

Relatively finite measure-preserving extensions and lifting multipliers by Rokhlin cocycles

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September 23, 2009

Dedicated to Stephen Smale in recognition of his contributions to topology and dynamical systems

2000 Mathematical Subject Classification: 37A05, 37A20, 37A40.

Key words and phrases: ergodicity, Rokhlin cocycle, joining, disjointness, non-singular automorphism, relatively finite measure preserving extension.

Abstract

We show that under some natural ergodicity assumptions extensions given by Rokhlin cocycles lift the multiplier property if the associated locally compact group extension has only countably many L^∞ -eigenvalues. We make use of some analogs of basic results from the theory of finite-rank modules associated to an extension of measure-preserving systems in the setting of a non-singular base.

Introduction

Our main concern in this paper is with measure-preserving automorphisms of a standard Borel probability space (X, \mathcal{B}, μ) . We will denote by $\text{Aut}_0(X, \mathcal{B}, \mu)$ the Polish group of all such automorphisms up to almost-everywhere agreement.

Assume that $T \in \text{Aut}_0(X, \mathcal{B}, \mu)$ is ergodic. If \bar{T} acting on $(\bar{X}, \bar{\mathcal{B}}, \bar{\mu})$ is an ergodic extension of T then the classical Abramov-Rokhlin Theorem states that we can assume that $(\bar{X}, \bar{\mathcal{B}}, \bar{\mu}) = (X, \mathcal{B}, \mu) \otimes (Y, \mathcal{C}, \nu)$ and

$$\bar{T}(x, y) = (Tx, \theta_x(y)),$$

*Research partially supported by MSRI (Berkeley) program “Ergodic Theory and Additive Combinatorics”

†Research partially supported by Polish MNiSzW grant N N201 384834; partially supported by Marie Curie “Transfer of Knowledge” EU program – project MTKD-CT-2005-030042 (TODEQ) and MSRI (Berkeley) program “Ergodic Theory and Additive Combinatorics”

where $\theta : X \rightarrow \text{Aut}_0(Y, \mathcal{C}, \nu)$ is a so-called *Rokhlin cocycle*.

Moreover, as noticed in [3], any $\text{Aut}_0(Y, \mathcal{C}, \nu)$ -valued cocycle θ is cohomologous to a special cocycle. Namely, there exist a locally compact amenable group G together with a cocycle $\varphi : X \rightarrow G$ and a measurable G -action $\mathcal{S} = (S_g)_{g \in G}$ on (Y, \mathcal{C}, ν) such that θ is cohomologous to the cocycle $x \mapsto S_{\varphi(x)}$. The resulting automorphism will be denoted by $T_{\varphi, \mathcal{S}}$: that is,

$$T_{\varphi, \mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y)).$$

It follows that the study of ergodic extensions of a given transformation T is reduced to that of systems of the form $T_{\varphi, \mathcal{S}}$.

Such extensions have recently been examined in the context of lifting disjointness properties [3], [6], [8], [14], [15]. In particular, the investigations [3] and [15] led to sufficient conditions on φ and \mathcal{S} so that if T and R are disjoint in Furstenberg's sense [4] (denoted $T \perp R$) then also $T_{\varphi, \mathcal{S}} \perp R$.

A more subtle problem concerns the lifting of the multiplier property. Given a class $\mathcal{F} \subset \text{Aut}_0(X, \mathcal{B}, \mu)$, we denote by \mathcal{F}^\perp the class of automorphisms disjoint from all systems in \mathcal{F} . An automorphism T is said to be a *multiplier* of \mathcal{F}^\perp (denoted $T \in \mathcal{M}(\mathcal{F}^\perp)$) if for each automorphism $S \in \mathcal{F}^\perp$ and any ergodic joining $T \vee S$ we also have $T \vee S \in \mathcal{F}^\perp$. A central result of [15] provided an example of an ergodic automorphism $T_{\varphi, \mathcal{S}} \in \text{WM}^\perp$ which was not a multiplier of WM^\perp , and which could then be used to answer negatively the question, originally posed by Glasner in [6], of whether the class WM^\perp is closed under ergodic joinings. In that construction T was an ergodic rotation and the skew product T_φ acting on $X \times G$ by the formula $T_\varphi(x, g) = (Tx, \varphi(x)g)$ (which preserves the **infinite** measure $\mu \otimes \lambda_G$, where λ_G denotes a Haar measure of G) had uncountably many L^∞ -eigenvalues. This property was crucial for the arguments of [15], all earlier constructions over a rotation for which T_φ had only countably many L^∞ -eigenvalues having led to multipliers of WM^\perp (see [8]). The relationship between [6], [8] and [15] was explained in [3], where the main result asserted that given an ergodic rotation T and an ergodic cocycle φ such that T_φ has countable L^∞ -spectrum, the automorphism $T_{\varphi, \mathcal{S}}$ is always a multiplier of WM^\perp .

In the present paper we use an alternative approach to the multiplier property to prove a generalization of the result from [3] (see Theorem 4 below):

Assume that T is ergodic and that $\varphi : X \rightarrow G$ is ergodic, and that these are such that $T_{\varphi, \mathcal{S}}$ is ergodic and T_φ has countably many L^∞ -eigenvalues. If in addition $T \in \mathcal{M}(\mathcal{F}^\perp)$ for some class $\mathcal{F} \subset \text{WM}$, then also $T_{\varphi, \mathcal{S}} \in \mathcal{M}(\mathcal{F}^\perp)$. In particular, the result holds for $\mathcal{F} = \text{WM}$.

We will also show that this theorem gives a criterion for the lifting of the *joining primeness property* introduced in [16], and from this that we can produce examples of systems with the joining primeness property which are not distally simple.

In the course of proving these results, we are naturally drawn into an examination of certain relatively finite measure preserving extensions of underlying

non-singular base systems. In this setting we make use of some analogs of basic results from the theory of finite-rank modules associated to an extension of measure-preserving systems (see, for example, Chapter 9 of Glasner [7]), whose proofs turn out to be easily transplantable to the setting of a non-singular base and which may be of some independent interest.

1 Relatively finite measure-preserving factors of singular automorphisms

The later sections of this paper will rely on some basic notions concerning not only measure-preserving systems, but also certain non-singular systems. We denote by $\text{Aut}(X, \mathcal{B}, \mu)$ the group of all non-singular automorphisms (considered up to almost-everywhere agreement) of a standard Borel probability space (X, \mathcal{B}, μ) .

Assume that $T \in \text{Aut}(X, \mathcal{B}, \mu)$. Denote by $\mathcal{P}(X)$ the set of probability measures on (X, \mathcal{B}) . Let

$$\alpha(x) = \frac{d\mu \circ T}{d\mu}(x)$$

denote the Radon-Nikodým derivative of $\mu \circ T$ with respect to μ (the non-singularity of T gives that $\mu \circ T \sim \mu$). Let

$$\mathcal{P}_\alpha = \{\lambda \in \mathcal{P}(X) : \frac{d\lambda \circ T}{d\lambda}(x) = \alpha(x) \text{ } \lambda - \text{a.e.}\}.$$

Then $\mu \in \mathcal{P}_\alpha$ and \mathcal{P}_α is a simplex whose subset of extremal points $\mathcal{E}_\alpha \subset \mathcal{P}_\alpha$ contains precisely the ergodic members of \mathcal{P}_α .

Each measure $\nu \in \mathcal{P}_\alpha$ gives rise to a non-singular automorphism (T, X, \mathcal{B}, ν) . As is well-known, the simplex \mathcal{P}_α now appears in the ergodic decomposition of (T, ν) .

Theorem 1 ([9]) *If $\nu \in \mathcal{P}_\alpha$ then the ergodic decomposition of (T, ν) is given by*

$$(1) \quad \nu = \int_{\mathcal{P}_\alpha} \epsilon dQ(\epsilon)$$

for some probability measure Q on \mathcal{P}_α . □

Returning to our original non-singular system (T, X, \mathcal{B}, μ) , assume that $\mathcal{A} \subset \mathcal{B}$ is an invariant σ -algebra (a *factor* of T). Following [3] we say that the factor $T|_{\mathcal{A}}$ is *relatively finite measure-preserving (r.f.m.p.)* if $\alpha = \frac{d\mu \circ T}{d\mu}$ is \mathcal{A} -measurable. Let (S, Y, \mathcal{C}, ν) be the factor system $(T|_{\mathcal{A}}, X/\mathcal{A}, \mathcal{A}, \mu|_{\mathcal{A}})$ given by

\mathcal{A} and let $\pi : X \rightarrow Y$ be the factor map. Then the fact that S is an r.f.m.p. factor of T is expressed by the equality (see [3])

$$(2) \quad \mu_{S\pi(x)} = \mu_{\pi(x)} \circ T^{-1},$$

where $\mu = \int_Y \mu_y d\nu(y)$ is the disintegration of μ over ν .

Notice that in the ergodic decomposition (1) the Radon-Nikodým derivative of Q -a.e. ergodic component ϵ is also α . From this we can quickly deduce the following.

Corollary 1 *If $T|_{\mathcal{A}}$ is ergodic then for Q -a.e. ergodic component ϵ in (1) the system $(T|_{\mathcal{A}}, \mu|_{\mathcal{A}})$ is also a factor of $(T, X, \mathcal{B}, \epsilon)$ and moreover the factor $\mathcal{A} \subset \mathcal{B}$ is also r.f.m.p. for $(T, X, \mathcal{B}, \epsilon)$.*

Proof.

We only need to prove that the action of T on \mathcal{A} with the measure $\mu|_{\mathcal{A}}$ is a factor of (T, ϵ) . As remarked in [3], this follows directly from the formula (2) and the uniqueness of ergodic decomposition. \square

We now recall a non-singular version of Rokhlin's Theorem (see [18]). Let T be an ergodic non-singular automorphism of a standard probability Borel space (X, \mathcal{B}, μ) and let S acting on (Y, \mathcal{C}, ν) be its factor (given by an invariant sub- σ -algebra $\mathcal{A} \subset \mathcal{B}$). Then, up to measure space isomorphism, $(X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu) \otimes (Z, \mathcal{D}, \rho)$ for another standard Borel space (Z, \mathcal{D}, ρ) and in these new "coordinates" we can express $T = S_{\Theta}$, where

$$S_{\Theta}(y, z) = (Sy, \Theta(y)(z))$$

and $\Theta : Y \rightarrow \text{Aut}(Z, \mathcal{D}, \rho)$ is a Rokhlin cocycle with values in the group of non-singular automorphisms of (Z, \mathcal{D}, ρ) . The automorphism S_{Θ} has still S as its factor. If S was an r.f.m.p. factor of T , it is now an r.f.m.p. factor of S_{Θ} . However the disintegration of $\nu \otimes \rho$ over ν is trivial and now the equality (2) asserts that $\rho = \rho \circ \Theta(y)^{-1}$ for ν -a.e. $y \in Y$ (that is, the fiber automorphisms are measure-preserving).

Corollary 2 *Under the above notation, if T is ergodic and S is an r.f.m.p. factor then (up to isomorphism) in the skew product representation of T over S the Rokhlin cocycle Θ takes values in the group $\text{Aut}_0(Z, \mathcal{D}, \rho)$ of measure-preserving automorphisms of (Z, \mathcal{D}, ρ) .* \square

2 Finite-rank modules over a non-singular base

Our later proofs will make use of a version of the theory of invariant finite-rank modules over a factor, adapted to the setting of an r.f.m.p. extension of a non-singular base system. This machinery is well-known as applied to extensions of measure-preserving systems, and much of it can simply be carried over to our scenario unchanged. For this purpose we refer the reader to Chapter 9 of Glasner's book [7] for a thorough account, and will first recall some basic results from there. Let us stress that in this section we will make repeated appeal to the fact that our transformation is ergodic. It seems likely that an analogous theory can be developed without this restriction (as has been done for the measure-preserving case in [2]), but we do not explore this rather technical subject here. Later we will apply the results of this section to the f.m.p. extensions described in the Introduction in a somewhat complicated way; in order to lighten our notation, for the duration of the present section we will instead let T be a non-singular transformation on (X, \mathcal{B}, μ) that is r.f.m.p. over $\mathcal{A} \subset \mathcal{B}$, with the warning that this does not correspond to the transformation T discussed in the Introduction.

Given a finite measure space (X, \mathcal{B}, μ) and a sub- σ -algebra $\mathcal{A} \subset \mathcal{B}$, an \mathcal{A} -module is a subspace $M \subset L^2(X, \mathcal{B}, \mu)$ such that whenever $f \in M$ and $h \in L^\infty(X, \mathcal{B}, \mu)$ is \mathcal{A} -measurable, then also $h \cdot f \in M$.

Our picture of such modules becomes rather clearer when we invoke the Rokhlin representation of $\mathcal{A} \subset \mathcal{B}$. For the remainder of this section let us write $(X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu) \otimes (Z, \mathcal{D}, \rho)$, so that our transformation T becomes S_Θ as above. We can now clearly identify the \mathcal{A} -measurable members of $L^\infty(X, \mathcal{B}, \mu)$ with $L^\infty(Y, \mathcal{C}, \nu)$, and we will henceforth abusively write simply that $L^\infty(Y, \mathcal{C}, \nu) \subset L^\infty(X, \mathcal{B}, \mu)$. Writing $\pi : X \rightarrow Y$ for the canonical factor map, we will also refer instead to a π -module.

A π -module M can now be easily identified with a measurable bundle of Hilbert subspaces $M_y \subset L^2(Z, \mathcal{D}, \rho)$ indexed by $y \in Y$, so that M is the space of measurable sections of this bundle. This module is of *finite-rank* if $\dim M_y < \infty$ almost everywhere, and is of *rank* r if $\dim M_y = r$ almost everywhere. For this special class of modules we have the following relativized ability to select orthonormal bases.

Lemma 1 (Lemma 9.4 of [7]) *If M is a π -module of rank $r < \infty$ then there is an r -tuple $\phi_1, \phi_2, \dots, \phi_r$ of members of M such that $\mathbb{E}(\phi_i \bar{\phi}_j | \mathcal{A}) \equiv \delta_{ij}$ for $1 \leq i, j \leq r$ and such that any $f \in M$ can be expressed as $\sum_{i=1}^r h_i \cdot \phi_i$ for some $h_1, h_2, \dots, h_r \in L^\infty(Y, \mathcal{C}, \nu)$.*

We refer to such a tuple of functions as a fiberwise orthonormal basis of M .

□

Our interest will be in finite-rank π -modules that are invariant under S_Θ . In

order to work with these in the setting of a non-singular base system (S, Y, \mathcal{C}, ν) we first introduce a more refined residence for them within $L^2(X, \mathcal{B}, \mu)$.

Definition 1 We write $L_A^\infty L^2(X, \mathcal{B}, \mu)$ for the subspace of those measurable functions $f \in L^2(X, \mathcal{B}, \mu)$ such that the conditional expectation satisfies $\mathbb{E}(f | \mathcal{A}) \in L^\infty(\mu|_{\mathcal{A}})$. We equip this function space with the norm

$$\|f\|_{L_A^\infty L^2} := \|\mathbb{E}(|f|^2 | \mathcal{A})\|_\infty.$$

Under the identification $(X, \mathcal{B}, \mu) = (Y, \mathcal{C}, \nu) \otimes (Z, \mathcal{D}, \rho)$, it is clear that $L_A^\infty L^2(X, \mathcal{B}, \mu)$ is identified with $L^\infty(Y, \mathcal{C}, \nu; L^2(Z, \mathcal{D}, \rho))$.

Now, we have represented the transformation T as S_Θ , and so its action on $L^\infty(Y, \mathcal{C}, \nu; L^2(Z, \mathcal{D}, \rho))$ can be written as

$$\xi \circ S_\Theta(y, z) = \xi(Sy, \Theta(y)(z)) = U_{\Theta(y)}(\xi(Sy, \cdot))(z),$$

where $U_{\Theta(y)} \in U(L^2(Z, \mathcal{D}, \rho))$ is the unitary operator associated to the transformation $\Theta(y) \in \text{Aut}_0(Z, \mathcal{D}, \rho)$ by the Koopman representation. In order to work with this representation, given a function $f \in L^2(X, \mathcal{B}, \mu)$ we will sometimes write $f|_y$ for the restriction $f(y, \cdot)$ to the fiber $\pi^{-1}\{y\}$, regarded as a member of $L^2(Z, \mathcal{D}, \rho)$.

The following is now immediate from this representation.

Lemma 2 *A π -submodule $M \subset L^2(X, \mathcal{B}, \mu)$ is S_Θ -invariant if and only if $M_{S_y} = U_{\Theta(y)}(M_y)$ for almost every y . \square*

Proposition 1 *If $M \subset L^2(X, \mathcal{B}, \mu)$ is an S_Θ -invariant finite-rank module over $\mathcal{A} \subset \mathcal{B}$, then its rank is almost surely constant on Y , and if $\phi_1, \phi_2, \dots, \phi_r$ is a fiberwise orthonormal basis for M then $\sum_{j=1}^r |\phi_j(x)|^2$ is almost surely equal to r . As a result any such M is contained in $L^\infty(X, \mathcal{B}, \mu)$.*

Proof.

The assertion that $\dim M_y$ is almost surely constant follows simply because on the one hand M_y and hence also $\dim M_y$ vary measurably with y , but on the other $\dim M_{S_y} = \dim(U_{\Theta(y)}(M_y)) = \dim M_y$ for almost every y , and so $\dim M_y$ is almost surely invariant under the transformation S , which we have assumed is ergodic.

This constancy of $\dim M_y$ now allows us to pick a fiberwise orthonormal basis $\phi_1, \phi_2, \dots, \phi_r$. In terms of this, we know that

$$U_{\Theta(y)}(\text{span}\{\phi_1|_{S_y}, \phi_2|_{S_y}, \dots, \phi_r|_{S_y}\}) = \text{span}\{\phi_1|_y, \phi_2|_y, \dots, \phi_r|_y\},$$

and so the unitary cocycle $U_{\Theta(\cdot)}$ specializes to give a measurable family of $r \times r$ unitary matrices $(U_{ij}(y))_{1 \leq i, j \leq r}$ such that

$$U_{\Theta(y)}(\phi_i|_{Sy})(z) = \sum_{j=1}^r U_{ij}(y)\phi_j|_y(z)$$

$\nu \otimes \rho$ -almost surely. However, the left-hand side of this equation is simply $\phi_i(Sy, \Theta(y)(z))$, and so squaring and summing in i we obtain

$$\sum_{i=1}^r |\phi_i(Sy, \Theta(y)(z))|^2 = \sum_{i=1}^r \left| \sum_{j=1}^r U_{ij}(y)\phi_j|_y(z) \right|^2 = \sum_{j=1}^r |\phi_j(y, z)|^2,$$

by the unitarity of $(U_{ij}(y))_{1 \leq i, j \leq r}$. It follows that the expression $\sum_{j=1}^r |\phi_j(y, z)|^2$ is invariant under $S_{\Theta} = T$, which we also assumed ergodic, and so is almost surely constant. Since by definition $\int_Z |\phi_i(y, z)|^2 \rho(dz) = 1$ for each $i = 1, 2, \dots, r$, by integrating over $z \in Z$ we see that this constant must equal r .

It follows in particular that each ϕ_i is bounded in L^∞ by this constant, and now it is immediate that membership of L^∞ persists under taking finite $L^\infty(Y, \mathcal{C}, \nu)$ -linear combinations. \square

Remark 1 The unitary cocycle U_{ij} constructed in the course of the above proof is often referred to as the *relative eigenvalue* associated to the finite-rank module M . Note that different choices of basis for M can give rise to different relative eigenvalues, but they are always cohomologous.

Finite-rank modules are lent particular importance in ergodic theory by their rôle in the classical dichotomy proved by Furstenberg [5] and Zimmer [22, 23] between relatively weakly mixing extensions and those containing a nontrivial subextension that can be coordinatized as a compact homogeneous skew-product extension. Although we will not need this result in the present work, its proof for extensions over a non-singular base is identical to that over a measure-preserving base, now that we have Proposition 1 at our disposal, and so we simply state the relevant definitions and result here for completeness.

Definition 2 Suppose that (X, \mathcal{B}, μ, T) is a non-singular system and $\mathcal{A} \subset \mathcal{B}$ a factor over which it is r.f.m.p. This factor is *relatively weakly mixing* if the non-singular system $(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes_{\mathcal{A}} \mu, T \times T)$ is ergodic (this is the usual construction of the relative product over \mathcal{A} , which is easily seen to give a non-singular system if (T, X, \mathcal{B}, μ) is non-singular; see, for instance, [19]). On the other hand, it is *isometric* if the space $L^\infty_{\mathcal{A}} L^2(X, \mathcal{B}, \mu)$ is spanned by its finite rank T -invariant \mathcal{A} -submodules.

Proposition 2 *The system (T, X, \mathcal{B}, μ) fails to be relatively weakly mixing over an r.f.m.p. factor $\mathcal{A} \subset \mathcal{B}$ if and only if it contains a nontrivial isometric subextension of $(T|_{\mathcal{A}}, X/\mathcal{A}, \mathcal{A}, \mu|_{\mathcal{A}})$. \square*

For our applications in this paper our particular interest will be in invariant modules of rank one, whose behaviour exhibits the following useful feature.

Proposition 3 *Any two distinct rank-one T -invariant π -modules M and N are fiberwise orthogonal, in that $\mathbb{E}(f\bar{g}|\mathcal{A}) = 0$ whenever $f \in M$ and $g \in N$.*

Proof.

We assume that M and N are rank-one T -invariant π -modules that are not fiberwise orthogonal and show that they must actually coincide. Indeed, in this case each admits a one-element fiberwise orthonormal basis: that is, there are $\phi \in M$ and $\psi \in N$ such that $M = L^\infty(Y, \mathcal{C}, \nu) \cdot \phi$ and $N = L^\infty(Y, \mathcal{C}, \nu) \cdot \psi$.

Now Proposition 1 gives that $|\phi|$ and $|\psi|$ are both equal to 1 almost everywhere, and that there are maps $\sigma, \tau : Y \rightarrow \mathbb{S}^1$ such that $U_{\Theta(y)}\phi|_{Sy} = \sigma(y)\phi|_y$ and $U_{\Theta(y)}\psi|_{Sy} = \tau(y)\psi|_y$. Defining a new function $\xi : Y \rightarrow \mathbb{C}$ by

$$\xi(y) := \langle \phi|_y, \psi|_y \rangle_{L^2(Z, \mathcal{D}, \rho)},$$

we compute from these relations and the unitarity of $U_{\Theta(y)}$ that

$$\xi(Sy) = \langle \phi|_{Sy}, \psi|_{Sy} \rangle_{L^2(Z, \mathcal{D}, \rho)} = \langle U_{\Theta(y)}\phi|_{Sy}, U_{\Theta(y)}\psi|_{Sy} \rangle_{L^2(Z, \mathcal{D}, \rho)} = \overline{\tau(y)}\sigma(y)\xi(y).$$

Since by assumption ϕ and ψ are not fiberwise orthogonal, it follows that ξ is non-zero on a subset of Y of positive measure. On the other hand, the above shows that $|\xi(y)|$ is S -invariant, and so in fact $|\xi|$ is a non-zero constant almost everywhere. Therefore we can define $\xi'(y) := \xi(y)/|\xi(y)| \in \mathbb{S}^1$, and now the above relations together imply that the function $\xi'(y)\overline{\phi(y, z)}\psi(y, z)$ is T -invariant:

$$\begin{aligned} \xi'(Sy)\overline{\phi(Sy, \Theta(y)(z))}\psi(Sy, \Theta(y)(z)) &= (\overline{\tau(y)}\sigma(y)\xi'(y))\overline{(\sigma(y)\overline{\phi(y, z)})}(\tau(y)\psi(y, z)) \\ &= \xi'(y)\overline{\phi(y, z)}\psi(y, z). \end{aligned}$$

Hence by the ergodicity of T it must be a constant, say $c \in \mathbb{S}^1$, and so rearranging gives $\phi(y, z) = c\xi'(y)\psi(y, z) \in N$. Exactly similarly we have $\psi \in M$, and so $N = M$, as asserted. \square

Remark 2 In order to work with finite-rank invariant modules of higher rank, it is necessary first to prove that any such module can be decomposed as a

direct sum of finite-rank invariant modules that are *irreducible*, in that they admit no further proper invariant submodules. This follows relatively easily by observing that given finite-rank invariant modules $N \subset M$, the fiberwise orthogonal complement $M \ominus N$ defined by $(M \ominus N)_y = M_y \ominus N_y$ is also an invariant module, and then performing a simple induction on rank to show that repeatedly choosing minimal sub-modules and then restricting to their orthogonal complements leads to the desired direct sum decomposition.

These irreducible finite-rank modules form the building blocks of all others, but it is classically known that Proposition 3 does not extend to general irreducible finite-rank invariant modules of higher rank, even in the setting of measure-preserving actions. In the Appendix we include for completeness an example that witnesses this failure.

Let us now bring Proposition 3 to bear on the problem of growth of the set of L^∞ -eigenvalues.

We know that T acts as an isometry on the space $L^\infty(X, \mathcal{B}, \mu)$. We let $e(T)$ denote the group of L^∞ -eigenvalues of T : $c \in \mathbb{S}^1$ belongs to $e(T)$ if there exists $0 \neq f \in L^\infty(X, \mathcal{B}, \mu)$ such that $f \circ T = cf$. Since T is also assumed ergodic, the modulus of this f is constant, whence in the ergodic case we can additionally assume that eigenfunctions have modulus 1. The group $e(T)$ is a Borel subgroup of \mathbb{S}^1 and the Borel structure is generated by a Polish topology (stronger than the induced Euclidean topology, see [10], [17]). Recall also that since L^∞ is not separable, in general $e(T)$ is uncountable (see [17], Chapter 15). Our next aim is to prove the following.

Theorem 2 *Assume that T is a non-singular ergodic automorphism of a standard probability Borel space (X, \mathcal{B}, μ) . Assume moreover that $\mathcal{A} \subset \mathcal{B}$ is a factor of T which is r.f.m.p. Then the quotient $e(T)/e(T|_{\mathcal{A}})$ is countable. In particular, if $e(T|_{\mathcal{A}})$ is countable, then $e(T)$ is also countable.*

In other words we want to prove that if T has uncountably many eigenfunctions then all its r.f.m.p. factors also have uncountably many eigenfunctions. We will see that this follows quickly from Proposition 3.

Proof.

Suppose that $f \in L^\infty(X, \mathcal{B}, \mu)$ is an L^∞ -eigenfunction of T with eigenvalue $c \in \mathbb{S}^1$. Then $M_f := L^\infty(Y, \mathcal{C}, \nu) \cdot f$ is a rank-1 \mathcal{A} -module, and since $(h \cdot f) \circ T = c \cdot (h \circ T) \cdot f$ we see that M is T -invariant. If now g is another L^∞ -eigenfunction of T with eigenvalue $c' \neq c$, then either $M_f = M_g$, in which case we have in particular that $f = h \cdot g$ for some $h \in L^\infty(Y, \mathcal{C}, \nu)$, hence $cf = c'(h \circ T)g = c'(h \circ T)\bar{h}f$ and so h is an L^∞ -eigenfunction of S with eigenvalue $\overline{cc'}$; or $M_f \neq M_g$, in which case by Proposition 3 they are fiberwise orthogonal in $L^\infty(Y, \mathcal{C}, \nu; L^2(Z, \mathcal{D}, \rho))$.

Let $\{c_i \in e(T) : f_i \circ T = c_i \cdot f_i, i \in I\}$ be a maximal family of eigenvalues so that for $i \neq j$ the rank-1 \mathcal{A} -modules M_{f_i} and M_{f_j} are fiberwise orthogonal. By fiberwise orthogonality, $\mathbb{E}(f_i \cdot \overline{f_j} | \mathcal{A}) = 0$ and in particular $f_i \perp f_j$ in $L^2(X, \mathcal{B}, \mu)$.

It follows that I is countable. Moreover, by the first part of the proof $e(T) = \bigcup_{i \in I} c_i e(T|_{\mathcal{A}})$ and the result follows. \square

Using Corollary 1 we also obtain the following.

Corollary 3 *Assume that T is a non-singular automorphism of a standard probability Borel space (X, \mathcal{B}, μ) . Assume that $\mathcal{A} \subset \mathcal{B}$ is an ergodic r.f.m.p. factor such that the group $e(T|_{\mathcal{A}})$ of L^∞ -eigenvalues is countable. Then for almost each ergodic component ε of T , $e(T, \varepsilon)$ is also countable.*

3 Cocycles, Mackey actions and invariant measures for Rokhlin cocycle extensions

We will now recall some basic facts about cocycles (see e.g. [1], [20]). Assume that T is an ergodic measure-preserving automorphism of a standard probability Borel space (X, \mathcal{B}, μ) , i.e. $T \in \text{Aut}_0(X, \mathcal{B}, \mu)$. Let G be a locally compact second countable (l.c.s.c) group. Each measurable map $\varphi : X \rightarrow G$ is called a *cocycle*; more precisely φ generates the cocycle $\varphi^{(n)}(\cdot)$ by the following formula

$$\varphi^{(n)}(x) = \begin{cases} \varphi(T^{n-1}x) \cdot \dots \cdot \varphi(x) & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ (\varphi(T^{-1}(x)) \cdot \dots \cdot \varphi(T^n x))^{-1} & \text{if } n < 0. \end{cases}$$

Denote by T_φ the corresponding skew product:

$$T_\varphi : (X \times G, \mathcal{B}(X \times G), \lambda_G) \rightarrow (X \times G, \mathcal{B}(X \times G), \lambda_G), \quad T_\varphi(x, g) = (Tx, \varphi(x)g),$$

where λ_G is a left-invariant Haar measure. Then $(T_\varphi)^n(x, g) = (T^n x, \varphi^{(n)}(x)g)$ and also $\varphi^{(n+m)}(x) = \varphi^{(n)}(T^m x) \varphi^{(m)}(x)$ for every $n, m \in \mathbb{Z}$. Let $\tau = (\tau_g)_{g \in G}$ denote the natural (right) G -action on $X \times G$: $\tau_g(x, h) = (x, hg^{-1})$. Then τ_g is non-singular with respect to λ_G and it commutes with T_φ . Fix a probability measure λ equivalent to λ_G and consider the σ -algebra \mathcal{I}_φ of T_φ -invariant subsets. Since $(X \times G, \mathcal{B}(X \times G), \mu \otimes \lambda)$ is a standard probability Borel space, the quotient space $((X \times G)/\mathcal{I}_\varphi, \mathcal{I}_\varphi, \mu \otimes \lambda|_{\mathcal{I}_\varphi})$ is well-defined (and is also standard). This space is called the *space of ergodic components* and it will be denoted by $(C_\varphi, \mathcal{B}_\varphi, \lambda_\varphi)$. Since τ preserves \mathcal{I}_φ it also acts on the space of ergodic components. This non-singular G -action is called the *Mackey action* associated to φ , and it is always ergodic. It will be denoted by $\tau^\varphi = (\tau_g^\varphi)_{g \in G}$.

A cocycle $\varphi : X \rightarrow G$ will be called *ergodic* if T_φ is ergodic. This is clearly equivalent to the fact that the Mackey action reduces to the one-point action. A cocycle φ is said to be *recurrent* if T_φ is conservative (i.e. it has no wandering sets of positive measure). This is equivalent to the fact that for a.e. ergodic

component of T_φ , the non-singular action of T_φ on it is *properly ergodic* (that is, it is not reduced to a single orbit).

Assume now that we have a measurable representation of G in the group $\text{Aut}_0(Y, \mathcal{C}, \nu)$ of measure-preserving automorphisms of (Y, \mathcal{C}, ν) : $g \mapsto S_g$, and we put $\mathcal{S} = (S_g)_{g \in G}$. We define the corresponding skew product $T_{\varphi, \mathcal{S}}$ acting on $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ by $T_{\varphi, \mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y))$. We are interested in $T_{\varphi, \mathcal{S}}$ -invariant probability measures whose projection on X is μ . The simplex of such measures will be denoted by $\mathcal{P}(T_{\varphi, \mathcal{S}}; \mu)$. On the product space $C_\varphi \times Y$ we can consider the diagonal G -action $\tau^\varphi \times \mathcal{S}$: $(\tau^\varphi \times \mathcal{S})_g = \tau_g^\varphi \times S_g$. Following [3] (and the earlier papers [14], [15]) we consider also the simplex

$$\begin{aligned} & \mathcal{P}(\tau^\varphi \times \mathcal{S}, \mathcal{B}_\varphi; \lambda_\varphi) \\ & := \{\rho \in \mathcal{P}(C_\varphi \times Y) : \rho|_{\mathcal{B}_\varphi} = \lambda_\varphi \text{ and } \mathcal{B}_\varphi \text{ is an r.f.m.p. factor of } \tau^\varphi \times \mathcal{S}\}. \end{aligned}$$

The main result about this situation that we need is the following.

Theorem 3 ([3],[15]) *The simplices $\mathcal{P}(T_{\varphi, \mathcal{S}}; \mu)$ and $\mathcal{P}(\tau^\varphi \times \mathcal{S}, \mathcal{B}_\varphi; \lambda_\varphi)$ are affine isomorphic. \square*

In what follows we will also use some elements of the proof of this theorem.

4 Joinings

Assume that $T \in \text{Aut}_0(X, \mathcal{B}, \mu)$ and $S \in \text{Aut}_0(Y, \mathcal{C}, \nu)$ are ergodic. Any $T \times S$ -invariant measure $\kappa \in \mathcal{P}(X \times Y, \mathcal{B} \otimes \mathcal{C})$ whose projections κ_X on X and κ_Y on Y are μ and ν respectively is called a *joining* of T and S , and we write $\kappa \in J(T, S)$. Each $\kappa \in J(T, S)$ defines a new automorphism which is $(T \times S, X \times Y, \mathcal{B} \otimes \mathcal{C}, \kappa)$. Sometimes, less formally, we will also denote the latter system as $T \vee S$. In general $\kappa \in J(T, S)$ is not ergodic, but its ergodic decomposition consists solely of joinings (e.g. [6]), and in particular the set $J^e(T, S)$ of ergodic joinings is always non-empty. Note that $\mu \otimes \nu \in J^e(T, S)$ if and only if $e(T) \cap e(S) = \{1\}$. Another easy example of an ergodic joining is available when T and S are isomorphic: indeed, if $W : (X, \mathcal{B}, \mu) \rightarrow (Y, \mathcal{C}, \nu)$ is an isomorphism then the measure μ_W defined by

$$\mu_W(B \times C) = \mu(B \cap W^{-1}(C))$$

is a member of $J^e(T, S)$. The resulting system $T \vee S$ is called a *graph joining*, and is isomorphic to T . Following [4], T and S are called *disjoint* if $J(T, S) = \{\mu \otimes \nu\}$, and in this case we write $T \perp S$.

We can also consider joinings of higher orders. If $T_i \in \text{Aut}_0(X_i, \mathcal{B}_i, \mu_i)$ is ergodic for $i = 1, \dots, n$ then each $T_1 \times \dots \times T_n$ -invariant $\kappa \in \mathcal{B}(X_1 \times \dots \times X_n)$ whose projections κ_{X_i} are μ_i for $i = 1, \dots, n$ is called a joining of T_1, \dots, T_n .

Such a joining is called *pairwise independent* [12] if $\kappa_{X_i \times X_j} = \mu_i \otimes \mu_j$ for $i \neq j$, $i, j = 1, \dots, n$. An ergodic $T \in \text{Aut}_0(X, \mathcal{B}, \mu)$ is called PID if every $\kappa \in J(T_1, \dots, T_n)$, with all $T_i = T$, which is pairwise independent is equal to $\mu^{\otimes n}$. The PID property was introduced as a joining counterpart to the problem of mixing of all orders [12] (see Chapter 11 in [7]).

Let us recall another definition in this connexion. Recall that an extension of systems is *distal* if it can be expressed as a (possibly transfinite) tower of isometric extensions (see, for instance, Chapter 10 of Glasner [7]). Given this, following [11] a PID automorphism T is called *distally simple* (DS) if for each $\kappa \in J^e(T, T) \setminus \{\mu \otimes \mu\}$ the system $(T \times T, X \times X, \mathcal{B} \otimes \mathcal{B}, \kappa)$ over the factor given by $\mathcal{B} \otimes \{\emptyset, X\} \subset \mathcal{B} \otimes \mathcal{B}$ (i.e. the extension $T \vee T \rightarrow T$) is distal.

Consider now an ergodic system T which has the property that whenever we take its ergodic joining with the Cartesian product of two weakly mixing automorphisms $S_1 \times S_2$, then in the joining $T \vee (S_1 \times S_2)$ one of S_i s, say S_1 , is independent from the joining $T \vee S_2$. Such T are said to have the *joining primeness* (JP) property, and their basic properties were studied in [16]. It follows from the definition that each JP system is PID, and a little work shows that each DS system enjoys the JP property [16].

It can be shown that under some mild assumptions on φ and \mathcal{S} the extension $T_{\varphi, \mathcal{S}} \rightarrow T$ is relatively weakly mixing [14], while any DS system must be distal over an arbitrary non-trivial factor [11], and so in general the DS property of T is not retained by $T_{\varphi, \mathcal{S}}$. An implicit question in [16] is whether under some natural assumptions on φ and \mathcal{S} the JP property persists under such extensions. We will identify some such assumptions in the next section, and so obtain natural examples of JP systems which do not enjoy the DS property (see also [13]).

5 Lifting multipliers of R^\perp

Assume that $T \in \text{Aut}_0(X, \mathcal{B}, \mu)$ is ergodic. We will now study ergodic properties of automorphisms of the form $T_{\varphi, \mathcal{S}} \vee S'$, i.e. joinings of $T_{\varphi, \mathcal{S}}$ with S' acting on (Y', \mathcal{C}', ν') , with a view towards proving the lifting result stated in the introduction. It is tempting to write such an automorphism as $(T \vee S')_{\varphi^{S'}, \mathcal{S}}$ with $\varphi^{S'}(x, y') = \varphi(x)$, but we must be aware that in this notation it is implicit that the “coordinate Y ” is independent of $X \times Y'$ which is of course not true in general. (To see this it is enough to take $S' = T_{\varphi, \mathcal{S}}$ and consider a graph self-joining.) Notice however that any joining κ of $T_{\varphi, \mathcal{S}}$ and S' is a member of $\mathcal{P}((T \vee S')_{\varphi^{S'}, \mathcal{S}}; \kappa_{X \times Y'})$, where $\kappa_{X \times Y'}$ is the projection of κ onto $X \times Y'$.

Remark 3 Assume that $\varphi : X \rightarrow G$ is ergodic and \mathcal{S} is uniquely ergodic. Then, as shown in [14],

$$\mathcal{P}(T_{\varphi, \mathcal{S}}; \mu) = \{\mu \otimes \nu\},$$

i.e. $T_{\varphi, \mathcal{S}}$ is relatively uniquely ergodic. Assume now that

$$(3) \quad \kappa \in J(T_{\varphi, \mathcal{S}}, S'), \kappa \neq \kappa_{X \times Y'} \otimes \nu.$$

Then the cocycle $\varphi^{S'} : X \times Y' \rightarrow G$ given by $\varphi^{S'}(x, y') = \varphi(x)$ and $(T \times S', \kappa_{X \times Y'})$ cannot be ergodic. Indeed, if the cocycle $\varphi^{S'}$ were ergodic we would have again that $(T \vee S')_{\varphi^{S'}, \mathcal{S}}$ is relatively uniquely ergodic which is a contradiction with (3), since $\kappa_{X \times Y'} \otimes \nu \in \mathcal{P}((T \vee S')_{\varphi^{S'}, \mathcal{S}}; \kappa_{X \times Y'})$.

We will now study joinings between a probability preserving system of the form $(T_{\varphi, \mathcal{S}}, \kappa')$ with $\kappa' \in \mathcal{P}(T_{\varphi, \mathcal{S}}; \mu)$ (which we will shortly denote $T'_{\varphi, \mathcal{S}}$) and a system R (acting on (Z, \mathcal{D}, ρ)) that is weakly mixing.

Lemma 3 *Assume that $\kappa \in J(T'_{\varphi, \mathcal{S}}, R)$. If for a.e. ergodic $c \in C_{\varphi}$ the non-singular automorphism (T_{φ}, c) has only countably many L^{∞} -eigenvalues then*

$$\kappa = \kappa_{X \times Y} \otimes \rho$$

provided $\kappa|_{X \times Z} = \mu \otimes \rho$ (of course $\kappa_{X \times Y} = \kappa'$).

Proof.

The proof will be a small modification of the proof of Proposition 6.1 from [15] (and also bears comparison with the proofs of Proposition 2.1 [15] and Proposition 6.1 [3]).

We use the notation φ^R for the cocycle φ treated as a function defined on $X \times Z$, and so serving as a cocycle for $(T \times R, \mu \otimes \rho)$; clearly $\kappa \in \mathcal{P}((T \times R)_{\varphi^R, \mathcal{S}}; \mu \otimes \rho)$. Let us write

$$\kappa = \int_{X \times Z} \delta_{(x, z)} \otimes \kappa_{(x, z)} d\mu(x) d\rho(z)$$

and put $\bar{\kappa}_{(x, g, z)} = S_g \kappa_{(x, z)}$; these measures define the measure $\bar{\kappa}$ by

$$\bar{\kappa} = \int_{X \times Z \times G} \bar{\kappa}_{(x, g, z)} d\kappa|_{X \times Z}(x, z) d\lambda(g).$$

Finally, the isomorphism in Theorem 3 sends κ to $\tilde{\kappa}$ which is the projection of $\bar{\kappa}$ on $C_{\varphi^R} \times Y$. The map

$$(x, g, z) \mapsto \bar{\kappa}_{(x, g, z)}$$

is $(T \times R)_{\varphi^R}$ -invariant (see the formula (10) in [15]), that is it is \mathcal{I}_{φ^R} -measurable. By assumption, on a.e. ergodic component $c \in C_{\varphi}$ the (non-singular) automorphism T_{φ} is ergodic and has only countably many L^{∞} -eigenvalues. Since R is weakly mixing, the non-singular Cartesian product automorphism $(T_{\varphi}|_c) \times R$ is still ergodic (this follows, for instance, from the spectral condition of Theorem 2.7.1 in [1], since the spectral type of our weakly mixing transformation R must annihilate the countable set $e(T_{\varphi}|_c)$). It follows that $\mathcal{I}_{\varphi^R} = \mathcal{I}_{\varphi} \otimes \{\emptyset, Z\}$. Hence the map $(x, g, z) \mapsto \bar{\kappa}_{(x, g, z)}$ is in fact a function of (x, g) alone, and upon integrating

$$\kappa_{(x, z)} = \int_G S_g^{-1} \bar{\kappa}_{(x, g, z)} d\lambda(g)$$

is a function of x alone, whence the result. \square

We have now reached the proof of the main result:

Theorem 4 *Assume that T is an ergodic automorphism of a standard probability Borel space (X, \mathcal{B}, μ) . Let $\varphi : X \rightarrow G$ be an ergodic cocycle such that $e(T_\varphi)$ is countable. Let $\mathcal{S} = (S_g)_{g \in G}$ be an ergodic representation of G in $\text{Aut}_0(Y, \mathcal{C}, \nu)$ (so $T_{\varphi, \mathcal{S}}$ is ergodic). Assume that R is a weakly mixing automorphism of (Z, \mathcal{D}, ρ) . If $T \in \mathcal{M}(\{R\}^\perp)$ then*

$$T_{\varphi, \mathcal{S}} \in \mathcal{M}(\{R\}^\perp).$$

Proof.

Take $\kappa \in J^e(T_{\varphi, \mathcal{S}}, S', R)$ where S' is an ergodic automorphism acting on (Y', \mathcal{C}', ν') and $S' \perp R$. Consider first the automorphism $(T \times S', \kappa|_{X \times Y'})_{\varphi^{S'}}$. Notice that T_φ is an r.f.m.p. factor (via the map $(x, y', g) \mapsto (x, g)$) and that the cocycle $\varphi^{S'}$ is recurrent. Since $e(T_\varphi)$ is countable, by Corollary 3 for a.e. ergodic component $c \in C_{\varphi^{S'}}$ the non-singular ergodic automorphism $((T \vee S')_{\varphi^{S'}}, c)$ has only countably many L^∞ -eigenvalues.

Consider κ as an element of $J^e((T \vee S')'_{\varphi^{S'}, \mathcal{S}}, R)$, where, by our standing assumption,

$$\kappa|_{X \times Y' \times Z} = \kappa|_{X \times Y'} \otimes \rho.$$

It now follows from Lemma 3 that $\kappa = \kappa|_{X \times Y' \times Y} \otimes \rho$ and the result follows. \square

This strengthens a result from [3] where the lifting multiplier theorem was proved for T an ergodic rotation.

Notice also that from the proof of Theorem 4 we easily deduce the following (see the end of Section 4).

Corollary 4 *Under the assumptions on φ and \mathcal{S} in Theorem 4, if T enjoys the JP (respectively, PID) property then so does $T_{\varphi, \mathcal{S}}$. \square*

A Appendix: Non-orthogonal irreducible rank-2 modules

We will show that Proposition 3 does not extend to irreducible rank-2 modules in the setting of measure-preserving systems.

We first construct the following extension of systems. Let (S, Y, \mathcal{C}, ν) be any aperiodic ergodic ν -preserving system. Then let $\theta : Y \rightarrow \mathrm{U}(2)$ (the group of 2×2 unitary matrices considered with Haar measure $\lambda_{\mathrm{U}(2)}$) be any ergodic cocycle. Let \mathbb{S}^3 denote the unit sphere in \mathbb{C}^2 . Take $z_0 = (1, 0) \in \mathbb{S}^3$ and fix $w \in \mathrm{U}(2)$, so that

$$(4) \quad w_{21} \neq 0$$

and also $\alpha := w_{11} = \langle z_0, wz_0 \rangle_{\mathbb{C}^2} \neq 0$. Then the set

$$\{(uz_0, uwz_0) : u \in \mathrm{U}(2)\} = \{(z_1, z_2) \in \mathbb{S}^3 \times \mathbb{S}^3 : \langle z_1, z_2 \rangle_{\mathbb{C}^2} = \alpha\} \subset \mathbb{S}^3 \times \mathbb{S}^3$$

is a hypersurface with the measure ρ which is the image of $\lambda_{\mathrm{U}(2)}$ via the map $u \mapsto (uz_0, uwz_0)$.

Form the extended space

$$(X, \mathcal{B}, \mu) := (Y, \mathcal{C}, \nu) \otimes (\mathrm{U}(2), \mathcal{B}(\mathrm{U}(2)), \lambda_{\mathrm{U}(2)})$$

and define the transformation T on X by

$$T(y, u) = (Sy, \theta(y)u).$$

It follows that T preserves $\mu = \nu \otimes \lambda_{\mathrm{U}(2)}$, and it is ergodic by our assumption on the cocycle θ .

Now let M and N be the finite-rank π -submodules of $L^2(X, \mathcal{B}, \mu)$ defined by the respective bases $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$, where

$$\phi_i(y, u) = (uz_0)_i \quad \text{and} \quad \psi_i(y, u) = (uwz_0)_i,$$

writing x_i for the i^{th} component of $x \in \mathbb{S}^3$ for $i = 1, 2$. It is now easily checked that the each of M and N is T -invariant, and their irreducibility follows from the fact that θ is ergodic.

Indeed, since M has rank 2, if it were not irreducible then it would contain a rank-1 submodule M' . Suppose that F is the base of M' . Then

$$F(y, u) = h_1(y)(uz_0)_1 + h_2(y)(uz_0)_2$$

and, by T -invariance, $F \circ T = f(y)F$ (with $|f| = 1$). However both sides of the latter equality depend only on (y, uz_0) , so if we consider $F = F(y, uz_0)$ then

$$F(Sy, (\theta(y)u)z_0) = f(y)F(y, uz_0).$$

Putting $H(y) = (\overline{h_1(y)}, \overline{h_2(y)})$ we can rewrite this as

$$F(y, u) = \langle uz_0, H(y) \rangle.$$

Hence

$$\langle \theta(y)uz_0, H(Sy) \rangle = f(y)\langle uz_0, H(y) \rangle.$$

Since uz_0 runs over all vectors of length 1 as u runs over $U(2)$, we must have $\theta(y)^{-1}H(Sy) = \overline{f(y)}H(y)$. This can be re-written for the function $\tilde{H}(y, u) = u^{-1}H(y)$ as $\tilde{H} \circ S_\theta = \overline{f(y)}\tilde{H}$. But

$$\tilde{H}(y, u) = (\tilde{H}_1(y, u), \tilde{H}_2(y, u)),$$

so $\tilde{H}_i \circ S_\theta = \overline{f(y)}\tilde{H}_i$ for $i = 1, 2$. Since f is of modulus one and S_θ is ergodic, there is a constant $c \in \mathbb{C}$ such that $\tilde{H}_1 = c\tilde{H}_2$ which is impossible because of the definition of \tilde{H} .

We will now show that as subspaces of $L^2(X, \mathcal{B}, \mu)$ we have $M \cap N = \{0\}$. If not, we can find g_1, g_2, h_1, h_2 elements of $L^\infty(Y, \mathcal{C}, \nu)$ such that

$$(5) \quad g_1(y)(uz_0)_1 + g_2(y)(uz_0)_2 = h_1(y)(uwz_0)_1 + h_2(y)(uwz_0)_2$$

a.e. for the product measure $\nu \otimes \lambda_{U(2)}$, so for ν -a.e. $y \in Y$ we have the above equality for $\lambda_{U(2)}$ -a.e. $u \in U(2)$. To show that (5) does not hold we take $(a, b) \in \mathbb{C}^2 \times \mathbb{C}^2$ and let

$$W := \{u \in U(2) : \langle (uz_0, uwz_0), (a, b) \rangle_{\mathbb{C}^4} = 0\}$$

$$= \{u \in U(2) : \langle ((u_{11}, u_{21}), (u_{11}w_{11} + u_{12}w_{21}, u_{21}w_{11} + u_{22}w_{21})), (a, b) \rangle_{\mathbb{C}^4} = 0\}.$$

We need to show that $\lambda_{U(2)}(W) = 0$ (unless $a = b = 0$). Suppose instead that $\lambda_{U(2)}(W) > 0$. By letting \mathbb{S}^1 act on $U(2)$ as left translations by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ we obtain that for a.e. $u \in W$ there are infinitely many $\lambda, |\lambda| = 1$, such that

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} u \in W.$$

Therefore

$$\langle ((u_{11}, \lambda u_{21}), (u_{11}w_{11} + u_{12}w_{21}, \lambda(u_{21}w_{11} + u_{22}w_{21}))), (a, b) \rangle_{\mathbb{C}^4} = 0.$$

Because this is linear in λ ,

$$\langle (u_{11}, u_{12}), (a_1 + \overline{w_{11}}b_1, \overline{w_{21}}b_1) \rangle = 0$$

and

$$\langle (u_{21}, u_{22}), (a_2 + \overline{w_{11}}b_2, \overline{w_{21}}b_2) \rangle = 0.$$

Since the latter two equalities are satisfied on a set of $u \in U(2)$ of positive measure, $a_1 + \overline{w_{11}}b_1 = 0 = \overline{w_{21}}b_1$ and also $a_2 + \overline{w_{11}}b_2 = 0 = \overline{w_{21}}b_2$. In view of (4) this implies $a_1 = a_2 = b_1 = b_2 = 0$.

Finally, the function

$$u \mapsto \overline{(uz_0)_1}(uwz_0)_1 + \overline{(uz_0)_2}(uwz_0)_2 = \langle uz_0, uwz_0 \rangle_{\mathbb{C}^2} = \alpha$$

is constant on $U(2)$ and therefore

$$\begin{aligned} & \langle \phi_1|_y, \psi_1|_y \rangle_{L^2(Z, \mathcal{D}, \rho)} + \langle \phi_2|_y, \psi_2|_y \rangle_{L^2(Z, \mathcal{D}, \rho)} \\ &= \int_{U(2)} \overline{(uz_0)_1} (uwz_0)_1 + \overline{(uz_0)_2} (uwz_0)_2 \lambda_{U(2)}(du) = \alpha \neq 0, \end{aligned}$$

so the modules M and N are not fiberwise orthogonal. This completes our example.

Notice that in the above example, although we have constructed two distinct irreducible modules M and N that are not fiberwise orthogonal, their associated relative eigenvalues (the unitary cocycles U_{ij}) are cohomologous (indeed, they are both equal to θ). In fact this is a general phenomenon: if two irreducible finite-rank modules have non-cohomologous relative eigenvalues, then it does hold that they must be fiberwise orthogonal. This is shown in the case of a measure-preserving base system by Thouvenot in [21]; we quickly recall the argument here for completeness.

We show that if M and N are irreducible but not fiberwise orthogonal and we adopt respective fiberwise orthonormal bases $\phi_1, \phi_2, \dots, \phi_r$ and $\psi_1, \psi_2, \dots, \psi_s$, then the relative eigenvalues obtained from expressing the restrictions of the overall cocycle U_Θ in terms of these bases are cohomologous. To see this, first let $P_y : M_y \rightarrow N_y$ be the restriction to M_y of the orthogonal projection $L^2(\mu_y) \rightarrow N_y$. This is easily seen to depend measurably on y . Our assumption that M and N are not fiberwise orthogonal now implies that P_y is not identically zero. However, we also clearly have $P_{S_y} \circ U_{\Theta(y)}|_{M_y} = U_{\Theta(y)}|_{N_y} \circ P_y$, as a consequence of the invariance of M and N , and from this it follows that $y \mapsto \text{im} P_y$ and $y \mapsto \ker P_y$ define invariant submodules of N and M respectively, which must therefore be trivial by irreducibility. Since P_y is non-zero, this implies that it is both almost surely surjective and almost surely injective, and hence that it is almost surely an isomorphism.

We can extend this conclusion about P_y as follows: for any Borel $I \subset \mathbb{R}$, the sum of the eigenspaces of $P_y^* P_y$ (respectively, of $P_y P_y^*$) corresponding to eigenvalues in I also defines an invariant submodule of M (respectively, of N), and so by irreducibility must be either full or trivial. This implies that both $P_y^* P_y$ and $P_y P_y^*$ are actually almost surely scalar multiples of the identity operator (on M_y and N_y , respectively), and hence that P_y is almost surely a scalar multiple of an isometry, say $P_y = \alpha_y \Phi_y$ with $\alpha_y > 0$. Now using this expression for P_y in the equation above gives $\alpha_{S_y} \Phi_{S_y} \circ U_{\Theta(y)}|_{M_y} = \alpha_y U_{\Theta(y)}|_{N_y} \circ \Phi_y$, and since $\Phi_{S_y} \circ U_{\Theta(y)}|_{M_y}$ and $U_{\Theta(y)}|_{N_y} \circ \Phi_y$ are both isometries this requires that $\alpha_{S_y} = \alpha_y$ and $\Phi_{S_y} \circ U_{\Theta(y)}|_{M_y} = U_{\Theta(y)}|_{N_y} \circ \Phi_y$. Finally, expressing this last equality in terms of the bases ϕ_i and ψ_j gives the desired cohomology, since $U_{\Theta(y)}|_{M_y}$ and $U_{\Theta(y)}|_{N_y}$ are expressed as the two relative eigenvalues and Φ_y becomes a unitary matrix, since it is an isometry expressed between two orthonormal bases.

Acknowledgements

This paper was written while both authors were visiting MSRI (Berkeley) in the Fall 2008. Our thanks go to them for their hospitality, and to Alexandre Danilenko for suggesting a valuable improvement to the proof of Theorem 2. We would also like to thank Emmanuel Lesigne for numerous helpful discussions and suggestions both on the content and on improving the presentation of the paper.

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