ON SOME SPECTRAL PROBLEMS IN ERGODIC THEORY

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Abstract. We survey selected advances in spectral theory of dynamical systems, present perspectives and discuss some open problems.

1. Introduction

Historically, spectral invariants were the first due to which, intuitively different measure-theoretic systems gained rigorous proofs of being non-isomorphic as dynamical systems. During decades a vision of spectral classification of dynamical systems considerably evolved: from a belief in serious restrictions imposed by dynamics to a present belief that it is rather common to have little restrictions (compared to general and well-known classification of unitary operators on separable Hilbert spaces). This, in practice, is observed in the prevalence of constructions and, using Anatole Katok's expression (shared by the author of the article): "a weak hope for theorems". Spectral theory does not only provide invariants for dynamics, but also serves as an "engine" for complicated measure-theoretic constructions of dynamical systems with interesting (sometimes "exotic") properties, perhaps [27] is the most illustrative article in this direction. Moreover, via regarding measures on product spaces as relevant Markov operators of L^2-spaces, spectral theory is closely related to the joining theory, hence Furstenberg's disjointness, of dynamical systems, the most modern tool in contemporary (abstract) ergodic theory.

Many interesting spectral questions turned out to be difficult and remain unsolved – our current understanding of spectral classification of dynamical systems is definitely unsatisfactory. In this article we discuss classical invariants of spectral theory like maximal spectral type and multiplicity function, but also concentrate on selected latest achievements and discuss perspectives and open problems concerning spectral properties of flows, especially in the smooth context. The present material is written on the base of my joint article [35] with Anatole Katok. It also uses survey articles by A.I. Danilenko [12] and myself [45]. For a nice introduction to spectral theory of dynamical system, see monographs [8], [49] and [55].

2. Spectral theory. Koopman operator

Assume that $(X, \mathcal{B}, \mu)$ is a probability, standard Borel space (non-atomic!), so up to a Borel isomorphism, we may think of $X = [0, 1]$ considered with the σ-algebra of Borel sets and Lebesgue measure. In particular, $L^2(X, \mathcal{B}, \mu)$ is separable. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an invertible, measure-preserving transformation; we will write $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$.

We can now define the associated *Koopman operator* $U_T\colon L^2(X,\mathcal{B},\mu)\to L^2(X,\mathcal{B},\mu)$ by the following formula:

$$U_Tf:=f\circ T \text{ for } f\in L^2(X,\mathcal{B},\mu). \tag{1}$$

In *spectral theory of dynamical systems* we study properties of U_T, that is, properties of T that are stable under **spectral isomorphism** in $\mathrm{Aut}(X,\mathcal{B},\mu)$.[1] Note that ergodicity, weak mixing, mixing **are** spectral properties while entropy is not (see Example 4.1) and whether mixing of all orders so is, remains an open problem. Note also that $\mathrm{Aut}(X,\mathcal{B},\mu)$ is a Polish group, when we consider the strong operator topology on it (indeed, the group $\mathrm{Aut}(X,\mathcal{B},\mu)$ is closed under pointwise convergence: if $U_{T_n}f\to Vf$ for each $f\in L^2(X,\mathcal{B},\mu)$, then V is also Koopman).

Before we enter into the subject, let us briefly discuss the "Koopman" operators on L^p-spaces for $p\neq 2$, which are defined by the same token as in (1) for $f\in L^p(X,\mathcal{B},\mu)$. On one hand side, based on the classical fact that there are not too many isometries of $L^p([0,1])$ for $p\neq 2$, the following result shows that this problem is too strong and in fact, L^p-spectral classification of measure-preserving automorphisms is the same as metric classification:

Proposition 2.1 ([28], [45])**.** *Assume that $1\leq p<+\infty$, $p\neq 2$. Let $T\in\mathrm{Aut}(X,\mathcal{B},\mu)$ and $S\in\mathrm{Aut}(Y,\mathcal{C},\nu)$ be ergodic. Assume that $V:L^p(X,\mathcal{B},\mu)\to L^p(Y,\mathcal{C},\nu)$ is an isometry such that $V\circ U_T=U_S\circ V$. Then T and S are measure-theoretically isomorphic.*[2]

On the other hand, there are interesting open problems when $p\neq 2$, from which perhaps mostly known is the following:

Problem 1 (Thouvenot's problem, 1986)**.** *Is is true that for each ergodic $T\in\mathrm{Aut}(X,\mathcal{B},\mu)$ there exists $f\in L^1(X,\mathcal{B},\mu)$ such that $L^1(X,\mathcal{B},\mu)=\overline{\mathrm{span}}\{f\circ T^k:\ k\in\mathbb{Z}\}$ for some $f\in L^1(X,\mathcal{B},\mu)$?* [3]

Iwanik [26], in 1991, proved that for each Bernoulli automorphism T and $1<p<+\infty$, each $n\geq 1$ and all $f_1,\ldots,f_n\in L^p(X,\mathcal{B},\mu)$, we have

$$\overline{\mathrm{span}}\{f_j\circ T^k:\ k\in\mathbb{Z}, j=1,\ldots,n\}\neq L^p(X,\mathcal{B},\mu)$$

which we can interpret as "Bernoulli automorphisms have infinite multiplicity" in all L^p spaces whenever $1<p<+\infty$.

3. Classification of unitary operators on separable Hilbert spaces

Spectral classification of unitary operators on separable Hilbert spaces is based on the following:

[1] If $T_i\in\mathrm{Aut}(X_i,\mathcal{B}_i,\mu_i)$, $i=1,2$, then a unitary operator $V:L^2(X_1,\mathcal{B}_1,\mu_1)\to L^2(X_2,\mathcal{B}_2,\mu_2)$ establishes a spectral isomorphism of T_1 and T_2 if $V\circ U_{T_1}=U_{T_2}\circ V$.

[2] The proof of the above result uses Lamperti's theorem to obtain that $(Vf)(y)=j(y)\cdot f(Jy)$, where $J:Y\to X$ is non-singular. Then the equivariance yields: $j(y)f(TJy)=j(Sy)f(JSy)$; take $f=1$, to obtain that $j=\mathit{const}$ by the ergodicity of S. Finally, we note that the image $J_*(\nu)$ is a T-invariant, ergodic measure satisfying $J_*(\nu)\ll\mu$, and we use the ergodicity of T to conclude.

[3] The problem is also open in the category of Bernoulli automorphisms.

Theorem 3.1 (Herglotz's theorem). *If $U : H \to H$ is a unitary operator on a Hilbert space H and $f \in H$, then the sequence $\mathbb{Z} \ni k \mapsto \langle U^k f, f\rangle$ is positive definite and therefore there exists a unique (positive, finite, Borel) measure σ_f on $\mathbb{S}^1$ such that*

$$\widehat{\sigma}_f(k) := \int_{\mathbb{S}^1} z^k \, d\sigma_f(z) = \langle U^k f, f\rangle \text{ for each } k \in \mathbb{Z}.$$

σ_f is called the *spectral measure* of f. Let us consider the following example: $H_\sigma := L^2(\mathbb{S}^1, \sigma)$, where σ is a (positive, Borel) finite measure on the circle, and $V_\sigma : H_\sigma \to H_\sigma$, $(V_\sigma f)(z) = zf(z)$ which is clearly unitary. We then have $H_\sigma = \mathbb{Z}(\mathbf{1}) := \overline{\mathrm{span}}\{V_\sigma^k \mathbf{1} : k \in \mathbb{Z}\}$ and one says that H_σ is equal to the **cyclic** space $\mathbb{Z}(\mathbf{1})$, where $\mathbf{1}(z) = 1$. V_σ is an example of a unitary operator (defined on a separable Hilbert space) with *simple spectrum.* Returning to the abstract setting, if H is a (separable) Hilbert space and U is a unitary operator on it such that $H = \mathbb{Z}(f)$ for some $f \in H$, then the map

$$U^k f \mapsto V_{\sigma_f}^k(\mathbf{1}) = z^k, \; k \in \mathbb{Z},$$

extends to a (linear) isometry intertwining U and V_{σ_f}. If the above holds, we say that U has *simple spectrum.*

Theorem 3.2 (Spectral theorem). *Assume that $U : H \to H$ is unitary on a separable Hilbert space H. There exists a decomposition, called a* spectral decomposition*, $H = \bigoplus_{n\geq 1} \mathbb{Z}(f_n)$* [4] *such that*

$$\sigma_{f_1} \gg \sigma_{f_2} \gg \dots \tag{2}$$

For any other spectral decomposition $H = \bigoplus_{n\geq 1} \mathbb{Z}(f'_n)$, we have

$$\sigma_{f_n} \equiv \sigma_{f'_n} \text{ for each } n \geq 1. \tag{3}$$

The sequence (2) is called a *spectral sequence.* The type[5] of σ_{f_1} is called the *maximal spectral type* of U and is denoted by σ_U. To obtain a spectral decomposition we choose any cyclic space $\mathbb{Z}(f)$ and extend it to a maximal cyclic space, say $\mathbb{Z}(f_1)$. Such an extension exists by separability and it has the property that its spectral measure σ_{f_1} dominates all other spectral measures. Then we pass to $\mathbb{Z}(f_1)^\perp$ and repeat the previous argument. Again, it is separability which yields the existence of a spectral decomposition.

Then, we define $M_U(z) := \sum_{n\geq 1} \mathbf{1}_{\mathrm{supp}\frac{d\sigma_{f_n}}{d\sigma_{f_1}}}(z)$ which is called the *spectral multiplicity* function of U; it is defined σ_U-a.e. and its values in $\mathbb{N} \cup \{+\infty\}$, where $\mathbb{N} = \{1, 2, \dots\}$. Note that when we know σ_U and M_U then we can reconstruct a spectral sequence (2) (up to equivalence of measures). It follows that:

- Any sequence $\sigma_1 \gg \sigma_2 \dots$ can be realized as a spectral sequence of some U (indeed, take $U = V_{\sigma_1} \oplus V_{\sigma_2} \oplus \dots$).

[4] Note that if a Hilbert space admits such a decomposition then it **must** be separable.

[5] By the *type* of σ we mean the class of measures equivalent (i.e. having the same null sets) to σ. In fact, unless a confusion may arise, in what follows, we will not distinguish between a spectral measure and its type. Note, for example, that the topological support of a measure σ is the same for all measures equivalent to σ.

- Equivalently, given σ and $M : \mathbb{S}^1 \to \mathbb{N} \cup \{+\infty\}$ which is measurable and defined σ-a.e., we can find U so that $\sigma = \sigma_U$ and $M = M_U$.
- Two unitary operators are (spectrally) isomorphic if and only if they have the same spectral sequence (2) (up to equivalence od spectral measures); equivalently, if and only if they have the same maximal spectral types and the same multiplicity functions.

Remark 3.3. For Koopman operators, we will rather use notation σ_T, M_T instead of σ_{U_T}, M_{U_T}, respectively.

4. Basic questions of spectral theory of Koopman operators

One of classical unsolved problems of spectral theory is the following:

Problem 2. *Which sequences $\sigma_1 \gg \sigma_2 \gg \ldots$ appear as spectral sequences of Koopman operators $U_T|_{L^2_0(X,\mathcal{B},\mu)}$ for an ergodic $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$?* [6]

Instead of considering Problem 2 whose solution is hard to see in a reasonable perspective, out of it, we can separate the following two natural questions:

Problem 3. *Which measures appear as the maximal spectral type of an ergodic automorphism?*

Problem 4. *Which subsets of $\mathbb{N}$ (see Remark 4.1 below) appear as the set* essval(M_T) [7] *of essential values of an ergodic automorphism T? (Such sets are called* Koopman realizable*).*

Example 4.1. (i) Assume that T is Bernoulli. Then $\lambda \equiv \lambda \equiv \ldots$ is its spectral sequence, so $\sigma_{U_T} = \sigma_T = [\lambda]$ and $\mathrm{essval}(M_T) = \{+\infty\}$. It follows that all Bernoulli automorphisms are spectrally isomorphic.
(ii) If T is an irrational rotation then its spectral sequence is given by: $\sigma \gg 0 \equiv 0 \equiv \ldots$, where $\sigma_T = \sigma = \sum_{\ell=1}^{\infty} \frac{1}{2^\ell}\delta_{e^{2\pi i\ell\alpha}}$ and $\mathrm{essval}(M_T) = \{1\}$. In particular, ergodic automorphisms with discrete spectrum have simple spectrum.

Remark 4.1. Of course, in Problem 4, we should consider subsets of $\mathbb{N}\cup\{+\infty\}$ as possible sets of essential values of the multiplicity function. However, as all constructions below will have singular maximal spectral types, by considering their product with a Bernoulli shift, we only add $+\infty$ as the new essential value (corresponding to Lebesgue measure), see [12].

We would like to emphasize that spectral problems are essentially zero entropy problems because of the classical Rokhlin-Sinai theorem: *If $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ then in the orthocomplement of $L^2(\pi(T))$, where $\pi(T) \subset \mathcal{B}$ stands for the Pinsker σ-algebra,*[8] *the spectrum is*

[6] By an obvious reason, we consider only the U_T-invariant subspace $L^2_0(X, \mathcal{B}, \mu)$ which is the orthocomplement of constant functions. Ergodicity is assumed to avoid "cheap" tricks with repetitions of the same space, say $L^2(X, \mathcal{B}, \mu) \oplus L^2(X, \mathcal{B}, \mu)$ when, for example, we consider the spectral multiplicity problem.

[7] Formally, $n \in \mathrm{essval}(M_T)$ if $\sigma_T(\{n\}) > 0$.

[8] Pinsker σ-algebra is the largest σ-algebra such that the corresponding factor automorphism has zero entropy.

countable Lebesgue.[9] For a generalization of this result to actions of other groups, see [17] (the result was also obtained independently by Thouvenot (unpublished)).

5. Maximal spectral type

5.1. **Restrictions.** The following result describes restrictions on the maximal spectral type of an ergodic automorphism T:

Proposition 5.1. *(i) Topological support* $\mathrm{supp}(\sigma_T)$ *of* σ_T *is* $\mathbb{S}^1$.
(ii) The measure $\widetilde{\sigma_T}(A) := \sigma_T(\overline{A})$ *is equivalent to* σ_T.
(iii) If $e^{2\pi i\alpha}$ *is an eigenvalue of* U_T *then the measure* $\sigma_{T,\alpha}(A) := \sigma_T(e^{2\pi i\alpha}\cdot A)$ *is equivalent to* σ_T.

To show the symmetry of the maximal spectral type, i.e. (ii), notice that each Koopman operator preserves the space of real functions. To obtain (iii) note that if $f \in L^2_0(X, \mathcal{B}, \mu)$ and g is an eigenfunction corresponding to an eigenvalue $c \in \mathbb{S}^1$ then $|g| = 1$, so $fg \in L^2(X, \mathcal{B}, \mu)$ and, for each $n \in \mathbb{Z}$, we have

$$\int (fg)\circ T^n \cdot \overline{(fg)}\, d\mu = c^n \int f \circ T^n \cdot \overline{f}\, d\mu,$$

so $\sigma_{fg} = \delta_c * \sigma_f$. Finally, (i) requires the following observations: $\mathrm{supp}(\sigma_T) = \{z \in \mathbb{C} : z\cdot Id - U_T \text{ is not a bijection}\}$ (i.e. the topological support is equal to Gelfand spectrum of U_T), then (because of normality of U_T) $\{z \in \mathbb{C} : z\cdot Id - U_T \text{ is not a bijection}\} = \{z \in \mathbb{C} : z \text{ is an approximative eigenvalue}\}$, so the problem is reduced to solving "inequalities" $\|U_T(f) - zf\|_{L^2} < \epsilon$ (more precisely: for each $z \in \mathbb{S}^1$ and $\epsilon > 0$ we seek $f \in L^2(X, \mathcal{B}, \mu)$ of norm 1) which can be done using Rokhlin lemma and considering $\sum_{i=0}^{h-1} z^i \mathbf{1}_{T^iF}$ for a Rokhlin tower $F, TF, \ldots, T^{h-1}F$ which almost fulfills the whole space.

5.2. **Exponentials of measures as maximal spectral types.** If σ is a continuous symmetric measure on the circle and $U = \exp(V_\sigma) := \bigoplus_{\ell\geq 0} V_\sigma^{\odot \ell}$, then U is a Koopman operator via the classical Gaussian construction. Indeed, take a centered, real stationary Gaussian process $(X_n)_{n\in\mathbb{Z}}$ with spectral measure σ, i.e. $\mathbb{E}(X_n \cdot X_0) = \widehat{\sigma}(n)$ for each $n \in \mathbb{Z}$, with joint distribution μ_σ on $\mathbb{R}^{\mathbb{Z}}$ and consider the shift T on $(\mathbb{R}^{\mathbb{Z}}, \mu_\sigma)$. The following is classical:

Theorem 5.2. *We have* $\sigma_{U_{T_\sigma}} = \sum_{n\geq 1} \frac{1}{2^n}\sigma^{*n}$.[10] *In particular,* $\sum_{n\geq 1}\frac{1}{2^n}\sigma^{*n}$ *is a measure of maximal spectral type for an ergodic (in fact, weakly mixing) automorphism.*

We also recall the following (for a new proof, see [42]):

Theorem 5.3 (Girsanov's theorem, 1950th)**.** *Either* $\mathrm{essval}(M_{U_{T_\sigma}}) = \{1\}$ *or* $M_{U_{T_\sigma}}$ *has to be unbounded.*

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[9]There are zero entropy automorphism with countable Lebesgue spectrum: time-1 map of horocycle flows, see Section 8.1, or the even factor of a Gaussian automorphism given by $\sigma \perp \lambda$ with $\sigma * \sigma \equiv \lambda$, see e.g. [46].

[10]σ^{*n} means the convolution $\sigma * \ldots * \sigma$ (n times).

We can have $\mathrm{essval}(M_{U_{T_\sigma}}) = \{1, +\infty\}$ (take $\sigma \perp \lambda$ with $\sigma * \sigma \equiv \lambda$). In 2010, Danilenko and Ryzhikov [15] proved that **every** multiplicative sub-semigroup of $\mathbb{N}$ has a Gaussian "realization". However, they also proved that, in general, $\mathrm{essval}(U_{T_\sigma})$ need not have such a multiplicative structure.[11] More than that, Ryzhikov in [63] showed that for an **arbitrary** $E \subset \mathbb{N}$, the set $E \cup \{+\infty\}$ is the set of essential values for the multiplicity function of a (mixing) Gaussian automorphism (it is unknown whether this can be done without ∞ as an essential value).

Remark 5.4. Gaussian systems are also useful to show that conditions (i)-(iii) for a measure σ are not sufficient to realize σ as the maximal spectral type of a Koopman operator. Indeed, we can find a (continuous, with full topological support) so called Kronecker measure σ, i.e. satisfying: For each $f \in L^2(\mathbb{S}^1, \sigma)$ and $\epsilon > 0$ there exists $k \in \mathbb{Z}$ such that $\|f - z^k\|_{L^2(\sigma)} < \epsilon$. Then the famous Foiaş-Stratila theorem from 1967 [19] tells us that whenever $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ is ergodic and a real-valued $f \in L^2(X, \mathcal{B}, \mu)$ has $\sigma_f \equiv \sigma$ then the process $(f \circ T^n)_{N \in \mathbb{Z}}$ has to be Gaussian; then f will be in the first chaos of the corresponding L^2-space, each σ^{*j} is also a spectral measure and also $\sigma \perp \sum_{j \geq 2} \frac{1}{2^j} \sigma^{*j}$, so σ cannot be a measure of maximal spectral type.

Note that, essentially, **our knowledge about the maximal spectral types of Koopman operators has not changed since the 1960th**!

6. Spectral multiplicity

6.1. **Maximal spectral multiplicity.** Because of limited number of examples, for quite a long time there had been a belief that the maximal spectral multiplicity (that is, the essential supremum of the multiplicity function M_T) is ether 1 or $+\infty$. However, in 1966 Oseledets [51] proved that: *There exists an ergodic* $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ *such that* $1 < \mathrm{esssup}(M_T) < +\infty$. In fact, in [51] he introduced the concept of interval exchange transformations (IETs), showed that IETs of d intervals have maximal spectral multiplicity $\leq d - 1$ and use the idea of double group extension of such to create a relevant example. Refining Oseledets construction, in 1983, Robinson (a student of Katok) proved the following:

Theorem 6.1 ([57]). *For each $n \geq 1$, there exists an ergodic T such that* $\mathrm{esssup}(M_T) = n$. [12]

Recall that for $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, $\mathrm{essval}(M_T)$ stands for the set of essential values of the multiplicity function M_T.

(4) **General multiplicity conjecture:** Each subset $E \subset \mathbb{N}$ is realizable as $\mathrm{essval}(M_T)$.

[11] For example, the double factorials set $\{(2n-1)!! : n \in \mathbb{N}\}$ has a Gaussian "realization". This shows that Proposition 6.4.4. claiming the multiplicativity of multiplicity function in the Gaussian case, given in the book Katok-Thouvenot [37] (also in [57]), is false.

[12] Robinson also used double group extensions constructions; moreover, Katok-Stepin's theory of periodic approximation to apply a generic type arguments has been used.

Recall also that, generally, we are interested to answer the question which sequences $\sigma_1 \gg \sigma_2 \gg \dots$ are realizable as spectral sequences of Koopman operators, cf. Problem 2. In such a sequence we have either $\equiv$ or $\gg$ (without equivalence). It is not hard to see that $n \in \mathrm{essval}(M_U)$ if and only if $\sigma_{f_n} \gg \sigma_{f_{n+1}}$ (which, at this moment, means that that we have absolute continuity without equivalence). We can hence reformulate (4) as follows:

Problem 5. *Which sequences $(s_n)_{n\geq 1} \in \{\equiv, \gg\}^J$ are Koopman realizable ($J \in \mathbb{N}$ or $J = \mathbb{N}$)?*

6.2. **Essential values of the multiplicity function.** $1 \in E$. The problem of Koopman realization of a subset E containing 1 has been solved in a series of papers by Robinson (1986), Goodson-Kwiatkowski-Lemańczyk-Liardet (1992) and Kwiatkowski (jr.)-Lemańczyk (1995), the latter with final result:

Theorem 6.2 ([58],[24],[44]). *Each subset $1 \in E \subset \mathbb{N}$ is Koopman realizable, i.e. there exists an ergodic T such that* $\mathrm{essval}(M_T) = E$.

While [58] still used the idea of double extension, in [24], there is only one group extension step to create some symmetries. We now detail on that. Let $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ and G be a compact, Abelian group. Assume that $\phi : X \to G$ a cocycle (a measurable function). We define the corresponding group extension:

$$T_\phi : X \times G \to X \times G,\ T_\phi(x, g) = (Tx, \phi(x) + g)$$

(considered with product measure $\mu \otimes m_G$, where m_G stands for Haar measure). We have

$$L^2(X \times G, \mu \otimes m_G) = \sum_{\chi \in \widehat{G}} L^2(X, \mu) \otimes \chi, \tag{5}$$

where $\widehat{G}$ denotes the group of characters of G. Note that the (closed) subspaces in the Fourier decomposition (5) are U_{T_ϕ}-invariant: $U_{T_\phi}(L^2(X, \mu) \otimes \chi) = L^2(X, \mu) \otimes \chi$. Assume that $S \in C(T)$, that is $S \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ with $ST = TS$, and for some continuous $v \in \mathrm{Aut}(G)$ we can solve the equation

$$\phi(Sx) - v(\phi(x)) = \xi(Tx) - \xi(x)$$

for a measurable $\xi : X \to G$. It is not hard to see that the above equation is equivalent to saying that the (measure-preserving) $S_{\xi,v}(x, g) := (Sx, \xi(x) + v(g))$ is an element of the centralizer $C(T_\phi)$ of T_ϕ. However, the T_ϕ-invariant subspaces in (5) are (in general) no longer $S_{\xi,v}$-invariant as

$$U_{S_{\xi,v}}(L^2(X, \mu) \otimes \chi) = L^2(X, \mu) \otimes (\chi \circ v).$$

It easily follows that the lengths of the orbits of the dual automorphism $\widehat{v}$ on $\widehat{G}$ yield a lower bound on the multiplicity. Finally, note that instead of T_ϕ, we can consider its so called *natural factor* $T_{\phi+H}$ acting on $X \times G/H$ by the formula $(x, g + H) \mapsto (Tx, \phi(x) + g + H)$ for a closed subgroup $H \subset G$. Passing to $L^2(X \times G/H)$, this yields LESS subspaces of the form $L^2(X, \mu) \otimes \chi$ to be involved and the key observation in [44] was the following algebraic fact:

Lemma 6.3 ([44]). *For each set $E \subset \mathbb{N}$ containing 1 there exist a countable (discrete) group $\mathscr{G}$, its algebraic automorphism V and a subgroup $\mathscr{H}$ such that*

$$E = L(\mathscr{G}, V, \mathscr{H}) := \{|\{V^j\chi : j \in \mathbb{Z}\} \cap \mathscr{H}| : \chi \in \mathscr{H}\}.$$

The data we need for a relevant construction are obtained from the lemma by duality (in particular, $\mathcal{H}$ is the annihilator of H in $\mathcal{G} = \widehat{G}$).

6.3. **Essential values of the multiplicity function. Rokhlin problem.** Let us recall first the classical

$$\begin{array}{c}\textbf{Rokhlin's homogeneous spectrum problem}:\\ \text{Is it true that for each } n \geq 2 \text{ there is an ergodic automorphism } T\\ \text{such that } \mathrm{essval}(M_T) = \{n\}?\end{array} \tag{6}$$

Essential ideas toward a solution of this problem are due to Katok who already in the mid 1980 proved that for a generic[13] automorphism T, $\mathrm{essval}(M_{T\times T}) \subset \{2,4\}$ (this was obtained via Katok's linked approximation theory). He also formulated the following:

(7) **Katok's conjecture**: Generically, we have $\mathrm{essval}(M_{T^{\times n}}) = \{n, n(n-1), \ldots, n!\}$

(note that for $n = 2$ it yields positive answer to Rokhlin's question). Katok's conjecture has been proved in 1999 by Ageev [2], [3] in the general case and Ryzhikov [59] for $n = 2$ (see also Anosov's presentation in [5]). In 2005, Ageev fully answered Rokhlin's question (6):

Theorem 6.4 ([4]). *For each $n \geq 2$ there is an ergodic $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ with homogenous spectrum of multiplicity n.*

To obtain the above result, Ageev exploited a new idea of using actions of non-Abelian groups and showing that a fixed "direction" automorphism fulfills our spectral requirement. The original proof of Ageev was simplified in [61] and [9]. In fact, Danilenko in 2006, combining methods leading to Theorems 6.2 and 6.4, obtained the following:

Theorem 6.5 ([9]). *For each $n \geq 2$, and $1 \in E \subset \mathbb{N}$ there is an ergodic $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ such that* $\mathrm{essval}(M_T) = n \cdot E$.

Remark 6.6. An interesting problem is finding Koopman realization (of the constructions from Theorem 6.5 and other) in the smooth category - see Question (5) in Section 10 [12]. I was informed by A. Danilenko that the question about smooth realization of the homogenous problem for $n = 2$ was originally asked by A. Katok. See a solution of this problem in the analytic category by Banerjee and Kunde [7].

6.4. **Essential values of the multiplicity function.** $2 \in E$. Realization of subsets containing 2 has been done in three papers, one by Katok-Lemańczyk in 2009 and two by Danilenko in 2010 and 2012, the latter two with final result:

Theorem 6.7 ([35], [10], [11]). *Each set $2 \in E \subset \mathbb{N}$ is Koopman realizable.*[14]

[13]Generic is meant topologically: a dense G_δ set in the Polish group $\mathrm{Aut}(X, \mathcal{B}, \mu)$.

[14]The papers [10] and [11] are about weakly mixing and mixing realizations, respectively.

The main idea behind the proof of the above result is to combine techniques giving a solution of Rokhlin's problem (6) for $n = 2$ and returning to Oseledets double group extension idea. Moreover, the technique of weak limits[15] is exploited to obtain simplicity of the spectrum of tensor products operators of the form $e^V \otimes W$ with a simultaneous control of homogenous multiplicity for the $W \otimes W$. It is also important that Danilenko was able to prove Lemma 6.3 without the awkward assumption $1 \in E$.

Remark 6.8. For other sets which are Koopman realizable: $\{k, \ell, k\ell\}$, $\{k, \ell, m, k\ell, km, \ell m, k\ell m\}$, etc.: see [61], [64].

7. Related results

7.1. **Infinite measure-preserving automorphisms.** In this context we study the L^2-space of a standard Borel space $(X, \mathcal{B}, \mu)$ with μ infinite (and σ-finite). We consider here Koopman operators on the **whole** $L^2(X, \mathcal{B}, \mu)$ as the constant functions are no longer integrable. Somewhat surprisingly Danilenko and Ryzhikov proved the following result:

Theorem 7.1 ([14],[15]). *For each subset $E \subset \mathbb{N} \cup \{+\infty\}$ there exists an ergodic infinite measure-preserving automorphism T such that* $\mathrm{essval}(M_T) = E$.[16]

7.2. **Flows.** The spectral theory for flows is similar to that for automorphisms: we assume that a flow $\mathcal{T} = (T_t)$ acting on a probability standard Borel space $(X, \mathcal{B}, \mu)$ is *measurable*, so the associated Koopman one-parameter group:

$$\mathbb{R} \ni t \mapsto U_{T_t}(f) = f \circ T_t$$

is continuous for each $f \in L^2(X, \mathcal{B}, \mu)$. Bochner's theorem tells us that the function $t \mapsto \int f \circ T_t \cdot \overline{f}\, d\mu$ is positive definite, so there exists a unique (positive, finite, Borel) measure σ_f on $\widehat{\mathbb{R}} = \mathbb{R}$ [17] such that

$$\widehat{\sigma}_f(t) := \int_{\mathbb{R}} e^{2\pi i t s}\, d\sigma_f(s) = \int_X f \circ T_t \cdot \overline{f}\, d\mu, \text{ for each } t \in \mathbb{R}.$$

The cyclic space $\mathbb{R}(f)$ is defined as the smallest, closed (T_t)-invariant subspace containing f, i.e. $\mathbb{R}(f) = \overline{\mathrm{span}}\{f \circ T_t : \ t \in \mathbb{R}\}$. As before, we obtain that on the unitary level all is determined by the maximal spectral type $\sigma_{\mathcal{T}}$ (a measure on $\mathbb{R}$) and the multiplicity function $M_{\mathcal{T}} : \mathbb{R} \to \mathbb{N} \cup \{+\infty\}$ defined $\sigma_{\mathcal{T}}$-a.e.

Interesting questions arise when we study relations between spectral properties of the flow and the Koopman operators corresponding to non-zero time automorphisms T_t. Some answers belong to folklore. As a sample, consider the multiplicity problem of time-automorphisms, in which the crucial object turns out to be the (Borel) group

$$H(\sigma_{\mathcal{T}}) = \{t \in \mathbb{R} : \ \delta_t * \sigma_{\mathcal{T}} \equiv \sigma_{\mathcal{T}}\}.$$

Now, if $t \in H(\sigma_{\mathcal{T}})$ then T_t has necessarily uniform countable multiplicity (and its maximal spectral type is given by the image $(e^{2\pi i t \cdot})_*(\sigma_{\mathcal{T}})$), see e.g. [47]. In particular, for flows

[15] A wide use of this technique has been originated by Moscow school.

[16] The papers [14] and [15] are about weakly mixing and mixing realizations, respectively.

[17] The natural isomorphism between $\widehat{\mathbb{R}}$ and $\mathbb{R}$ is given by $t \mapsto e^{2\pi i t \cdot}$, for $t \in \mathbb{R}$.

having Lebesgue measure as the maximal spectral type, we obtain that each non-zero time automorphism have countable Lebesgue spectrum. One more sample of such a result is the observation from [23] that if a flow $\mathcal{T}$ has simple singular spectrum then a typical time automorphism has also simple spectrum.

Another result of the folklore kind is to consider induced (unitary) representations: in this classical construction we pass from unitary operators ($\mathbb{Z}$-actions) to one-parameter groups of unitary operators ($\mathbb{R}$-actions). In fact, if $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ then induced (from U_T) representation is also Koopman and given by the classical construction of the suspension flow (special flow under the constant function 1). However, inducing introduces always $1 \in \mathbb{R}$ as an eigenvalue (for the one-parameter group). We hence obtain the following:

Corollary 7.2 (e.g. [13]). *For each $E \subset \mathbb{N}$ which is Koopman realizable (i.e. $E = U_T$) there is an ergodic flow $\mathcal{T}$ such that* $\mathrm{essval}(\mathcal{T}) = E \cup \{1\}$.

However, the ergodic flow in the above corollary cannot be replaced by a weakly mixing flow, and to obtain stronger (and more natural) results, we must leave the case of folklore results and rather try to adapt proofs. In [13], some methods from the $\mathbb{Z}$-action case have been adapted to $\mathbb{R}$-actions and they lead to the extensions of Theorems 6.2 and 6.7 to flows. Moreover, Danilenko and Solomko [16] answered positively Rokhlin's question (6) in the class of $\mathbb{R}$-actions.

It would be interesting to understand some continuity in possible variations of spectral properties of Koopman operators corresponding to non-zero time automorphisms. For example, in [47] it has been proved that the function

$$t \mapsto \mathrm{esssup}(M_{T_t})$$

is of second Baire class answering a question of Thouvenot from the 1990th.

Remark 7.3. See also the article [62] by Ryzhikov about "unusual" behaviour of the maximal spectral multiplicity for powers of a weakly mixing automorphism.

8. Other spectral problems raised by Anatole Katok. Time changes of flows

8.1. **Time changes of horocycle flows.** Let Γ be a lattice in $SL(2, \mathbb{R})$. We consider $X = SL(2, \mathbb{R})/\Gamma$ with the image μ (which is finite) of Haar measure of $SL(2, \mathbb{R})$. The action

$$h_t(x\Gamma) := \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x\Gamma \text{ for } x \in SL(2, \mathbb{R})$$

is the corresponding *horocycle flow*. Horocycle flows have Lebesgue spectrum of infinite multiplicity (Parasyuk, 1953 [53]). Given a function $v : X \to \mathbb{R}^+$, we can consider the corresponding time change flow $(h_t^v)_{t\in\mathbb{R}}$.[18] Smooth time changes of horocycle flows are mixing (Kushnirenko [43], Marcus [48]).

In 2006, Katok and Thouvenot [37] formulated the following:

(8) **Katok-Thouvenot's Conjecture**: All flows obtained by a sufficiently smooth time change of horocycle flows have countable Lebesgue spectrum.

Maximal spectral type Lebesgue part was proved by Forni and Ulcigrai in 2012 [20], and (the absolute continuity of the maximal spectral type) independently by Tiedra de Aldecoa (2012) [66]. However, the problem of multiplicity turned out to require a new approach as the standard representation theory of $SL(2,\mathbb{R})$ (which is used to prove infinite multiplicity for horocycle flows themselves) cannot be used after a time change. Finally, countable Lebesgue spectrum for smooth time changes has been proved by Fayad, Forni and Kanigowski (in 2019) [18].

8.2. **Time changes of linear flow on the torus.** We now consider special flows over irrational rotations, so let $Tx = x + \alpha$ for $x \in \mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}^+$ be piecewise smooth, with the sum of jumps different from zero. The weak mixing property of the special flow T^f was already proved by von Neumann in 1932(!) [50]. In what follows, we refer to flows obtained as above as to von Neumann flows. Recently, they became of particular interest after a discovery in [22] that if α has bounded partial quotients then they enjoy a (variation of) celebrated Ratner's property. This latter property[19] was discovered by Ratner in the 1980th in [56] for horocycle flows. In 2004, A. Katok formulated the following conjecture:

(9) **Katok's Conjecture** Von Neumann's special flows have finite multiplicity.[20]

This conjecture is still open. However, Kanigowski and Solomko in 2016 [31] proved that these flows have no finite rank, so there is no "easy" way to prove possible finite maximal spectral multiplicity by seeing a kind of measurable approximation by finitely many towers.

9. Summarizing comments and questions

This section is in large part (except for the last paragraphs) taken, sometimes *in extenso*, from [35]. Returning to the multiplicity problem, we notice that in all known constructions, appearance of nonsimple finite multiplicity spectrum is due to some symmetries:

- symmetry of double skew products with a group structure in the second extension, first noticed by Oseledets [51], originally systematically explored by Robinson and further developed Goodson-Kwiatkowski-Lemańczyk-Liardet [24],

[18] If (R_t) is a (measurable) flow on $(Z,\mathcal{D},\kappa)$ and $v : Z \to \mathbb{R}^+$, $v \geq \varepsilon_0 > 0$, is integrable, then, for κ-a.e. $z \in Z$ and all $t \in \mathbb{R}$ there is a unique solution $u = u(t,z)$ of $\int_0^u v(R_s z)\, ds = t$. Then we set $R_t^v(z) = R_{u(t,z)}(z)$ and we obtain a new flow (R_t^v) which preserves the measure $\left(\frac{v}{\int v\, d\kappa}\right) d\kappa$. Note that special flows over an automorphism T are time changes of the suspension of T.

[19] This property roughly describes a polynomial way of divergence of nearby points which are not in the same orbit.

- the obvious symmetry of the Cartesian powers, first used in the unpublished version of Katok's notes (cf. [34]) which has circulated since mid-eighties, and brought to the final form by Ageev [3] and Ryzhikov [59] and
- symmetry involving a certain non-Abelian finite extension of a cyclic group discovered by Ageev [4].

These symmetries, especially when they meet together in constructions, lead to a considerable progress in our understanding of the multiplicity function of Koopman operators but the following cases certainly need new ideas:

Problem 6. *Are the simplest unsolved cases* $\{3,4\}$, $\{3,5\}$, $\{3,7\}$ *realizable?*

Let us now come back to classical questions of spectral theory:

Problem 7. *Can the maximal spectral type be absolutely continuous but not Lebesgue?*

Problem 8. *Can the maximal spectral type* $\sigma = \sigma_T$ *for* U_T *be absolutely continuous with respect to its convolution* $\sigma * \sigma$ *but not equivalent to it?*

Recall that Kolmogorov's group property conjecture of the spectrum[21] was disproved in Katok and Stepin's article [36] in 1967, but see also Oseledets [52] and Stepin [65]. Notice that for $\sigma = \sigma_T$ there are three known possibilities:

- σ is equivalent to $\sigma * \sigma$, as for Lebesgue spectrum or for Gaussian systems;
- σ and $\sigma*\sigma$ are mutually singular, as for a generic measure preserving transformation T;
- σ and $\sigma * \sigma$ have a common part but neither is absolutely continuous with respect to the other, as for $T \times T$ for a generic T (as $\sigma_{T\times T} = \sigma_T + \sigma_T^{*2} + \sigma_T^{*3} + \sigma_T^{*4}$, and typically all convolutions of the maximal spectral type are mutually singular [65]).

Note that Problem 8 might be related to Problem 7 as for each absolutely continuous measure its certain self-convolution is equivalent to Lebesgue measure.

Returning to Proposition 5.1 (i), it is natural to ask the following:

Problem 9. *Is it true that all spectral types in a spectral sequence of a measure preserving transformation with continuous spectrum are dense?*

Frączek (unpublished) answered positively this question for some group extensions of rotations.

The following problem is closely related to the famous Banach problem on the existence of simple Lebesgue spectrum automorphism.

Problem 10. *Does there exist an ergodic measure preserving transformation whose maximal spectral type is absolutely continuous but the spectrum is not Lebesgue with countable multiplicity?*

The difference from Problem 7 is that it is conceivable that the maximal spectral type is Lebesgue while not all others are.

[21] Kolmogorov's group property for $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ states that $\sigma_T \ll \sigma_T * \sigma_T$.

An account of research around Banach problem was published in 2008 [45]: the main achievement consisted in exhibiting constructions with all possible **even** multiplicities of Lebesgue component in the spectrum. Not much has changed since then. Recall also that Guenais [25] proved that there is a generalized Morse sequence for which the associated dynamical system (subshift) has a Lebesgue component[22] if and only if there exists a sequence of L^1-ultraflat[23] trigonometric polynomials $P_{n_k}(z)=\sum_{j=-n_k}^{n_k} a_j^{(k)} z^j$ $(|z|=1)$ with $a_j^{(n)}=\pm 1$. Recently, on arXiv there appeared the paper [6] in which the authors prove Littlewood conjecture on the existence of flat (in the uniform sense) trigonometric polynomials: there are $c, C>0$ such that for each $n\geq 1$ we can find a polynomial $Q_n(z)=\sum_{j=-n}^{n} b_j^{(n)} z^j$ with $b_j^{(n)}=\pm 1$, such that $c\sqrt{n}\leq |Q_n(z)|\leq C\sqrt{n}$ for each $|z|=1$. This remarkable progress does not seem however to imply the existence of a sequence of L^1-ultraflat trigonometric polynomials with coefficients ± 1.

While there was no a substantial progress in Banach problem in the framework of Koopman operators, there were advances in a similar problem for Koopman flows. Prikhodko in [54] published a construction of a rank one flow having Lebesgue spectrum. As rank one implies simplicity of the spectrum, the result yields solution of Banach problem for Koopman flows. To do this, he proved the following version of Littlewood conjecture: *For all* $[a,b]\subset\mathbb{R}^+$ *and* $n\geq 1$, *one can find polynomials* $P_n(t)=\sum_{j=0}^{n-1} e^{2\pi i w_j^{(n)} t}$ *for some real* $w_j^{(n)}$, *so that* $\|P_n\|_{L^1([a,b])}/\sqrt{n}\to 1$. However, although the article has been published in 2013, some details were rather cryptic and no further developments of methods/results/ideas from [54] have appeared so far. [24] Of course, as we have noticed in Section 7.2, a solution of Banach problem for flows does not imply it for automorphisms (as all non-zero time automorphisms have countable Lebesgue spectrum). **It seems to be an important task for the ergodic community to convincingly explain the status of Banach problem.** [25]

Lebesgue spectrum (plus a discrete component) is characteristic for algebraic systems (affine transformations, nil-systems, horocycle flows, or more generally, unipotent actions), hence continuous singular part is absent. Whether one can expect some restrictions on spectral types by looking at smooth systems in dimension 2, say, flows on surfaces or billiards, is definitely less clear. Typically, these dynamical systems have special representations as special flows over irrational rotations or (more generally) interval exchange transformations. Roof functions have singularities which correspond to singularities of the original flows on surfaces. For the simplest case - when the roof function is of bounded variation,

[22] All subshifts given by generalized Morse sequences have simple spectrum (and they have a discrete component in the spectrum). Therefore, if ultraflat polynomials with coefficients ± 1 do exist, also there are dynamical systems with Lebesgue component of multiplicity 1.

[23] That is, $\|P_{n_k}\|_{L^1}/\sqrt{n_k}\to 1$ when $k\to\infty$.

[24] This is to be compared with a solution of Rokhlin's problem for $n=2$, after which Anosov's book [5] appeared and explained all details.

[25] The status of [1] claiming solution of the original Banach problem, i.e. of the existence of an ergodic **infinite** measure-preserving automorphism whose Koopman operator has simple Lebesgue spectrum, posted firstly on arXiv in 2015 is even less clear.

mixing is excluded as shown by Kochergin [39] and Katok [33]. In fact, when the base is an irrational rotation, the flows are **spectrally disjoint** with all mixing flows as shown by Frączek-Lemańczyk [21]. The absence of mixing persists for flows with symmetric logarithmic singularities over a typical IET as proved by Ulcigrai [67].[26]. On the other hand, mixing is "typical" in the asymmetric case, e.g. Kochergin [41] and Khanin-Sinai [38]. Also, mixing appears in case of so called power singularities [40]. While it was rather expected (including the author of the article) that the spectrum in dimension 2 is singular, Fayad-Forni-Kanigowski [18] showed that for some class of special flows with power singularities (and over irrational rotations) the spectrum is countable Lebesgue. One can of course speculate that flows with such singularities are simply either horocycle flows or their smooth time changes in "different coordinates" [27], but it follows from [29] and [32] that special flows with power singularities are not isomorphic to horocycles flows. [28] Moreover, in the recent paper of Kanigowski-Lemańczyk-Ulcigrai [30] it was shown that (for some class of) flows under a function with logarithmic singularities are disjoint with horocycle flows. **Full understanding of spectral theory of smooth flows on surfaces seems to be one of main challenges in ergodic theory in forthcoming years.**

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