A new proof of Alexeyev's Theorem

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Abstract

We give a new proof of Alexeyev's Theorem on realization of the maximal spectral type by a bounded function.

Introduction

In [1], Alexeyev has proved that whenever U is a unitary operator of the Hilbert space $L^2(X,\mathcal{B},\mu)$ with $(X,\mathcal{B},\mu)$ being a standard probability Borel space then there exists a bounded function $f \in L^\infty(X,\mathcal{B},\mu)$ whose spectral measure realizes the maximal spectral type of U. The proof in [1] uses spectral theory and some arguments from the classical theory of analytic functions of one complex variable. Later, using the same idea, Frączek [2] extended Alexeyev's Theorem for realization of the maximal spectral type by functions as "smooth" as is the "smoothness" structure of the underlying space X is. In particular, all such results are satisfied for Koopman representations.

In this note we will show that Alexeyev's Theorem holds for actions of locally compact second countable (l.c.s.c.) Abelian groups. The proof looks new, even in case of unitary representations of $\mathbb{Z}$; it is based on an idea different from [1] and uses only a basic measure theory. As for generality and "smoothness" of results, we will follow [2].

For the spectral theory of unitary representations of l.c.s.c. Abelian groups we refer the reader to [3].

1 Alexeyev's Theorem for group actions

Assume that F is Fréchet space. Then for each sequence $(x_n) \subset F$ there is a sequence (a_n) of positive numbers such that the series $\sum_{l=1}^{\infty} a_{n_l} x_{n_l}$ converges in F for each sequence $n_1 < n_2 < \ldots$. Indeed, if $|\!|\cdot|\!|$ denotes the corresponding

F-norm of F, then using the continuity of multiplication by scalars, for each $n \geq 1$ there is $b_n > 0$ such that $\|a_n x_n\| < \frac{1}{2^n}$ for each $0 < a_n < b_n$, $n \geq 1$. Since F is complete and each numerical series $\sum_{l=1}^{\infty} \|a_{n_l} x_{n_l}\|$ converges, the series $\sum_{l=1}^{\infty} a_{n_l} x_{n_l}$ converges in F.

Assume that $\mathbb{A}$ is an l.c.s.c. Abelian group. Assume that $\mathcal{U} = (U_a)_{a \in A}$ is a continuous unitary representation of $\mathbb{A}$ in a separable Hilbert space H. Recall that, given $x \in H$, by the Bochner-Herglotz Theorem there exists a unique positive finite Borel measure σ_x on $\widehat{\mathbb{A}}$ such that for each $a \in \mathbb{A}$

$$\widehat{\sigma}_x(a) := \int_{\widehat{\mathbb{A}}} \chi(a)\, d\sigma_x(\chi) = \langle U_a x, x \rangle.$$

The measure σ_x is called the *spectral measure* of x. Among spectral measures there are maximal ones with respect to the relation of absolute continuity of measures. All maximal spectral measures are equivalent. The equivalence class of maximal spectral measures is called the *maximal spectral type* of U and is denoted by σ_U. Let

$$\mathbb{A}(x) = \overline{span}\left(\{U_a x :\ a \in \mathbb{A}\}\right).$$

Recall that $\mathcal{U}|_{\mathbb{A}(x)}$ is spectrally equivalent to the representation $\mathcal{V}_{\sigma_x} = ((V_{\sigma_x})_a)_{a \in \mathbb{A}}$ acting on $L^2(\widehat{\mathbb{A}}, \sigma_x)$ by the formula

$$(V_{\sigma_x})_a (f)(\chi) = \chi(a) f(\chi).$$

Theorem 1 (Alexeyev's Theorem) *Let $\mathcal{U} = (U_a)_{a \in \mathbb{A}}$ be a (continuous) unitary representation of $\mathbb{A}$ in a separable Hilbert space H. Assume that $F \subset H$ is a dense linear subspace. Assume moreover that together with a certain F-norm $\|\cdot\|$, stronger than the norm $\|\cdot\|$ given by the scalar product of H, F becomes a Fréchet space. Then for each spectral measure σ ($\ll \sigma_{\mathcal{U}}$) there exists $y \in F$ such that $\sigma_y \gg \sigma$. In particular, there exists $y \in F$ realizing the maximal spectral type of U.*

Proof.

Assume $\sigma = \sigma_x$ is the spectral measure of an $x \in H$. In view of Spectral Theorem, there exists an isomorphism I, determined by $Ix = 1 = 1_{\widehat{\mathbb{A}}}$, from $H = \mathbb{A}(x) \oplus G$ to $L^2(\widehat{\mathbb{A}}, \sigma) \oplus G'$ establishing equivalence of the relevant unitary representations.

Fix $\delta > 0$. Then there exists $z \in F$ such that $\|x - z\| < \delta$. Hence $\|1 - Iz\| < \delta$, and $Iz = f' + g'$, where $f' \in L^2(\widehat{\mathbb{A}}, \sigma)$ and $g' \in G'$. Observe that $\|1 - Iz\|^2 = \|1 - f'\|^2 + \|g'\|^2$, so in particular $\|1 - f'\| < \delta$ and $\|g'\| < \delta$. Let (ε_n) be an arbitrary sequence of positive numbers decreasing to zero. Then, by selecting $\delta_n > 0$ sufficiently small, we can find a sequence $(z_n) \subset F$ (with the decomposition $Iz_n = f_n + g_n$, $f_n \in L^2(\widehat{\mathbb{A}}, \sigma)$, $g_n \in G'$) such that $\|x - z_n\| < \delta_n$ and

$$\sigma\left(\{\chi \in \widehat{\mathbb{A}};\ |1 - f_n(\chi)| < \varepsilon_n\}\right) \to \sigma(\widehat{\mathbb{A}}), \tag{1}$$

when $n \to \infty$.

Let (a_n) be a sequence of positive numbers such that the series $\sum_{l=1}^{\infty} a_{n_l} z_{n_l}$ is convergent in F for an arbitrary subsequence $n_1 < n_2 < \ldots$ Note that, without loss of generality, we can assume that the series $\sum_{n=1}^{\infty} a_n$ converges. We claim that the subsequence (n_l) can be selected so that

$$\sum_{l=1}^{\infty} a_{n_l} f_{n_l}(\chi) \neq 0 \ \text{ for } \sigma\text{-a.e. } \chi \in \widehat{\mathbb{A}}. \tag{2}$$

The proof of (2) is contained in Lemma 1 below. We set $f = \sum_{l=1}^{\infty} a_{n_l} f_{n_l}$, $g = \sum_{l=1}^{\infty} a_{n_l} g_{n_l}$ (the two series are convergent in $L^2(\widehat{\mathbb{A}}, \sigma) \oplus G'$, since the sequence $(\|f_n\|)$ is bounded and $\|g_n\| \to 0$, when $n \to \infty$). In view of (2), the spectral measure of the function f (for the representation $\mathcal{V}_\sigma$) is equivalent to the measure σ. For the element $y = I^{-1}(f + g)$ we have $\sigma_y = \sigma_f + \sigma_g$, so

$$\sigma_y \geq \sigma_f \equiv \sigma.$$

Finally $I^{-1}(f + g) = y = \sum_{l=1}^{\infty} a_{n_l} z_{n_l} \in F$, because the series $\sum_{l=1}^{\infty} a_{n_l} z_{n_l}$ is convergent in F and the F-norm $\|\cdot\|$ is stronger than the norm in H (so the sum of the two series represents the same element in H). □

We now prove the technical (standard) lemma.

Lemma 1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Assume that (j_n) is a sequence of real measurable functions defined on Ω. Assume moreover that $j_n \to 0$ in measure P. Let (a_n) be a sequence of positive real numbers converging to zero and let $0 < \varepsilon_n \to 0$.*

$$P\left(\{\omega \in \Omega;\ |j_n(\omega) - a_n| < \varepsilon_n a_n\}\right) \to 1, \tag{3}$$

when $n \to \infty$. Then there exists a subsequence (n_k) such that

$$\sum_{k\geq 1} j_{n_k}(\omega) \neq 0 \quad \textit{dla } P\textit{-p.w. } \omega \in \Omega.$$

Proof.

Step 1. We claim that there exist a set $B \in \mathcal{F}$ and an increasing sequence of natural numbers $(n_k^{(1)})$ such that

$$P(B) > \frac{2}{3}, \tag{4}$$

$$\begin{array}{l} \omega \in B \ \Rightarrow \ \sum_l j_{n_{k_l}}(\omega) \neq 0 \\ \text{for an arbitrary subsequence } (n_{k_l}) \text{ of } (n_k^{(1)}). \end{array} \tag{5}$$

To this end select first $n_1^{(1)}$ so that

$$P\left(\{|j_{n_1^{(1)}} - a_{n_1^{(1)}}| < \varepsilon_{n_1^{(1)}} a_{n_1^{(1)}}\}\right) > \frac{3}{4}.$$

Set $A_1^{(1)} = \{|j_{n_1^{(1)}} - a_{n_1^{(1)}}| < \varepsilon_{n_1^{(1)}} a_{n_1^{(1)}}\}$ and choose $n_2^{(1)} > n_1^{(1)}$ in such a way that

$$P\left(\{|j_{n_2^{(1)}} - a_{n_2^{(1)}}| < \varepsilon_{n_2^{(1)}} a_{n_2^{(1)}}\} \cap A_1^{(1)}\right) > (1-\delta_2)P(A_1^{(1)}),$$

where δ_2 is a small positive number. Setting $A_2^{(1)} = \{|j_{n_2^{(1)}} - a_{n_2^{(1)}}| < \varepsilon_{n_2^{(1)}} a_{n_2^{(1)}}\} \cap A_1^{(1)}$ and having one more small positive number δ_3 select $n_3^{(1)} > n_2^{(1)}$ in such a way that

$$P\left(\{|j_{n_3^{(1)}} - a_{n_3^{(1)}}| < \varepsilon_{n_3^{(1)}} a_{n_3^{(1)}}\} \cap A_2^{(1)}\right) > (1-\delta_3)P(A_2^{(1)})$$

and set $A_3^{(1)} = \{|j_{n_3^{(1)}} - a_{n_3^{(1)}}| < \varepsilon_{n_3^{(1)}} a_{n_3^{(1)}}\} \cap A_2^{(1)}$. By continuing this process it is clear that for a relevant choice of sufficiently small numbers $\delta_k > 0$ and a relevant choice of $n_k^{(1)}$ it is sufficient to define $B = \bigcap_{k=1}^{\infty} A_k^{(1)}$ so that (4) and (5) were satisfied.

Set $B_1 = B^c$.

Step 2. We claim that there exist a subset $\mathcal{F} \ni B_2 \subset B_1$ and a subsequence $(n_k^{(2)})$ of $(n_k^{(1)})$ $(n_1^{(2)} > n_1^{(1)})$ such that

$$P(B_2) > \frac{2}{3}P(B_1), \tag{6}$$

$$\omega \in B_2 \ \Rightarrow \ j_{n_1^{(1)}}(\omega) + \sum_l j_{n_{k_l}}(\omega) \neq 0 \quad \text{for arbitrary subsequence } (n_{k_l}) \text{ of } (n_k^{(2)}). \tag{7}$$

To justify the above assertion, consider the function $j_{n_1^{(1)}}$ on B_1. There exists $\delta > 0$ such that

$$P\left(\{j_{n_1^{(1)}} = 0 \vee |j_{n_1^{(1)}}| > \delta\}\right) > \frac{4}{5}P(B_1)$$

and let $A_0^{(2)} := \{j_{n_1^{(1)}} = 0 \vee |j_{n_1^{(1)}}| > \delta\} \cap B_1$. Select $n_1^{(2)} > n_1^{(1)}$ so that

$$P\left(\{|j_{n_1^{(2)}} - a_{n_1^{(2)}}| < \varepsilon_{n_1^{(2)}} a_{n_1^{(2)}}\}\right) > \frac{3}{4}P(A_0^{(2)}).$$

Set $A_1^{(2)} = \{|j_{n_1^{(2)}} - a_{n_1^{(2)}}| < \varepsilon_{n_1^{(2)}} a_{n_1^{(2)}}\}$ and choose $n_2^{(2)} > n_1^{(2)}$ in such a way that

$$P\left(\{|j_{n_2^{(2)}} - a_{n_2^{(2)}}| < \varepsilon_{n_2^{(2)}} a_{n_2^{(2)}}\} \cap A_1^{(2)}\right) > (1-\delta_2)P(A_1^{(2)}),$$

where δ_2 is a small positive number. Setting $A_2^{(2)} = \{|j_{n_2^{(2)}} - a_{n_2^{(2)}}| < \varepsilon_{n_2^{(2)}} a_{n_2^{(2)}}\} \cap A_1^{(2)}$ and having one more small positive number δ_3 we select $n_3^{(2)} > n_2^{(2)}$ in such a way that

$$P\left(\{|j_{n_3^{(2)}} - a_{n_3^{(2)}}| < \varepsilon_{n_3^{(2)}} a_{n_3^{(2)}}\} \cap A_2^{(2)}\right) > (1-\delta_3)P(A_2^{(2)})$$

an we set $A_3^{(2)} = \{|j_{n_3^{(2)}} - a_{n_3^{(2)}}| < \varepsilon_{n_3^{(2)}} a_{n_3^{(2)}}\} \cap A_2^{(2)}$. By continuing this process it is clear that for a relevant choice of sufficiently small numbers $\delta_k > 0$ and a relevant choice of numbers $n_k^{(2)}$ it is enough to define $B_2 = \bigcap_{k=1}^{\infty} A_k^{(2)}$ so that (6) and (7) were satisfied.

Set $B_3 = (B \cup B_2)^c$.

Step 3. We claim that there exist $\mathcal{F} \ni B_4 \subset B_3$ and a subsequence $(n_k^{(3)})$ of $(n_k^{(2)})$ $(n_1^{(3)} > n_1^{(2)})$ such that

$$P(B_4) > \frac{2}{3}P(B_3), \tag{8}$$

$$\omega \in B_4 \ \Rightarrow \ j_{n_1^{(1)}}(\omega) + j_{n_1^{(2)}}(\omega) + \sum_l j_{n_{k_l}}(\omega) \neq 0 \quad \text{for each subsequence } (n_{k_l}) \text{ of } (n_k^{(3)}). \tag{9}$$

To justify this assertion consider the function $j_{n_1^{(1)}} + j_{n_1^{(2)}}(\omega)$ on B_3. There exists $\delta > 0$ such that

$$P\left(\{j_{n_1^{(1)}} + j_{n_1^{(2)}} = 0 \vee |j_{n_1^{(1)}} + j_{n_1^{(2)}}| > \delta\}\right) > \frac{4}{5}P(B_3)$$

and let $A_0^{(3)} := \{j_{n_1^{(1)}} + j_{n_1^{(2)}} = 0 \vee |j_{n_1^{(1)}} + j_{n_1^{(1)}}| > \delta\} \cap B_3$. Select $n_1^{(3)} > n_1^{(2)}$ so that

$$P\left(\{|j_{n_1^{(3)}} - a_{n_1^{(3)}}| < \varepsilon_{n_1^{(3)}} a_{n_1^{(3)}}\} \cap A_0^{(3)}\right) > \frac{3}{4}P(A_0^{(3)}).$$

Proceeding as above we will construct a set of full measure, namely $B \cup B_2 \cup B_4 \cup \ldots$, on which, for the sequence $(n_1^{(k)})_{k \geq 1}$, the assertion of the lemma holds. □

Applying now Frączek's analysis [2] we can deduce that if H has an additional structure then we can find elements belonging to specific subspaces and realizing the maximal spectral type. For example, if $H = L^2(X, \mu)$, where $(X, \mathcal{B}, \mu)$ is a measure space with finite or infinite measure, then we can find a function $y \in L^\infty(X, \mu) \cap L^2(X, \mu)$ realizing the maximal spectral type of U. If X is a compact metric space and μ is a positive Borel measure then the maximal spectral type of U is realized by a continuous function. Similar assertion are obtained on manifolds, see [2].

Remark 1 By a small modification of the proof of Theorem 1 we can obtain that if $\sigma = \sigma_u$ and $\varepsilon > 0$ then $y \in F$ in Theorem 1 can be found in the ε-neighborhood of u.

Remark 2 It is possible to adapt the original Alexeyev's proof from [1] to obtain another proof of Theorem 1.

Remark 3 A similar proof of Alexeyev's Theorem to the one presented here has also been obtained by Andres del Junco.

References

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