,j) \in A \times A \mid i \cdot j \leqslant 2\}. \end{aligned}$$

Denote $\|\cdot\|_1$ the ℓ_1-metric on $\mathbb{Z}^2$ defined by $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$ and consider subset

$$\Omega = \{\omega \in A^{\mathbb{Z}^2} \mid (\omega(x), \omega(y)) \notin G \text{ if } \|x - y\|_1 = 1\}.$$

It is translation invariant and $T|_\Omega$ is the required action.

Subset $M \subset \mathbb{Z}^2$ is called connected if for every pair $x, y \in M$ there exists a path $x = x_0, x_1, \ldots, x_k = y$ in M such that $\|x_i - x_{i-1}\|_1 = 1$, $1 \leqslant i \leqslant n$. Finite subset $M \subset \mathbb{Z}^2$ is called embracing if $0 \in M$ and both $M, \mathbb{Z}^2 \setminus M$ are connected. Notation:

$$\partial M = \{x \in M \mid \exists y \in \mathbb{Z}^2 \setminus M : \|x - y\|_1 = 1\}.$$

The following sufficient condition for the phase transition was given in [1].

Theorem 3.44.1. *Let N_l be the number of embracing subsets $M \subset \mathbb{Z}^2$ with $|\partial M| = l$. If n satisfies the inequality $\sum_{l=1}^{\infty} N_l (\frac{32}{n})^l < 1$ then $T|_\Omega$ possesses exactly two probability measures of maximal entropy*

Introducing ℓ_∞-metric on $\mathbb{Z}^2$, $\|(x_1, x_2)\|_\infty = \max(|x_1|, |x_2|)$, we define subset $M \subset \mathbb{Z}^2$ to be $*$-connected if for every pair $x, y \in M$ there exists a path $x = x_0, x_1, \ldots, x_k = y$ in M, such that $\|x_i - x_{i-1}\|_\infty = 1$, $1 \leqslant i \leqslant n$. Our propose here is
1) to indicate that for every embracing subset M its boundary ∂M is $*$-connected, and
2) to use 1) for estimating N_l.

Notations:

$$\begin{aligned} B_\infty(x, z) &= \{y \in \mathbb{Z}^2 | \ \|y - x\|_\infty \leqslant z\}, \\ B'_\infty(x, z) &= B_\infty(x, z) \setminus \{x\|. \end{aligned}$$

Lemma 3.44.1. *For every embracing set M, $\#M > 1$, and every $x_0 \in \partial M$, $B'_\infty(x_0, 1) \cap \partial G \neq \emptyset$.*

One more notation: for $M \subset \mathbb{Z}^2$ let $\operatorname{int} M$ be the set of $x \in \mathbb{Z}^2$ such that there are no connected path from x to ∞ which does not intersect M.

Lemma 3.44.2. *If for $*$-connected paths γ_1 and γ_2, $\operatorname{int}\gamma_1 \cup \operatorname{int}\gamma_2$ is connected and $\operatorname{int}\gamma_1 \cap \operatorname{int}\gamma_2 \subset \gamma_1 \cup \gamma_2$, then $\gamma_1 \cup \gamma_2$ is $*$-connected.*

From two lemmas above it follows the

Proposition 3.44.1. *If $M \subset F^2$ is embracing then ∂M is $*$-connected.*

In fact together with an estimate (cf. [2]) for the number of connected subgroups of size l leads to the basic

Lemma 3.44.3. $N_l \leqslant \frac{l}{2}(9e)^l$.

The proof follows the arguments given in [1] for the estimate $N_l \leqslant \frac{l}{2}(49e)^l$. Further improvement seems plausible if one takes into account Dobrushin's arguments in [3].

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3.45 Bernd O. Stratmann
Weak multifractal formalism for conformal measures

For a large class of fractal sets the idea of an iterated function system has turned out to be a very convenient and efficient concept. Traditionally, the development of fractal geometry was always very much inspired by various phenomena which appear in conformal analysis and number theory. The talk continued this tradition by explaining how methods from metrical Diophantine number theory can be used to study certain tame parabolic iterated function systems. These results generalize results for geometrically finite Kleinian groups with parabolic elements (obtained in [1] [2] [3]) and for parabolic rational rational functions (obtained in [4] [5] [6] [7]), which represent complex analytic analogies of Jarník's classical number theoretical theorem on the Hausdorff dimension of well-approximable numbers.

More precisely, let $\mathcal{S} = \{\phi_i : i \in I\}$ be a tame parabolic finite iterated function system which satisfies the super strong open set condition (SSOSC) (see [6]). For x_i denoting the fixed point of one of the parabolic generators, for $\delta > 0$ and for a hyperbolic word $\omega \in I$, define

$$B_\omega(i) = B_\omega = \overline{B}(\phi_\omega(x_i), |\phi'_\omega(x_i)|) \quad \text{and} \quad B^\delta_\omega(i) = B^\delta_\omega = \overline{B}(\phi_\omega(x_i), (|\phi'_\omega(x_i)|)^{1+\delta}).$$

The main interest of this talk focused on the sets

$$J_i^\delta := \bigcap_{q\geq 1} \bigcup_{n\geq q} \bigcup_{|\omega|=n} B^\delta_\omega, \quad J^\delta := \bigcup_{i\in\Omega} J_i^\delta.$$

We gave an outline of how to derive the following theorem of [SU3]. Furthermore, we explained in which way this result leads to a complete description of the 'weak-multifractal spectra' (see [S2]) of the h-conformal measure which is canonically associated to $\mathcal{S}$ (where h denotes the Hausdorff dimension of the limit set of the iterated function system $\mathcal{S}$).

Theorem 3.45.1. **(a)** *If $h \leq 1$, then*

$$\mathrm{HD}(J^\delta) = \frac{h}{1+\delta}.$$

(b) *If $h \geq 1$, then*

$$\mathrm{HD}(J_i^\delta) = \begin{cases} \frac{h}{1+\delta} & \text{if } \delta \geq h-1 \\ \frac{h+\delta p_i}{1+\delta(1+p_i)} & \text{if } \delta \leq h-1, \end{cases}$$

where p_i denotes the number of petals at the parabolic fixed point x_i. In particular, with $p_{min} := \min\{p_i : i\}$, we have that

$$\mathrm{HD}(J^\delta) = \begin{cases} \frac{h}{1+\delta} & \text{if } \delta \geq h-1 \\ \frac{h+\delta p_{min}}{1+\delta(1+p_{min})} & \text{if } \delta \leq h-1. \end{cases}$$

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3.46 Yuri Tomilov[21]

Strong stability of bounded evolution families and its relation to ergodic theorems

One very important task in the study of a linear nonautonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = A(t)u(t), & t \geq s \geq 0, \\ u(s) = x, & x \in X, \end{cases} \tag{3.1}$$

in a Banach space X, is the study of asymptotic behaviour of its solutions. Among the most interesting types of asymptotic behaviour we would like to mention stability in the sense that solutions vanish at infinity (see Definition below). However, a few characterizations (if any) of this kind of asymptotics of solutions to (3.1) are known.

In the case of the autonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = x, & x \in X, \end{cases} \tag{3.2}$$

quite a lot stability criterias have already been obtained. For example, assuming that (3.2) is well-posed and that the operator A generates a bounded C_0-semigroup

[21] This report is based on the joint work with C. J. K. Batty and R. Chill

$(T(t))_{t\geq 0}$, there is the Arendt-Batty-Lyubich-Vu Theorem and all its generalizations to bounded individual solutions or bounded uniformly continuous functions. One typical assumption in results of this type is the countability of some spectrum, for example, countability of the boundary spectrum $\sigma(A) \cap i\mathbb{R}$.

Another group of results, based on resolvent estimates rather than simple spectral conditions, has developed recently, coming closer to a characterization of stable semigroups. By means of unitary dilations (in Hilbert spaces) and limit isometric groups there have been obtained growth conditions on the resolvent near the imaginary axis which are sufficient for stability, and which are close to being necessary.

We gave several characterizations of stability of evolution families such as arise from well-posed, nonautonomous Cauchy problems. The characterizations are of terms of bounded complete trajectories of the dual family, stability of associated evolution semigroups and spectral properties of the generator of an evolution semigroup. To formulate them, we have to introduce some concepts and notation.

Recall that a two-parameter family $\mathbf{U} = (U(t,s))_{t\geq s\geq 0} \subset \mathbf{L}(X)$ is called an *evolution family* if it satisfies the following three conditions:

(i) $U(t,t) = I$ for all $t \geq 0$.

(ii) $U(t,s)U(s,r) = U(t,r)$ for all $0 \leq r \leq s \leq t$.

(iii) $U(\cdot,\cdot)$ is strongly continuous from $\{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t\}$ into $\mathbf{L}(X)$.

Evolution families usually appear in the context of the nonautonomous Cauchy problem (3.1). Well-posedness of the Cauchy problem (3.1) is equivalent to the existence of an evolution family $\mathbf{U}$ such that for all $s \geq 0$ and for all $x \in X$ the unique mild solution u of (3.1) is given by $u(t) = U(t,s)x$. In the autonomous case, i.e. when $A(t) = A$ is constant, well-posedness is equivalent to the condition that A generates a C_0-semigroup $(T(t))_{t\geq 0}$. In that case we have $U(t,s) = T(t-s)$ for all $t \geq s$.

Definition 3.46.1. We call an evolution family $(U(t,s))_{t\geq s\geq 0}$ on a Banach space X *(strongly) stable* if for all $s \geq 0$ and for all $x \in X$ one has $\lim_{t\to\infty} \|U(t,s)x\| = 0$.

Next we introduce the concepts of *complete trajectory* and *evolution semigroup* which are basic in our study of stability.

A function $g : \mathbb{R}_- \to X^*$ is called a *complete trajectory for* $(U(-s,-t)^*)_{s\leq t\leq 0}$ whenever it satisfies the condition $U(-s,-t)^*g(s) = g(t)$ for all $s \leq t \leq 0$. One can imagine a complete trajectory as a backward continuation of a trajectory of an evolution family to the negative time. Note that this definition of a complete trajectory differs from that in the literature in that g is only defined on the half-line $\mathbb{R}_-$. However, in the autonomous case when $U(t,s) = T(t-s)$, a complete trajectory in our sense can be uniquely extended to a complete trajectory on $\mathbb{R}$ by defining $g(t) = T(t)^*g(0)$ for $t \geq 0$. We call a complete trajectory $g : \mathbb{R}_- \to X^*$ *nontrivial* if g is not identically 0.

Further, let $1 \leq p \leq \infty$, and let $E_p := L^p(\mathbb{R}_+; X)$ if $1 \leq p < \infty$, and $E_\infty := C_{00}(\mathbb{R}_+; X)$ (the space of continuous functions vanishing at 0 and at infinity). It is well known that the family $(\mathbf{T}_p(t))_{t\geq 0}$ defined by

$$(\mathbf{T}_p(t)f)(s) = \begin{cases} U(s,s-t)f(s-t), & s \geq t, \\ 0, & s < t, \end{cases} \qquad t,s \geq 0, f \in E_p,$$

is a C_0-semigroup on the Banach space E_p. We call $(\mathbf{T}_p(t))_{t\geq 0}$ the *evolution semigroup associated with* $(U(t,s))_{t\geq s\geq 0}$ *on the space* E_p, and we denote by $\mathbf{G}_p$ its generator. The notion of an evolution semigroup allows to reduce the study of a nonautonomous Cauchy problem to an autonomous one.

Finally, for an evolution family $\mathbf{U} = (U(t,s))_{t\geq s\geq 0}$ and a function $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$, let

$$(U * f)(t) := \int_0^t U(t,s)f(s)\, ds \text{ for all } t \geq 0.$$

When $U(t,s) = T(t-s)$ for a semigroup $\mathbf{T}$, $U * f$ is the *convolution* of $\mathbf{T}$ and f in the usual sense.

Our first main result is the following theorem which we believe is new even in case X is finite-dimensional space.

Theorem 3.46.1. *Let* $(U(t,s))_{t\geq s\geq 0}$ *be a bounded evolution family on a Banach space* X, *and let* $(\mathbf{T}_p(t))_{t\geq 0}$ *be the evolution semigroup associated with* $(U(t,s))_{t\geq s\geq 0}$ *on* E_p $(1 \leq p \leq \infty)$. *Then the following assertions are equivalent:*
(1) *The evolution family* $(U(t,s))_{t\geq s\geq 0}$ *is strongly stable.*
(2) *The semigroup* $(\mathbf{T}_p(t))_{t\geq 0}$ *is stable for some* $1 \leq p \leq \infty$.
(3) *The semigroup* $(\mathbf{T}_p(t))_{t\geq 0}$ *is stable for all* $1 \leq p \leq \infty$.
(4) *The range* $\operatorname{Rg}\mathbf{G}_1$ *of* $\operatorname{Rg}\mathbf{G}_1$ *is dense in* $L^1(\mathbb{R}_+; X)$.
(5) *The set*

$$F := \{f \in L^1(\mathbb{R}_+; X) : U * f \in L^1(\mathbb{R}_+; X)\} \tag{3.3}$$

is dense in $L^1(\mathbb{R}_+; X)$.

The equivalences (1)⇔(4)⇔(5) should be compared to famous Datko's characterization of *uniformly exponentially* stable evolution families in which density of $\operatorname{Rg}\mathbf{G}_1$ has been replaced by surjectivity of $\mathbf{G}_p$.

Now, observing that, $P\sigma(\mathbf{G}_p)$, the point spectrum of $\mathbf{G}_p$ is empty, we obtain the following *ergodic* characterization of stability.

Corollary 3.46.1 (stability versus ergodicity). *Let* $(U(t,s))_{t\geq s\geq 0}$ *be a bounded evolution family on a Banach space* X, *and let* $(\mathbf{T}_1(t))_{t\geq 0}$ *be the evolution semigroup associated with* $(U(t,s))_{t\geq s\geq 0}$ *on* E_1. *Then the evolution family* $(U(t,s))_{t\geq s\geq 0}$ *is strongly stable iff the evolution semigroup* $(\mathbf{T}_1(t))_{t\geq 0}$ *is mean ergodic on* E_1.

Thus, the study of strong stability of solutions to (1) can be reduced to the study of mean ergodicity of the corresponding evolution semigroup in E_1. Note the specific role of the Banach space E_1 in this conclusion. For example, the corollary cannot be true for E_p where $p \in (1,\infty)$. In fact, if $p \in (1,\infty)$ and X is reflexive, then E_p is reflexive. The fact that the point spectrum of $\mathbf{G}_p$ is empty and the Mean Ergodic Theorem imply that $\mathbf{G}_p$ has dense range in E_p, even if $(U(t,s))_{t\geq s\geq 0}$ is not a stable evolution family. Similarly, whereas Datko's Theorem is true for every $p \in [1,\infty]$, the equivalence (1)⇔(5) cannot be true for $p \in (1,\infty]$.

The proof of the above results is based, in particular, on the following extension of the well-known Lin-Derriennic theorem which has an independent interest.

Proposition 3.46.1. *Let $(U(t,s))_{t\geq s\geq 0}$ be a bounded evolution family on a Banach space X. Let B^* be the closed unit ball in the dual space X^* and define for fixed $s \geq 0$*

$$J_s^* := \bigcap_{t\geq s} U(t,s)^*(B^*).$$

Then:
(i) *For every $x \in X$ we have*

$$\frac{1}{M} \limsup_{t\to\infty} \|U(t,s)x\| \leq \sup_{x^*\in J_s^*} |\langle x, x^*\rangle| \leq \liminf_{t\to\infty} \|U(t,s)x\|, \tag{3.4}$$

where $M = \sup_{t\geq s\geq 0} \|U(t,s)\|$.
(ii) $\lim_{t\to\infty} \|U(t,s)x\| = 0$ *if and only if x annihilates J_s^*.*
(iii) *For every $x^* \in J_0^*$ there exists a bounded complete trajectory g for the evolution family $(U(-s,-t)^*)_{s\leq t\leq 0}$ such that $g(0) = x^*$.*

So, $(U(t,s))_{t\geq s\geq 0}$ is strongly stable iff $(U(-s,-t)^*)_{s\leq t\leq 0}$ does not admit a bounded nontrivial complete trajectory. To get Corollary one has to observe that $\operatorname{Ker} \mathbf{G}_1^*$ can be identified with the set of all complete bounded trajectories for $(U(-s,-t)^*)_{s\leq t\leq 0}$.

Specializing the above technique for the semigroup case, i.e. when $U(t,s) = T(t-s)$ for all $t \geq s$, we obtain the criteria for strong stability of bounded C_0-semigroup in terms of some algebraic properties of its generator. Before stating the result we define $A_+(\mathbb{R};X)$ to be the image under Fourier transform of the space $L^1(\mathbb{R}_+;X)$, i.e. the space of all functions $f : \mathbb{R} \to X$ for which there exists $g \in L^1(\mathbb{R}_+;X)$ such that $f(\beta) = \int_0^\infty e^{-i\beta s} g(s)\,ds =: \mathbf{F}g(\beta)$ $(\beta \in \mathbb{R})$. By injectivity of the Fourier transform the function g is uniquely determined and we put $\|f\|_{A_+} := \|g\|_1$

Theorem 3.46.2. (i) *Assume that $U(t,s) = T(t-s)$ for some bounded C_0-semigroup $(T(t))_{t\geq 0}$ on X, and denote by A the generator of $(T(t))_{t\geq 0}$. Then the strong stability of $(T(t))_{t\geq 0}$ is equivalent to:*
$P\sigma(A) \cap i\mathbb{R} = \emptyset$ and the multiplication operator $\mathbf{M}$ defined by

$$\begin{aligned} D(\mathbf{M}) &:= \{f \in A_+(\mathbb{R};X) : f(\beta) \in \operatorname{Rg}(i\beta - A) \text{ and} \\ &\qquad\qquad \beta \mapsto (i\beta - A)^{-1} f(\beta) \in A_+(\mathbb{R};X)\} \\ \mathbf{M}f(\beta) &:= (i\beta - A)^{-1} f(\beta) \end{aligned}$$

is densely defined.
(ii) *The condition*
the space $\bigcap_{\beta\in\mathbb{R}} \operatorname{Rg}(i\beta - A)$ is dense in X
implies strong stability of $(T(t))_{t\geq 0}$.

Whether the converse to the second statement (ii) is true? The answer is not known.

3.47 Reinhard Winkler

The dynamical behaviour of Kronecker sequences and compactifications of integers

Our Question: Given $\alpha \in \mathbf{T} = \mathbf{R}/\mathbf{Z}$ and a segment $S \subseteq \mathbf{R}/\mathbf{Z}$. Let $T = T(\alpha, S)$ denote the set of all integers k with $k\alpha \in S$. More generally we can consider Kronecker sequences. This means that we have a vector $a = (\alpha_1, \ldots, \alpha_n) \in (\mathbf{R}/\mathbf{Z})^n$, a rectangle $R = S_1 \times \ldots \times S_n \subseteq \mathbf{T}^n$ and the set $T = T(a, R)$ of all k with $ka \in R$. Given just the sets $T(\alpha, S)$ or $T(a, R)$, how much do we know about α resp. a?

Sander's number theoretic result: By the means of Kronecker's approximation theorem, J. Sander, cf. [2], proved that, with some obvious exceptions, $T(\alpha, S)$ and $T(a, R)$ cannot coincide. (In fact Sander's result is much more general w.r.t. S and R.) This means that in some way the sets T carry the essential information about α and a. We are investigating how to get this information.

An abstract setting: The natural generalization of our question is to replace $\mathbf{T}$ or $\mathbf{T}^n$ and the embeddings $k \mapsto k\alpha$ resp. $k \mapsto ka$ by arbitrary compactifications (C, ι) of the integers. This means that C is a compact group and $\iota : \mathbf{Z} \to C$ is a homomorphism such that $\iota(\mathbf{Z})$ is dense in C. Let μ_C denote the normalized Haar measure on C. The role of S and R is played by so-called continuity sets $M \subseteq C$, i.e. sets whose topological boundary δM satisfies $\mu_C(\delta M) = 0$. The corresponding sets $T = \iota^{-1}(M)$ have been investigated and called Hartman measurable sets in [4] and [3]. A very important fact is that they form a Boolean set algebra $\mathcal{H}$ on $\mathbf{Z}$ and $T = \iota^{-1}(M) \in \mathcal{H}$ has a density coinciding with $\mu_C(M)$.

Furthermore the system of all compactifications has a natural structure. We write $(C_1, \iota_1) \leq (C_2, \iota_2)$ whenever there exists a continuous homomorphism $\varphi : C_2 \to C_1$ with $\varphi\iota_2 = \iota_1$. If furthermore $(C_2, \iota_2) \leq (C_1, \iota_1)$ then φ is an algebraic and topological isomorphism and we identify the compactifications. The resulting system $\mathcal{COMP}$ is partially ordered by $\leq$. Its maximal element is the Bohr compactification. Any compactification is a factor of the Bohr compactification.

Using Pontrjagin's duality: By Pontrjagin's duality, factors correspond to subgroups of the dual. The dual of the Bohr compactification of $\mathbf{Z}$ is $\mathbf{T}_d$, the one dimensional torus with the discrete topology. Thus, if $\mathcal{SUB}$ denotes the system of all subgroups of $\mathbf{T}_d$, one can prove $(\mathcal{SUB}, \leq) \cong (\mathcal{COMP}, \subseteq)$ by the isomorphism Φ, sending the subgroup $A \subseteq \mathbf{T}_d$ the compactification (C_A, ι_A), $\iota_A : k \mapsto (k\alpha)_{\alpha \in A} \in C_A \subseteq \mathbf{T}^A$.

Furthermore we look at the structure $\mathcal{FILT}$ consisting of all filters $\mathcal{F}$ on $\mathbf{Z}$ coming from a $(C, \iota) \in \mathcal{COMP}$ in the sense that $\mathcal{F} = \mathcal{F}(C, \iota) = \Sigma(C, \iota)$ ($\Sigma : \mathcal{COMP} \to \mathcal{FILT}$) consists of all $\iota^{-1}(U)$, $U \subseteq C$ neighbourhood of $0 \in C$. Clearly $\mathcal{FILT}$ is partially ordered by set theoretic inclusion. For $\mathcal{F} \in \mathcal{FILT}$ let $\Psi(\mathcal{F})$ be the set of all $\alpha \in \mathbf{T}_d$ such that we have filter convergence $\lim_{k \in F \in \mathcal{F}} k\alpha = 0 \in \mathbf{T}$. It turns out that Φ, Σ and Ψ form a commutative triangle diagram of isomorphisms of partially ordered sets.

An auxiliary function: Given $(C, \iota) \in \mathcal{COMP}$ and a continuity set $M \subseteq C$, the function $f = f_M : C \to \mathbf{R}$, $c \mapsto \mu_C(M \triangle (M + c))$ is continuous and the set $Z(M) = f^{-1}(0)$ of zeros is a subgroup of C. The use of the function f is based on the fact that, since $\iota(\mathbf{Z})$ is dense in C, f is determined by the values on $\iota(\mathbf{Z})$ and can be computed only by means of $T = \iota^{-1}(M)$: $f(\iota(k))$ must coincide with the density $d_k(T)$ of $T \triangle (T + k)$.

The filter induced by a Hartman sequence: The essential step to get information from T is to use the above numbers $d_k(T)$ to define the sets $F(T, \varepsilon) = \{k \in \mathbf{Z} : d_k(T) < \varepsilon\}$, $\varepsilon > 0$, generating the filter $\mathcal{F}(T)$. It turns out that $\mathcal{F}(T) \in \mathcal{COMP}$ and in many cases coincides with $\mathcal{F}(C, \iota)$.

The results: Using the continuity of f_M, it is now an easy exercise to show that $\mathcal{F}(T) \subseteq \mathcal{F}(C, \iota)$. The converse inclusion holds under the assumption that M is aperiodic in the sense that $Z(M) = \{0\}$ is trivial. Also the proof of this fact is now a routine work of 5 lines. If $Z(M)$ is arbitrary one gets $\mathcal{F}(T) = \mathcal{F}(C/Z(M), \kappa\iota)$ with the canonical mapping $\kappa : c \mapsto c + Z(M)$. Non degenerated sectors S or rectangles R are aperiodic, hence this yields Sander's result in the above stated version as a consequence. Here we make use of the isomorphisms $(\mathcal{COMP}, \leq) \cong (\mathcal{FILT}, \subseteq) \cong (\mathcal{SUB}, \subseteq)$. More generally it should be mentioned that there are aperiodic continuity sets $M \subseteq C$ if and only if the corresponding group $\Psi(\mathcal{F}(C, \iota)) \in \mathcal{SUB}$ is countable. Full proofs of these results and further references are contained in [4].

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3.48 Michiko Yuri
Weak Gibbs measures for certain non-hyperbolic systems

In this talk, first we present a new method for the construction of conformal measures ν for infinite-to-one piecewise C^0-invertible Markov systems associated to potentials ϕ which may fail both summable variation and bounded distortion, but satisfy the weak bounded variation (see below). Next we show the existence of equilibrium states μ for potentials ϕ of weak bounded variation which is equivalent to the conformal measures ν. The equilibrium states may fail the Gibbs property in the sense of Bowen but satisfy a version of Gibbs property (so-called *weak Gibbs*) under certain condition. In particular, we can observe the weak Gibbs property of equilibrium states for typical mathematical models of intermittence which is a common phenomenon in the transition to turbulence, i.e., piecewise C^1- invertible maps with indifferent periodic points (Manneville-Pomeau maps, Brun's map, Inhomogeneous Diophantine algorithm, a complex continued fraction algorithm etc).

Definition 3.48.1. We say that a triple $(T, X, Q = \{X_a\}_{a\in I})$ is a *piecewise* C^0-*invertible system* if X is a compact metric space, $T : X \to X$ is a noninvertible map which is not necessarily continuous, and $Q = \{X_a\}_{a\in I}$ is a countable disjoint partition $Q = \{X_a\}_{a\in I}$ of X such that $\bigcup_{a\in I} intX_a$ is dense in X and satisfy the following properties.

(01) For each $a \in I$ with $\mathrm{int}\, X_a \neq \emptyset, T|_{\mathrm{int}\, X_a} : \mathrm{int}\, X_a \to T(intX_a)$ is a homeomorphism and $(T|_{intX_a})^{-1}$ extends to a homeomorphism ψ_a on $cl(T(intX_a))$.

(02) $T(\bigcup_{intX_a=\emptyset} X_a) \subset \bigcup_{intX_a=\emptyset} X_a$.

(03) $\{X_a\}_{a\in I}$ generates $\mathcal{F}$, the sigma algebra of Borel subsets of X.

Definition 3.48.2. We say that ϕ is a potential of *weak bounded variation* (WBV) if there exists a sequence of positive numbers $\{C_n\}_{n\geq 1}$ satisfying $\lim_{n\to\infty}(1/n) \log C_n = 0$ and $\forall n \geq 1, \forall X_{a_1\dots a_n} \in \bigvee_{i=0}^{n-1} T^{-i}Q$,

$$\frac{\sup_{x\in X_{a_1\dots a_n}} \exp(\sum_{i=0}^{n-1} \phi(T^i x))}{\inf_{x\in X_{a_1\dots a_n}} \exp(\sum_{i=0}^{n-1} \phi(T^i x))} \leq C_n.$$

Definition 3.48.3. A Borel probability measure ν is called a *weak Gibbs measure for ϕ with a constant $-P$* if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n)\log K_n = 0$ such that ν-a.e.x,

$$K_n^{-1} \leq \frac{\nu(X_{a_1\dots a_n}(x))}{\exp(\sum_{i=0}^{n-1} \phi T^i(x) - nP)} \leq K_n,$$

where $X_{a_1\dots a_n}(x)$ denotes the cylinder containing x.

Our method is based on the existence of a derived map T^* (Schweiger's jump transformation) which is uniformly expanding and guarantees a weak Hölder-type property of the potential ϕ^* associated to ϕ. For the construction of conformal measures ν, we observe a good relation between the topological pressure for ϕ and the topological pressure associated to ϕ^* with respect to T^*. The key to the proof of the weak Gibbs property of equilibrium states μ for ϕ is to clarify the order of divergence of the invariant density $d\mu/d\nu$ near indifferent periodic points. Lastly, we establish a version of the local product structure (*weak local product structure*) for ergodic measures $\overline{\mu}$ which are the invertible extension of the ergodic weak Gibbs measures μ. As a special case, $\overline{\mu}$ possesses asymptotically "almost" local product structure in the sense of Barreira-Pesin-Schmeling ([1]).

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3.49 Henryk Żołądek *Multi-dimensional Jouanolou system*

Ordinary differential equations with polynomial right-hand side, when considered in the complex domain, define holomorphic foliations of the projective space whose leaves are open Riemann surfaces. It is known that a typical such foliation (in the space of foliations of fixed degree) reveals a chaotic behaviour; the leaves are dense in the phase space and the entropy is positive. Another aspect of this theory is the problem of integrability of polynomial differential equations; here by integrability we mean existence of one (or more) first integral expressed in resonable terms (in quadratures, in elementary functions etc.) The necessary condition for such integrability is the existence of invariant algebraic varietes (e.g. hypersurfaces, curves).

J.-P. Jouanolou was the first who has shown that holomorphic foliations of the projective plane without any invariant algebraic curve (including the line at infinity) form a dense subset in the space of foliations of fixed degree. (Later A. Lins-Neto has shown that this subset is also open.) By standard arguments, which use Zariski closeness of the space of foliations with an invariant algebraic curve of fixed degree, the proof of the Jouanolou's theorem relies on finding an example of a polynomial vector field without algebraic leaves. He has chosen the following system, called the Jouanolou system, written in the homogeneous coordinates in $\mathbf{C}P^n$

$$\dot{x}_0 = x_n^s,\ \dot{x}_1 = x_0^s, \ldots, \dot{x}_n = x_{n-1}^s$$

Here $s \geq 2$ is the degree of the corresponding foliation and $n = 2$ in the Jouanolou's book "Equations de Pfaff algébriques" .

We have obtained the following generalization of that result to arbitrary dimension.

Theorem 3.49.1. *If $n \geq 2$ and $s \geq 2$, then the Jouanolou system has no invariant algebraic hypersurfaces.*

This implies that the space of foliations of $\mathbf{C}P^n$ without invariant algebraic hypersurfaces is dense in the space of all foliations of given degree.

There exist other results about invariant algebraic varietes. A. J. Maciejewski, J. Moulin Ollagnier, A. Nowicki and J.-M. Strelcyn have proved that the thesis of the above theorem holds when $n+1$ is a prime number and $s > 2n/(n-1)$. A. Lins Neto and M. Soares have shown that a generic foliation of $\mathbf{C}P^n$, $n \geq 2$ does not have algebraic leaves and M. Soares has proved that a generic foliation of $\mathbf{C}P^3$ does not have invariant algebraic surfaces (he has not proved that the Jouanolou foliation has this property). Lins Neto and Soares use methods of differential geometry (characteristic classes); the methods of Maciejewski et al. are essentially algebraic.

In the proof of the theorem we use methods of analytic geometry. Firstly, following the paper of Maciejewski et al., the problem is reduced to showing that another system $\dot{y}_0 = y_0(sy_n - y_0), \ldots, \dot{y}_n = y_n(sy_{n-1} - y_n)$ does not have homogeneous polynomial first integrals. Next, we expand the alleged first integral into powers of the last variable y_n ,with polynomial coefficients which satisfy a system of differential equations. The latter (reduced) system, restricted to the hyperplane $y_n = 0$, is integrable with $n-1$ rational first integrals. This allows us to solve partially the mentioned system of equations for the polynomials–coefficients. By a careful estimation of degrees of some variables in these coefficients we obtain a contradiction.

4

Open problems

I Vitaly Bergelson

See page 30. See also Section 5 in: *Ergodic Ramsey theory - an update*, Ergodic Theory of $\mathbb{Z}^d$-Actions (edited by M. Pollicott and K. Schmidt), London Math. Soc. Lecture Note Series **228** (1996), 1-61.

II Maurice Courbage

See page 37.

III Yves Derriennic

See page 46.

IV Sébastien Ferenczi

See page 52

V Sergey Gefter

Let G be a non-discrete locally compact second countable group with left Haar measure μ, and Γ be a countable dense subgroup of G. Γ acts on the measure space (G, μ) by left translations. Let us denote the Γ-orbital equivalence relation by R_Γ.

Question V.1. Let G be a non-amenable as a discrete group. To which extent is G determined by equivalence relation R_Γ?

It is known that if G is compact and Γ contains a dense subgroup in G with T-property and ICC-property, R_Γ determine G with precision up to a subgroup of a finite index (see Theorem 3.9 in the paper S. L. Gefter and V. Ya. Golodets, *Fundamental groups for ergodic actions and actions with unit fundamental groups*, Publ. RIMS, Kyoto Univ., **24** (1988), 821-847).

VI Eugene Gutkin

In this contribution we formulate a few open questions in billiard dynamics. The billiard flow takes place in a bounded, connected, planar domain, Y (the *billiard table*) with a piecewise C^1 boundary, $X = \partial Y$. The qualitate features of the dynamics crucially depend on the shape of the table. Three types of geometric shapes have attracted most attention. These are the smooth and strictly convex tables, the piecewise concave ones, and the polygons. We will refer to the corresponding dynamics (and the billiard tables) as *elliptic, hyperbolic*, and *parabolic* respectively. It is standard to pass from the billiard flow to the *billiard map*. See [7]. The phase space of the billiard map is $\Phi = X \times [0, \pi]$. In standard coordinates the Liouville measure, μ, is given by $d\mu = \sin\theta \, ds \, d\theta$ [7]. It is invariant under the billiard map.

For the elliptic billiard dynamics, the study of periodic orbits goes back to G. D. Birkhoff. He obtained a quadratic lower bound on their number. The first question concerns an upper bound of sorts on periodic orbits. Let $\Phi_n \subset \Phi$ be the set of periodic points of the prime period n.

Problem VI.1. Prove that $\mu(\Phi_n) = 0$ for all n.

For $n = 2$ this is straightforward. For $n = 3$ this is a theorem of M. Rychlik [5]. Simpler proofs were obtained by L. Stoyanov, Ya. Vorobets, and M. Wojtkowski. See [7]. For $n \geq 4$ the question is open. By a result of V. Ivrii, a positive answer to the question has nontrivial implications for the spectral asymptotics of the Dirichlet Laplacean of Y. Although typically the set Φ_n is finite for each n, N. Innami gave examples when Φ_3 is a continuum.

This class of hyperbolic billiard tables arose from the work of Ya. Sinai [6] which was motivated by the ergodic hypothesis for the Boltzmann gas. It is much wider than the original family of dispersing tables of Sinai [2]. The billiard dynamics in these tables has strong chaotic properties [1]. Many open questions for hyperbolic billiard tables have to do with the *decay of correlations*. We will discuss the statistics of periodic orbits in dispersing billiards. Let $f_Y(n)$ be the number of k-segment periodic orbits, with $k \leq n$. What is the asymptotics of this counting function? By theorems of L. Stoyanov and N. Chernov, there are asymptotic bounds $c_- e^{h_- n} \leq f_Y(n) \leq c_+ e^{h_+ n}$, as $n \to \infty$.

Problem VI.2. Does the limit

$$h = \lim_{n \to \infty} \frac{\log f_Y(n)}{n}$$

exists? Is h the *topological entropy* of the billiard map?

We refer to [3] for the background on polygonal billiards. A polygon Y is *rational* if the angles between the sides of Y are of the form $\pi m/n$. Then the billiard flow decomposes into a one-parameter family of *directional billiard flows* b_θ^t, $0 \leq \theta \leq 2\pi$. In the topological space of n-gons the set $\mathcal{E}(n)$ of ergodic n-gons is residual in the sense of Baire category [4]. By a theorem of Vorobets, $\mathcal{E}(n)$, which is a bounded subset in a Euklidean space, contains all polygons whose angles admit a superexponentially fast approximation by rationals. Let μ be the Lebesgue measure in the ambient Euclidean space.

Problem VI.3. Is $\mu(\mathcal{E}(n)) > 0$?

The question is open even for $n = 3$, i.e., for triangles. The space of triangles is identified with the unit square in $\mathbb{R}^2$. Acute triangles correspond to the mechanical systems of three elastic point masses confined to a circle. The question above corresponds to the following: Is the system of three elastic masses on a circle ergodic for almost all masses?

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VII Mariusz Lemańczyk

Let $T : (X, \mathcal{B}, \mu) \longrightarrow (X, \mathcal{B}, \mu)$ be an ergodic automorphism of a probability standard Borel space. By $J(T)$ ($J^e(T)$) we denote the set of all self-joinings (ergodic self-joinings) of T that is all $T \times T$-invariant measures defined on $(X \times X, \mathcal{B} \otimes \mathcal{B})$ whose both natural projections are equal to μ. To each element $\lambda \in J(T)$ we associate a Markov operator $\Phi_\lambda : L^2(X, \mu) \longrightarrow L^2(X, \mu)$ given by $\int_X \Phi_\lambda(f)\overline{g}\, d\mu = \int_{X \times X} f\overline{g}\, d\lambda$. We also have $\Phi_\lambda T = T\Phi_\lambda$. Moreover, each Markov operator on $L^2(X, \mu)$ that commutes with T is necessarily of the form Φ_λ. This introduces a semigroup law on $J(T)$. Together with the weak topology and the natural simplex structure on $J(T)$ we obtain that $J(T)$ is a compact semitopological affine semigroup.

In 1995, del Junco, Lemańczyk and Mentzen ([2]) introduced a notion of semisimplicity. We say that T is semisimple if for any $\lambda \in J^e(T)$ the extension $(T \times T, \lambda)$ over (T, μ) (given by the projection on the first coordinate) is relatively weakly mixing (see [1] for definition of relative weak mixing). The notion of semisimplicity generalizes the notion of minimal self-joinings ([5]) and simplicity ([3],[6]). Moreover some Gaussian automorphisms turn out to be semisimple ([4]). It follows from basic properties of relative products that $J^e(T)$ is stable under composition whenever T is semisimple.

Problem VII.1. Is it true that T is semisimple if and only if $J^e(T)$ is a subsemigroup of $J(T)$?

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VIII Emmanuel Lesigne

See page 76.

IX Christian Mauduit

See page 94.

X Miłosz Michalski

See page 97.

XI Mahendra Nadkarni

See page 98.

XII Ryszard Rudnicki

Problem XII.1. Does there exist a (smooth) continuous time dynamical system in $\mathbb{R}^3$ which has a dense trajectory?

Problem XII.2. Let $T : X \longrightarrow X$ be a bounded linear operator on a Banach space X. Assume that there exists a probability Borel measure μ invariant with respect to T. Assume that X is the linear support of μ. Does there exist a Gaussian measure ν invariant under T with topological support X?

Remark. Linear topological support of a probability Borel measure μ is the smallest closed linear subspace Y of X such that $\mu(Y) = 1$. The topological support of μ is the smallest closed set $F \subset X$ such that $\mu(F) = 1$.

Problem XII.3. Let $\{S_t\}_{t\geq 0}$ be a semidynamical system generated by an evolution equation $x'(t) = Ax$ in a Banach space X, i.e. A is the infinitesimal generator of the continuous semigroup of linear operators $\{S_t\}_{t\geq 0}$. Give sufficient (and necessary) conditions for the existence of an invariant Gaussian measure μ for $\{S_t\}_{t\geq 0}$ with $\operatorname{supp}\mu = X$.

In order to formulate next problems we need some definitions.

Let X be a compact space, m be a given Borel measure on X and $T : X \longrightarrow X$ be a nonsingular transformation with respect to measure m. Let P be the Frobenius-Perron operator corresponding to T, i.e. $\int_A Pf\, dm = \int_{T^{-1}(A)} f\, dm$ for every Borel set A and every $f \in L^1(X,m)$. The system $(X, \mathcal{B}(X), m; T)$ is called *completely mixing* if for every $f \in L^1(X,m)$ with $\int f\, dm = 0$ we have $\lim_{n\to\infty} \|P^n f\| = 0$, where $\|\cdot\|$ is the norm in $L^1(X,m)$. A probability Borel measure μ is called *limit measure* for the system $(X, \mathcal{B}(X), m; T)$ if

$$\lim_{n\to\infty} \int \varphi(x) P^n f(x)\, m(dx) = \int \varphi(x)\, \mu(dx) \int f(x)\, m(dx)$$

for every $\varphi \in C(X)$ and $f \in L^1(X,m)$. If the limit measure μ exists and is non-trivial, i.e. $\operatorname{supp}\mu$ contains at least two points, then the system $(X, \mathcal{B}(X), m; T)$ is called *chaotic*. If the system $(X, \mathcal{B}(X), m; T)$ is completely mixing and chaotic with the limit measure μ then the pair $(\operatorname{supp}\mu, \mu)$ is called the *stochastic attractor* for the system $(X, \mathcal{B}(X), m; T)$. A probability Borel measure μ is called *Bowen-Ruelle measure* for the system $(X, \mathcal{B}(X), m; T)$ if for every $\varphi \in C(X)$ we have $n^{-1}\sum_{k=0}^{n-1} \varphi(T^k(x)) \longrightarrow \int \varphi\, d\mu$ for m-a.e. x.

Problem XII.4. Let $(X, \mathcal{B}(X), m; T)$ be a completely mixing system on a compact metric space X. Does there exist the limit measure for this system?

Problem XII.5. Let $(\operatorname{supp}\mu, \mu)$ be a stochastic attractor for $(X, \mathcal{B}(X), m; T)$. Is μ the Bowen-Ruelle measure?

Problems XII.4 and XII.5 are given in the paper: R. Rudnicki, *On a one-dimensional analogue of the Smale horseshoe*, Annales Polon. Math. **54** (1991), 147-153.

XIIa Gerhard Keller — answers to problems by R. Rudnicki

Completely mixing maps without limit measure

ABSTRACT. We combine some results from the literature to give examples of completely mixing interval maps without limit measure.

Let X be a compact metric space with Borel σ-algebra $\mathcal{B}$ and equipped with some Borel measure m. Consider a transformation $T : X \to X$ which is non-singular with respect to m, which means that $m(T^{-1}A) = 0$ if and only if $m(A) = 0$ for each Borel set A. Let $P : L^1_m \to L^1_m$ be the Frobenius-Perron operator of T so that

$$\int \varphi \cdot P^n f \, dm = \int (\varphi \circ T^n) \cdot f \, dm \qquad \forall f \in L^1_m \ \forall \varphi \in L^\infty_m .$$

We adopt the following definitions:

- The system $(X, \mathcal{B}, m, T)$ is *completely mixing*, if $\lim_{n\to\infty} \|P^n f\| = 0$ for each $f \in L^1_m$ with $\int f \, dm = 0$.
- A probability measure μ on $\mathcal{B}$ is a *limit measure* for the system $(X, \mathcal{B}, m, T)$, if for each probability density $h \in L^1_m$ the measures $P^n h \cdot m$ converge weakly to μ, in other words, if

$$\lim_{n\to\infty} \int \varphi \cdot P^n f \, dm = \int \varphi \, d\mu \cdot \int f \, dm \qquad \forall f \in L^1_m \ \forall \varphi \in C(X) .$$

- If a system $(X, \mathcal{B}, m, T)$ is completely mixing and has a nontrivial limit measure μ (i.e. μ is not a one point mass), then μ is called a *stochastic attractor* for the system.
- A probability measure μ on $\mathcal{B}$ is a *Sinai-Ruelle-Bowen measure* for the system $(X, \mathcal{B}, m, T)$, if for each $\varphi \in C(X)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k x) = \int \varphi \, d\mu \quad m - \text{a.e. } x .$$

The problems

Rudnicki [8] posed the following problems:

Problem 1 Does each completely mixing system have a limit measure? If T has an invariant probability density the answer is obviously "yes". In general, however, this is not true. In fact, we provide counterexamples in the class of quadratic interval maps.

Problem 2 Is a stochastic attractor necessarily a Bowen-Ruelle-Sinai measure? We give a counterexample in the class of piecewise C^2 interval maps with two surjective branches and two neutral fixed points.

The counterexamples

1. *A completely mixing quadratic interval map without limit measure*

For $0 < a \le 4$ denote by $T_a : [0,1] \to [0,1]$ the map $T_a(x) = ax(1-x)$. Given a parameter a we denote by I the dynamical interval $[T_a^2(\frac{1}{2}), T_a(\frac{1}{2})]$ and consider henceforth the restriction of T_a to I.

The first ingredient to the construction of our counterexamples is a recent result by Bruin and Hawkins [2, Theorem 4.2]. It says that if $T_a : I \to I$ is topologically mixing, then it is Lebesgue exact, i.e. the tail-σ-algebra $\mathcal{T} = \bigcap_{n=0}^{\infty} T_a^{-n}\mathcal{B}$ contains only sets of Lebesgue measure zero or full Lebesgue measure.[1]

The second ingredient is an old result of Lin [6], see also [1, Theorem 1.3.3]. It says that a system $(X, \mathcal{B}, m, T)$ is exact if and only if it is completely mixing.[2] Hence, if T_a is topologically mixing, then it is completely mixing.

The third ingredient are real quadratic maps without asymptotic measure constructed by Hofbauer and Keller [3]. Denote by $\bar{\omega}_a(m)$ the set of all weak accumulation points of the sequence of probability measures $\left(\frac{1}{n}\sum_{k=0}^{n-1} m \circ T^{-k}\right)_{n>0}$ where m denotes the normalized Lebesgue measure on I. Theorem 1 of [3] provides an uncountable family of parameters a for which the set of ergodic measures in $\bar{\omega}_a(m)$ is infinite.[3] Such maps are, in particular, not completely mixing, because $m \circ T^{-k} = P^k 1 \cdot m$ so that complete mixing of T_a (in the sense of the above definition) would imply $\bar{\omega}_a(m) = \{\mu\}$.

The missing link that combines these results to produce examples of completely mixing maps without limit measure is the observation that the maps constructed in [3,4] are topologically mixing. For a unimodal interval map topological mixing is equivalent to the nondecomposability of its kneading sequence (equivalently to the nonrenormalizability of the map).[4] But this follows readily from equations (3.6) and (3.7) in [3].

Finally we remark that we have obtained a bit more than only a negative answer to the above problem. We showed:

Theorem There are uncountably many maps T_a in the quadratic family which are completely mixing with respect to Lebesgue measure, but for which $\bar{\omega}_a(m)$ is the set of all T_a-invariant probability measures. In particular, the sequence of measures $\left(\frac{1}{n}\sum_{k=0}^{n-1} m \circ T^{-k}\right)_{n>0}$ does not converge weakly for these parameters.

2. A stochastic attractor which is not a Sinai-Ruelle-Bowen measure

The second example uses interval maps with two indifferent fixed points where the contact of the graph of the map to the diagonal is of higher than second order. To be definite we consider the map $T : [0,1] \to [0,1]$,

$$T(x) = \begin{cases} x + 4x^3 & \text{for } x \in [0, \frac{1}{2}) \\ x - 4(1-x)^3 & \text{for } x \in [\frac{1}{2}, 1]\ . \end{cases}$$

T has a smooth σ-finite invariant density with non-integrable singularities at $x = 0$ and $x = 1$. Thaler [9, Theorem 1] proved that such maps are Lebesgue exact, so by the result of Lin again, they are completely mixing. Since $\lim_{n\to\infty} \int_\delta^{1-\delta} P^n 1\, dm = 0$ for all $\delta > 0$, the set of weak accumulation points of the measures $P^n 1 \cdot m$ is contained

[1] More precisely, Bruin and Hawkins assume that the map T_a has no Cantor attractor in the sense of Milnor. But Lyubich [7] showed that a topologically mixing quadratic map T_a never has such an attractor. For our construction, however, this deep result need not be invoked, because the denseness of the critical orbit in the examples below excludes the existence of a Cantor attractor.

[2] The reader should be warned that Lin [6] uses a different terminology concerning the notion of complete mixing. The terminology used in this note is adopted from [8].

[3] In [4] this construction is modified in such a way that, for uncountably many parameters a, $\bar{\omega}_a(m)$ is even the set of *all* invariant probability measures of T_a.

[4] Recall that we restricted T_a to its dynamical interval.

in $\{a\delta_0 + (1-a)\delta_1 : 0 \le a \le 1\}$. Since T has the symmetry $T(x) = 1 - T(1-x)$, it maps symmetric densities h (i.e. $h(x) = h(1-x)$) to symmetric ones. In particular, all $P^n 1$ are symmetric. Hence $P^n 1 \cdot m \to \mu := \frac{1}{2}(\delta_0 + \delta_1)$ weakly so that μ is a stochastic attractor for $(X, \mathcal{B}, m, T)$. On the other hand, a recent result of Inoue [5, Corollary 2.2] shows that μ is not a Sinai-Ruelle-Bowen measure for the system. Indeed, he proves

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_A(T^k x) = 1 \quad m - \text{a.e. } x$$

for all intervals $A = (0, \delta)$ and $A = (1-\delta, 1)$ and all $\delta > 0$.

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Motivated by [1] and [2], with the notation as in the abstract 3.40 (page 122) we say that homomorphism $\pi : (X, T) \longrightarrow (Y, T)$ has *entirely positive entropy* (e.p.e.

for short) if

$$R_\pi \setminus \Delta_X = \bigcup_{m \in M(X,T)} E_{m|Y}.$$

In such a case we also say that (X, T) is of relatively entirely positive entropy with respect to (Y, T) (rel. e.p.e. for short). The above condition is apparently stronger than rel. u.p.e. from [1]. The rel. u.p.e. does not imply neither rel. c.p.e. nor rel. w.m. (see [1]).

Question XIV.1. Does rel. e.p.e imply rel. c.p.e. and/or rel. w.m.?

If yes, rel. e.p.e. would be a good counterpart for the notion of topological K-homomorphism.

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