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Semisimple extensions of irrational rotations

by

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Abstract. We show that semisimple actions of l.c.s.c. Abelian groups and cocycles with values in such groups can be used to build new examples of semisimple automorphisms ($\mathbb{Z}$-actions) which are relatively weakly mixing extensions of irrational rotations.

Introduction. It is an important problem in ergodic theory to study classes of automorphisms with a "given" set of self-joinings (see [24]). Historically, such an approach was first presented in [20] by D. Rudolph, where automorphisms with a minimal structure of self-joinings (called MSJ) were shown to exist. A generalization of this notion appeared in [25] and then in [9]. In these two articles the notion of 2-fold simplicity was introduced and studied. An ergodic automorphism is called 2-fold simple if its only ergodic self-joinings are either graphs or the product measure. A further generalization was proposed in [8], where the notion of semisimplicity was introduced. An ergodic automorphism is said to be semisimple if for each of its ergodic self-joinings the automorphism corresponding to the self-joining is relatively weakly mixing with respect to both marginal σ-algebras. As proved in [8], such automorphisms have still strong ergodic properties, and in particular the structure of their factors can be easily described. Based on some earlier results of J.-P. Thouvenot, it was already remarked in [8] that some Gaussian automorphisms are semisimple (Gaussian automorphisms are never 2-fold simple). In [16], F. Parreau, J.-P. Thouvenot and the first author developed a far reaching study of Gaussian automorphisms with a minimal (in the category of Gaussian automorphisms) set of self-joinings. All such Gaussian systems turn out to be semisimple.

Almost all historical examples of automorphisms presented above are weakly mixing. In fact, the only exception are ergodic rotations which are 2-fold simple but not weakly mixing. More precisely, the MSJ property implies weak mixing, while in the class of 2-fold simple automorphisms we have:

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either such an automorphism is weakly mixing or it is a rotation. In the class of semisimple automorphisms, there has been the question of whether the existence of a discrete part in the spectrum forces a decomposition into a direct product of the form "discrete spectrum automorphism × weakly mixing automorphism". The question is natural because, as noticed in [8], an ergodic distal automorphism is semisimple if and only if it is a rotation. Actually more is true: if an ergodic automorphism is semisimple then it is relatively weakly mixing over its Kronecker factor (see Section 1.2).

In this article we will construct semisimple weakly mixing extensions of irrational rotations answering the above question. The main idea of the construction comes from papers by D. Rudolph and E. Glasner and B. Weiss [7]. Roughly, we fix a simple (or even semisimple) action of an Abelian l.c.s.c. group which will serve as fibre automorphisms of a skew product whose base is an irrational rotation. Under some assumptions on the relevant fibre cocycle, the skew product turns out to be semisimple (it cannot be 2-fold simple). In order to see that we have constructed a completely new class (in particular, no aforementioned direct product decomposition exists) of semisimple automorphisms we use some recent results from [14]: the class we will consider is disjoint in the sense of Furstenberg from all weakly mixing automorphisms, on the other hand the automorphisms from this class are relatively weakly mixing extensions of the base irrational rotation.

A slightly technical part of our paper is to show the existence of some cocycles over irrational rotations, taking values in Abelian l.c.s.c. groups and having strong ergodic properties (see Section 3). Here, we consider two well known examples of cocycles (one real-valued and the other integer-valued) over the rotation by an irrational α, where α has bounded partial quotients.

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1. Preliminaries

1.1. *$\mathbb{Z}$-action cocycles taking values in Abelian l.c.s.c. groups.* Assume that $(X, \mathcal{B}, \mu)$ is a standard probability space and $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic automorphism. Let G be an Abelian locally compact second countable (l.c.s.c.) group. Assume moreover that $\varphi : X \to G$ is a cocycle. More precisely, this means that φ is measurable and the formula

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{n-1}x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -(\varphi(T^n x) + \ldots + \varphi(T^{-1}x)) & \text{if } n < 0, \end{cases}$$

defines a cocycle for the $\mathbb{Z}$-action given by $n \mapsto T^n$ ($n \in \mathbb{Z}$). We will say that φ is *ergodic* if the corresponding cylinder flow

$$T_\varphi : (X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes m_G) \to (X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes m_G),$$
$$T_\varphi(x, g) = (Tx, \varphi(x) + g),$$

is ergodic. Here $\mathcal{B}(G)$ denotes the σ-algebra of Borel subsets of G and m_G stands for an (infinite whenever G is not compact) Haar measure on $\mathcal{B}(G)$. As follows from [22], ergodicity of φ is "controlled" by the group $E(\varphi)$ of *essential values* of φ. More precisely, let $\overline{G} = G \cup \{\infty\}$ be the one-point compactification of G (if G is compact then $\overline{G} = G$). We define $g \in \overline{E}(\varphi)$ if for each open neighbourhood $U \ni g$ in $\overline{G}$, for each $A \in \mathcal{B}$ of positive measure, there exists $N \in \mathbb{Z}$ such that $\mu(A \cap T^{-N} A \cap [\varphi^{(N)} \in U]) > 0$. Then we put $E(\varphi) := \overline{E}(\varphi) \cap G$. It turns out ([22]) that $E(\varphi)$ is a closed subgroup of G and:

(i) φ is ergodic iff $E(\varphi) = G$,

(ii) φ is a *coboundary* (i.e. $\varphi(x) = f(x) - f(Tx)$ for a measurable map $f : X \to G$) iff $\overline{E}(\varphi) = \{0\}$.

We say that two cocycles $\varphi, \psi : X \to G$ are *cohomologous* if $\varphi - \psi$ is a coboundary. In this case $\overline{E}(\varphi) = \overline{E}(\psi)$. Given a cocycle $\varphi : X \to G$, let $\varphi^* : X \to G/E(\varphi)$ be the corresponding quotient cocycle.

Lemma 1 ([22]). $E(\varphi^*) = \{0\}$. ∎

Following [22], we say that the cocycle φ is *regular* if $\overline{E}(\varphi^*) = \{0\}$. Then φ is cohomologous to a cocycle $\psi : X \to E(\varphi)$ and the latter is ergodic as a cocycle with values in the closed subgroup $E(\varphi)$ (see [22]).

In particular, if $E(\varphi)$ is cocompact then φ is regular and as a direct consequence we find that all cocycles taking values in compact groups are regular.

The following proposition appeared in [17].

Proposition 1. *Let T be ergodic. Assume that G, H are Abelian l.c.s.c. groups and let $\pi : G \to H$ be a continuous group homomorphism. If $\varphi : X \to G$ is a cocycle, then*

$$\overline{\pi(E(\varphi))} \subset E(\pi \circ \varphi).$$

Moreover, $\overline{\pi(E(\varphi))} = E(\pi \circ \varphi)$ whenever φ is regular. ∎

Given $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ and $\varphi : X \to G$ we denote by $\varphi_*\mu$ the image of μ on G via φ. Recall also that an increasing sequence (q_n) of integers is called a *rigidity time* for T if $T^{q_n} \to \mathrm{Id}$ weakly. We will make use of the following essential value criterion.

Proposition 2 ([17]). *Assume that T is ergodic and let $\varphi : X \to G$ be a cocycle with values in an Abelian l.c.s.c. group G. Let (q_n) be a rigidity time for T. Suppose that $(\varphi^{(q_n)})_*\mu \to \nu$ weakly on $\overline{G}$. Then* $\mathrm{supp}(\nu) \subset \overline{E}(\varphi)$. ∎

1.2. *Self-joinings of ergodic automorphisms.* Let $(X,\mathcal{B},\mu)$ be a standard probability space. We denote by $\mathrm{Aut}(X,\mathcal{B},\mu)$ the group of all measure-preserving automorphisms of $(X,\mathcal{B},\mu)$. Let $T\in\mathrm{Aut}(X,\mathcal{B},\mu)$ be ergodic. By the *centralizer* $C(T)$ of T we mean the subgroup $\{S\in\mathrm{Aut}(X,\mathcal{B},\mu): ST=TS\}$. Any T-invariant sub-σ-algebra $\mathcal{A}$ of $\mathcal{B}$ is called a *factor* of T (more precisely the corresponding factor is the quotient action of T on $(X/\mathcal{A},\mathcal{A},\mu)$). If there is no ambiguity on T, we shall also say that $\mathcal{B}$ (or T) *has* $\mathcal{A}$ *as its factor* or that $\mathcal{B}$ *is an extension of* $\mathcal{A}$, which we denote by $\mathcal{B}\to\mathcal{A}$. The maximal factor of T with discrete spectrum (which does exist) is called the *Kronecker factor* of T.

Let $T_i:(X_i,\mathcal{B}_i,\mu_i)\to(X_i,\mathcal{B}_i,\mu_i)$, $i=1, 2$, be two automorphisms (of probability standard spaces). We denote by $J(T_1,T_2)$ the set of all *joinings* of them. More precisely, $\lambda\in J(T_1,T_2)$ if λ is a $T_1\times T_2$-invariant probability measure on $(X_1\times X_2,\mathcal{B}_1\otimes\mathcal{B}_2)$ whose marginals are equal to μ_i. We should emphasize that by a joining we will also mean the corresponding automorphism $T_1\times T_2$ on $(X_1\times X_2,\mathcal{B}_1\otimes\mathcal{B}_2,\lambda)$ and the notation $(T_1\times T_2,\lambda)$ will often appear. The subset $J^{\mathrm{e}}(T_1,T_2)$ of *ergodic joinings* consists of those λ for which $(T_1\times T_2,\lambda)$ is ergodic. If each T_i is ergodic then $J^{\mathrm{e}}(T_1,T_2)$ is the set of extremal points of $J(T_1,T_2)$ and the ergodic decomposition of each joining consists of elements of $J^{\mathrm{e}}(T_1,T_2)$. In case $T_1=T_2=T$ we speak about *self-joinings* and use the notation $J(T)$, $J^{\mathrm{e}}(T)$. In this case, if $\mathcal{A}\subset\mathcal{B}$ is a factor of T then we denote by $J(T,\mathcal{A})$ (resp. $J^{\mathrm{e}}(T,\mathcal{A})$) the set of self-joinings (resp. ergodic self-joinings) of the factor action of T on $(X/\mathcal{A},\mathcal{A},\mu)$.

If $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is an ergodic automorphism, then to each $S\in C(T)$ we associate the *graph self-joining* μ_S given by

$$\mu_S(A\times B)=\mu(A\cap S^{-1}B)\quad\text{for each } A,B\in\mathcal{B}. \tag{1}$$

Clearly, μ_S is ergodic. If $S=\mathrm{Id}$ then instead of μ_{Id} we will also write $\varDelta_X$. Following [9], we say that T is 2-*fold simple* if each ergodic self-joining of T is either a graph or equals $\mu\otimes\mu$. If T is 2-fold simple and $C(T)=\{T^i: i\in\mathbb{Z}\}$ then T has the 2-*fold minimal self-joining* (MSJ) property. If $\mathcal{A}$ is a factor of T then the *relative product over* $\mathcal{A}$ is the self-joining $\mu\otimes_{\mathcal{A}}\mu$ in $J(T)$ given by

$$\mu\otimes_{\mathcal{A}}\mu(A\times B)=\int\limits_{X/\mathcal{A}}E(A|\mathcal{A})E(B|\mathcal{A})\,d\mu\qquad\text{for each } A,B\in\mathcal{B}. \tag{2}$$

This self-joining need not be ergodic. We say that an automorphism T is *relatively weakly mixing over* $\mathcal{A}$, or that $\mathcal{B}\to\mathcal{A}$ *is relatively weakly mixing*, if $\mu\otimes_{\mathcal{A}}\mu\in J^{\mathrm{e}}(T)$. The relative product is a particular case of the following construction. Assume that $\lambda\in J(T,\mathcal{A})$, that is, λ is a self-joining of a factor $\mathcal{A}$ of T. Then the self-joining $\widehat{\lambda}$ of T given by

$$\widehat{\lambda}(A\times B)=\int\limits_{X/\mathcal{A}\times X/\mathcal{A}} E(A|\mathcal{A})(\overline{x})E(B|\mathcal{A})(\overline{y})\,d\lambda(\overline{x},\overline{y})\qquad (A,B\in\mathcal{B})$$

is called the *relatively independent extension* of λ.

In the case of two automorphisms $T_i:(X_i,\mathcal{B}_i,\mu_i)\to(X_i,\mathcal{B}_i,\mu_i)$, $i=1,2$, easy extensions of formulas (1) and (2) allow us to define joinings between T_1 and T_2 when an isomorphism $S:(X_1,\mathcal{B}_1,\mu_1,T_1)\to(X_2,\mathcal{B}_2,\mu_2,T_2)$ is given, or, more generally, when there is an isomorphism between a non-trivial factor of T_1 and a factor of T_2 (in the latter case we say that T_1 and T_2 *have a common factor*).

A notion complementary to weak mixing is distality (see [26] for the definition). Given a factor $\mathcal{A}\subset\mathcal{B}$ there exists exactly one factor $\widehat{\mathcal{A}}$ such that $\mathcal{A}\subset\widehat{\mathcal{A}}\subset\mathcal{B}$, $\mathcal{B}\to\widehat{\mathcal{A}}$ is relatively weakly mixing and $\widehat{\mathcal{A}}\to\mathcal{A}$ is distal (see [26] or [4, Th. 6.17 and the final remark on page 139]). The decomposition $\mathcal{B}\to\widehat{\mathcal{A}}\to\mathcal{A}$ is called the *Furstenberg–Zimmer decomposition* of $\mathcal{B}\to\mathcal{A}$. It follows that, given a factor $\mathcal{A}$, there exists a smallest factor $\widehat{\mathcal{A}}\supset\mathcal{A}$ such that T is relatively weakly mixing over $\widehat{\mathcal{A}}$. If $\mathcal{A}$ is trivial, then $\widehat{\mathcal{A}}=\mathcal{D}$ is the maximal distal factor of T.

Following [8], we say that an ergodic automorphism T is *semisimple* if for each ergodic $\lambda\in J^{\mathrm{e}}(T)$ the extension $(\mathcal{B}\otimes\mathcal{B},\lambda)\to(\mathcal{B}\times X,\lambda)$ is relatively weakly mixing (clearly, $(\mathcal{B}\times X,\lambda)$ can be identified with $(\mathcal{B},\mu)$). It has been noticed in [8] that an ergodic distal automorphism is semisimple iff it is isomorphic to a rotation. Moreover, if T is semisimple and $\mathcal{B}\to\mathcal{A}$ is relatively weakly mixing then $\mathcal{A}$ is also semisimple ([8]). It follows that if T is semisimple and $\mathcal{D}$ stands for its maximal distal factor then $\mathcal{D}$ is semisimple because $\mathcal{B}\to\mathcal{D}$ is relatively weakly mixing. We have shown the following.

PROPOSITION 3. *If T is semisimple then it is a relatively weakly mixing extension of its Kronecker factor.* ■

Two automorphisms $T_i:(X_i,\mathcal{B}_i,\mu_i)\to(X_i,\mathcal{B}_i,\mu_i)$, $i=1,2$, are said to be *disjoint* if $J(T_1,T_2)=\{\mu_1\otimes\mu_2\}$ ([3]). We will then write $T_1\perp T_2$.

1.3. *Actions of Abelian locally compact second countable groups.* Assume that G is an Abelian l.c.s.c. group and let $\mathcal{G}=\{R_g\}_{g\in G}$ be a Borel action of this group on a Borel space $(Y,\mathcal{C})$ (we always suppose that such a space is standard, that is, up to isomorphism, Y is a Polish space, while $\mathcal{C}$ stands for the σ-algebra of Borel sets), meaning that the map $G\times Y\ni(g,y)\mapsto R_gy\in Y$ is measurable. If now ν is a probability measure invariant under the action of G then the notions defined in the previous section for $\mathbb{Z}$-actions can be extended to corresponding notions for actions of G on $(Y,\mathcal{C},\nu)$ (see also [9]). We say that $\mathcal{G}$ is *mildly mixing* if it has no non-trivial rigid factors, that is, whenever for $A\in\mathcal{C}$ there exists $(g_n)\subset G$, $g_n\to\infty$ and $\nu(R_{g_n}A\triangle A)\to 0$ $(n\to\infty)$, then $\nu(A)=0$ or 1 (see [23], also [5], [14], [15]).

Some of our results will require the use of Gaussian actions. We refer to [2] for a general theory of Gaussian $\mathbb{Z}$-actions. It is easily generalized to the case of actions of Abelian l.c.s.c. groups. In [16], Gaussian systems whose ergodic self-joinings remain Gaussian (GAG) are studied. In particular, all Gaussian systems with simple spectrum are GAG. It follows from [16] that the GAG systems are semisimple.

1.4. *Rokhlin cocycle extensions.* Assume that $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is an ergodic automorphism. Let $\mathcal{G}=\{R_g\}_{g\in G}$ be an action of G on $(Y,\mathcal{C},\nu)$, where G is Abelian l.c.s.c. Assume that $\varphi:X\to G$ is a cocycle. We define

$$T_{\varphi,\mathcal{G}}:(X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\otimes\nu)\to(X\times Y,\mathcal{B}\otimes\mathcal{C},\mu\otimes\nu),$$
$$T_{\varphi,\mathcal{G}}(x,y)=(Tx,R_{\varphi(x)}(y)).$$

We call $T_{\varphi,\mathcal{G}}$ a *Rokhlin cocycle extension* of T. We will make use of some recent results from [14].

PROPOSITION 4 ([14]). *If $\mathcal{G}$ is ergodic and φ is ergodic, then $T_{\varphi,\mathcal{G}}$ is ergodic.* ∎

PROPOSITION 5 ([14]). (i) *If $\mathcal{G}$ is mildly mixing and $T_{\varphi,\mathcal{G}}$ is ergodic, then the extension $T_{\varphi,\mathcal{G}}\to T$ is relatively weakly mixing.*

(ii) *If $\mathcal{G}$ is weakly mixing, $T_{\varphi,\mathcal{G}}$ is ergodic and the maximal spectral type of $\mathcal{G}$ satisfies the group property then the extension $T_{\varphi,\mathcal{G}}\to T$ is relatively weakly mixing. In particular, the assertion holds whenever the action $\mathcal{G}$ is Gaussian.* ∎

We will also make use of the following relative unique ergodicity result for Rokhlin cocycle extensions.

PROPOSITION 6 ([14]). *Assume that φ is ergodic and $\mathcal{G}$ is a Borel action on $(Y,\mathcal{C})$. Suppose that ϱ is an ergodic $T_{\varphi,\mathcal{G}}$-invariant measure (on $\mathcal{B}\otimes\mathcal{C}$) whose projection on $\mathcal{B}$ equals μ. Then $\varrho=\mu\otimes\nu'$, where ν' is $\mathcal{G}$-invariant and ergodic.* ∎

The following disjointness result has recently been proved by F. Parreau and the first author.

THEOREM 1 ([15]). *Suppose that W is an ergodic automorphism. If $T\perp W$, $\varphi:X\to G$ is ergodic and the action $\mathcal{G}$ is mildly mixing, then $T_{\varphi,\mathcal{G}}\perp W$.* ∎

1.5. *Irrational rotations.* By the *circle* we mean $\mathbb{T}=[0,1)$ with addition mod 1. Given $t\in\mathbb{R}$ we denote by $\{t\}$ its fractional part. Given an irrational α let $[0:a_1,a_2,\ldots]$ denote the continued fraction expansion of α (see e.g. [10]). We say that α has *bounded partial quotients* if the sequence (a_n) is bounded. We denote by (q_n) the sequence of denominators of α, that is,

$q_0 = 1, q_1 = a_1$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$, $n \geq 2$. We will make use of the following result of C. Kraaikamp and P. Liardet (see also [13]).

THEOREM 2 ([12]). *If α has bounded partial quotients, then*:

(i) *for each real $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$ the set of accumulation points* (*in* $\mathbb{T}$) *of* $(\{q_n\beta\})$ *is infinite*;

(ii) *for each real $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$ there exists $0 < c < 1$ in the set of accumulation points* (*in* $\mathbb{T}$) *of* $(\{q_n\beta\})$.

The following well known result follows from the classical Koksma inequality (e.g. [11]) and elementary properties of denominators of α.

PROPOSITION 7. *If $Tx = x + \alpha$ ($x \in \mathbb{T}$), $f : \mathbb{T} \to \mathbb{R}$ has bounded variation and $\int_{\mathbb{T}} f(t)\,dt = 0$ then $|f^{(q_n)}(t)| \leq 2\,\mathrm{Var}(f)$ for each $t \in \mathbb{T}$ and each $n \geq 1$.* ∎

1.6. *Description of closed subgroups of* $\mathbb{R}^2$. Closed subgroups of $\mathbb{R}^n$ are described in [19, Chapter II]. In particular, if $E \subset \mathbb{R}^2$ is a closed subgroup then it has one of the following forms:

(a) $E = \{n\vec{v}_1 + m\vec{v}_2 : n, m \in \mathbb{Z}\}$, where $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$,
(b) $E = \{t\vec{v} :\ t \in \mathbb{R}\}$ where $\vec{v} \in \mathbb{R}^2$,
(c) $E = \{t\vec{v}_1 + k\vec{v}_2 : t \in \mathbb{R},\ k \in \mathbb{Z}\}$, where $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2 \setminus \{(0,0)\}$,
(d) $E = \mathbb{R}^2$.

Note that subgroups of the form (c) (and also (a), (d)) are cocompact.

2. Self-joinings of Rokhlin cocycles extensions for regular cocycles

2.1. *Invariant measures for subactions of a product action.* Let $\mathcal{G} = \{R_g\}_{g\in G}$ be a weakly mixing action of G on $(Y, \mathcal{C}, \nu)$, where G is Abelian l.c.s.c. Given a closed subgroup $H \subset G \times G$ satisfying

$$\overline{\pi_1(H)} = G = \overline{\pi_2(H)}, \tag{3}$$

where $\pi_i(g_1, g_2) = g_i$, $i = 1, 2$, we denote by $\mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$ the set of all probability measures ϱ on $\mathcal{C} \otimes \mathcal{C}$ which are $R_{g_1} \times R_{g_2}$-invariant for all $(g_1, g_2) \in H$ and satisfy $\varrho(C \times Y) = \varrho(Y \times C) = \nu(C)$ for all $C \in \mathcal{C}$. If $\varrho \in \mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$ and

$$\varrho = \int \varrho_\gamma\, dP(\gamma)$$

denotes the H-ergodic decomposition of ϱ, then $\nu(\cdot) = \int \varrho_\gamma(\cdot \times Y)\, dP(\gamma)$ and in view of (3), for P-a.e. γ, $\varrho_\gamma(\cdot \times Y)$ is $\mathcal{G}$-invariant. Since ν is an extremal point in the simplex of all $\mathcal{G}$-invariant measures, $\varrho_\gamma \in \mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$ for P-a.e. γ. It easily follows that $\mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H)$ is a simplex whose set of extremal points coincides with $\mathcal{M}^{\mathrm{e}}(Y\times Y,\mathcal{C}\otimes\mathcal{C};H)$, the set of ergodic members of $\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};H)$.

The role of our assumption of the G-action $\mathcal{G}=\{R_g\}_{g\in G}$ being weakly mixing is easily seen in the following.

Proposition 8. *An ergodic G-action is weakly mixing iff for each subgroup $H\subset G\times G$ satisfying* (3), *the corresponding H-action $\mathcal{H}$ on $(Y\times Y,\nu\otimes\nu)$ is still ergodic.*

Proof. Only the "only if" part requires a proof. Let $p:\widehat{G}\times\widehat{G}\to\widehat{H}$ be the dual homomorphism corresponding to the natural embedding of H in $G\times G$. Define

$$\Gamma=H^{\perp}=\{(\gamma_1,\gamma_2)\in\widehat{G}\times\widehat{G}:(\gamma_1,\gamma_2)(H)=1\}.$$

In view of (3),

$$\Gamma\cap(\widehat{G}\times\{1\})=\Gamma\cap(\{1\}\times\widehat{G})=\{(1,1)\}. \tag{4}$$

The maximal spectral type of the product action $\mathcal{G}\times\mathcal{G}$ equals $\tau\otimes\tau$, where τ stands for the maximal spectral type of $\mathcal{G}$. Since $\mathcal{G}$ is weakly mixing, $\tau=\delta_0+\tau_c$, where τ_c is a continuous measure on $\widehat{G}$ and τ_c is the maximal spectral type of the unitary action of $\mathcal{G}$ on the space $L^2_0(Y,\mathcal{C},\nu)$ of zero mean functions. Now, the maximal spectral type of the product action $\mathcal{G}\times\mathcal{G}$ on $L^2_0(Y\times Y,\mathcal{C}\otimes\mathcal{C},\nu\otimes\nu)$ equals

$$\delta_0\otimes\tau_c+\tau_c\otimes\delta_0+\tau_c\otimes\tau_c.$$

Since the maximal spectral type of $\mathcal{H}$ on any $\mathcal{G}\times\mathcal{G}$-invariant subspace of $L^2(Y\times Y,\mathcal{C}\otimes\mathcal{C},\nu\otimes\nu)$ is the image via p of the maximal spectral type of $\mathcal{G}\times\mathcal{G}$ on that space, all we need to show is that the measures

$$p_*(\delta_0\otimes\tau_c),\quad p_*(\tau_c\otimes\delta_0),\quad p_*(\tau_c\otimes\tau_c)$$

are singular with respect to δ_0. However, directly from (4) it follows that for each $\gamma\in\widehat{G}$,

$$\Gamma\cap(\widehat{G}\times\{\gamma\})=\Gamma\cap(\{\gamma\}\times\widehat{G})\quad\text{has at most one element.}$$

Now, if σ is any finite Borel measure on $\widehat{G}$, then

$$p_*(\tau_c\otimes\sigma)(\{0\})=(\tau_c\otimes\sigma)(\Gamma)=\int_{\widehat{G}}\tau_c(\Gamma\cap(\widehat{G}\times\{\gamma\}))\,d\sigma(\gamma)=0,$$

so the result follows easily. ∎

The following proposition describes the simplex $\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};H)$ in some cases.

Proposition 9. (i) $\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};G\times G)=\{\nu\otimes\nu\}$.

(ii) $\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};H)=\{\nu\otimes\nu\}$ *whenever H is cocompact.*

(iii) $\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};\Delta_G)=J(\mathcal{G})$.

Proof. (i) This is a standard argument using spectral disjointness of the trivial identity G-action and any ergodic G-action.

(ii) Assume that $\varrho_0 \in \mathcal{M}^{\mathrm{e}}(Y\times Y, \mathcal{C}\otimes\mathcal{C}; H)$. Then for each $(g_1, g_2) \in G\times G$,

$$(R_{g_1} \times R_{g_2})_* \varrho_0 \in \mathcal{M}(Y \times Y, \mathcal{C} \otimes \mathcal{C}; H).$$

We put

$$\varrho = \int_{(G\times G)/H} (R_{g_1} \times R_{g_2})_* \varrho_0 \, d((g_1, g_2)H). \tag{5}$$

Then ϱ is a well defined $\mathcal{G} \times \mathcal{G}$-invariant measure, and therefore by (i), $\varrho = \nu \otimes \nu$. We regard the decomposition (5) as a decomposition of an $\mathcal{H}$-invariant measure $\nu \otimes \nu$ which by Proposition 8 is $\mathcal{H}$-ergodic. Since in this decomposition all measures are in $\mathcal{M}(Y \times Y, \mathcal{C} \times \mathcal{C}; H)$, by extremality, $(R_{g_1} \times R_{g_2})_* \varrho_0$ is one and the same measure a.e. and therefore $\varrho_0 = \nu \otimes \nu$.

(iii) This is just the definition of $J(\mathcal{G})$. ∎

2.2. *Self-joinings for regular cocycles.* Assume that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic automorphism and let $\varphi : X \to G$ be a cocycle. Throughout we suppose that the G-action $\mathcal{G} = \{R_g\}_{g\in G}$ is weakly mixing. Assume that H is a closed subgroup of G. Furthermore, assume that $\psi : X \to H$ is a cocycle cohomologous to φ, i.e. for some measurable $f : X \to G$, $\varphi = f - f \circ T + \psi$. Then

$$T_{\varphi,\mathcal{G}} \text{ and } T_{\psi,\mathcal{H}} \text{ are relatively isomorphic,} \tag{6}$$

that is, there exists an isomorphism which is the identity on $\mathcal{B} \times Y$. Indeed, the map $X \times Y \ni (x, y) \mapsto (x, R_{f(x)}(y)) \in X \times Y$ establishes a relative isomorphism.

Given $\lambda \in J^{\mathrm{e}}(T)$ denote by $J(T_{\varphi,\mathcal{G}}; \lambda)$ the set of self-joinings of $T_{\varphi,\mathcal{G}}$ whose restriction to $\mathcal{B} \otimes \mathcal{B}$ equals λ. Given a cocycle $\varphi : X \to G$ and $\lambda \in J^{\mathrm{e}}(T)$, consider the cocycle $\varphi \times \varphi = (\varphi \times \varphi)_\lambda$, where

$$(\varphi \times \varphi)(x_1, x_2) = (\varphi(x_1), \varphi(x_2)) \ (\in G \times G).$$

It is considered as a cocycle for the $\mathbb{Z}$-action given by $(T \times T, \lambda)$. Let $H_\lambda \subset G \times G$ be the group of essential values of $(\varphi \times \varphi)_\lambda$ (i.e. $H_\lambda = E((\varphi \times \varphi)_\lambda)$). Denote by $\mathcal{H}_\lambda$ the action corresponding to the H_λ-subaction of the product $G \times G$-action $\{R_{g_1} \times R_{g_2}\}_{g_1,g_2\in G}$.

THEOREM 3. *Let $\varphi : X \to G$ be an ergodic cocycle. Assume that $\lambda \in J^{\mathrm{e}}(T)$ and that $(\varphi \times \varphi)_\lambda$ is regular. Then there exists a measurable function $f = (f_1, f_2) : X \times X \to G \times G$ (defined λ-a.e.) such that the map Λ_f given by*

$$\begin{aligned} X \times Y \times X \times Y &\ni (x_1, y_1, x_2, y_2) \\ &\mapsto (x_1, x_2, R_{f_1(x_1,x_2)} \times R_{f_2(x_1,x_2)}(y_1, y_2)) \in X \times X \times Y \times Y \end{aligned}$$

establishes an affine isomorphism of $J(T_{\varphi,\mathcal{G}};\lambda)$ *and*

$$\lambda\otimes\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};H_\lambda):=\{\lambda\otimes\varrho:\varrho\in\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};H_\lambda)\}.$$

More precisely, there exists an ergodic cocycle $\Theta:X\times X\to H_\lambda$ *for* $(T\times T,\lambda)$ *such that for each* $\widetilde{\lambda}\in J(T_{\varphi,\mathcal{G}};\lambda)$, Λ_f *establishes an isomorphism of* $(T_{\varphi,\mathcal{G}}\times T_{\varphi,\mathcal{G}},\widetilde{\lambda})$ *and* $((T\times T)_{\Theta,\mathcal{H}_\lambda},\lambda\otimes\varrho)$, *where* $\lambda\otimes\varrho=(\Lambda_f)_*(\widetilde{\lambda})$.

Proof. Since $(\varphi\times\varphi)_\lambda$ is regular, there exists a measurable function $f:X\times X\to G\times G$ and an ergodic cocycle $\Theta:X\times X\to H_\lambda$ (both maps are defined λ-a.e.) such that

$$(\varphi\times\varphi)(x_1,x_2)=f(x_1,x_2)-f(Tx_1,Tx_2)+\Theta(x_1,x_2)\qquad\lambda\text{-a.e.}$$

It then follows from Proposition 1 and the fact that φ is ergodic that H_λ has dense projections.

Assume that $\widetilde{\lambda}\in J^{\mathrm{e}}(T_{\varphi,\mathcal{G}};\lambda)$. If we denote by $\widetilde{\lambda}_1$ the image of $\widetilde{\lambda}$ via the map Λ_f then clearly the commutation relation

$$\Lambda_f(T_{\varphi,\mathcal{G}}\times T_{\varphi,\mathcal{G}})=(T\times T)_{\Theta,\mathcal{H}_\lambda}\Lambda_f$$

gives rise to a measure-theoretic isomorphism of the systems $(T_{\varphi,\mathcal{G}}\times T_{\varphi,\mathcal{G}},\widetilde{\lambda})$ and $((T\times T)_{\Theta,\mathcal{H}_\lambda},\widetilde{\lambda}_1)$. However Θ is ergodic and the projection of $\widetilde{\lambda}_1$ on $X\times X$ equals λ, so by the relative unique ergodicity property (Proposition 6) we have $\widetilde{\lambda}_1=\lambda\otimes\varrho$, where ϱ is $\mathcal{H}_\lambda$-invariant and ergodic.

Furthermore, the maps

$$X\times Y\ni(x_i,y_i)\overset{s_i}{\mapsto}(x_i,R_{f_i(x_1,x_2)}(y_i))\in X\times Y,$$

$i=1,2$, have the property that $(s_i)_*(\mu\otimes\nu)=\mu\otimes\nu$. It follows that the projections of ϱ on Y are equal to ν and therefore $\varrho\in\mathcal{M}^{\mathrm{e}}(Y\times Y,\mathcal{C}\otimes\mathcal{C};H_\lambda)$. Since for each $\varrho\in\mathcal{M}(Y\times Y,\mathcal{C}\otimes\mathcal{C};H_\lambda)$, $(\Lambda_f^{-1})_*(\lambda\otimes\varrho)\in J(T_{\varphi,\mathcal{G}};\lambda)$, the result follows. ∎

Remark 1. The above proof tells us that the isomorphism Λ_f of $(T_{\varphi,\mathcal{G}}\times T_{\varphi,\mathcal{G}},\widetilde{\lambda})$ and $((T\times T)_{\Theta,\mathcal{H}_\lambda},(\Lambda_f)_*(\widetilde{\lambda}))$ is "relative" over $T_{\varphi,\mathcal{G}}$ in the sense that

$$\Lambda_f(\mathcal{B}\otimes\mathcal{C}\otimes\{\emptyset,X\times Y\})=\mathcal{B}\otimes\{\emptyset,X\}\otimes\mathcal{C}\otimes\{\emptyset,Y\}$$

and the action of $((T\times T)_{\Theta,\mathcal{H}_\lambda},(\Lambda_f)_*(\widetilde{\lambda}))$ restricted to $\mathcal{B}\otimes\{\emptyset,X\}\otimes\mathcal{C}\otimes\{\emptyset,Y\}$ is isomorphic to $T_{\varphi,\mathcal{G}}$. It follows that the relative properties of the two automorphisms over $T_{\varphi,\mathcal{G}}$ are the same.

We will now study some particular cases of λ in which $(\varphi\times\varphi)_\lambda$ is indeed regular.

Corollary 1 (relative self-joinings). *If* $\varphi:X\to G$ *is ergodic then the map* $\Lambda_0:(x_1,y_1,x_2,y_2)\mapsto(x_1,x_2,y_1,y_2)$ *gives rise to an affine isomorphism of* $J^{\mathrm{e}}(T_{\varphi,\mathcal{G}};\Delta_X)$ *and* $\Delta_X\otimes J^{\mathrm{e}}(\mathcal{G})$.

Proof. We have $H_{\Delta_X} = \Delta_G$, $f = (0,0)$, and $(\varphi \times \varphi)_{\Delta_X}$ is regular. It follows that the map Λ_f is the identity, and we apply Theorem 3. ■

The above corollary allows us to give the list of relative factors of $T_{\varphi,\mathcal{G}}$, that is, all factors that contain $\mathcal{B} \times Y$. The result below generalizes the well known compact group extension case.

Corollary 2. *Assume that $\varphi : X \to G$ is ergodic. Let $\mathcal{B} \times Y \subset \widetilde{\mathcal{A}} \subset \mathcal{B} \otimes \mathcal{C}$ be a factor of $T_{\varphi,\mathcal{G}}$. Then there exists $\mathcal{D} \subset \mathcal{C}$ which is a $\mathcal{G}$-factor and $\widetilde{\mathcal{A}} = \mathcal{B} \otimes \mathcal{D}$.*

Proof. It follows from Corollary 1 that

$$(\mu \otimes \nu) \otimes_{\widetilde{\mathcal{A}}} (\mu \otimes \nu) = \int_{J^{\mathrm{e}}(\mathcal{G})} (\Lambda_0^{-1})_*(\Delta_X \otimes \varrho)\, dP(\varrho).$$

Therefore, this relative product is invariant under $\mathrm{Id}_X \times R_g \times \mathrm{Id}_X \times R_g$, $g \in G$, which means that $\widetilde{\mathcal{A}}$ is invariant for the action $\{\mathrm{Id}_X \times R_g\}_{g \in G}$. Since $\mathcal{B} \times Y \subset \widetilde{\mathcal{A}}$, we obtain a measurable family $\{Q_x\}_{x \in X}$ of partitions of Y such that $\{\{x\} \times Q_x\}_{x \in X}$ generates $\widetilde{\mathcal{A}}$. Let $\mathcal{C}_x \subset \mathcal{C}$ denote the σ-algebra generated by Q_x. Since $\widetilde{\mathcal{A}}$ is $\mathrm{Id}_X \times R_g$-invariant, $\mathcal{C}_x$ is a $\mathcal{G}$-factor for μ-a.e. $x \in X$. But $\widetilde{\mathcal{A}}$ is also $T_{\varphi,\mathcal{G}}$-invariant, so $R_{\varphi(x)} Q_x = Q_{Tx}$ for a.e. $x \in X$ and therefore the map $x \mapsto L^2(\mathcal{C}_x)$ is T-invariant. Since the map $x \mapsto E(\cdot|\mathcal{C}_x)$ is measurable, $Q_x = \text{const}$ for a.a. $x \in X$ and the result follows. ■

Corollary 3 (relative simplicity). *If $\mathcal{G}$ is additionally* 2*-fold simple, then $T_{\varphi,\mathcal{G}}$ is relatively* 2*-fold simple, that is, the only ergodic self-joining of $T_{\varphi,\mathcal{G}}$ that projects onto Δ_X is either a graph or the relatively independent extension of Δ_X.*

Proof. Assume that $\widetilde{\lambda} \in J^{\mathrm{e}}(T_{\varphi,\mathcal{G}}; \Delta_X)$. If $\widetilde{\lambda}$ is not the relative product, then $\widetilde{\lambda} = (\Lambda_0^{-1})_*(\Delta_X \otimes \nu_W)$, where $W \in C(\mathcal{G})$. Then clearly $\widetilde{\lambda} = (\mu \otimes \nu)_{\mathrm{Id} \times W}$. ■

We will now consider a more general situation of an ergodic self-joining of $T_{\varphi,\mathcal{G}}$ whose projection on $X \times X$ equals μ_S for some $S \in C(T)$. Denote by $\mathcal{M}(T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}})$ the set of all probability $T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}}$-invariant measures on $\mathcal{B} \otimes \mathcal{C} \otimes \mathcal{C}$ whose restrictions to $\mathcal{B} \otimes \mathcal{C} \otimes \{\emptyset, Y\}$ and to $\mathcal{B} \otimes \{\emptyset, Y\} \otimes \mathcal{C}$ equal $\mu \otimes \nu$ (here $\varphi \times \varphi \circ S : X \to G \times G$, $(\varphi \times \varphi \circ S)(x) = (\varphi(x), \varphi(Sx))$). Then the map Λ given by

$$X \times Y \times X \times Y \ni (x, y_1, Sx, y_2) \mapsto (x, y_1, y_2) \in X \times Y \times Y$$

establishes an affine isomorphism of $J(T_{\varphi,\mathcal{G}}; \mu_S)$ and $\mathcal{M}(T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}})$. More precisely, for each $\widetilde{\lambda} \in J(T_{\varphi,\mathcal{G}}; \mu_S)$,

$$\Lambda \text{ establishes an isomorphism of } (T_{\varphi,\mathcal{G}} \times T_{\varphi,\mathcal{G}}, \widetilde{\lambda}) \text{ and } (T_{\varphi \times \varphi \circ S, \mathcal{G} \times \mathcal{G}}, \Lambda_*(\widetilde{\lambda})).$$

Moreover, this isomorphism is the identity on the first two coordinates, so it is relative with respect to $T_{\varphi,\mathcal{G}}$. Therefore, in what follows we will identify $J(T_{\varphi,\mathcal{G}};\mu_S)$ with $\mathcal{M}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}})$.

Assume now that additionally $\varphi\times\varphi\circ S : X\to G\times G$ is regular. By replacing $J(T_{\varphi,\mathcal{G}};\mu_S)$ by $\mathcal{M}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}})$ in Theorem 3, we obtain the following.

PROPOSITION 10. *Assume that $\varphi : X\to G$ is ergodic, $S\in C(T)$ and $\varphi\times\varphi\circ S : X\to G\times G$ is regular, say*

$$(\varphi,\varphi\circ S)=f\circ(T\times T)-f+\Theta,$$

where $\Theta : X\to H_{\mu_S}$. Let $\widetilde{\lambda}\in\mathcal{M}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}})$. Then

(i) *$\widetilde{\lambda}$ is ergodic iff $(\Lambda_f)_*(\widetilde{\lambda})$ is ergodic*;

(ii) *$\widetilde{\lambda}$ is a one-point extension of $T_{\varphi,\mathcal{G}}$ (that is, $\widetilde{\lambda}$ is a graph) iff $(\Lambda_f)_*(\widetilde{\lambda})$ is a one-point extension of $T_{\varphi,\mathcal{G}}$*;

(iii) *$\widetilde{\lambda}=\mu\otimes\nu\otimes\nu$ iff $(\Lambda_f)_*(\widetilde{\lambda})=\mu_S\otimes\nu\otimes\nu$*;

(iv) *$(T_{\varphi,\mathcal{G}}\times T_{\varphi,\mathcal{G}},\widetilde{\lambda})\to T_{\varphi,\mathcal{G}}$ is relatively weakly mixing iff so is*

$$((T\times T)_{\Theta,\mathcal{H}_{\mu_S}},(\Lambda_f)_*(\widetilde{\lambda}))\to T_{\varphi,\mathcal{G}}. \ \blacksquare$$

REMARK 2. Note that if $S=T^k$ then the cocycle $(\varphi\times\varphi)_{\mu_S}$ is regular. Indeed, in that case $(\varphi\times\varphi)_{\mu_S}$ is cohomologous to $(\varphi\times\varphi)_{\Delta_X}$ since clearly $\varphi\circ T^k$ is T-cohomologous to φ ($\varphi\circ T^k=\varphi+\varphi^{(k)}\circ T-\varphi^{(k)}$). See the next section for examples of T and φ for which $\varphi\times\varphi\circ S$ is regular for each $S\in C(T)$.

We will also need the following.

LEMMA 2. *Assume that $\mathcal{G}$ is weakly mixing and φ is ergodic. Let $\mathcal{D}\subset\mathcal{C}$ be a $\mathcal{G}$-factor. Then $T_{\varphi,\mathcal{G}}$ is relatively weakly mixing over $\mathcal{B}\otimes\mathcal{D}$ if and only if $\mathcal{G}$ is a relatively weakly mixing action over $\mathcal{D}$.*

Proof. First notice that directly from the definition of conditional expectation, if $\mathcal{D}\subset\mathcal{C}$ then

$$(\mu\otimes\nu)\otimes_{\mathcal{B}\otimes\mathcal{D}}(\mu\otimes\nu)=(\Lambda_0^{-1})_*(\Delta_X\otimes\nu\otimes_{\mathcal{D}}\nu).$$

It follows that the relative product $T_{\varphi,\mathcal{G}}\times_{\mathcal{B}\otimes\mathcal{D}}T_{\varphi,\mathcal{G}}$ over the $T_{\varphi,\mathcal{G}}$-factor $\mathcal{B}\otimes\mathcal{D}$ is isomorphic to $T_{\varphi\times\varphi,\{R_g\times R_g\}_{g\in G}|\mathcal{C}\otimes_{\mathcal{D}}\mathcal{C}}$, where $\varphi\times\varphi : X\to G\times G$, $(\varphi\times\varphi)(x)=(\varphi(x),\varphi(x))$.

Since $(\varphi,\varphi) : X\to\Delta_G$ is ergodic, $T_{\varphi\times\varphi,\{R_g\times R_g\}_{g\in G}|\mathcal{C}\otimes_{\mathcal{D}}\mathcal{C}}$ is ergodic if and only if the diagonal Δ_G-action on $\mathcal{C}\otimes_{\mathcal{D}}\mathcal{C}$ is ergodic itself (see Proposition 4). $\blacksquare$

3. Cocycles over irrational rotations. In this section we put $X = \mathbb{T} = [0,1)$ and consider an irrational rotation $Tx = x + \alpha \pmod 1$ on X. By μ we denote Lebesgue measure on $\mathbb{T}$. Throughout this section α is assumed to have bounded partial quotients.

3.1. *A real-valued ergodic cocycle φ for which $\varphi \times \varphi \circ S$ is regular for each $S \in C(T)$.* We will consider the real cocycle $\varphi(x) = \{x\} - 1/2$.

Let (q_n) be the sequence of denominators of α. Assume that $\beta \in [0,1)$ and that

$$\{q_{n_k}\beta\} \to c \quad (c \in [0,1)).$$

Consider the sequence (ν_k) of probability measures on $\mathbb{R}^2$ defined by

$$\nu_k := ((\varphi \times \varphi \circ S)^{(q_{n_k})})_*\mu = (\varphi^{(q_{n_k})} \times \varphi^{(q_{n_k})} \circ S)_*\mu,$$

where $Sx = x + \beta \pmod 1$. Since

$$(\forall x, y \in [0,1)) \quad |\varphi^{(q_n)}(x) - \varphi^{(q_n)}(y)| \le 4 \operatorname{Var} \varphi = 4$$

and $\int_X \varphi\, d\mu = 0$, we have $\operatorname{Im}(\varphi \times \varphi \circ S)^{(q_n)} \subset [-4,4] \times [-4,4]$. It follows that we can select a subsequence of (ν_k) which converges weakly to a probability measure ν (which is also concentrated on the above square). No harm arises if we assume that $\nu_k \to \nu$.

We will now show in what kind of subsets of $\mathbb{R}^2$ the support of ν is contained. To this end note that

$$\varphi^{(q_n)}(x) = q_n x + \frac{q_n(q_n - 1)}{2}\alpha - \frac{q_n}{2} + M(x), \quad \text{where } M(x) \in \mathbb{Z}.$$

It follows that $\varphi^{(q_n)}(x+\beta) = \varphi^{(q_n)}(x) + q_n\beta + M(x+\beta) - M(x)$ if $x + \beta < 1$, while $\varphi^{(q_n)}(\{x + \beta\}) = \varphi^{(q_n)}(x) + (q_n\beta - q_n) + M(\{x + \beta\}) - M(x)$ if $1 \le x + \beta < 2$.

Consider now the image of the measure $(\varphi^{(q_{n_k})} \times \varphi^{(q_{n_k})} \circ S)_*\mu$ via

$$F : \mathbb{R} \times \mathbb{R} \to \mathbb{T}, \quad F(x,y) = e^{2\pi i(y-x)},$$

that is, we send ν_k to the circle. But $F \circ (\varphi \times \varphi \circ S)^{(q_{n_k})}(x) = e^{2\pi i q_{n_k}\beta}$, whence $F_*\nu_k$ is the Dirac measure concentrated at $e^{2\pi i q_{n_k}\beta}$. Since $\nu_k \to \nu$ weakly, $F_*\nu_k \to F_*\nu$ (since all these measures are concentrated on a bounded subset of $\mathbb{R}^2$). Since $F_*\nu_k = \delta_{e^{2\pi i q_{n_k}\beta}}$ and $e^{2\pi i q_{n_k}\beta} \to e^{2\pi i c}$, we have $F_*\nu = \delta_{e^{2\pi i c}}$. It follows that

$$\operatorname{supp} \nu \subset \{(x,y) \in \mathbb{R}^2 : e^{2\pi i(y-x)} = e^{2\pi i c}\} = \{(x,y) \in \mathbb{R}^2 : y - x - c \in \mathbb{Z}\}.$$

LEMMA 3. *If α has bounded partial quotients, then the cocycle $\varphi \times \varphi \circ S$ is ergodic for each $\beta \in [0,1)$ satisfying $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$.*

Proof. In view of Proposition 2, $\operatorname{supp}(\nu) \subset E(\varphi \times \varphi \circ S)$. It follows from [17] that each accumulation point of the sequence $((\varphi^{(q_n)})_*\mu)$ is an absolutely continuous measure (more precisely, it is a measure whose image via exp is Lebesgue measure on the circle). Thus the support of ν which is contained in the union of lines of the form $y = x - c - k$ (with parameter $k \in \mathbb{Z}$) has "absolutely continuous" projections on both coordinates.

Due to Theorem 2(i), if α has bounded partial quotients then for each $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$, the set of accumulation points of the sequence $(\{q_n\beta\})$ is infinite.

Since the projections of ν are absolutely continuous and the number of $c \in [0,1)$ under consideration is infinite, $E(\varphi \times \varphi \circ S)$ cannot be of the form (a)–(c) (see 1.6). It follows that $E(\varphi \times \varphi \circ S) = \mathbb{R}^2$ and thus $\varphi \times \varphi \circ S$ is ergodic. ∎

LEMMA 4. *If α has bounded partial quotients and $\beta \in (\mathbb{Q}\alpha + \mathbb{Q}) \setminus (\mathbb{Z}\alpha + \mathbb{Z})$ then $E(\varphi \times \varphi \circ S)$ is cocompact. More precisely, if $\beta = (u/v)\alpha + s/w$, then*

$$E(\varphi \times \varphi \circ S) = \{(t, t + k/M) : t \in \mathbb{R},\, k \in \mathbb{Z}\},$$

where M divides vw. In particular, $\varphi \times \varphi \circ S$ is a regular cocycle.

Proof. It follows from Theorem 2(ii) that there exists $c \in (0,1)$ and a subsequence (q_{n_k}) such that $\{q_{n_k}\beta\} \to c$. As in the proof of Lemma 3, we deduce that there are uncountably many essential values of $\varphi \times \varphi \circ S$ in the union of the straight lines $y = x - c - k$ (k as before is an integer-valued parameter). It follows directly that the group $E(\varphi \times \varphi \circ S)$ is either of the form (c) or (d), hence cocompact. In particular, $\varphi \times \varphi \circ S$ is regular.

Let $\omega : \mathbb{R}^2 \to \mathbb{R}$, $\omega(s,t) = t - s$. Clearly $\omega(\varphi \times \varphi \circ S)(x) = \varphi(Sx) - \varphi(x)$. Since $\varphi \times \varphi \circ S$ is regular, $\overline{\omega(E(\varphi \times \varphi \circ S))} = E(\omega(\varphi \times \varphi \circ S))$.

Let $r \in E(\pi(\varphi \times \varphi \circ S))$. By taking open intervals in the definition of an essential value we easily see that there exists a sequence $x_k \in \mathbb{T}$, $k \geq 1$, and a rigidity time $(n_k)_{k\geq 1}$ such that $\varphi^{(n_k)}(Sx_k) - \varphi^{(n_k)}(x_k) \to r$. Equivalently, $n_k\beta + N(x_k) \to r$, where $N(x_k) \in \mathbb{Z}$, $k \geq 1$. Since $\beta = (u/v)\alpha + s/w$, we get $(u/v)n_k\alpha + l_k/w \to r$. But $(n_k)_{k\geq 1}$ is a rigidity time, so $\{n_k\alpha\} \to 0$. Therefore $r = \frac{l}{vw} \in \frac{1}{vw}\mathbb{Z}$. Thus $E(\varphi \circ S - \varphi) = (1/M)\mathbb{Z}$ for some M which divides vw. This implies

$$E(\varphi \times \varphi \circ S) = \{(t, t + k/M) : t \in \mathbb{R},\, k \in \mathbb{Z}\}$$

(recall that we have already noticed that $E(\varphi \times \varphi \circ S)$ is cocompact and it is not discrete). ∎

LEMMA 5. *For each α and $\beta \in \mathbb{Z}\alpha$ the cocycle $\varphi \times \varphi \circ S$ is cohomologous to $\varphi \times \varphi$. In particular, it is regular.*

Proof. For each $n \in \mathbb{Z}$, we simply have $\varphi \times \varphi \circ T^n - \varphi \times \varphi = 0 \times \varphi^{(n)} \circ T - 0 \times \varphi^{(n)}$. The cocycle $\varphi \times \varphi$ is ergodic as a cocycle taking values in the subgroup $\Delta_{\mathbb{R}} = \{(t,t) : t \in \mathbb{R}\}$ since φ is ergodic. ∎

Collecting the results contained in Lemmas 3–5 we have proved the following.

THEOREM 4. *If α has bounded partial quotients and $\beta \in [0,1)$ then $\varphi \times \varphi \circ S$ is a regular cocycle. Moreover, for each $S \in C(T)$, $E(\varphi \times \varphi \circ S)$ is not discrete and it is either cocompact or equals $\Delta_{\mathbb{R}}$.* ∎

3.2. *An integer-valued ergodic cocycle φ for which $\varphi \times \varphi \circ S$ is regular for each $S \in C(T)$.* In this subsection we will prove that there are cocycles $\varphi : X \to \mathbb{Z}$ over irrational rotations such that:

(A) φ is ergodic,
(B) $(\forall S \in C(T))\ \varphi \times \varphi \circ S : X \to \mathbb{Z} \times \mathbb{Z}$ is regular,
(C) $(\forall S \in C(T))\ \varphi \times \varphi \circ S : X \to \mathbb{Z} \times \mathbb{Z}$ is NOT ergodic.

Let $G = \{(m,n) \in \mathbb{Z}^2 : m - n \text{ is even}\}$. Then:

- G is a subgroup of $\mathbb{Z}^2$,
- G has index 2 in $\mathbb{Z}^2$; in particular, G is cocompact,
- G is generated by $\{(1,1),(1,-1)\}$,
- $\Delta_{\mathbb{Z}} \subset G$.

Let
$$\varphi : X \to \mathbb{Z}, \qquad \varphi(x) = \begin{cases} 1, & x \in [0,1/2), \\ -1, & x \in [1/2,1). \end{cases}$$
The ergodicity of φ has been shown in [1].

For each $\beta \in [0,1)$, $\mathrm{Im}(\varphi \times \varphi \circ S) \subset G$, where $Sx = x + \beta$; therefore $\varphi \times \varphi \circ S : X \to \mathbb{Z} \times \mathbb{Z}$ cannot be ergodic ($E(\varphi \times \varphi \circ S) \subset G$).

THEOREM 5. *There exists an uncountable set $\Sigma \subset [0,1)$ of irrational numbers such that for each $\alpha \in \Sigma$, $\varphi \times \varphi \circ S$ is regular for each $S \in C(T)$ and*:
$$E(\varphi \times \varphi \circ S) = G \qquad \forall \beta \notin \mathbb{Z}\alpha \cup \{1/2\},$$
$$E(\varphi \times \varphi \circ S) = \Delta_{\mathbb{Z}} \qquad \forall \beta \in \mathbb{Z}\alpha,$$
$$E(\varphi \times \varphi \circ S) = \widetilde{\Delta}_{\mathbb{Z}} \qquad \textit{if } \beta = 1/2, \ \textit{where } \widetilde{\Delta}_{\mathbb{Z}} = \{(n,-n) : n \in \mathbb{Z}\}.$$

The proof of Theorem 5 will be done in several steps. First of all we define the set Σ.

A number α is in Σ if the following conditions are satisfied:

(i) α is irrational with bounded partial quotients;
(ii) $|\alpha - p_n/q_n| \le 1/(3q_n^2)$ for each $n \ge 1$;
(iii) q_n is odd for each $n \ge n_0$;
(iv) infinitely many of the p_n's are odd, and infinitely many are even.

It is clear that Σ is uncountable.

Fix $\alpha \in \Sigma$. Assume that $\beta \in [0,1)$, $Sx = x + \beta$. Suppose that (q_{n_k}) is a subsequence of the sequence of denominators of α for which
$$\{q_{n_k}\beta\} \to c \quad \text{with } 0 < c < 1.$$

We have
$$q_{n_k}\left(\beta-\frac{r_k}{q_{n_k}}\right)\to c,\quad \text{where}\quad r_k=[q_{n_k}\beta].$$
Hence for $\varepsilon>0$ ($\varepsilon\ll c$) and k large enough,
$$(c-\varepsilon)\,\frac{1}{q_{n_k}}<\beta-\frac{r_k}{q_{n_k}}<(c+\varepsilon)\,\frac{1}{q_{n_k}},$$
or equivalently (see Figure 1)
$$\beta\in\left(\frac{r_k}{q_{n_k}}+(c-\varepsilon)\,\frac{1}{q_{n_k}},\frac{r_k}{q_{n_k}}+(c+\varepsilon)\,\frac{1}{q_{n_k}}\right).$$

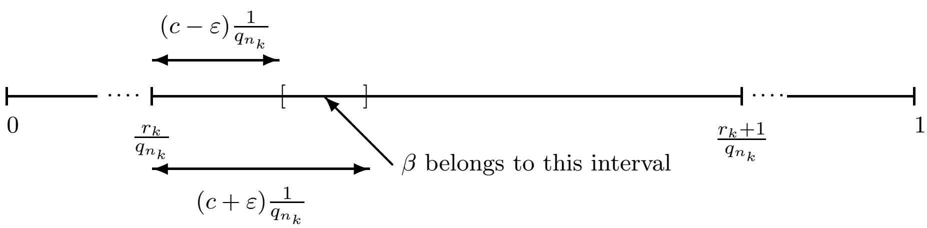

Fig. 1

CASE I: $0<c<1/2$. We assume additionally

(7) $$n_k \text{ are even for } k\ge k_0$$

(for such n_k we have $\alpha>p_{n_k}/q_{n_k}$). We fix δ such that

(8) $$0<\delta<1/2-c,$$

(9) $$\delta<1/6.$$

Choose δ' satisfying $0<\delta'<(1/2-c)-\delta$ and then $\varepsilon\ll c$ so that
$$0<\varepsilon<\frac{1}{100}\left(\frac{1}{2}-c-\delta-\delta'\right).$$
For each k large enough ($k\ge k_1$ and k_1 will be specified by the argument below) define
$$A_k^{(i)}=\left[\frac{i}{q_{n_k}},\frac{i}{q_{n_k}}+\delta\,\frac{1}{q_{n_k}}\right),\qquad i=0,1,\ldots,q_{n_k}-1.$$
For each $j=0,1,\ldots,q_{n_k}-1$, we have $j/(3q_{n_k}^2)\le 1/(3q_{n_k})$, so in view of (9), the interval $T^jA_k^{(i)}$

(10) $$\begin{cases}\text{is contained in an interval } \left[\frac{s}{q_{n_k}},\frac{s}{q_{n_k}}+\frac{1}{2}\cdot\frac{1}{q_{n_k}}\right)\\ \text{and the map } \{0,1,\ldots,q_{n_k}-1\}\ni j\mapsto s\in\{0,1,\ldots,q_{n_k}-1\} \text{ is 1-1}\end{cases}$$

(see Figure 2).

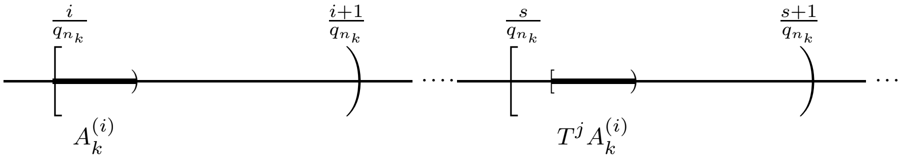

Fig. 2

In view of (8), the interval $SA_k^{(i)}$ is contained in an interval $\left[\frac{\widetilde{s}}{q_{n_k}}, \frac{\widetilde{s}}{q_{n_k}} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}\right)$ (Figure 3).

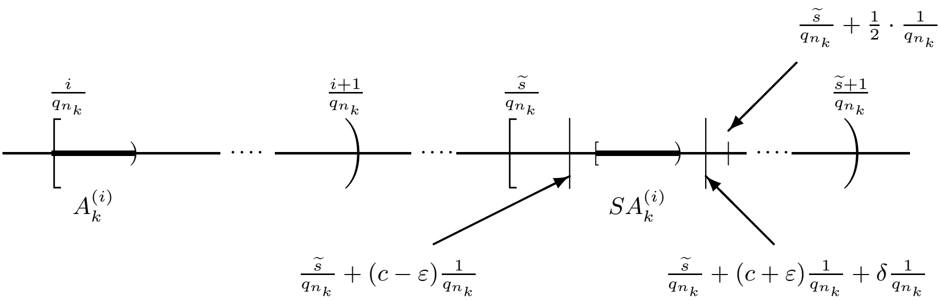

Fig. 3

For each $j = 0, 1, \ldots, q_{n_k} - 1$, the interval $T^j SA_k^{(i)}$

$$(11) \quad \begin{cases} \text{is contained in an interval } \left[\frac{t}{q_{n_k}}, \frac{t}{q_{n_k}} + \frac{1}{q_{n_k}}\right) \\ \text{and the map } \{0, 1, \ldots, q_{n_k} - 1\} \ni j \mapsto t \in \{0, 1, \ldots, q_{n_k} - 1\} \text{ is 1-1.} \end{cases}$$

For each $i \in \{0, 1, \ldots, q_{n_k} - 1\}$, define

$$b_k(i) = j \ \Leftrightarrow\ T^j SA_k^{(i)} \subset \left[\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{q_{n_k}}, \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}\right)$$

(note that since q_{n_k} is odd, the last interval equals $[r'/q_{n_k}, (r'+1)/q_{n_k})$, where $r' = q_{n_k} - 1/2$). By (11), the function b_k is well defined on $\{0, 1, \ldots, q_{n_k} - 1\}$ with values in $\{0, 1, \ldots, q_{n_k} - 1\}$. In fact,

$$(12) \qquad b_k \text{ is a bijection.}$$

Indeed, if $j = b_k(i_1) = b_k(i_2)$, then the intervals $T^j SA_k^{(i_1)}$ and $T^j SA_k^{(i_2)}$ are both contained in an interval of length $1/q_{n_k}$; it follows that $A_k^{(i_1)}$ and $A_k^{(i_2)}$ are contained in an interval of length $1/q_{n_k}$, which is an obvious contradiction.

We say that $i \in \{0, 1, \ldots, q_{n_k} - 1\}$ is *good* if $0 \le b_k(i) \le 3\delta' q_{n_k}$. In view of (12),

$$(13) \qquad \#\{i = 0, 1, \ldots, q_{n_k} - 1 : i \text{ is good}\} \ge \delta' q_{n_k}.$$

We will show that if

$$i \text{ is good}, \ x \in A_k^{(i)} \ \Rightarrow \ (\varphi^{(q_{n_k})}(x), \varphi^{(q_{n_k})}(x+\beta)) = (1,1). \tag{14}$$

Indeed, $\varphi^{(q_{n_k})}(x) = 1$ follows directly from (10) (and this is true for each $i = 0, 1, \ldots, q_{n_k} - 1$). We have

$$T^{b_k(i)}(x+\beta) \in \left[\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{q_{n_k}}, \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}\right)$$

and the distance between $T^{b_k(i)}(x+\beta)$ and $\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{q_{n_k}}$ is estimated by (see Figure 3)

$$(c+\varepsilon)\,\frac{1}{q_{n_k}} + \delta\,\frac{1}{q_{n_k}} + \frac{b_k(i)}{3q_{n_k}^2} \le (c+\varepsilon+\delta+\delta')\,\frac{1}{q_{n_k}} < \frac{1}{2} \cdot \frac{1}{q_{n_k}}$$

(by our choice of ε, for k large enough), so $\varphi^{(q_{n_k})}(x+\beta) = 1$.

It follows from (13) and (14) that

$$\mu\{x \in [0,1) : (\varphi^{(q_{n_k})}(x), \varphi^{(q_{n_k})}(x+\beta)) = (1,1)\} \ge \delta\delta'$$

for each k sufficiently large, so

$$(1,1) \in E(\varphi \times \varphi \circ S). \tag{15}$$

We now show that $(1,-1) \in E(\varphi \times \varphi \circ S)$. We have assumed that α has bounded partial quotients, so for some $C > 0$,

$$\left|\alpha - \frac{p_n}{q_n}\right| \ge \frac{1}{Cq_n^2}$$

for all $n \ge 1$. Since (ii) holds, for some subsequence $(q_{n_{k_l}})$ we have

$$q_{n_{k_l}}^2 \left|\alpha - \frac{p_{n_{k_l}}}{q_{n_{k_l}}}\right| \xrightarrow[l\to\infty]{} \frac{1}{D},$$

where $D \ge 3$. But the n_{k_l}'s are still even, so without loss of generality, we simply assume that

$$q_{n_k}^2 \left|\alpha - \frac{p_{n_k}}{q_{n_k}}\right| \xrightarrow[k\to\infty]{} \frac{1}{D}. \tag{16}$$

Fix

$$0 < \delta'' < \min\left(\frac{c}{2}, \frac{1}{2} - \frac{1}{D}, \frac{1}{D}\right). \tag{17}$$

Let $0 < \varepsilon < \delta''/100$. For k sufficiently large and $i = 0, 1, \ldots, q_{n_k} - 1$ put

$$B_k^{(i)} = \left[\frac{i}{q_{n_k}} + \left(\frac{1}{2} - \frac{1}{D}\right)\frac{1}{q_{n_k}} - \frac{\delta''}{q_{n_k}}, \frac{i}{q_{n_k}} + \left(\frac{1}{2} - \frac{1}{D}\right)\frac{1}{q_{n_k}} - \varepsilon\,\frac{1}{q_{n_k}}\right)$$

(see Figure 4). In view of (16), $q_{n_k}(\alpha - p_{n_k}/q_{n_k}) \le (1/D + \varepsilon)(1/q_{n_k})$ for each k large enough, so we have: for each $j = 0, 1, \ldots, q_{n_k} - 1$, for the

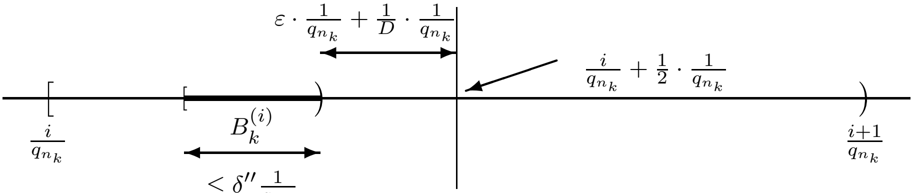

Fig. 4

interval $T^j B_k^{(i)}$,

(18) (10) holds.

The interval $SB_k^{(i)}$ is contained in an interval $[\overline{s}/q_{n_k}, \overline{s}/q_{n_k} + 1/q_{n_k})$ and for each $j = 0, 1, \ldots, q_{n_k} - 1$, for the interval $T^j SB_k^{(i)}$,

(19) (11) holds.

It follows that the formula

$$c_k(i) = j \ \Leftrightarrow\ T^j SB_k^{(i)} \subset \left[\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{q_{n_k}}, \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}\right)$$

defines a bijection $c_k : \{0, 1, \ldots, q_{n_k} - 1\} \to \{0, 1, \ldots, q_{n_k} - 1\}$. Notice that in view of (17), $ST^{q_{n_k}-1}B_k^{(i)}$ is contained in an interval $[u/q_{n_k}, (u+1)/q_{n_k})$. In fact it is contained in the right half of that interval and more precisely, using (16), the distance of $ST^{q_{n_k}-1}B_k^{(i)}$ from $\frac{u}{q_{n_k}} + \frac{1}{2} \cdot \frac{1}{q_{n_k}}$ is at least $\frac{c}{3} \cdot \frac{1}{q_{n_k}}$ (for k large enough)—see Figure 5. It now follows from (16) that the interval

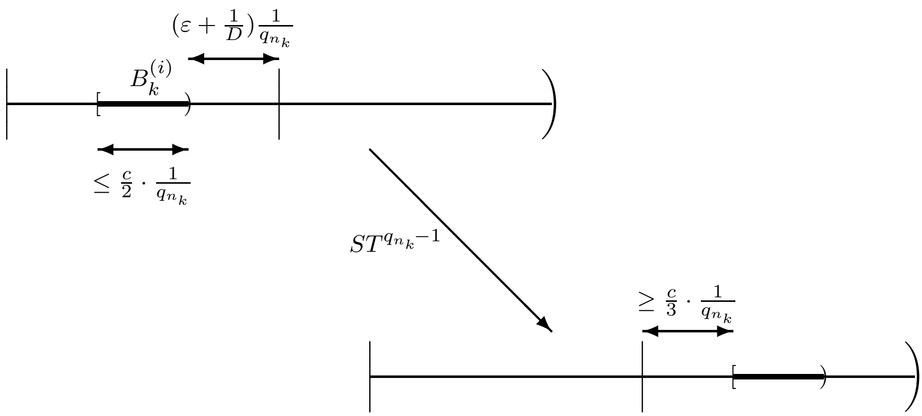

Fig. 5. $T^{q_{n_k}-1}$ shifts $B_k^{(i)}$ close to $i'/q_{n_k} + (1/2) \cdot (1/q_{n_k})$, while S shifts $T^{q_{n_k}-1}B_k^{(i)}$ into the right half of $[u/q_{n_k}, (u+1)/q_{n_k})$.

$ST^{q_{n_k}-j}B_k^{(i)}$ is contained in the right half of $[u_j/q_{n_k}, (u_j+1)/q_{n_k})$ whenever

$$j\left(\alpha - \frac{p_{n_k}}{q_{n_k}}\right) < \frac{c}{3} \cdot \frac{1}{q_{n_k}},$$

and thus (using (ii)), for all $j \in \{0,1,\ldots,q_{n_k}-1\}$ satisfying

$$0 < j < cq_{n_k}.$$

We say that $i \in \{0,1,\ldots,q_{n_k}-1\}$ is *good* if $c_k(i) = q_{n_k} - j$ with $0 < j < cq_{n_k}$. The number of good i's is at least cq_{n_k}.

It follows from (18), (19) and the above discussion that

$$\mu\{x \in [0,1) : (\varphi^{(q_{n_k})}(x), \varphi^{(q_{n_k})}(x+\beta)) = (1,-1)\} \ge c(\delta'' - \varepsilon),$$

so $(1,-1) \in E(\varphi \times \varphi \circ S)$.

In order to conclude case I we have to consider the situation when n_k is odd for all k large enough. First put

$$C_k^{(i)} = \left[\frac{i}{q_{n_k}} + \frac{1}{2}\cdot\frac{1}{q_{n_k}} - \delta_1 \frac{1}{q_{n_k}}, \frac{i}{q_{n_k}} + \frac{1}{2}\cdot\frac{1}{q_{n_k}}\right)$$

(see Figure 6). By considerations similar to those used before, it follows that

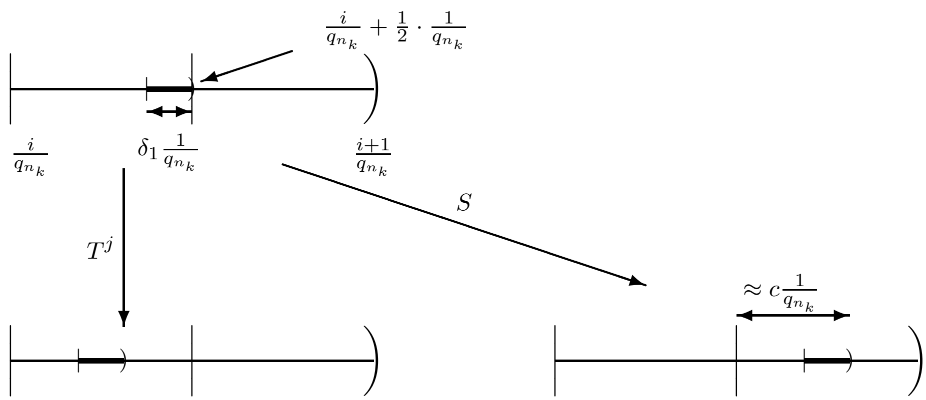

Fig. 6

$(1,-1) \in E(\varphi \times \varphi \circ S)$. If we put

$$D_k^{(i)} = \left[\frac{i}{q_{n_k}} + (1-c-\varepsilon)\frac{1}{q_{n_k}} - \delta_2 \frac{1}{q_{n_k}}, \frac{i}{q_{n_k}} + (1-c-\varepsilon)\frac{1}{q_{n_k}}\right)$$

then similar arguments show that $(-1,-1) \in E(\varphi \times \varphi \circ S)$ (see Figure 7).

CASE II: $1/2 < c < 1$. We replace β by $-\beta$ (that is, S by S^{-1}) and by case I we obtain

$$(1,1), (1,-1) \in E(\varphi \times \varphi \circ S^{-1}) \tag{20}$$

because $\{q_{n_k}(-\beta)\} \to -c$ (and $-c = 1-c$, $0 < 1-c < 1/2$). In fact, to obtain (20) we have

$$((\varphi \times \varphi \circ S^{-1})^{(q_{n_k})})_*\mu \to \nu$$

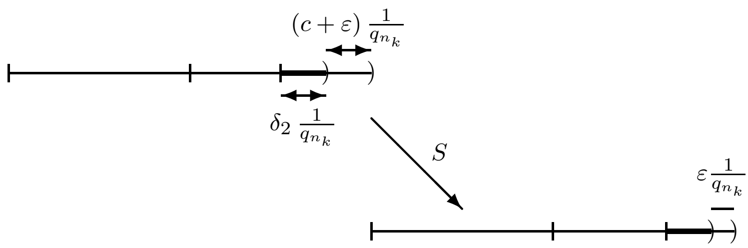

Fig. 7

and $\nu\{(1,-1)\} > 0$. In other words, there exists a $\kappa > 0$ such that for k large enough,

$$(\exists Y_k \subset [0,1),\ \mu(Y_k) \geq \kappa)(\forall x \in Y_k) \quad (\varphi^{(q_{n_k})}(x), \varphi^{(q_{n_k})}(S^{-1}x)) = (1,-1).$$

Put $Y'_k := S^{-1}Y_k$. For $x \in Y'_k$, we have

$$\varphi^{(q_{n_k})}(x) = -1, \quad \varphi^{(q_{n_k})}(Sx) = 1$$

for all k large enough. It follows that for any limit measure ν' of the distributions $((\varphi \times \varphi \circ S)^{(q_{n_k})})_*\mu$ we have $\nu'\{(-1,1)\} \geq \kappa$ and therefore $(-1,1) \in E(\varphi \times \varphi \circ S)$. We show similarly that $(1,1) \in E(\varphi \times \varphi \circ S)$, so finally $E(\varphi \times \varphi \circ S) = G$.

Case III: $c = 1/2$. It is clear that the method described in case I (and case II) gives $(1,-1) \in E(\varphi \times \varphi \circ S)$.

Case IV: $c = 0$. In this case one shows (by the method of I and II) that $(1,1) \in E(\varphi \times \varphi \circ S)$.

Conclusion of (I)–(IV):

$$\text{(21)} \qquad \begin{cases} \text{If } c \neq 0, 1/2 \text{ belongs to the set of accumulation points} \\ \text{of the sequence } \mathcal{A}_\alpha(\beta) = \{\{q_n\beta\} : n \geq 1\} \text{ then } E(\varphi \times \varphi \circ S) = G, \end{cases}$$

in particular

$$\text{(22)} \qquad \text{if } \beta \notin \mathbb{Q}\alpha + \mathbb{Q}, \text{ then } E(\varphi \times \varphi \circ S) = G$$

(indeed, in this case the sequence $A_\alpha(\beta)$ has infinitely many limit points). Furthermore,

$$\text{(23)} \qquad \begin{cases} \text{if } \{0, 1/2\} \text{ is contained in the set of limit points of } A_\alpha(\beta), \\ \text{then } E(\varphi \times \varphi \circ S) = G. \end{cases}$$

Case V: $\beta = 1/2$. In this case, $\varphi \circ S = -\varphi$ $(Sx = x + 1/2)$ and $\mathrm{Im}(\varphi \times \varphi \circ S) \subset \widetilde{\Delta}_{\mathbb{Z}}$, and since $1/2$ belongs to the set of accumulation points of $A_\alpha(\beta)$ (in view of (iii)), $(1,-1) \in E(\varphi \times \varphi \circ S)$, so $\varphi \times \varphi \circ S$ is ergodic as a cocycle taking values in $\widetilde{\Delta}_{\mathbb{Z}}$.

Case VI: $\beta = 1/2^l$, $l \geq 2$ (also $\beta = i/2^l$ with i odd). Because of (iii), $q_n = 2\widetilde{q}_n + 1$ and

$$q_n\beta = \frac{\widetilde{q}_n}{2^{l-1}} + \frac{1}{2^l}.$$

Since the set of accumulation points of $\{\{\widetilde{q}_n/2^{l-1}\} : n \geq n_0\}$ is contained in $\{i/2^{l-1} : \ i = 0, 1, \ldots, 2^{l-1} - 1\}$,

$$(24)\quad (\exists c \neq 0, 1/2) \quad c \text{ belongs to the set of accumulation points of } A_\alpha(\beta),$$

so by (21), $E(\varphi \times \varphi \circ S) = G$.

Case VII: $\beta = u/v$, $v \neq 2^l$. It follows that $v = 2^l w$, where $w \geq 3$ is odd. The set of accumulation points of $A_\alpha(\beta)$ is contained in $\{i/(2^l w) : i = 0, 1, \ldots, 2^l w - 1\}$. But w cannot divide all denominators q_n, $n \geq n_0$, because two consecutive denominators are relatively prime. Thus (24) also holds.

Case VIII: $\beta = (1/2^l)\alpha$, $l \geq 2$ (and $\beta = (1/2^l)\alpha + 1/2$). First note that

$$\left| q_n \frac{\alpha}{2^l} - \frac{p_n}{2^l} \right| = \frac{q_n}{2^l} \left| \alpha - \frac{p_n}{q_n} \right| \leq \frac{1}{2^l} \cdot \frac{1}{3q_n},$$

so the set of accumulation points of $A_\alpha(\beta)$ is the same as that of $\{p_n/2^l : n \geq 1\}$ and we are in the situation of case VI by (iv).

Case IX: $\beta = (u/v)\alpha$ with u, v relatively prime, $v \neq 2^l$, $l \geq 1$ (and $\beta = (u/v)\alpha + 1/2$). This reduces to the study of the set of accumulation points of $\{p_n \cdot u/v : n \geq 1\}$ and a reasoning as in case VII applies.

Case X: $\beta = \alpha/2$ (and $\beta = \alpha/2 + 1/2$). In this case, we consider $\{p_n \cdot 1/2 : n \geq 1\}$ (and $\{(p_n + q_n) \cdot 1/2 : n \geq 1\}$) and due to (iv), both 0 and 1/2 are in the set of accumulation points of $A_\alpha(\beta)$; then we apply (23).

If none of the above cases holds, then it remains to consider the following:

Case XI: $\beta = (u/v)\alpha + s/v$, where

- $(u, v, s) = 1$,
- $1 \leq u, s < v$,
- $v \geq 3$.

We study the set of accumulation points of

$$\left\{ \frac{up_n + sq_n}{v} : n \geq 1 \right\}$$

which is contained in $\{i/v : i = 0, 1, \ldots, v - 1\}$. If the only accumulation points are 0 or 1/2, then

$$(\exists N)(\forall n \geq N)(\exists k_n \in \mathbb{Z}) \quad 2up_n + 2sq_n = k_n v.$$

Hence for $n \geq N$ we have

$$v\begin{pmatrix} k_n \\ k_{n+1} \end{pmatrix} = \begin{pmatrix} p_n & q_n \\ p_{n+1} & q_{n+1} \end{pmatrix} \begin{pmatrix} 2u \\ 2s \end{pmatrix}$$

where

$$\det \begin{bmatrix} p_n & q_n \\ p_{n+1} & q_{n+1} \end{bmatrix} = 1 \text{ or } -1.$$

It follows that

$$v\begin{pmatrix} p_n & q_n \\ p_{n+1} & q_{n+1} \end{pmatrix}^{-1} \begin{pmatrix} k_n \\ k_{n+1} \end{pmatrix} = \begin{pmatrix} 2u \\ 2s \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} p_n & q_n \\ p_{n+1} & q_{n+1} \end{smallmatrix}\right)^{-1}$ is integer-valued, v divides $2u$ and $2s$. However, $v \geq 3$, so we obtain a contradiction. Hence (24) must hold in this case.

The proof of Theorem 5 is complete.

4. Semisimple automorphisms. In this section, we return to a general study of automorphisms of the form $T_{\varphi,\mathcal{G}}$.

We prove a theorem giving rise to new classes of semisimple automorphisms.

THEOREM 6. *Let $\mathcal{G} = \{R_g\}_{g\in G}$ be a mildly mixing action of G. Assume that T is an irrational rotation and $\varphi : X \to G$ is an ergodic cocycle such that for each $S \in C(T)$, the cocycle $\varphi \times \varphi \circ S : X \to G \times G$ is regular. Assume moreover that for each $S \in C(T)$, $E(\varphi \times \varphi \circ S)$ is either cocompact or equal to Δ_G.*

(i) *If $\mathcal{G}$ is 2-fold simple then each ergodic self-joining of $T_{\varphi,\mathcal{G}}$ is either a graph or the relatively independent extension of a graph joining of T. Moreover $T_{\varphi,\mathcal{G}}$ is semisimple.*

(ii) *If $\mathcal{G}$ is semisimple then $T_{\varphi,\mathcal{G}}$ is semisimple.*

Proof. (i) Take $\widetilde{\lambda} \in J^{\mathrm{e}}(T_{\varphi,\mathcal{G}})$. Hence, for some $S \in C(T)$, $\widetilde{\lambda}$ is an extension of μ_S and therefore (see the discussion before Proposition 10) we can assume that $\widetilde{\lambda} \in \mathcal{M}^{\mathrm{e}}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}})$.

Assume first that $\varphi\times\varphi\circ S$ is ergodic as a G^2-cocycle. The corresponding $\mathcal{G} \times \mathcal{G}$-action $\{R_{g_1} \times R_{g_2}\}_{g_1,g_2\in G}$ is uniquely ergodic in the sense of Proposition 9(i). By Theorem 3 and Proposition 10(iii), $\widetilde{\lambda} = \mu \otimes \nu \otimes \nu$. In view of Proposition 10(iv), it remains to show that the extension

$$(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}}, \widetilde{\lambda}) \to T_{\varphi,\mathcal{G}}$$

is relatively weakly mixing. But if we put $W = T_{\varphi,\mathcal{G}}$ and consider the cocycle $\varphi \circ S$ as $\varphi \circ S : X \times Y \to G$ (that is, as a cocycle for W) then $T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}}$ and $W_{\varphi\circ S,\mathcal{G}}$ are relatively isomorphic (over the common factor $T_{\varphi,\mathcal{G}}$) and moreover $W_{\varphi\circ S,\mathcal{G}}$ is relatively weakly mixing over the base W because the G-action is mildly mixing and $W_{\varphi\circ S,\mathcal{G}}$ is ergodic (see Proposition 5(i)).

Consider now a more general case: $E(\varphi\times\varphi\circ S) = H$ is a proper subgroup of $G\times G$. Then the extension

$$(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}}, \mu\otimes\nu\otimes\nu)\to T_{\varphi,\mathcal{G}}$$

is relatively weakly mixing iff so is

$$(T_{\Theta,\mathcal{H}}, \mu\otimes\nu\otimes\nu)\to T_{\varphi,\mathcal{G}},$$

where Θ is an ergodic cocycle with values in H cohomologous to $\varphi\times\varphi\circ S$.

Suppose additionally that H is cocompact. Then, in the sense of Proposition 9(ii), the corresponding $\mathcal{H}$-action is still uniquely ergodic. Hence $\widetilde{\lambda} = \mu\otimes\nu\otimes\nu$ and therefore the same argument as in the previous case shows that $(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}}, \widetilde{\lambda})$ is relatively weakly mixing over $T_{\varphi,\mathcal{G}}$.

It remains to consider the case $H = \Delta_G$. It follows that $\widetilde{\lambda} = \mu\otimes\varrho$ (recall that we still identify $\widetilde{\lambda}$ with an element of $\mathcal{M}^{\mathrm{e}}(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}})$), where $\varrho\in J^{\mathrm{e}}(\mathcal{G})$. Since $\mathcal{G}$ is 2-fold simple, either ϱ is the product measure or it is a graph measure. If $\varrho = \nu\otimes\nu$ then we are in the situation already considered. Otherwise ϱ is a graph and by Proposition 10(ii), so must be $\widetilde{\lambda}$.

(ii) The proof is along the same lines as the one of (i) except the case of $H = \Delta_G$. We have to show that the extension

$$(T_{\Theta,\Delta_{\mathcal{G}}}, \mu\otimes\varrho)\to T_{\Theta,\{(R_g,R_g)_{g\in G}\}}$$

is relatively weakly mixing. As usual we consider $T_{\varphi,\mathcal{G}}$ as a factor of the system $T_{\Theta,\{(R_g,R_g)_{g\in G}\}}$ "sitting" on the first two coordinates (as a σ-algebra it is equal to $\mathcal{B}\otimes\mathcal{C}\times Y$). By considering Θ, $\mathcal{D} = \mathcal{C}\times Y$ and ϱ, we are in the situation of Lemma 2. Because $\mathcal{G}$ is semisimple and $\varrho\in J^{\mathrm{e}}(\mathcal{G})$, $(\mathcal{C}\otimes\mathcal{C},\varrho)$ is relatively weakly mixing over $\mathcal{D}$, whence $(T_{\varphi\times\varphi\circ S,\mathcal{G}\times\mathcal{G}}, \widetilde{\lambda})$ is relatively weakly mixing over $T_{\varphi,\mathcal{G}}$ and the result follows. ∎

Remark 3. It is now easy to describe the smallest natural family (in the sense of [8]) of semisimple automorphisms arising from Theorem 6. Indeed, such a family consists of all factors of $T_{\varphi,\mathcal{G}}$ relative to which $T_{\varphi,\mathcal{G}}$ is weakly mixing. First of all, note that T is a maximal distal factor of $T_{\varphi,\mathcal{G}}$. It follows that if $\mathcal{A}$ is a factor relative to which $T_{\varphi,\mathcal{G}}$ is weakly mixing then $\mathcal{A}$ contains the "first coordinate". We then apply Corollary 2. We find that factors of $T_{\varphi,\mathcal{G}}$ relative to which $T_{\varphi,\mathcal{G}}$ is weakly mixing are of the form $\mathcal{B}\otimes\mathcal{D}$, where $\mathcal{D}$ is a $\mathcal{G}$-factor. If $\mathcal{D}$ is non-trivial then it is determined by a compact subgroup K of $C(\{R_g\}_{g\in G})$ ([9]); consequently, $\mathcal{B}\otimes\mathcal{D}$ is determined by $\{\mathrm{Id}\}\times K$, which is a compact subgroup of $C(T_{\varphi,\mathcal{G}})$, and it follows that $T_{\varphi,\mathcal{G}}$ is not relatively weakly mixing over $\mathcal{B}\otimes\mathcal{D}$ unless $\mathcal{D} = \mathcal{C}$. We have shown that the smallest natural family equals $\{T, T_{\varphi,\mathcal{G}}\}$.

5. Final remarks. The examples of semisimple automorphisms given by Theorem 6 are weakly mixing extensions of rotations and each such example is disjoint from any weakly mixing automorphism (see Proposition 5 and Theorem 1). There are Gaussian actions which are mildly mixing and semisimple (see [16]). By looking at the proof of Theorem 6, it is clear that the mild mixing assumption can be replaced by being Gaussian semisimple.

If we consider extensions $T_{\varphi,\mathcal{G}}$ of irrational rotations in which $G=\mathbb{Z}$ and φ is given by Theorem 5 then one more assumption on $\{R^n\}_{n\in\mathbb{Z}}$ has to be added. This is caused by the fact that $\widetilde{\Delta}_{\mathbb{Z}}$ appears as the group of essential values of $\varphi\times\varphi\circ S$ for some $S\in C(T)$. This gives rise to the study of $J(R,R^{-1})$. In order to obtain Theorem 6 it is sufficient to assume that the $\mathbb{Z}$-action R satisfies either

(i) $R\perp R^{-1}$ (see [8] for the case of MSJ), or

(ii) R is isomorphic to R^{-1} (which is always the case whenever Gaussian actions are considered).

In Theorem 3 we deal with self-joinings of order 2. It is clear however that the same results hold for self-joinings of higher degrees. Given $n\geq 2$ denote by $J_n(T_{\varphi,\mathcal{G}})$ the set of n-self-joinings of $T_{\varphi,\mathcal{G}}$. Then Corollary 1 yields the following.

PROPOSITION 11. *If $\varphi : X\to G$ is ergodic then the map*

$$\Lambda_0^n : (x_1,y_1,x_2,y_2,\ldots,x_n,y_n)\mapsto(x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n)$$

gives rise to an affine isomorphism of $J_n(T_{\varphi,\mathcal{G}};\Delta_X)$ and $\{\Delta_X\otimes\varrho : \varrho\in J_n(\mathcal{G})\}$. ∎

In [9] the following problem has been formulated: Is it true that for each ergodic zero entropy automorphism $W:(Z,\mathcal{E},\kappa)\to(Z,\mathcal{E},\kappa)$, if $\varrho\in J_3(W)$ is pairwise independent then $\varrho=\kappa\otimes\kappa\otimes\kappa$? (The problem is open in the weakly mixing case.) An affirmative answer would imply that each 2-fold mixing automorphism is 3-fold mixing (the latter being Rokhlin's well known open problem). We have been unable to answer del Junco–Rudolph's question. However, the method of the present paper yields a negative answer to the *relative* version of their problem. Indeed, let T be an ergodic rotation, $\varphi:X\to\mathbb{Z}$ an ergodic cocycle and R a Bernoulli automorphism. It is well known that there exists $\varrho\in J_3^{\mathrm{e}}(R)$ which is pairwise independent but it is not the product measure $\nu\otimes\nu\otimes\nu$. By the above proposition, $(\Lambda_0^3)_*^{-1}(\mu\otimes\varrho)$ is an ergodic element of $J_3(T_{\varphi,\mathcal{G}})$ (here $T_{\varphi,\mathcal{G}}(x,y)=(Tx,R^{\varphi(x)}(y))$) which is relatively pairwise independent, but is different from $(\Lambda_0^3)_*^{-1}(\mu\otimes(\nu\otimes\nu\otimes\nu))$. Moreover, $T_{\varphi,\mathcal{G}}$ being disjoint from the class of weakly mixing transformations, the entropy of $T_{\varphi,\mathcal{G}}$ equals zero and hence also the relative entropy of $T_{\varphi,\mathcal{G}}$ over T equals zero.

References

[1] J. Aaronson and M. Keane, *The visits to zero of some deterministic random walks*, Proc. London Math. Soc. 44 (1982), 535–553.

[2] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinaĭ, *Ergodic Theory*, Springer, New York, 1982.

[3] H. Furstenberg, *Disjointness in ergodic theory, minimal sets and diophantine approximation*, Math. Systems Theory 1 (1967), 1–49.

[4] —, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, Princeton 1981.

[5] H. Furstenberg and B. Weiss, *The finite multipliers of infinite ergodic transformations*, in: Lecture Notes in Math. 688, Springer, 1978, 127–132.

[6] E. Glasner, *On the multipliers of* $\mathcal{W}^{\perp}$, Ergodic Theory Dynam. Systems 14 (1994), 129–140.

[7] E. Glasner and B. Weiss, *Processes disjoint from weak mixing*, Trans. Amer. Math. Soc. 316 (1989), 689–703.

[8] A. del Junco, M. Lemańczyk and M. K. Mentzen, *Semisimplicity, joinings and group extensions*, Studia Math. 112 (1995), 141–164.

[9] A. del Junco and D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory Dynam. Systems 7 (1987), 229–248.

[10] A. Ya. Khinchin, *Continued Fractions*, Nauka, Moscow, 1978 (in Russian).

[11] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York, 1974.

[12] C. Kraaikamp and P. Liardet, *Good approximation and continued fractions*, Proc. Amer. Math. Soc. 112 (1991), 303–309.

[13] G. Larcher, *A convergence problem connected with continued fractions*, ibid. 103 (1988), 718–722.

[14] M. Lemańczyk and E. Lesigne, *Ergodicity of Rokhlin cocycles*, J. Anal. Math. 85 (2001), 43–86.

[15] M. Lemańczyk and F. Parreau, *Rokhlin's extensions and lifting disjointness*, Ergodic Theory Dynam. Systems, to appear.

[16] M. Lemańczyk, F. Parreau and J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self-joinings are Gaussian*, Fund. Math. 164 (2000), 253–293.

[17] M. Lemańczyk, F. Parreau and D. Volný, *Ergodic properties of real cocycles and pseudo-homogeneous Banach spaces*, Trans. Amer. Math. Soc. 348 (1996), 4919–4938.

[18] C. Moore and K. Schmidt, *Coboundaries for non-singular actions and a problem of H. Helson*, Proc. London Math. Soc. 40 (1980), 443–475.

[19] S. A. Morris, *Pontryagin Duality and the Structure of Locally Compact Abelian Groups*, London Math. Soc. Lecture Note Ser. 29, Cambridge Univ. Press, 1977.

[20] D. Rudolph, *An example of a measure-preserving map with minimal self-joinings, and applications*, J. Anal. Math. 35 (1979), 97–122.

[21] —, $\mathbb{Z}^n$ *and* $\mathbb{R}^n$ *cocycle extensions and complementary algebras*, Ergodic Theory Dynam. Systems 6 (1986), 583–599.

[22] K. Schmidt, *Cocycles of Ergodic Transformation Groups*, Lecture Notes in Math. 1, Mac Millan Co. of India, 1977.

[23] K. Schmidt and P. Walters, *Mildly mixing actions of locally compact groups*, Proc. London Math. Soc. (3) 45 (1982), 506–518.

[24] J.-P. Thouvenot, *Some properties and applications of joinings in ergodic theory*, in: Ergodic Theory and its Connections with Harmonic Analysis (Alexandria, 1993), Cambridge Univ. Press, 1995, 207–238.

[25] W. A. Veech, *A criterion for a process to be prime*, Monatsh. Math. 94 (1982), 335–341.
[26] R. Zimmer, *Ergodic actions with generalized discrete spectrum*, Illinois J. Math. 20 (1976), 555–588.

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