On invariant measures for $\mathscr{B}$-free systems

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Abstract

We show that the $\mathscr{B}$-free subshift $(S, X_{\mathscr{B}})$ associated to a $\mathscr{B}$-free system is intrinsically ergodic, i.e. it has exactly one measure of maximal entropy. Moreover, we study invariant measures for such systems. It is proved that each ergodic invariant measure is of joining type, determined by a joining of the Mirsky measure of a $\mathscr{B}'$-free subshift contained in $(S, X_{\mathscr{B}})$ and an ergodic invariant measure of the full shift on $\{0,1\}^{\mathbb{Z}}$. Moreover, each ergodic joining type measure yields a measure-theoretic dynamical system with infinite rational part of the spectrum corresponding to the above Mirsky measure. Finally, we show that, in general, hereditary systems may not be intrinsically ergodic.

Contents

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Introduction

Assume that $\mathscr{B} = \{b_1, b_2, \dots\} \subset \{2, 3, \dots\}$ is such that

$$(b_i, b_j) = 1 \text{ whenever } i \neq j \text{ and } \sum_{i\geq 1} 1/b_i < +\infty. \tag{0.1}$$

For example, we can take $\mathscr{B} = \{p_i^2 : i \geq 1\}$, where $p_i \in \mathscr{P}$ stands for the ith prime number. To $\mathscr{B}$ we associate a two-sided sequence $\eta \in \{0,1\}^{\mathbb{Z}}$ by setting

$$\eta(n) := \begin{cases} 1 & \text{if } b_i \nmid n \text{ for all } i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then let

$$X_\eta := \{y \in \{0,1\}^{\mathbb{Z}} : \text{each block occurring on } y \text{ occurs on } \eta\}.$$

We will also write $X_{\mathscr{B}}$ instead of X_η. Let S stand for the shift transformation on $\{0,1\}^{\mathbb{Z}}$ and notice that X_η is closed and S-invariant (shortly, X_η is a subshift). We will call X_η the *$\mathscr{B}$-free subshift.* When $b_i = p_i^2$, $i \geq 1$, the corresponding subshift is called the *square-free.*

When $\mathscr{B}$ satisfies (0.1), it follows by [1] that the topological entropy $h_{top}(S, X_\eta)$ of the subshift (S, X_η) is positive. A natural question arises whether there is only one invariant measure ν whose entropy $h_\nu(S, X_\eta)$ attains the value of topological entropy, i.e. whether (S, X_η) is intrinsically ergodic [17]. In Section 1, we will give a simple proof of the following result which for the square-free subshift has been obtained by Peckner in [12].

Theorem 0.0.1. *For each $\mathscr{B} \subset \mathbb{N}$ satisfying* (0.1)*, the corresponding $\mathscr{B}$-free subshift (S, X_η) is intrinsically ergodic.*

The $\mathscr{B}$-free subshifts turn out to be hereditary systems [8], i.e. they have the following property:

$$\text{whenever } x \in X_\eta,\ y \in \{0,1\}^{\mathbb{Z}} \text{ and } y \leq x \text{ (coordinatewise) then } y \in X_\eta. \tag{0.2}$$

In Section 3.1, we show how to adapt the method used to prove Theorem 0.0.1 to obtain that other natural hereditary systems are intrinsically ergodic (e.g. Sturmian hereditary systems).

In Section 2, we study the set $\mathcal{P}(S, X_\eta)$ of invariant measures for (S, X_η) which is completely determined by the subset $\mathcal{P}^e(S, X_\eta)$ of ergodic measures. Among them, the most natural non-trivial member of $\mathcal{P}^e(S, X_\eta)$ is the so called Mirsky measure $\nu_{\mathscr{B}}$ (see Section 1.1) which yields the (ergodic) dynamical system with purely discrete spectrum whose group of eigenvalues consists of all roots of unity of degree $b_1 \cdot \ldots \cdot b_k$, $k \geq 1$, see [1], [4], [13]. A basic observation (see Proposition 2.2.2) is that whenever $\rho \neq \delta_{(\ldots,0,0,\ldots)}$ is an ergodic measure for (S, X_η) then the corresponding measure-theoretic system has infinite rational discrete spectrum generated by b'_k-roots of unity for some

$$1 < b'_k | b_k \text{ for each } k \geq 1. \tag{0.3}$$

As (S, X_η) may contain, as a subsystem, another $\mathscr{B}'$-free subshift, the measure-theoretic dynamical system $(S, X_\eta, \nu_{\mathscr{B}'})$ may have essentially smaller spectrum than that determined by $\nu_{\mathscr{B}}$. A natural question arises whether all sequences of (b'_k) satisfying (0.3) are "realizable". We provide a complete answer in Section 2.2.4: the spectrum of the dynamical system of each (non-trivial) ergodic invariant measure contains the group of $b'_1 \cdot \ldots \cdot b'_k$-roots of unity, $k \geq 1$, where in addition to (0.3) we have

$$\sum_{k \geq 1} 1/b'_k < +\infty. \tag{0.4}$$

Moreover, the Mirsky measure $\nu_{\mathscr{B}'}$ determined by $\mathscr{B}' = \{b'_k : k \geq 1\}$ yields exactly such spectrum. As a corollary, we obtain that, for example, the square-free subshift has no ergodic invariant measure for which the spectrum of the associated dynamical systems consists of all $p_1 \cdot \ldots \cdot p_k$-roots of unity, $k \geq 1$.

In general, the set $\mathcal{P}^e(S, X_\eta)$ is quite rich. Te see more members of $\mathcal{P}^e(S, X_\eta)$ other than simply Mirsky measures of free subsystems of (S, X_η), we consider *joining type measures* obtained in the following way. Let $M\colon X_\eta \times \{0,1\}^{\mathbb{Z}} \to X_\eta$ be given by $M(x, u)(n) := x(n) \cdot u(n)$ for $n \in \mathbb{Z}$ (the values of M are in X_η because of (0.2)). Let λ be an ergodic joining of the Mirsky measure $\nu_{\mathscr{B}'}$ of a $\mathscr{B}'$-free subshift contained in (S, X_η) and an invariant measure κ for the full shift $(S, \{0,1\}^{\mathbb{Z}})$[1]. Then the image $M_*(\lambda)$ of λ via M belongs to $\mathcal{P}^e(S, X_\eta)$. Such a measure is called a joining type measure[2]. One of the main results of the paper states that each member of $\mathcal{P}^e(S, X_\eta)$ is a joining type measure:

Theorem 0.0.2. *For any $\nu \in \mathcal{P}^e(S, X_\eta)$ there exist a $\mathscr{B}'$-free system and $\widetilde{\rho} \in \mathcal{P}^e(S \times S, X_\eta \times \{0,1\}^{\mathbb{Z}})$ such that $X_{\eta'} \subset X_\eta$, $\widetilde{\rho}|_{X_\eta} = \nu_{\mathscr{B}'}$ and $M_*(\widetilde{\rho}) = \nu$.*

We also take a closer look at the dynamical systems given by ergodic invariant measures. We prove that the measure with maximal entropy yields a system which, up to isomorphism, is the Cartesian product of the discrete spectrum automorphism given by $\nu_{\mathscr{B}}$ and the Bernoulli system with the entropy $\log 2 \cdot \Pi_{i\geq 1}(1 - 1/b_i)$. Moreover, we show that whenever $\kappa \in \mathcal{P}(S, \{0,1\}^{\mathbb{Z}})$ yields a system doubly disjoint[3] [6] from the system given by $\nu_{\mathscr{B}}$, then the map M is an isomorphism of corresponding measure-theoretic dynamical systems, i.e. given by $\nu_{\mathscr{B}} \otimes \kappa$ and $\nu_{\mathscr{B}} * \kappa$.

Finally, in the last section of the paper, we answer negatively a question raised in [9] whether each hereditary system is intrinsically ergodic.

[1]This means that λ is an $S \times S$-invariant ergodic measure on $X_\eta \times \{0,1\}^{\mathbb{Z}}$ such that $\lambda|_{X_\eta} = \nu_{\mathscr{B}'}$ and $\lambda|_{\{0,1\}^{\mathbb{Z}}} = \kappa$.

[2] When $\lambda = \nu_{\mathscr{B}'} \otimes \kappa$ then $M_*(\lambda)$ is called to be of product type; it will be denoted $\nu_{\mathscr{B}'} * \kappa$ as it can be viewed as convolution of measures defined on monoids with the natural coordinate **multiplication**.

[3]This forces κ to have zero entropy.

1 Intrinsic ergodicity of $\mathscr{B}$-free systems

1.1 Basic properties

We will recall here some known facts about the dynamical systems associated to $\mathscr{B}$-free numbers. Set $\Omega := \Pi_{i\geq 1}\mathbb{Z}/b_i\mathbb{Z}$. With the product topology and the coordinatewise addition Ω becomes a compact metrizable Abelian group. Let $\mathbb{P}$ stand for the (normalized) Haar measure of Ω (which is the product of uniform measures on $\mathbb{Z}/b_i\mathbb{Z}$). Denote by $T\colon \Omega\to\Omega$ the homeomorphism given by

$$T\omega = \omega + (1,1,\dots) = (\omega(1)+1, \omega(2)+1,\dots),$$

where $\omega = (\omega(1),\omega(2),\dots)$. The dynamical system (T,Ω) is uniquely ergodic and the measure-theoretic system $(T,\Omega,\mathcal{B}(\Omega),\mathbb{P})$ has discrete spectrum (with the group of eigenvalues equal to the $b_1\cdot\ldots\cdot b_k$-roots of unity, $k\geq 1$). In particular,

$$(T,\Omega,\mathcal{B}(\Omega),\mathbb{P}) \text{ has zero entropy.} \tag{1.1}$$

Let $\varphi\colon \Omega\to\{0,1\}^{\mathbb{Z}}$ be defined as

$$\varphi(\omega)(n) = \begin{cases} 1 & \text{if } (\forall i\geq 1) \quad \omega(i)+n\neq 0 \bmod b_i, \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

and let $\eta := \varphi(0,0,\dots)$. It is easy to check that η corresponds to the characteristic function of the set $\{m\in\mathbb{Z} : (\forall i\geq 1) \quad b_i\nmid m\}$ of $\mathscr{B}$-free numbers.

Following [13], call a subset $A\subset\mathbb{Z}$ *admissible* (more precisely, *$\mathscr{B}$-admissible*) if $|A \bmod b_i| < b_i$ for each $i\geq 1$. A point $y\in\{0,1\}^{\mathbb{Z}}$ is said to be *admissible* if its support

$$\operatorname{supp}(y) := \{m\in\mathbb{Z} : y(m)=1\}$$

is admissible.

Lemma 1.1.1 ([1], [13])**.** *We have $X_\eta = X_{\mathscr{B}} = \{y\in\{0,1\}^{\mathbb{Z}} : y \text{ is admissible}\}$. Moreover, $\varphi(\Omega)\subset X_\eta$.*

It follows immediately that the subshift X_η is hereditary[4] (see [9] for basic properties of hereditary systems). This means that if $x\in X_\eta$ and $y\in\{0,1\}^{\mathbb{Z}}$ with $\operatorname{supp}(y)\subset\operatorname{supp}(x)$ then $y\in X_\eta$. Whenever $\operatorname{supp}(y)\subset\operatorname{supp}(x)$, we will write $y\leq x$.

Lemma 1.1.2 ([1], cf. [12], cf. [13])**.** *We have $h_{top}(S,X_\eta) = \log 2\cdot\Pi_{i=1}^{\infty}\left(1-\frac{1}{b_i}\right)$.*

Observe that the map φ is equivariant, i.e. $\varphi\circ T = S\circ\varphi$. Moreover, φ is Borel but not continuous. Let $\nu_{\mathscr{B}} := \varphi_*(\mathbb{P})$ be the image of $\mathbb{P}$ via φ. Then $\nu_{\mathscr{B}}$ is S-invariant. It is called the Mirsky measure of (S,X_η), cf. [11]. Let $A\subset\mathbb{Z}$ be non-empty and finite, and set

$$C_A^j := \{x\in X_\eta : (\forall n\in A) \;\; x(n)=j\},\; j=0,1. \tag{1.3}$$

As shown in [1],

$$\nu_{\mathscr{B}}(C_A^1) = \Pi_{i\geq 1}\left(1-\frac{|A \bmod b_i|}{b_i}\right).$$

[4]This observation was communicated to us by T. Downarowicz.

Using Lemma 1.1.2, it follows that

$$\nu_{\mathscr{B}}(C_0^1) = \Pi_{i\geq 1}\left(1-\frac{1}{b_i}\right) = h_{top}(S, X_\eta)/\log 2. \tag{1.4}$$

Set

$$Y := \{x \in X_\eta : (\forall i \geq 1)\ \ |\mathrm{supp}(x) \bmod b_i| = b_i - 1\}. \tag{1.5}$$

Notice that Y is a Borel set and $SY = Y$. Finally, we have the following:

Lemma 1.1.3 (cf. [12])**.** *Any measure ν with maximal entropy is concentrated on Y.*[5]

1.2 A few observations

Our aim is to show that we can define an "inverse" of φ on Y. A difficulty is that the image $\varphi(\Omega)$ of the map $\varphi\colon \Omega \to X_\eta$ is not "quite" included in Y and the map itself is not 1-1. Indeed, for example the all 0 sequence which does not belong to Y can be arranged to come about by assigning to each $n \in \mathbb{Z}$ some index k_n in a 1-1 manner and then choosing $\omega(k_n) \in \mathbb{Z}/b_{k_n}\mathbb{Z}$ so that $\omega(k_n) + n = 0 \bmod b_{k_n}$ (hence, the fiber $\varphi^{-1}((\ldots, 0, 0, \ldots))$ is uncountable). We now show how to bypass this difficulty.

Following [1], given $k \geq 1$ and $z \in \mathbb{Z}/b_k\mathbb{Z}$, we set

$$\Omega_{k,z} := \{\omega \in \Omega : \omega(k) = z\},$$
$$E_{k,z} := \{\omega \in \Omega : (\forall s \geq 1)\ \varphi(\omega)(-z + sb_k) = 0\}.$$

Then $\Omega_{k,z} \subset E_{k,z}$ and

$$T\Omega_{k,z} = \Omega_{k,z+1},\ TE_{k,z} \subset E_{k,z+1}. \tag{1.6}$$

Moreover,

$$\omega \notin E_{k,z} \text{ if and only if } -z + sb_k \in \mathrm{supp}(\varphi(\omega)) \text{ for some } s \geq 1. \tag{1.7}$$

Let

$$\Omega_0' := \bigcap_{k\geq 1}\ \bigcap_{z\in\mathbb{Z}/b_k\mathbb{Z}} \left(E_{k,z}^c \cup \Omega_{k,z}\right) \text{ and } \Omega_0 := \bigcap_{k\in\mathbb{Z}} T^k\Omega_0'.$$

Clearly, Ω_0 is a Borel T-invariant subset of Ω. We have the following result.

Lemma 1.2.1 (Proposition 3.2 in [1])**.** *We have $\mathbb{P}(\Omega_0) = 1$ and $\varphi|_{\Omega_0}$ is 1-1.*

Define a Borel map $\theta\colon Y \to \Omega$ (cf. [12]) by setting

$$\theta(y) = \omega \text{ if } -\omega(i) \notin \mathrm{supp}(y) \bmod b_i \text{ for all } i \geq 1. \tag{1.8}$$

Lemma 1.2.2. *We have:*

(i) θ is equivariant, i.e. $T \circ \theta = \theta \circ S$.

(ii) For each $y \in Y$ we have $y \leq \varphi(\theta(y))$.

[5] In [12], the proof is given for $\mathscr{B} = \{p_i^2 : i \geq 1\}$. In the general case, the proof goes along the same lines.

(iii) $\varphi(\Omega_0) \subset Y$ *(in particular,* $\theta \circ \varphi|_{\Omega_0} = id_{\Omega_0}$*).*

Proof. (i) Note that $\mathrm{supp}(Sx) = \mathrm{supp}(x) - 1$. Hence $-\omega(i) \notin \mathrm{supp}(x) \bmod b_i$ if and only if $-(\omega(i)+1) \notin \mathrm{supp}(Sx) \bmod b_i$.

(ii) Suppose that for some $n \in \mathbb{Z}$ we have $y(n) = 1$. Then, by (1.8), $\theta(y)(k) \neq -n \bmod b_k$ for all $k \geq 1$. In other words, $\theta(y)(k) + n \neq 0 \bmod b_k$ for all $k \geq 1$, i.e. $\varphi(\theta(y))(n) = 1$.

(iii) Fix $k \geq 1$ and $\omega \in \Omega_0$. We need to prove that $|\mathrm{supp}(\varphi(\omega)) \bmod b_k| = b_k - 1$. We have $\omega \in \bigcap_{z \in \mathbb{Z}/b_k\mathbb{Z}} \left(E^c_{k,z} \cup \Omega_{k,z}\right)$. Clearly, $\omega \in \Omega_{k,\omega(k)}$ (in particular, $-\omega(k) \notin \mathrm{supp}(\varphi(\omega))$). Moreover, since $E^c_{k,z} \cap \Omega_{k,z} = \emptyset$, we also have

$$\omega \in \bigcap_{z \in \mathbb{Z}/b_k\mathbb{Z} \setminus \{-\omega(k)\}} E^c_{k,z}. \tag{1.9}$$

It follows from (1.7) that, given $z \in \mathbb{Z}/b_k\mathbb{Z} \setminus \{-\omega(k)\}$, for some $s \geq 1$, $-z + sb_k \in \mathrm{supp}(\varphi(\omega))$, whence $-z \in \mathrm{supp}(\varphi(\omega)) \bmod b_k$ which completes the proof. □

Remark 1.2.3. When $\omega \in \Omega_0$ then, of course, $T^{b_k}\omega \in \Omega_0$. We also have $(T^{b_k}\omega)(k) = \omega(k)$. It follows by (1.9) that

$$\omega, T^{b_k}\omega \in \bigcap_{z \in \mathbb{Z}/b_k\mathbb{Z} \setminus \{-\omega(k)\}} E^c_{k,z}.$$

In view of (1.7) (applied to $T^{b_k}\omega$), for each $z \neq -\omega(k)$ there exists $s \geq 1$ such that

$$\varphi(\omega)(-z + (s+1)b_k) = S^{b_k}(\varphi(\omega))(-z + sb_k) = \varphi(T^{b_k}\omega)(-z + sb_k) = 1.$$

By considering $T^{mb_k}\omega$, $m \geq 1$, we conclude: for each $z \in \mathbb{Z}/b_k\mathbb{Z} \setminus \{-\omega(k)\}$

$$\omega \in \Omega_0 \;\Rightarrow\; \left|\{s \geq 1 : -z + sb_k \in \mathrm{supp}\,\varphi(\omega)\}\right| = \infty. \tag{1.10}$$

It follows that

$$\omega \in \Omega_0 \;\Rightarrow\; (\mathrm{supp}(\varphi(\omega)) \setminus E) \bmod b_k = \mathbb{Z}/b_k\mathbb{Z} \setminus \{-\omega(k)\} \tag{1.11}$$

for each finite set $E \subset \mathbb{Z}$ and each $k \geq 1$. In particular, for $\omega \in \Omega_0$,

$$\text{if } y \leq \varphi(\omega) \text{ and } \left|\{r \in \mathbb{Z} : y(r) \neq \varphi(\omega)(r)\}\right| < \infty \text{ then } y \in Y. \tag{1.12}$$

Finally, notice that

$$\text{if } y \in Y \text{ and } y \leq \varphi(\omega) \text{ then } \theta(y) = \omega. \tag{1.13}$$

Remark 1.2.4. We can repeat the proof of (ii) in Lemma 1.2.2 to obtain the following:

$$\text{for each } x \in X_\eta \text{ there is } \omega \in \Omega \text{ such that } x \leq \varphi(\omega). \tag{1.14}$$

Indeed, since $x \in X_\eta$ is admissible, for each $k \geq 1$, choose

$$a_k \in (\mathbb{Z}/b_k\mathbb{Z}) \setminus (\mathrm{supp}(x) \bmod b_k)$$

and set $\omega := (-a_1, -a_2, \ldots)$. Then $x \leq \varphi(\omega)$. It follows that

$$X_\eta = \{x \in \{0,1\}^{\mathbb{Z}} : (\exists \omega \in \Omega)\;\; x \leq \varphi(\omega)\}. \tag{1.15}$$

Less formally, we can phrase this by saying that X_η is the hereditary system generated by $\varphi(\Omega)$, that is, generated by the symbolic "model" $(S, \varphi(\Omega))$ of the odometer (T, Ω).

Remark 1.2.5. The odometer (T, Ω) is entirely determined by its group of eigenvalues: the group of $b_1 \cdot \ldots \cdot b_k$-roots of unity, $k \geq 1$. Note that we can replace $\{b_k : k \geq 1\}$ with $\mathscr{B}' = \{b'_j : j \geq 1\}$ so that the corresponding odometers (T, Ω) and (T', Ω') are topologically conjugate and (0.1) holds for $\mathscr{B}'$. For example, $b'_1 = b_1 \cdot \ldots b_{i_1}$, $b'_2 = b_{i_1+1} \cdot \ldots \cdot b_{i_2}$, ... In this way we obtain a new hereditary system $(S, X_{\eta'})$, cf. (1.15), which in general will not be conjugated to (S, X_η) because $h_{top}(S, X_{\eta'})$ will be different from $h_{top}(S, X_\eta)$, see Lemma 1.1.2. In this way, we can obtain a hereditary system $(S, X_{\eta'})$ generated by a symbolic model $(S, \varphi'(\Omega'))$ of the odometer (T, Ω) which has the entropy arbitrarily close to $\log 2$.

Notice also that

(1.16) whenever $\omega \neq \omega''$ are two points from Ω_0 then $\varphi(\omega)$ and $\varphi(\omega'')$ are not $\leq$-comparable.

Indeed, there exists $k \geq 1$ such that $\omega(k) \neq \omega''(k)$. Moreover,

$$\omega, \omega'' \in \bigcap_{z \in \mathbb{Z}/b_k\mathbb{Z}} (E^c_{k,z} \cup \Omega_{k,z}).$$

Hence, $\omega \in \Omega_{k,\omega(k)}$, $\omega'' \in E^c_{k,\omega(k)}$. It follows that $\varphi(\omega)(-\omega(k) + sb_k) = 0$ for each $s \geq 1$, while there exists $s_0 \geq 1$ such that $\varphi(\omega'')(-\omega(k) + s_0 b_k) = 1$, see (1.7). Hence, if $\varphi(\omega), \varphi(\omega'')$ are $\leq$-comparable, it must be $\varphi(\omega) \leq \varphi(\omega'')$, and, by symmetry, we obtain equality. Finally, $\omega = \omega''$ since φ is 1-1 on Ω_0.

Fix a measure ν on Y with maximal entropy: $h_\nu(S, X_\eta) = \log 2 \cdot \Pi_{i \geq 1}\left(1 - \frac{1}{b_i}\right)$ (cf. Lemma 1.1.3).

Lemma 1.2.6. *We have* $\theta_*(\nu) = \mathbb{P}$.

Proof. This follows directly from Lemma 1.2.2 (i) and the fact that (T, Ω) is uniquely ergodic. □

Let $Y_0 := \theta^{-1}(\Omega_0)$. Then, by Lemma 1.2.2 (i), Y_0 is an S-invariant Borel subset of Y. Now, by Lemma 1.2.1 and Lemma 1.2.6, we have

$$\nu(Y_0) = \theta_*(\nu)(\Omega_0) = \mathbb{P}(\Omega_0) = 1. \tag{1.17}$$

Let $Q = (Q_0, Q_1)$ be the partition of Y according to the value at the zero coordinate, i.e. $Q_j = C_0^j \cap Y$, $j = 0, 1$. This is a generating partition. Set

$$Q^- := \bigvee_{-\infty}^{-1} S^j Q \text{ and } \mathcal{A} := \theta^{-1}(\mathcal{B}(\Omega)).$$

Remark 1.2.7. Since Q is a generating partition, the σ-algebra $\bigcap_{m \geq 0} S^{-m} Q^-$ is the Pinsker σ-algebra of $(S, Y, \mathcal{B}(Y), \nu)$ (see e.g. [7], Thm. 18.9).

Lemma 1.2.8. *We have* $\mathcal{A} \subset \bigcap_{m \geq 0} S^{-m} Q^-$ *modulo* ν.

Proof. In view of Remark 1.2.7, the result follows from (1.1) and from Lemma 1.2.6. □

It follows that a.e. atom of the partition corresponding to the Pinsker σ-algebra of $(S, Y, \mathcal{B}(Y), \nu)$ is contained in an atom of the partition of Y corresponding to $\mathcal{A}$. We also have

$$\mathcal{A} \subset S^{-m}Q^{-} \text{ for } m \geq 0, 1$$

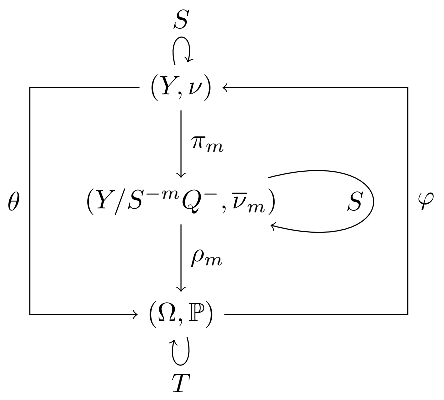
so, in other words, after removing a set of ν-measure zero from Y, for the remaining points in Y we have the following: for each $m \geq 1$

$$[y_1, y_2 \in Y, \ (\forall j \leq -m) \ \ y_1(j) = y_2(j)] \ \Rightarrow \ \theta(y_1) = \theta(y_2). \tag{1.18}$$

Fix $m \geq 0$. Let π_m be the natural quotient map from Y to $Y/S^{-m}Q^{-}$. Let $\overline{\nu}_m := (\pi_m)_*(\nu)$. Notice that $S^{-1}(S^{-m})Q^{-} \subset S^{-m}Q^{-}$, so S acts naturally on the quotient space $Y/S^{-m}Q^{-}$ as an endomorphism preserving $\overline{\nu}_m$. Moreover, the map π_m is equivariant, i.e.

$$\pi_m \circ S = S \circ \pi_m.$$

Using (1.18), we may also define the quotient map $\rho_m \colon Y/S^{-m}Q^{-} \to \Omega$ which is equivariant as well. Then $(\rho_m)_*(\overline{\nu}_m) = \mathbb{P}$. In other words, we have the following commuting diagram (in which θ, π_m and ρ_m are measure-preserving while $\varphi \colon \Omega \to Y$ is defined $\mathbb{P}$-a.e. and is not measure-preserving):



Remark 1.2.9. Since the maps $\pi_m \colon Y \to Y/S^{-m}Q^{-}$, $\rho_m \colon Y/S^{-m}Q^{-} \to \Omega$ and $\varphi \colon \Omega \to Y$ are equivariant, it follows immediately that $S^k \circ \varphi \circ \rho_m = \varphi \circ \rho_m \circ S^k$. Therefore, if $\overline{z} \in Y/S^{-m}Q^{-}$ then

$$\varphi \circ \rho_m(\overline{z})(m + k) = \varphi(\rho_m(S^k \overline{z}))(m) \ \text{ for every } \ k \in \mathbb{Z}. \tag{1.19}$$

We will identify points in $Y/S^{-m}Q^{-}$ with their $Q(-\infty, -m-1]$-names: for $y \in Y$, let $\overline{y}$ be the atom of the partition associated to $S^{-m}Q^{-}$ which contains y, i.e.

$$\overline{y} = \ldots i_{-1} i_0 \iff y \in S^{-m-1}Q_{i_0} \cap S^{-m-2}Q_{i_{-1}} \cap \ldots.$$

The following observation is well-known.

Lemma 1.2.10. *For each $m \geq 0$, for each $r = 0, 1, \ldots, 2m$ and $\overline{\nu}_m$-a.e. $\overline{y} \in Y/S^{-m}Q^{-}$, we have*

$$\begin{aligned} &\overline{\nu}_m \big(S^{m-r} Q_{i_{m-r}} | \\ &\quad S^{m-r-1} Q_{i_{m-r-1}} \cap \ldots \cap S^{-m+1} Q_{i_{-m+1}} \cap S^{-m} Q_{i_m} \cap S^{-m} Q^{-} \big) (\overline{y}) \\ &\qquad = \overline{\nu}_m \big(S^{-m} Q_{i_{m-r}} | S^{-m} Q^{-} \big) (\overline{y} i_{-m} \ldots i_{m-r-1}) \end{aligned} \tag{1.20}$$

for each choice of $i_k \in \{0, 1\}$, $-m \leq k \leq m$.

Proof. Assume that $\overline{y} = \dots j_{-m-2} j_{-m-1}$. For $\overline{\nu}$-a.e. such a $\overline{y}$, by stationarity, we have

$$\begin{aligned}
&\overline{\nu}_m \left(S^{m-r} Q_{i_{m-r}} | S^{m-r-1} Q_{i_{m-r-1}} \cap \dots \cap S^{-m+1} Q_{i_{-m+1}} \cap S^{-m} Q_{i_m} \right. \\
&\qquad \left. \cap S^{-m} Q^- \right) (\overline{y}) \\
&= \lim_{t\to\infty} \nu \left(S^{m-r} Q_{i_{m-r}} | S^{m-r-1} Q_{i_{m-r-1}} \cap \dots \cap S^{-m+1} Q_{i_{-m+1}} \cap S^{-m} Q_{i_m} \right. \\
&\qquad \left. \cap S^{-m} \left(S^{-1} Q_{j_{-m-1}} \cap \dots \cap S^{-t} Q_{j_{-m-t}} \right) \right) \\
&= \lim_{t\to\infty} \nu \left(S^{-m} Q_{i_{m-r}} | S^{-m} (S^{-1} Q_{i_{m-r-1}} \cap \dots \cap S^{-2m+r} Q_{i_m} \right. \\
&\qquad \left. \cap S^{-2m+r-1} Q_{j_{-m-1}} \cap \dots \cap S^{-2m+r-t} Q_{j_{-m-t}}) \right) \\
&= \overline{\nu}_m \left(S^{-m} Q_{i_{m-r}} | S^{-m} Q^- \right) (\dots j_{-m-2} j_{-m-1} i_{-m} \dots i_{m-r-1}) \\
&= \overline{\nu}_m \left(S^{-m} Q_{i_{m-r}} | S^{-m} Q^- \right) (\overline{y} i_{-m} \dots i_{m-r-1})
\end{aligned}$$

which completes the proof. □

1.3 Proof of Theorem 0.0.1

1.3.1 Outline of the proof

Let ν be a measure of maximal entropy for (S, X_η). In order to prove Theorem 0.0.1, we will show that the conditional measures ν_ω in the disintegration (cf. Lemma 1.2.6)

$$\nu = \int_\Omega \nu_\omega \, d\mathbb{P}(\omega) \tag{1.21}$$

of ν over $\mathbb{P}$ given by the mapping $\theta\colon Y \to \Omega$ are unique $\mathbb{P}$-a.e. This will yield intrinsic ergodicity for (S, X_η). In fact, we will show that $\nu = \mu$, where the measure μ is defined in the following way. Recall first that for $\omega \in \Omega_0$, we have $\varphi(\omega) \in Y$. Moreover, in view of Lemma 1.2.2 (ii), $\varphi(\omega)$ is the largest element in $\theta^{-1}(\omega)$. In particular, for each $y \in \theta^{-1}(\omega)$, $y[-k,k] \leq \varphi(\omega)[-k,k]$. Therefore, there are at most 2^m blocks $u = (u_{-k}, \dots, u_k)$ on $\theta^{-1}(\omega)$, $m = \sum_{j=-k}^{k} \varphi(\omega)(j)$, obtained by replacing some of the 1s in $\varphi(\omega)[-k,k]$ by 0s. In fact, in view of (1.12) and (1.13) all such blocks do occur on $\theta^{-1}(\omega)$. For $u = (u_{-k}, \dots, u_k) \in \{0,1\}^{2k+1}$ denote by $[u]$ the corresponding cylinder set. If u is such that $u \leq \varphi(\omega)[-k,k]$, we set $\mu_\omega([u]) := 2^{-m}$, where m has been defined above. Thus, the measure μ_ω is equidistributed on all $(2k+1)$-blocks which occur on $\theta^{-1}(\omega)$ for $\omega \in \Omega_0$. Finally, we set

$$\mu = \int_\Omega \mu_\omega \, d\mathbb{P}(\omega).$$

In a less formal way, a random point distributed according to μ is obtained by first choosing an $\omega \in \Omega$ according to $\mathbb{P}$ and then for each $n \in \mathbb{Z}$, where $\varphi(\omega)(n) = 1$ changing the 1 to 0 with probability 1/2, independently for all such n.

We will show that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mu_\omega = \nu_\omega$. In order to do it, we will show that for A belonging to a countable dense family of subsets in $\mathcal{B}$, we have

$$\nu_\omega(A) = \mu_\omega(A) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{1.22}$$

Recall that

$$\nu_\omega(A) = \mathbb{E}^\nu(A|\Omega)(\omega). \tag{1.23a}$$

To get the equality (1.22), we will step by step make use of the equality

$$\mathbb{E}^\nu(A|\Omega)(\omega) = \mathbb{E}^\nu\left(\mathbb{E}^\nu(A|Y/S^{-m}Q^-)(\overline{y}_m)\big|\Omega\right)(\omega) \tag{1.23b}$$

where $A \in \bigvee_{t=-m}^{m} S^t Q$, $m \geq 0$ and show that

$$\mathbb{E}^\nu(A|Y/S^{-m}Q^-)(\overline{y}_m) = \mu_\omega(A) \tag{1.23c}$$

for all $\overline{y}_m$ having the same ρ_m-projection ω (for this, we use (1.4) and a convexity argument on the entropy). The proof will go as follows:

- we first show that (1.22) holds for $A \in Q$, that is, for $m = 0$;
- we show that (1.22) is satisfied for $A \in \bigvee_{t=-m}^{m} S^t Q$ for any $m \geq 0$.

The first of the above steps is not necessary – it can be seen as a toy model for the second step. However, we include it to increase readability. In what follows we identify Y with Y_0 and Ω with Ω_0.

1.3.2 Toy model: $\nu_\omega = \mu_\omega$ on Q

Let

$$\widehat{C}_0^j := \varphi^{-1}(C_0^j) = \{\omega \in \Omega : \varphi(w)(0) = j\} \text{ for } j = 0, 1 \tag{1.24}$$

(recall that the sets C_0^j were defined in (1.3)). Then $\Omega = \widehat{C}_0^0 \cup \widehat{C}_0^1$. Moreover,

$$Y = \theta^{-1}(\Omega) = \theta^{-1}(\widehat{C}_0^0) \cup \theta^{-1}(\widehat{C}_0^1). \tag{1.25}$$

It follows from (1.24) that for $j = 0, 1$, we have

$$\theta^{-1}(\widehat{C}_0^j) = \bigcup_{\omega \in \Omega, \varphi(\omega)(0) = j} \theta^{-1}(\omega). \tag{1.26}$$

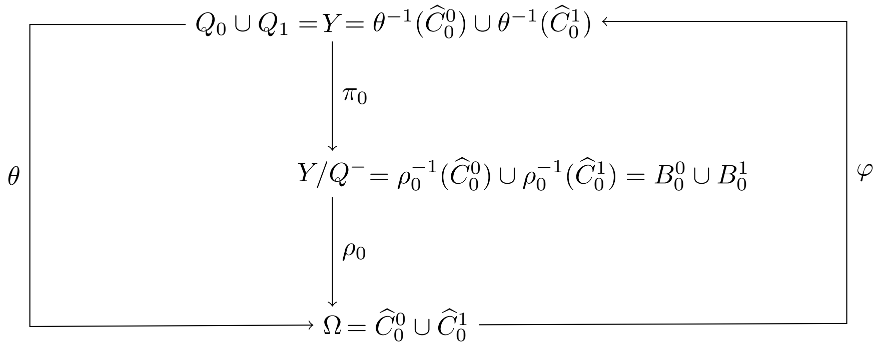
In other words, (1.25) is the partition of Y given by the fibers $\theta^{-1}(\omega)$ of θ, according to the value at the zero coordinate of the biggest element $\varphi(\omega)$ in the fiber, cf. Lemma 1.2.2 (ii). Finally, let

$$B_0^j := \rho_0^{-1}(\widehat{C}_0^j) \text{ for } j = 0, 1,$$

i.e. we have

$$Y/Q^- = B_0^0 \cup B_0^1.$$

This can be summarized in the following diagram (φ is not measure-preserving):



We claim that $\theta^{-1}(\widehat{C}_0^0) \subset Q_0$. Indeed, in view of (1.26), if $y \in \theta^{-1}(\widehat{C}_0^0)$ then $y \in \theta^{-1}(\omega)$, where $\varphi(\omega)(0) = 0$, i.e. $\varphi(\omega) \in Q_0$. Since $y \leq \varphi(\omega)$, $y \in Q_0$. It follows that for $\overline{y} \in B_0^0 = \rho_0^{-1}(\widehat{C}_0^0)$ we have

$$\pi_0^{-1}(\overline{y}) \subset \pi_0^{-1}\rho_0^{-1}(\widehat{C}_0^0) = \theta^{-1}(\widehat{C}_0^0) \subset Q_0$$

and therefore

$$(Q_0, Q_1) \cap \pi_0^{-1}(\overline{y}) = (\pi_0^{-1}(\overline{y}), \emptyset).$$

Since the measure $\overline{\nu}_0(\cdot|Q^-)(\overline{y})$ is concentrated on $\pi_0^{-1}(\overline{y})$, we obtain for each $\overline{y} \in B_0^0$

$$\big(\overline{\nu}_0(Q_0|Q^-)(\overline{y}), \overline{\nu}_0(Q_1|Q^-)(\overline{y})\big) = (1, 0) =: (\lambda_0(Q_0), \lambda_0(Q_1)). \tag{1.27}$$

Therefore, $H_\nu(Q|Q^-)(\overline{y}) = 0$ whenever $\overline{y} \in B_0^0$.

Now, in view of Lemma 1.2.6 and the definition of $\nu_{\mathscr{B}}$, we obtain

$$\nu_{\mathscr{B}}(C_0^1) = \mathbb{P}(\widehat{C}_0^1) = \nu(\theta^{-1}(\widehat{C}_0^1)).$$

Hence, using additionally (1.4), we have

$$\begin{aligned}
\nu(\theta^{-1}(\widehat{C}_0^1)) \log 2 = h_{top}(S, X_\eta) &= h_\nu(S, X_\eta) \\
&= \int_{Y/Q^-} H_\nu(Q|Q^-)(\overline{y})\ d\overline{\nu}_0(\overline{y}) \\
&= \int_{B_0^0} H_\nu(Q|Q^-)(\overline{y})\ d\overline{\nu}_0(\overline{y}) + \int_{B_0^1} H_\nu(Q|Q^-)(\overline{y})\ d\overline{\nu}_0(\overline{y}) \\
&= \int_{B_0^1} H_\nu(Q|Q^-)(\overline{y})\ d\overline{\nu}_0(\overline{y}) \leq \overline{\nu}_0(B_0^1) \log 2 = \nu(\theta^{-1}(\widehat{C}_0^1)) \log 2.
\end{aligned}$$

It follows that for $\overline{\nu}_0$-a.e. $\overline{y} \in B_0^1$, we have $H_\nu(Q|Q^-)(\overline{y}) = \log 2$, or, equivalently

$$\big(\overline{\nu}_0(Q_0|Q^-)(\overline{y}), \overline{\nu}_0(Q_1|Q^-)(\overline{y})\big) = (1/2, 1/2) =: (\lambda_1(Q_0), \lambda_1(Q_1)). \tag{1.28}$$

Both (1.27) and (1.28) do not depend on $\overline{y}$ itself but only on the value $\varphi(\rho_0(\overline{y}))(0)$ which allows us to make use of (1.23c). We now use (1.23a), (1.23b) and (1.23c) to conclude that in the disintegration (1.21) of ν over $\mathbb{P}$ (via θ), for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\nu_\omega(Q_i) = \mu_\omega(Q_i)$ for $i = 0, 1$ (in view of (1.27) and (1.28)).

1.3.3 General case: $\nu_\omega = \mu_\omega$ on $\bigvee_{t=-m}^{m} S^t Q$

Now, fix $m \geq 0$ and let

$$\widehat{C}_m^j := \varphi^{-1}(C_m^j) = \{\omega \in \Omega : \varphi(w)(m) = j\} \text{ for } j = 0, 1.$$

Then $\Omega = \widehat{C}_m^0 \cup \widehat{C}_m^1$ and

$$Y = \theta^{-1}(\Omega) = \theta^{-1}(\widehat{C}_m^0) \cup \theta^{-1}(\widehat{C}_m^1).$$

We have

$$\theta^{-1}(\widehat{C}_m^j) = \cup_{\omega \in \Omega, \varphi(\omega)(m) = j}\, \theta^{-1}(\omega). \tag{1.29}$$

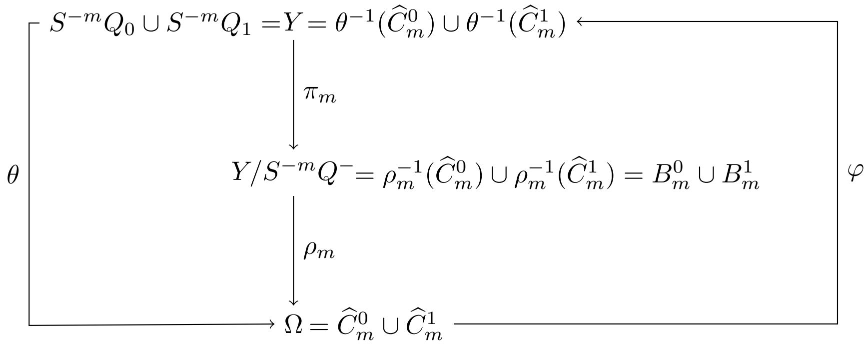
Finally, let

$$B_m^j := \rho_m^{-1}(\widehat{C}_m^j) \text{ for } j = 0, 1,$$

i.e. we have

$$Y/S^{-m}Q^- = B_m^0 \cup B_m^1.$$

This can be summarized in the following diagram:



As in the toy model case, we obtain that $\theta^{-1}(\widehat{C}_m^0) \subset S^{-m}Q_0$, whence, for each $\overline{y} \in B_m^0$,

$$\begin{aligned}(1.30) \quad &\big(\overline{\nu}_m(S^{-m}Q_0|S^{-m}Q^-)(\overline{y}), \overline{\nu}_m(S^{-m}Q_1|S^{-m}Q^-)(\overline{y})\big) \\ &\qquad = (1,0) = (\lambda_0(Q_0), \lambda_0(Q_1)).\end{aligned}$$

Therefore, $H_\nu(S^{-m}Q|S^{-m}Q^-)(\overline{y}) = 0$ whenever $\overline{y} \in B_m^0$. Since $S^{-m}Q$ is a generating partition whose past is equal to $S^{-m}Q^-$, the computation of

$$\int_{Y/S^{-m}Q^-} H_\nu(S^{-m}Q|S^{-m}Q^-)\, d\overline{\nu}_m$$

similar to the toy model case leads to

$$\begin{aligned}(1.31) \quad &\big(\overline{\nu}_m(S^{-m}Q_0|S^{-m}Q^-)(\overline{y}), \overline{\nu}_m(S^{-m}Q_1|S^{-m}Q^-)(\overline{y})\big) \\ &\qquad = (1/2,1/2) = (\lambda_1(Q_0), \lambda_1(Q_1))\end{aligned}$$

for $\overline{\nu}_m$-a.e. $\overline{y} \in B_m^1$.

We claim that

$$(1.32) \qquad \nu_\omega = \mu_\omega \text{ on } \bigvee_{t=-m}^{m} S^t Q.$$

In order to prove this, choose $(i_{-m}, \dots, i_0, \dots, i_m) \in \{0,1\}^{2m+1}$. By the chain rule for conditional probabilities and Lemma 1.2.10, we obtain

$$\begin{aligned}&\overline{\nu}_m(S^m Q_{i_m} \cap \dots \cap Q_{i_0} \cap S^{-1}Q_{i_{-1}} \cap \dots \cap S^{-m}Q_{i_{-m}} \big| S^{-m}Q^-)(\overline{y}) \\ &\quad = \Pi_{r=0}^{2m} \overline{\nu}_m(S^{m-r}Q_{i_{m-r}} \big| S^{m-r-1}Q_{i_{m-r-1}} \cap \dots \cap S^{-m}Q_{i_{-m}} \cap S^{-m}Q^-)(\overline{y}) \\ &\quad = \Pi_{r=0}^{2m} \overline{\nu}_m(S^{-m}Q_{i_{m-r}} \big| S^{-m}Q^-)(\overline{y} i_{-m} \dots i_{m-r-1}).\end{aligned}$$

It follows from (1.30) and (1.31) that for ν_m-a.e. $\overline{y}$

$$\overline{\nu}_m(S^{-m}Q_{i_{m-r}} \big| S^{-m}Q^-)(\overline{y} i_{-m} \dots i_{m-r-1}) = (\lambda_{j_r}(Q_0), \lambda_{j_r}(Q_1)),$$

where $j_r = \varphi(\rho_m(\overline{y}i_{-m}\dots i_{m-r-1}))(m)$ (see the definition of B_m^r). Using (1.19), we hence obtain $j_r = \varphi(\rho_m(\overline{y}))(m+r)$. As in the toy model, using (1.23a), (1.23b) and (1.23c), we obtain (1.32).

Carrying this out for all $m \in \mathbb{N}$, we will show that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mu_\omega = \nu_\omega$ and hence $\mu = \nu$ as required. The proof of Theorem 0.0.1 is complete.

2 Invariant measures for $\mathscr{B}$-free systems

Recall that $M\colon X_\eta \times \{0,1\}^{\mathbb{Z}} \to X_\eta$ is given by

$$M(x \cdot u)(n) = x(n) \cdot u(n),\ n \in \mathbb{Z}.$$

Since M is equivariant, for each $\rho \in \mathcal{P}^e(S \times S, X_\eta \times \{0,1\}^{\mathbb{Z}})$, we have $M_*(\rho) \in \mathcal{P}^e(S, X_\eta)$. In particular, in the above construction, we can consider measures ρ whose projection onto the first coordinate is the Mirsky measure $\nu_{\mathscr{B}}$. In fact, instead of $\nu_{\mathscr{B}}$, we can also use the Mirsky measures $\nu_{\mathscr{B}'}$, where $\mathscr{B}'$ is such that the corresponding free system $X_{\eta'}$ is a subsystem of X_η, see Examples 1 and 2 below. We will call the measures of the form $M_*(\rho)$, where $\rho \in \mathcal{P}^e(S \times S, X_\eta \times \{0,1\}^{\mathbb{Z}})$ $\rho|_{X_\eta} = \nu_{\mathscr{B}'}$ for some $\mathscr{B}'$-free subshift $X_{\mathscr{B}'} \subset X_{\mathscr{B}}$, to be of *joining type*[6] (see also footnote 2).

The natural question arises whether $\mathcal{P}^e(S, X_\eta)$ consists only of measures of joining type. We will give a positive answer to this question in Sections 2.1.1 and 2.2.4.

2.1 Invariant measures on Y

2.1.1 Ergodic invariant measures on Y are of joining type

The main result in this section is the following:

Theorem 2.1.1. *For any $\nu \in \mathcal{P}^e(S, Y)$ there exists $\widetilde{\rho} \in \mathcal{P}^e(S \times S, X_\eta \times \{0,1\}^{\mathbb{Z}})$ such that $\widetilde{\rho}|_{X_\eta} = \nu_{\mathscr{B}}$ and $M_*(\widetilde{\rho}) = \nu$.*

Remark 2.1.2. Some of the objects occurring in the proof will be very similar to the their "one-sided versions" described in [12].

Proof of Theorem 2.1.1. Notice first that $\nu \neq \delta_{(\dots,0,0,0,\dots)}$. In particular,

$$\nu(\{y \in Y : |\mathrm{supp}(y) \cap (-\infty, 0)| = |\mathrm{supp}(y) \cap (0, \infty)| = \infty\}) = 1. \tag{2.1}$$

For $x, z \in \{0,1\}^{\mathbb{Z}}$ with $|\mathrm{supp}\, z \cap (-\infty, 0)| = |\mathrm{supp}\, z \cap (0, \infty)| = \infty$, let $\widehat{x}_z$ be the sequence obtained by reading consecutive coordinates of x which are in supp z, and such that

$$\widehat{x}_z(0) = x(\min\{k \geq 0 : k \in \mathrm{supp}\, z\}).$$

Let $\Theta\colon Y \to \Omega \times \{0,1\}^{\mathbb{Z}}$ be given by

$$\Theta(y) = (\theta(y), \widehat{y}_{\varphi(\theta(y))}).$$

We have

$$(\Theta \circ S)y = (\theta S y, \widehat{Sy}_{\varphi(\theta(Sy))}) = (T\theta y, \widehat{Sy}_{S\varphi(\theta(y))}). \tag{2.2}$$

[6] Notice that ρ, as a member of $\mathcal{P}^e(S \times S, X_\eta \times \{0,1\}^{\mathbb{Z}})$, is an ergodic joining of $\nu_{\mathscr{B}'} \in \mathcal{P}^e(S, X_\eta)$ and $\rho|_{\{0,1\}^{\mathbb{Z}}} \in \mathcal{P}^e(S, \{0,1\}^{\mathbb{Z}})$.

Notice that for $x, z \in \{0,1\}^{\mathbb{Z}}$, the value of $\widehat{Sx}_{Sz}$ depends on $z(0)$ in the following way:

$$\widehat{Sx}_{Sz} = \begin{cases} \widehat{x}_z & \text{if } z(0) = 0, \\ S\widehat{x}_z & \text{if } z(0) = 1 \end{cases} \tag{2.3}$$

(we illustrate this in Figure 1). Therefore, it follows from (2.2) that

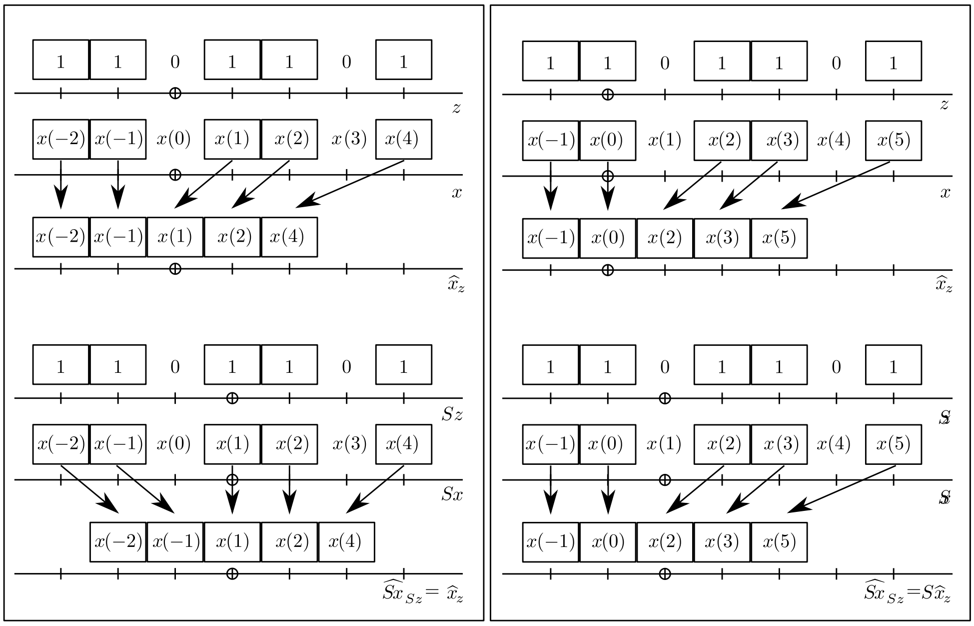


Figure 1: Illustration of formula (2.3). Case $z(0) = 0$ and $z(0) = 1$ (the symbol $\oplus$ stands for the zero coordinate).

$$\Theta \circ S|_{Y_\infty} = \widetilde{T} \circ \Theta|_{Y_\infty},$$

where $\widetilde{T} \colon \Omega \times \{0,1\}^{\mathbb{Z}} \to \Omega \times \{0,1\}^{\mathbb{Z}}$ is given by

$$\widetilde{T}(\omega, x) = \begin{cases} (T\omega, x) & \text{if } \varphi(\omega)(0) = 0, \\ (T\omega, Sx) & \text{if } \varphi(\omega)(0) = 1 \end{cases}$$

and $Y_\infty := \{y \in Y : |\text{supp } \varphi(\theta(y)) \cap (-\infty, 0)| = |\text{supp } \varphi(\theta(y)) \cap (0, \infty)| = \infty\}$. Then, since $y \leq \varphi(\theta(y))$, it follows by (2.1) that $\nu(Y_\infty) = 1$ for any $\nu \in \mathcal{P}^e(S, Y)$. Thus, $\Theta \circ S = \widetilde{T} \circ \Theta$ holds a.e. with respect to any $\nu \in \mathcal{P}^e(S, Y)$:

$$\begin{array}{ccc} Y & \xrightarrow{\quad S \quad} & Y \\ \Big\downarrow \Theta & & \Big\downarrow \Theta \\ \Omega \times \{0,1\}^{\mathbb{Z}} & \xrightarrow{\quad \widetilde{T} \quad} & \Omega \times \{0,1\}^{\mathbb{Z}}. \end{array} \tag{2.4}$$

Notice that $\Theta(y)$ contains complete information about each $y \in Y_\infty$:

- the first coordinate, i.e. $\theta(y)$, contains, for each k, the information about the missing residue classes in supp y mod b_k,
- the second coordinate, i.e. $\widehat{y}_{\varphi(\theta(y))}$, contains the information about y along supp $\varphi(\theta(y))$.

This allows us to define $\Phi\colon \Theta(\Omega) \to X_\eta$ such that $\Phi \circ \widetilde{T} = S \circ \Phi$. We do this in the following way: $\Phi(\omega, x)$ is the unique element in X_η such that

(i) $\Phi(\omega, x) \leq \varphi(\omega)$,

(ii) $\widehat{\Phi(\omega, x)}_{\varphi(\omega)} = x$, i.e. the consecutive coordinates of x can be found in $\Phi(\omega, x)$ along $\varphi(\omega)$.

Notice that if follows from (1.10)[7] that $\Omega_0 \times \{0,1\}^{\mathbb{Z}} \subset \Theta(Y_\infty)$, i.e. $\Phi(\omega, x)$ is well-defined on $\Omega_0 \times \{0,1\}^{\mathbb{Z}}$. We will show now that the following diagram commutes:

$$\begin{array}{ccc}
\Omega_0 \times \{0,1\}^{\mathbb{Z}} & \xrightarrow{\ \widetilde{T}\ } & \Omega_0 \times \{0,1\}^{\mathbb{Z}} \\
\Big\downarrow \Phi & & \Big\downarrow \Phi \\
X_\eta & \xrightarrow{\quad S \quad} & X_\eta.
\end{array}$$

Indeed, in view of the definition of Φ and $\widetilde{T}$, $\Phi \circ \widetilde{T}(\omega, x)$ is the unique element in X_η such that:

- $\Phi \circ \widetilde{T}(\omega, x) \leq \varphi(T\omega)$,
- $(\Phi \circ \widetilde{T}(\omega, x))\widehat{\ }_{\varphi(T\omega)} = \begin{cases} x & \text{if } \varphi(T\omega)(0) = 0 \\ Sx & \text{if } \varphi(T\omega)(0) = 1. \end{cases}$

Moreover, by (2.3), we have

- $S \circ \Phi(\omega, x) \leq S\varphi(\omega) = \varphi(T\omega)$,
- $(S \circ \Phi(\omega, x))\widehat{\ }_{S\varphi(\omega)} = \begin{cases} (\Phi(\omega, x))\widehat{\ }_{\varphi(\omega)} & \text{if } \varphi(\omega)(0) = 0 \\ S((\Phi(\omega, x))\widehat{\ }_{\varphi(\omega)}) & \text{if } \varphi(\omega)(0) = 1, \end{cases}$

  i.e. $(\Phi \circ \widetilde{T}(\omega, x))\widehat{\ }_{\varphi(T\omega)} = \begin{cases} x & \text{if } \varphi(T\omega)(0) = 0 \\ Sx & \text{if } \varphi(T\omega)(0) = 1. \end{cases}$

Thus, we have obtained $S \circ \Phi = \Phi \circ \widetilde{T}$. Since

$$\Theta^{-1}(\Omega_0 \times \{0,1\}^{\mathbb{Z}}) = \theta^{-1}(\Omega_0) = Y_0,$$

where, by (1.17), $\nu(Y_0) = 1$ for any $\nu \in \mathcal{P}^e(S, Y)$, it follows that the composition $\Phi \circ \Theta$ is well-defined a.e. with respect to any $\nu \in \mathcal{P}^e(S, Y)$. We claim that

$$\Phi \circ \Theta = id_{Y_0}. \tag{2.5}$$

Indeed, we have $\Phi \circ \Theta(y) = \Phi(\theta(y), \widehat{y}_{\varphi(\theta(y))})$. Moreover, $\Phi(\theta(y), \widehat{y}_{\varphi(\theta(y))})$ is the unique element such that

$$\Phi(\theta(y), \widehat{y}_{\varphi(\theta(y))}) \leq \varphi(\theta(y)) \text{ and } (\Phi(\theta(y), \widehat{y}_{\varphi(\theta(y))}))\widehat{\ }_{\varphi(\theta(y))} = \widehat{y}_{\varphi(\theta(y))}.$$

[7] In a similar way as in (1.10), we have $\left|\{s \leq -1 : -z + sb_k \in \operatorname{supp} \varphi(\omega)\}\right| = \infty$ for $\omega \in \Omega_0$.

However, since $y \le \varphi(\theta(y))$, it follows immediately that $\Phi(\theta(y), \widehat{y}_{\varphi(\theta(y))}) = y$, which yields (2.5). Hence, for each $\nu \in \mathcal{P}^e(S, Y)$, we have

$$\nu = \Phi_*(\Theta_*\nu), \text{ where } \Theta_*\nu \in \mathcal{P}^e(\widetilde{T}, \Omega \times \{0,1\}^{\mathbb{Z}}).$$

Notice that we have also the following commuting diagram:

$$\begin{array}{ccc} \Omega_0 \times \{0,1\}^{\mathbb{Z}} & \xrightarrow{T \times S} & \Omega_0 \times \{0,1\}^{\mathbb{Z}} \\ \downarrow \Psi & & \downarrow \Psi \\ \Omega_0 \times \{0,1\}^{\mathbb{Z}} & \xrightarrow{\widetilde{T}} & \Omega_0 \times \{0,1\}^{\mathbb{Z}} \end{array} \tag{2.6}$$

where $\Psi(\omega, x) = (\omega, \widehat{x}_{\varphi(\omega)})$. Indeed, using (2.3), we obtain

$$\begin{aligned} \Psi \circ (T \times S)(\omega, x) &= \Psi(T\omega, Sx) \\ &= (T\omega, \widehat{Sx}_{\varphi(T\omega)}) = (T\omega, \widehat{Sx}_{S\varphi(\omega)}) = \begin{cases} (T\omega, \widehat{x}_{\varphi(\omega)}) & \text{if } \varphi(\omega)(0) = 0 \\ (T\omega, S\widehat{x}_{\varphi(\omega)}) & \text{if } \varphi(\omega)(0) = 1, \end{cases} \end{aligned}$$

whereas

$$\widetilde{T} \circ \Psi(\omega, x) = \widetilde{T}(\omega, \widehat{x}_{\varphi(\omega)}) = \begin{cases} (T\omega, \widehat{x}_{\varphi(\omega)}) & \text{if } \varphi(\omega)(0) = 0 \\ (T\omega, S\widehat{x}_{\varphi(\omega)}) & \text{if } \varphi(\omega)(0) = 1. \end{cases}$$

Notice that $\emptyset \neq \Psi^{-1}(\omega, y) \subset \{\omega\} \times \{0,1\}^{\mathbb{Z}}$. Moreover, given $(\omega, x) \in \Psi^{-1}(\omega, y)$, all other points in $\Psi^{-1}(\omega, y)$ are obtained by changing in an arbitrary way these coordinates in x which are not in the support of $\varphi(\omega)$. In particular, each fiber $\Psi^{-1}(\omega, y)$ is infinite. For $k_1 < \cdots < k_s$ and $(i_1, \ldots, i_s) \in \{0,1\}^s$ we define the following cylinder set:

$$C = C_{k_1,\ldots,k_s}^{i_1,\ldots,i_s} := \{x \in \{0,1\}^{\mathbb{Z}} : x(k_j) = i_j, 1 \le j \le s\}. \tag{2.7}$$

For each such C and for $A \in \mathcal{B}(\Omega)$ we put

$$\lambda_{(\omega,y)}(A \times C) := \mathbb{1}_A(\omega) \cdot 2^{-m}, \text{ where } m = |\{1 \le j \le s : \varphi(\omega)(k_j) = 0\}|,$$

whenever $\Phi(\omega, y)$ agrees with C along $\varphi(\omega)$, i.e.

$$\Phi(\omega, y)(k_j) \cdot \varphi(\omega)(k_j) = C(k_j) \cdot \varphi(\omega)(k_j)$$

(otherwise we set $\lambda_{(\omega,y)}(A \times C) := 0$). In view of part (i) of the definition of Φ, this is equivalent to

$$\Phi(\omega, y)(k_j) = i_j \text{ whenever } \varphi(\omega)(k_j) = 1.$$

We claim that the following is true:

(a) the map $F : (\omega, y) \mapsto \lambda_{(\omega,y)}$ is measurable,

(b) $(T \times S)_*\lambda_{(\omega,y)} = \lambda_{\widetilde{T}(\omega,y)}$.

For (a), it suffices to show that sets of the form

$$V_{A,C,a,\varepsilon} = \{(\omega, y) \in \Omega_0 \times \{0,1\}^{\mathbb{Z}} : |\lambda_{(\omega,y)}(A \times C) - a| < \varepsilon\}.$$

are measurable for any $A \in \mathcal{B}(\Omega)$, any cylinder C as in (2.7), any $a \in \mathbb{R}$ and $\varepsilon > 0$. Indeed, for $\underline{\lambda} \in \mathcal{P}(\Omega_0 \times \{0,1\}^{\mathbb{Z}})$ and $a = \underline{\lambda}(A \times C)$

$$V_{A,C,a,\varepsilon} = F^{-1}(\{\lambda \in \mathcal{P}(\Omega \times \{0,1\}^{\mathbb{Z}}) : |\lambda(A \times C) - \underline{\lambda}(A \times C)| < \varepsilon\}).$$

Notice that each $V_{A,C,a,\varepsilon}$ is an at most countable union of sets of the form

$$V_{A,C,b} := \{(\omega, y) : \lambda_{(\omega,y)}(A \times C) = b\},$$

where $b \in \{0\} \cup \{2^{-m} : m \geq 0\}$. Let

$$V_C := \{(\omega, y) : (\Phi(\omega, y)(k_j) - C(k_j)) \cdot \varphi(\omega)(k_j) = 0, 1 \leq j \leq s\}.$$

Then

$$V_{A,C,0} = (A^c \times \{0,1\}^{\mathbb{Z}}) \cup V_C^c$$

and for $m \geq 0$,

$$V_{A,C,2^{-m}} = (A \times \{0,1\}^{\mathbb{Z}}) \cap V_C \cap \Big\{(\omega, y) : \sum_{j=-k}^{k} \varphi(\omega)(j) = 2k + 1 - m\Big\}.$$

This implies measurability of the sets $V_{A,C,a}$ as φ and Φ are measurable. To see that also (b) holds, notice first that we have

$$(T \times S)_* \lambda_{(\omega,y)}(A \times C) = \lambda_{(\omega,y)}(T^{-1}A \times S^{-1}C)$$

and

$$\lambda_{\widetilde{T}(\omega,y)}(A \times C) = \begin{cases} \lambda_{(T\omega,x)}(A \times C) & \text{if } \varphi(\omega)(0) = 0 \\ \lambda_{(T\omega,Sx)}(A \times C) & \text{if } \varphi(\omega)(0) = 1. \end{cases}$$

We have

$$\begin{aligned} \Phi \circ \widetilde{T}(\omega, y)(k_j) = C(k_j)\varphi(T\omega)(k_j) \\ \iff S \circ \Phi(\omega, y)(k_j) = C(k_j) S\varphi(\omega)(k_j) \\ \iff \Phi(\omega, y)(k_j + 1) = C(k_j)\varphi(\omega)(k_j + 1). \end{aligned}$$

Moreover, clearly $T\omega \in A \iff \omega \in T^{-1}A$. Finally, we also have

$$|\{1 \leq j \leq s : \varphi(\omega)(k_j + 1) = 0\}| = |\{1 \leq j \leq s : \varphi(T\omega)(k_j) = 0\}|.$$

This ends the proof of (b) in view of the definition of measures $\lambda_{(\omega,y)}$. Therefore, for $\rho \in \mathcal{P}^e(\widetilde{T}, \Omega \times \{0,1\}^{\mathbb{Z}})$, we have

$$\widetilde{\rho} := \int \lambda_{(\omega,y)} \, d\rho(\omega, y) \in \mathcal{P}(T \times S, \Omega \times \{0,1\}^{\mathbb{Z}}) \text{ with } \Psi_* \widetilde{\rho} = \rho.$$

The last step in the proof is to notice that

$$M \circ (\varphi \times id_{\{0,1\}^{\mathbb{Z}}}) = \Phi \circ \Psi.$$

It follows that for any $\nu \in \mathcal{P}(S,Y)$[8] we have

$$\nu = \Phi_*\Theta_*\nu = \Phi_*\Psi_*\widetilde{\Theta_*\nu} = M_*(\varphi \times id_{\{0,1\}^{\mathbb{Z}}})_*\widetilde{\Theta_*\nu},$$

which completes the proof as $\widetilde{\Theta_*\nu} \in \mathcal{P}(T\times S, \Omega\times\{0,1\}^{\mathbb{Z}})$ and

$$\varphi \times id_{\{0,1\}^{\mathbb{Z}}} \colon \Omega\times\{0,1\}^{\mathbb{Z}} \to X_\eta \times \{0,1\}^{\mathbb{Z}}$$

induces an isomorphism between $\mathcal{P}(T\times S,\Omega\times\{0,1\}^{\mathbb{Z}})$ and the simplex of probability $S\times S$-invariant measures on $X_\eta\times\{0,1\}^{\mathbb{Z}}$ whose projection onto the first coordinate is $\nu_{\mathscr{B}}$. □

We will show later, see Section 2.2.4, that Theorem 2.1.1 is valid for each member of $\mathcal{P}^e(S,X_\eta)$ (with $\nu_{\mathscr{B}}$ replaced by a Mirsky measure of a subsystem). We postpone the proof of that fact to see first some introductory concepts and examples for a better understanding of the final result and its consequences.

Remark 2.1.3. The language introduced in the course of the proof of Theorem 2.1.1 can be used to provide another proof of Theorem 0.0.1. This proof is a simplification of the one presented in [12].

Proof of Theorem 0.0.1. Consider the transformation $\widetilde{T}_{C\times\{0,1\}^{\mathbb{Z}}}$ obtained by inducing $\widetilde{T}$ on the set $C\times\{0,1\}^{\mathbb{Z}}$. Notice that each point from $C\cap\Omega_0$ returns to $C\cap\Omega_0$ via T. In other words, the induced map on $C\times\{0,1\}^{\mathbb{Z}}$ is well-defined up to a set of measure zero for any measure $\nu\in\mathcal{P}(\widetilde{T},\Omega\times\{0,1\}^{\mathbb{Z}})$, since

$$\nu(C\times\{0,1\}^{\mathbb{Z}}\cap\Omega\times\{0,1\}^{\mathbb{Z}}) = \mathbb{P}(C\cap\Omega_0) = \mathbb{P}(C) = \nu(C\times\{0,1\}^{\mathbb{Z}}).$$

Moreover (see [12]), $\widetilde{T}_{C\times\{0,1\}^{\mathbb{Z}}}$ is a product transformation almost everywhere, with respect to any invariant measure. Since the first coordinate of $\widetilde{T}_{C\times\{0,1\}^{\mathbb{Z}}}$, i.e. T_C, is a uniquely ergodic map of zero entropy, it follows that $\widetilde{T}_{C\times\{0,1\}^{\mathbb{Z}}}$ is intrinsically ergodic, with topological entropy equal to $\log 2$. Therefore $\widetilde{T}$ is also intrinsically ergodic, with topological entropy equal to $\mathbb{P}(C)\log 2 > 0$. Moreover, it follows from (2.5) that, in particular, Θ is 1-1. Hence, $\Theta_*\colon\mathcal{P}(S,Y)\to\mathcal{P}(\widetilde{T},\Omega\times\{0,1\}^{\mathbb{Z}})$ is also 1-1 and for any $\nu\in\mathcal{P}(S,Y)$, $h_\nu(S,Y) = h_{\Theta_*\nu}(\widetilde{T},\Omega\times\{0,1\}^{\mathbb{Z}})$. The result follows now from Lemma 1.1.3. □

2.1.2 Product type measures supported on Y

An important subset of joining type measures are product type measures which are "ordinary convolutions", see footnote 2. In this section, we will deal with measures of the form

$$\nu_{\mathscr{B}} * \kappa := M_*(\nu_{\mathscr{B}}\otimes\kappa) \in \mathcal{P}^e(S,X_\eta).$$

Clearly, whenever $\kappa\in\mathcal{P}^e(S,\{0,1\}^{\mathbb{Z}})$ is such that $(S,\{0,1\}^{\mathbb{Z}},\kappa)$ has no eigenvalue which is a b_k-root of unity (for all $k\geq 1$) then $\nu_{\mathscr{B}}*\kappa$ is ergodic. We will give now a condition on κ which implies that the corresponding product type measure $\nu_{\mathscr{B}}*\kappa$ is supported on Y:

[8] If $\widetilde{\Theta_*\nu}$ is not ergodic, we consider its ergodic decomposition and replace $\widetilde{\Theta_*\nu}$ with any of the ergodic components.

Proposition 2.1.4. *Suppose that for any natural numbers $t_1 < t_2 < \ldots$, the measure $\kappa \in \mathcal{P}^e(S, \{0,1\}^{\mathbb{Z}})$ satisfies the following condition:*

$$\kappa\big(\{v \in \{0,1\}^{\mathbb{Z}} : v(t_1) = v(t_2) = \ldots = 0\}\big) = 0. \tag{2.8}$$

Then $(\nu_{\mathscr{B}} * \kappa)(Y) = 1$.

Proof. We have

$$\begin{aligned}(\nu_{\mathscr{B}} * \kappa)(Y) &= (\nu_{\mathscr{B}} \otimes \kappa)(M^{-1}(Y)) = \mathbb{P} \otimes \kappa((\varphi \times Id)^{-1} M^{-1}(Y)) \\ &= \mathbb{P} \otimes \kappa((\Omega_0 \times \{0,1\}^{\mathbb{Z}}) \cap ((\varphi \times Id)^{-1} M^{-1}(Y))).\end{aligned}$$

Moreover, for each $\omega \in \Omega_0$, we have $-\omega(k) \notin \operatorname{supp}(\varphi(\omega)) \bmod b_k$, so the more, for each $u \in \{0,1\}^{\mathbb{Z}}$, $-\omega(k) \notin \operatorname{supp}(\varphi(\omega)) \cdot u \bmod b_k$. On the other hand, by Remark 1.2.3 (see (1.10)), if $z \in \mathbb{Z}/b_k\mathbb{Z} \setminus \{-\omega(k)\}$ then there is an infinite sequence $s_1 < s_2 < \ldots$ such that $\varphi(\omega)(-z + s_i b_k) = 1$ for each $i \geq 1$. Therefore, in view of (2.8), for κ-a.e. $u \in \{0,1\}^{\mathbb{Z}}$ there is $i_0 = i_0(u)$ such that $u(-z + s_{i_0} b_k) = 1$. Hence $(\varphi(\omega) \cdot u)(-z + s_{i_0} b_k) = 1$, whence $M \circ (\varphi \times Id)(\omega, u) \in Y$. The result follows by Fubini's theorem. □

Remark 2.1.5. Notice that each Bernoulli measure $B(p, 1-p)$ satisfies condition (2.8). More generally, condition (2.8) will be satisfied in each system $(S, \{0,1\}^{\mathbb{Z}}, \kappa)$ which is mixing of all orders.

2.1.3 Disintegration of product type measures on Y

Let $\mathcal{L}_k$ be the family of blocks occurring on X_η at $[-k, k]$. Fix $C \in \mathcal{L}_k$. Then

$$\begin{aligned}\nu_{\mathscr{B}} * \kappa(C) &= \nu_{\mathscr{B}} \otimes \kappa(M^{-1}(C)) \\ &= \nu_{\mathscr{B}} \otimes \kappa\big(\{(x,z) \in X_\eta \times \{0,1\}^{\mathbb{Z}} : xz \in C\}\big) \\ &= \int_{X_\eta} \kappa(x^{-1}C)\, d\nu_{\mathscr{B}}(x) = \int_{\Omega_0} \kappa(\varphi(\omega)^{-1}C)\, d\mathbb{P}(w),\end{aligned}$$

where $\varphi(\omega)^{-1}C := \{D \in \mathcal{L}_k : \varphi(\omega) \cdot D = C\}$. Note that $\kappa(\varphi(\omega)^{-1}C) > 0$ only if $C \leq \varphi(\omega)[-k, k]$. Moreover, whenever $\varphi(\omega)(s) = 0$ then at the sth position of D we can have 0 or 1. It follows that

$$\nu_{\mathscr{B}} * \kappa = \int_{\Omega_0} \widetilde{\kappa}_\omega \, d\mathbb{P}(\omega), \tag{2.9}$$

where

$$\widetilde{\kappa}_\omega(C) = \sum_{D \in \mathcal{L}_k : \varphi(\omega) \cdot D = C} \kappa(D). \tag{2.10}$$

Remark 2.1.6. Notice that in order to conclude that (2.9) represents a disintegration of $\nu_{\mathscr{B}} * \kappa$ over $\mathbb{P}$, we need to know that $(T, \Omega, \mathbb{P})$ is a factor (via θ) of the system determined by the convolution measure. For this it suffices that $(\nu_{\mathscr{B}} * \nu)(Y) = 1$, see Proposition 2.1.4.

2.1.4 Product type measures on Y isomorphic to direct products

Remark 2.1.7. Note that (2.10) says that if we want to see the distribution of $\widetilde{\kappa}_\omega$ on blocks, we need to look at the distribution of κ on the cylinder sets

$$C^{i_1,\dots,i_m}_{j_1,\dots,j_m},\ \ i_r\in\{0,1\},$$

where $-k\le j_1<\ldots<j_m\le k$ are all positions t at which $\varphi(\omega)(t)=1$ and we copy this distribution to the family of all blocks smaller than or equal to $\varphi(\omega)[-k,k]$. Notice that if κ is a Bernoulli measure, we can "squeeze" (cf. Section 2.1.1) these positions and take the Bernoulli distribution on blocks of length m (in other words, we change 1 to 0 with probability $1-p$ when $\kappa=B(p,1-p)$). In particular, when $\kappa=B(1/2,1/2)$, we can see that $\widetilde{\kappa}_\omega=\mu_\omega$, where μ_ω is as in Section 1.3.1, i.e.

the measure of maximal entropy for (X_η,S)
is of product type: $\nu_{\mathscr{B}}*B(1/2,1/2)$.

Proposition 2.1.8 (cf. [12] for the square-free system)**.** *Let $\nu\in\mathcal{P}(S,X_\eta)$ be the measure of maximal entropy. Then (S,X_η,ν) is isomorphic to the direct product $(T,\Omega,\mathbb{P})\times(R,Z,\mathcal{D},\rho)$, where R is a Bernoulli automorphism with entropy* $\log 2\cdot\Pi_{i=1}^{\infty}\left(1-\frac{1}{b_i}\right)$.

Proof. By Remark 2.1.7, $\nu=\nu_{\mathscr{B}}*B(1/2,1/2)$, so we have the following sequence of factors maps

$$\left(S\times S,X_\eta\times\{0,1\}^{\mathbb{Z}},\nu_{\mathscr{B}}\otimes B(1/2,1/2)\right)\overset{M}{\to}(S,X_\eta,\nu)\overset{\theta}{\to}(T,\Omega,\mathbb{P})\overset{\varphi}{\to}(S,X_\eta,\nu_{\mathscr{B}})$$

with the last one being an isomorphism. Now,

$$\left(S\times S,X_\eta\times\{0,1\}^{\mathbb{Z}},\nu_{\mathscr{B}}\otimes B(1/2,1/2)\right)\overset{M\circ\theta\circ\varphi}{\longrightarrow}(S,X_\eta,\nu_{\mathscr{B}})$$

is relatively Bernoulli, so by Thouvenot's relative Bernoulli theory [15], also

$$(S,X_\eta,\nu)\overset{\theta\circ\varphi}{\longrightarrow}(S,X_\eta,\nu_{\mathscr{B}})$$

is relatively Bernoulli, in other words the factor $(S,X_\eta,\nu_{\mathscr{B}})$ splits off. □

Consider now the case $\kappa=B(p,1-p)$, $0<p<1$, i.e. κ is a Bernoulli measure. Fix $\omega\in\Omega$. By Remark 2.1.7, for the Bernoulli measures, we have

$$\text{dist}_{\widetilde{\kappa}_\omega}\left(\bigvee_{j=0}^{n-1}S^jQ\right)=\text{dist}_\kappa\left(\bigvee_{\ell=0}^{m(\omega)-1}S^\ell Q\right), \tag{2.11}$$

where $m(\omega):=|\{0\le k\le n-1:\varphi(\omega)(k)=1\}|$. Hence, by (2.11) and independence,

$$\frac{1}{n}H_{\widetilde{\kappa}_\omega}\left(\bigvee_{j=0}^{n-1}S^jQ\right)=\frac{1}{n}H_\kappa\left(\bigvee_{\ell=0}^{m(\omega)-1}S^\ell Q\right)=\frac{m(\omega)}{n}H_\kappa(Q).$$

It follows that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\lim_{n\to\infty} \frac{1}{n} H_{\widetilde{\kappa}_\omega}\left(\bigvee_{j=0}^{n-1} S^j Q\right) = \nu_{\mathscr{B}}(C_0^1) H_\kappa(Q)$. Since $h_{\nu_{\mathscr{B}}*\kappa}(S,Q)$ is equal to the relative entropy with respect to the $(T,\Omega,\mathbb{P})$ factor (as the latter has zero entropy),

$$h_{\nu_{\mathscr{B}}*\kappa}(S,Q) = \lim_{n\to\infty} \frac{1}{n} \int_\Omega H_{\widetilde{\kappa}_\omega}\left(\bigvee_{j=0}^{n-1} S^j Q\right) d\mathbb{P}(\omega),$$

and we obtain the following result:

Proposition 2.1.9. *If $\kappa = B(p, 1-p)$ then*

$$h(S, X_\eta, \nu_{\mathscr{B}} * B(p,1-p)) = -(p\log p + (1-p)\log(1-p))\Pi_{i\geq 1}(1-1/b_i).$$

Remark 2.1.10. It follows by the above that:

- For each value $0 \leq h \leq \log 2 \cdot \Pi_{i\geq 1}(1-1/b_i)$ there is an ergodic measure κ such that $h(S, X_\eta, \nu_{\mathscr{B}} * \kappa) = h$.
- Similarly as in the case $\kappa = B(1/2, 1/2)$, cf. Proposition 2.1.8, we obtain that the dynamical system $(S, X_\eta, \nu_{\mathscr{B}} * B(p,1-p))$ is isomorphic to the direct product of $(T,\Omega,\mathbb{P})$ and a Bernoulli automorphism with the entropy $-(p\log p + (1-p)\log(1-p))\Pi_{i\geq 1}(1-1/b_i)$.

Question 1. Can we obtain a general entropy formula for the product type measures $\nu_{\mathscr{B}} * \kappa$, e.g. where κ satisfies (2.8)? Is it true that entropy of the product type measure is positive whenever the entropy of κ is positive? Is the entropy of $\nu_{\mathscr{B}} * \kappa$ always smaller than the entropy of κ provided that the entropy of κ is positive?

Remark 2.1.11. Notice that except for the situation when $\kappa = \delta_{(\dots 11\dots)}$, the map $\theta\colon (S, X_\eta, \nu_{\mathscr{B}} * \kappa) \to (T,\Omega,\mathbb{P})$ cannot be an isomorphism. Indeed, if so then the conditional measures are Dirac measures, and in particular the distribution of $\widetilde{\kappa}_\omega$ on blocks of length 1 must be trivial. However this distribution is given by the distribution of κ on blocks of length 1 which cannot be trivial if $\kappa \neq \delta_{(\dots 11\dots)}$. Therefore, if κ yields a K-automorphism, then $(S, X_\eta, \nu_{\mathscr{B}} * \kappa) \to (T,\Omega,\mathbb{P})$ is relatively K, hence the entropy of $\nu_{\mathscr{B}} * \kappa$ is positive.

2.2 Invariant measures on X_η

2.2.1 Zero entropy measures and filtering

As we have seen in Section 2.1.4, if $\kappa = B(p, 1-p)$ then

$$h_{\nu_{\mathscr{B}}\otimes\kappa}(S\times S, X_\eta \otimes \{0,1\}^{\mathbb{Z}}) > h_{\nu_{\mathscr{B}}*\kappa}(S, X_\eta).$$

In particular, the map M cannot be an isomorphism. Clearly, if $\kappa = \delta_{(\dots 11\dots)}$ then $\nu_{\mathscr{B}} * \kappa = \nu_{\mathscr{B}}$, i.e. M **is** an isomorphism. A general question arises whether M can be an isomorphism of $(S\times S, X_\eta \times \{0,1\}^{\mathbb{Z}}, \nu_{\mathscr{B}} \otimes \kappa)$ and $(S, X_\eta, \nu_{\mathscr{B}} * \kappa)$ for $\kappa \neq \delta_{(\dots 11\dots)}$. In particular, we will be interested in the situation when κ yields a zero entropy system.

Now, we will look and the product type measures from the point of view of the filtering problem in ergodic theory ([3], [5], [6]). For this, we will need some notation (partially borrowed from [1]) and some tools.

Let $C \subset \Omega$ be given by $C := \varphi^{-1}(C_0^1)$, where $C_0^1 = \{x \in X_\eta : x(0) = 1\}$, i.e. $C = \{\omega \in \Omega : (\forall k \geq 1)\ \ \omega(k) \neq 0\}$. Then

$$\varphi(\omega) = (f(T^n\omega))_{n\in\mathbb{Z}}, \tag{2.12}$$

where $f(\omega) = \mathbb{1}_C(\omega)$.

Lemma 2.2.1. *The partition $\{C, \Omega \setminus C\}$ is a generating partition.*

Proof. This is just a reformulation of the fact the φ is $\mathbb{P}$-a.e. 1-1 (see Lemma 1.2.1). □

Recall also that

$$\nu_{\mathscr{B}} * \kappa = (M \circ (\varphi \times Id))_*(\mathbb{P} \otimes \kappa). \tag{2.13}$$

Furthermore, for each $\omega \in \Omega$, $z \in \{0,1\}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} M \circ (\varphi \times Id)(\omega, z)(n) &= \varphi(\omega)(n) \cdot z(n) \\ &= \mathbb{1}_C(T^n\omega) \cdot \mathbb{1}_{C_0^1}(S^n z) = \mathbb{1}_{C \times C_0^1}((T \times S)^n(\omega, z)). \end{aligned}$$

Now, if $p_0 : X_\eta \to \{0,1\}$ denotes the projection on the zero coordinate,

$$\mathbb{1}_{C \times C_0^1} = \mathbb{1}_C \otimes \mathbb{1}_{C_0^1} = p_0 \circ M \circ (\varphi \times Id).$$

Let $\mathcal{G} := (M \circ (\varphi \times Id))^{-1}(\mathcal{B}(X_\eta))$. It follows that the set

$$C \times C_0^1 \text{ is } \mathcal{G}\text{-measurable.} \tag{2.14}$$

Moreover,

$$\begin{aligned} &(S, X_\eta, \nu_{\mathscr{B}} * \kappa) \text{ is measure-theoretic isomorphic to} \\ &(T \times S, \Omega \times \{0,1\}^{\mathbb{Z}}/\mathcal{G}, \mathcal{G}, \mathbb{P} \otimes \kappa). \end{aligned} \tag{2.15}$$

We will also need some ergodic theory results coming from [3], concerning the filtering problem. The following result can be proved by repeating almost verbatim the proof of Proposition 5 in [3].

Proposition 2.2.2 (cf. [3])**.** *Assume that T and S are ergodic automorphisms of probability standard Borel spaces $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$, respectively. Assume that for each ergodic self-joinings λ of T and ρ of S, we have*

$$(T \times T, X \times X, \lambda) \perp (S \times S, Y \times Y, \rho).^9 \tag{2.16}$$

Assume that $\mathcal{F} \subset \mathcal{B} \otimes \mathcal{C}$ is a factor of $(T \times S, X \times Y, \mu \otimes \nu)$. Then there exist factors $\mathcal{B}_1 \subset \mathcal{B}$, $\mathcal{C}_1 \subset \mathcal{C}$ and compact subgroups $\mathcal{H} \subset C(T|_{\mathcal{B}_1})$,[10] $\mathcal{H}' \subset C(S|_{\mathcal{C}_1})$ with a continuous group isomorphism $\mathcal{H} \ni W \mapsto W' \in \mathcal{H}'$ such that

$$\mathcal{F} = \mathrm{Fix}\big(\{W \times W' : W \in \mathcal{H}\}\big).^{11} \tag{2.17}$$

[9] We write $\perp$ between two measure-theoretic automorphisms if they are disjoint, i.e. if the only joining between them is product measure [7].

[10] We denote the action of T on the factor $(X/\mathcal{B}_1, \mathcal{B}_1, \mu|_{\mathcal{B}_1})$ by $T|_{\mathcal{B}_1}$. Given an automorphism T, $C(T)$ stands for its centralizer.

[11] Given an automorphism T acting on $(X, \mathcal{B}, \mu)$ and $\mathcal{H} \subset C(T)$, we set

$$\mathrm{Fix}(\mathcal{H}) := \{A \in \mathcal{B} : (\forall W \in \mathcal{H})\ \ WA = A\}.$$

Clearly, $\mathrm{Fix}(\mathcal{H})$ is a factor of T.

In particular,

$$\mathcal{F} \supset \mathrm{Fix}(\mathcal{H}) \otimes \mathrm{Fix}(\mathcal{H}'). \tag{2.18}$$

Corollary 2.2.3. *Under the assumptions of Proposition 2.2.2, suppose additionally that $\mathcal{F}$ contains a "rectangle" $C \times J \in \mathcal{F}$, where the partitions $\{C, X \setminus C\}$, $\{J, Y \setminus J\}$ are generating for T and S, respectively. Then $\mathcal{F} = \mathcal{B} \otimes \mathcal{C}$.*

Proof. The set $C \times J$ is fixed by all elements $W \times W'$, $W \in \mathcal{H}$, whence $C \in \mathrm{Fix}(\mathcal{H})$ and $J \in \mathrm{Fix}(\mathcal{H}')$. Hence $C \times J \in \mathrm{Fix}(\mathcal{H}) \otimes \mathrm{Fix}(\mathcal{H}')$. The latter factor is a product factor, so it is invariant under the product $\mathbb{Z}^2$-action $\{T^m \times S^n : m, n \in \mathbb{Z}\}$, i.e. $(T^m \times S^n)(C \times J) \in \mathrm{Fix}(\mathcal{H}) \otimes \mathrm{Fix}(\mathcal{H}')$ for each $m, n \in \mathbb{Z}$, and the result follows. □

Now, consider $\kappa \in \mathcal{P}^e(S, \{0,1\}^{\mathbb{Z}})$ such that the following holds:

(2.19) Every ergodic self-joining ρ of $(S, \{0,1\}^{\mathbb{Z}}, \kappa)$ yields an ergodic system which has no $b_1 \cdot \ldots \cdot b_k$-root of unity, $k \geq 1$, in its spectrum.

(For example, if each such joining is totally ergodic, then (2.19) can be applied to an arbitrary $\mathscr{B}$-free system.) We recall that (2.19) forces κ to have zero entropy (by Smorodinsky-Thouvenot's theorem [14]).

Since every ergodic self-joining of $(T, \Omega, \mathbb{P})$ is a graph joining, (2.19) yields (2.16) for the relevant systems. Note also that (2.19) is the double disjointness condition of $(S, \{0,1\}^{\mathbb{Z}}, \kappa)$ with $(T, \Omega, \mathbb{P})$ from [6]. Thus, we have shown the following:

Corollary 2.2.4. *If κ satisfies (2.19) then $(S, X_\eta, \nu_{\mathscr{B}} * \kappa)$ is isomorphic to the Cartesian product $(T \times S, X_\eta \times \{0,1\}^{\mathbb{Z}}, \nu_{\mathscr{B}} \otimes \kappa)$.*

Remark 2.2.5. If $(S, \{0,1\}^{\mathbb{Z}}, \kappa)$ represents an irrational rotation, (2.19) is clearly satisfied, but there are many weakly mixing systems satisfying (2.19), e.g.: Gaussian systems GAG [10], simple systems [7] and factors of such systems, in particular, horocycle flows [16].

Remark 2.2.6. We will give now a direct proof of the fact that whenever κ represents an irrational rotation then we can filter out both coordinates, i.e. M is an isomorphism. Indeed, we take for $J \subset \mathbb{T}$ an interval. Then the rectangle $C \times J$ is in the smallest invariant σ-algebra $\mathcal{G}$ which makes the map

$$(\omega, z) \mapsto (\mathbb{1}_{C \times J}((T \times R_\alpha)^n(\omega, z)))_{n \in \mathbb{Z}}$$

measurable (cf. (2.14)). Given $\varepsilon > 0$ we can ε-approximate, whenever $k \geq 1$ is large enough, the set C by the levels of a T-tower (unique up to cyclic permutation of the levels) of height $M_k := b_1 \cdot \ldots \cdot b_k$ which fulfills the whole space. If we fix such a k and take any $1 \leq m < M_k$ then we can find a sequence $(n_i)_{i \geq 1}$ such that

$$n_i = m \mod M_k \text{ and } n_i \alpha \to 0. \tag{2.20}$$

Indeed, this is a consequence of the fact that $(\ell M_k + m)\alpha$ is close to zero if and only if $\ell(M_k \alpha)$ is close to $-m\alpha$ and the rotation by $M_k \alpha$ is minimal. Since

$(T^{n_i} \times S^{n_i})(C \times J) \in \mathcal{G}$, it easily follows by (2.20) that $\Omega \times J \in \mathcal{G}$, which means that we can filter out the second coordinate.

In order to obtain $C \times \mathbb{T} \in \mathcal{G}$ we proceed as follows. We ε-approximate the set C by the levels of the tower of height M_k. Now, consider $R_{M_k\alpha}$. Since J is an interval, we can find $n_1 < \ldots < n_r$, so that $\lambda_{\mathbb{T}}(\bigcup_{j=1}^{r} R^{n_j}_{M_k\alpha} J) > 1-\varepsilon$; here it is important that r depends only on J and not on $M_k\alpha$, r is "comparable" with $1/|J|$. Since r is fixed, we can easily see that whatever the numbers $n_1 < \ldots < n_r$ are, the set $\bigcap_{j=1}^{r} T^{n_j M_k} C$ will be ε-close to C. In this way, we obtain that $C \times \mathbb{T} \in \mathcal{G}$ and hence M is an isomorphism.

2.2.2 Rational discrete spectrum

We begin this section with two examples, showing the basic relations between the Mirsky measures for various free systems, under some additional assumptions on the sequences determining these systems.

Example 1. Let X_η and $X_{\eta'}$ be two free systems, with $\mathscr{B} = \{b_k : k \geq 1\}$ and $\mathscr{B}' = \{b'_k : k \geq 1\}$ respectively, and assume that $b'_k | b_k$ for each $k \geq 1$. Then clearly $\eta' \leq \eta$. In particular, each block that occurs on η' is dominated by a block that occurs on η, whence

$$X_{\eta'} \subset X_\eta. \tag{2.21}$$

Therefore, $\nu_{\mathscr{B}'} \in \mathcal{P}^e(S, X_\eta)$ is a measure which yields a dynamical system whose spectrum is "incomplete" in the sense that it is smaller than the whole group of b_k-roots of unity, $k \geq 1$.

Example 2. Now, assume that $\mathscr{B} = \{b_k : k \geq 1\}$ is a free system and take a subset $\overline{\mathscr{B}} = \{\overline{b}_k : k \geq 1\}$ with $\overline{b}_k = b_{n_k}$. It follows that

$$X_\eta \subset X_{\overline{\eta}}. \tag{2.22}$$

Now, we observe a different phenomenon than in Example 1. A larger $\overline{\mathscr{B}}$-free system has an invariant measure, namely the Mirsky measure of (S, X_η), which yields a system whose spectrum is larger than the "expected" one. In fact, the larger system has a smaller underlying odometer: $(\overline{T}, \overline{\Omega}, \overline{\mathbb{P}})$ is a factor of $(T, \Omega, \mathbb{P})$.

Remark 2.2.7. In view of the above two examples, one might expect that the condition that $X_\eta \subset X_{\eta'}$ can be expressed in terms of some relation between the sets $\mathscr{B}$ and $\mathscr{B}'$. This is indeed the case, see Proposition 2.3.1 and 2.3.5 for a complete charaterization.

Proposition 2.2.8. *Assume that $\mathcal{P}^e(S, X_\eta) \ni \nu \neq \delta_{(\ldots,0,0,\ldots)}$. The dynamical system (S, X_η, ν) has an infinite rational discrete spectrum. More precisely, the discrete spectrum part contains, for each $k \geq 1$, all b'_k-roots of unity for some $1 < b'_k | b_k$.*[12]

In order to prove the above proposition, we will use a refinement of the approach taken in [12]. Let us introduce first some notation which will be also

[12]Notice that $(b'_k, b'_\ell) = 1$ whenever $k \neq \ell$ by (0.1).

used later. Fix $\delta_{(\dots,0,0,\dots)} \neq \nu \in \mathcal{P}^e(S, X_\eta)$. Given $k \geq 1$ and $1 \leq s_k \leq b_k - 1$, set

$$Y_{k,s_k} := \{x \in X_\eta : |\text{supp}(x) \bmod b_k| = b_k - s_k\}.$$

Then Y_{k,s_k} is Borel and $SY_{k,s_k} = Y_{k,s_k}$. By ergodicity, for each $k \geq 1$ there is exactly one s_k such that $\nu(Y_{k,s_k}) = 1$. Now, for $a_i \in \mathbb{Z}/b_k\mathbb{Z}$, $i = 1, \dots, s_k$, with $a_i \neq a_j$ whenever $i \neq j$, we set

$$Y_{k,s_k;a_1,\dots,a_{s_k}} := \{x \in X_\eta : \text{supp}(x) \bmod b_k = \mathbb{Z}/b_k\mathbb{Z} \setminus \{a_1, \dots, a_{s_k}\}\} \subset Y_{k,s_k}.$$

For each $k \geq 1$, any two sets of such form are either disjoint or they coincide. Moreover, their union gives Y_{k,s_k}. It follows that there exists $(a_1^k, \dots, a_{s_k}^k)$ such that $\nu(Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}) > 0$. Since $\text{supp}(Sx) = \text{supp}(x) - 1$, we have

$$SY_{k,s_k;a_1^k,\dots,a_{s_k}^k} = Y_{k,s_k;a_1^k-1,\dots,a_{s_k}^k-1}. \tag{2.23}$$

Let

$$b_k' := \min\{j \geq 1 : \{a_1^k, \dots, a_{s_k}^k\} = \{a_1^k - j, \dots, a_{s_k}^k - j\}\} \tag{2.24}$$

and note that $b_k' \geq 2$. Clearly, $S^{b_k'}Y_{k,s_k;a_1^k,\dots,a_{s_k}^k} = Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}$ and the sets

$$Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}, SY_{k,s_k;a_1^k,\dots,a_{s_k}^k}, \dots, S^{b_k'-1}Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}$$

are pairwise disjoint. Moreover, by ergodicity,

$$\nu\Big(\bigcup_{j=0}^{b_k'-1} S^j Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}\Big) = 1.$$

Since $S^{b_k}Y_{k,s_k;a_1^k,\dots,a_{s_k}^k} = Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}$, we have $b_k'|b_k$. Finally, for $\underline{s} = (s_k)_{k\geq 1}$, we set $Y_{\underline{s}} := \bigcap_{k\geq 1} Y_{k,s_k}$.

Proof of Proposition 2.2.8. It suffices to notice that for $1 \leq s_k \leq b_k - 1$ and $\{a_1^k, \dots, a_{s_k}^k\}$ chosen so that $\nu(Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}) > 0$, and b_k' given by (2.24), the partition of Y_{k,s_k} into sets

$$S^j Y_{k,s_k;a_1^k,\dots,a_{s_k}^k},\ 0 \leq j \leq b_k' - 1$$

is a Rokhlin tower fulfilling the whole space, whence the b_k'-root of unity is an eigenvalue of (S, X_η, ν). □

We will give now another proof of Proposition 2.2.8. For this, we will need the following lemma:

Lemma 2.2.9. *Assume that $\mathcal{P}^e(S, X_\eta) \ni \nu \neq \delta_{(\dots,0,0,\dots)}$. Denote by R_k the rotation $z \mapsto z+1$ on $\mathbb{Z}/b_k\mathbb{Z}$ (considered as an ergodic system). Then (S, X_η, ν) is not disjoint with R_k.*

Proof. Suppose that (S, X_η, ν) is disjoint (see [5], [7]) with R_k. Let $y \in X_\eta$ be a generic point for ν. Since y is admissible, we can pick $a_k \in \mathbb{Z}/b_k\mathbb{Z}$ which does not belong to the support of y mod b_k. Let $z \in \{0, 1\}^{\mathbb{Z}}$ be such that

$$z(n) = 0 \iff n = a_k \bmod b_k.$$

This point is clearly generic for the periodic measure $\Delta_k := \frac{1}{b_k}\sum_{j=0}^{b_k-1}\delta_{S^j z}$ and the resulting dynamical system is isomorphic to R_k. Moreover, since $y(a_k + \ell b_k) = 0$ for each $\ell \in \mathbb{Z}$,

$$y \leq z. \tag{2.25}$$

By the disjointness assumption, $(y, z) \in X_\eta \times \{0,1\}^{\mathbb{Z}}$ is a generic point for the product measure $\nu \otimes \Delta_k$. But $\nu(C_0^1) > 0$ (since $\nu \neq \delta_{(\dots,0,0,\dots)}$) and $\Delta_k(C_0^0) > 0$ so the product measure of $C_0^1 \times C_0^0$ is positive while, by (2.25), no point $(S^i y, S^i z)$ belongs to $C_0^1 \times C_0^0$, a contradiction. □

Second proof of Proposition 2.2.8. It follows from Lemma 2.2.9 that for each $k \geq 1$ we have no disjointness of (S, X_η, ν) with R_k. This means that (S, X_η, ν) must have, for each $k \geq 1$, a nontrivial common factor with R_k, equivalently a common nontrivial eigenvalue. □

Remark 2.2.10. Consider $b_k = p_k^2$, $k \geq 1$ and then the corresponding square free system. By Proposition 2.2.8, any nontrivial ergodic measure must have at least all p_k-roots of unity in the spectrum of the corresponding dynamical system. A natural question arises whether there is a measure which yields the dynamical system with precisely such a spectrum.[13] We will show later that such a measure cannot exist, see Corollary 2.2.27.

In connection with the above remark, we consider the following example:

Example 3. Let $\mathscr{B} = \{p_{i_k} \in \mathscr{P} : k \geq 1\}$, so that $\sum_{k\geq 1} 1/p_{i_k} < +\infty$. Then

$$\mathscr{P} \setminus \mathscr{B} = \{q_i : i \geq 1\}.$$

Let $\kappa \in \mathcal{P}^e(S, \{0,1\}^{\mathbb{Z}})$ be such that $(S, \{0,1\}^{\mathbb{Z}}, \kappa)$ has discrete spectrum with the group of eigenvalues equal to the $q_1 \cdot \ldots \cdot q_i$-roots of unity, $i \geq 1$ (such κ exists by Krieger's theorem [7]). Now, $(S, \{0,1\}^{\mathbb{Z}}, \kappa)$ has discrete spectrum, so each ergodic self-joining of it is a graph joining and therefore (2.19) is satisfied. Now, by Corollary 2.2.4, the measure $\rho := \nu_{\mathscr{B}} * \kappa$ is such that the spectrum of $(S, \{0,1\}^{\mathbb{Z}}, \rho)$ is equal to all roots of unity of order $p_1 \cdot \ldots \cdot p_k$, $k \geq 1$.

2.2.3 Filtering $\mathbb{P}$ from $\nu_{\mathscr{B}} * \kappa$

Recall that since we have an equivariant Borel map $\theta\colon Y \to \Omega$, it follows immediately that for any $\nu \in \mathcal{P}^e(S, Y)$ the corresponding dynamical system (S, X_η, ν) has $(T, \Omega, \mathbb{P})$ as its factor. A natural question arises whether each measure $\nu \in \mathcal{P}^e(S, X_\eta)$ such that the point spectrum of (S, X_η, ν) contains the $b_1 \cdot \ldots \cdot b_k$-roots of unity, $k \geq 1$ must be concentrated on Y. We will see in Example 4 below that this is not the case.

Remark 2.2.11. Note that the Mirsky measure $\nu_{\mathscr{B}}$ is concentrated on $Y = \bigcap_{k\geq 1} Y_{k,1}$. Assume that $1 < b_k' | b_k$, $b_k/b_k' \geq 2$, $k \geq 1$, so that $\mathscr{B}' := \{b_k' : k \geq 1\}$ satisfies (0.1). Then

$$\nu_{\mathscr{B}'}\Big(\bigcap_{k\geq 1} Y_{k,s_k}\Big) = 1, \tag{2.26}$$

[13]Notice that this question cannot be answered following the path taken in Example 1 since $\sum_{k\geq 1} 1/p_k = \infty$.

where $s_k \geq b_k/b'_k \geq 2$, $k \geq 1$. Indeed, if $a \notin \operatorname{supp}(y) \bmod b'_k$ then $a + jb'_k \notin \operatorname{supp}(y) \bmod b_k$ for $j = 0, 1, \ldots, b_k/b'_k - 1$. Moreover, $\nu_{\mathscr{B}'}(Y) = 0$ since $\mathscr{B}' \neq \mathscr{B}$.

Remark 2.2.12. Notice that the Mirsky measures $\nu_{\mathscr{B}'}$ in Example 1 vanish on the set $Y = Y(X_\eta)$. Note however, that $\nu_{\mathscr{B}}(\overline{Y}) = 1$ for $\overline{Y} = Y(X_{\overline{\eta}})$ in Example 2.

Example 4. Consider $\mathscr{B} = \{b_k : k \geq 1\}$ a free system in which $b_k = b'_k \cdot c_k$, $(b'_k, c_k) = 1$, $b'_k \geq 2$ for $k \geq 1$,

$$\sum_{k \geq 1} 1/b'_k < +\infty \text{ and } \sum_{k \geq 1} 1/c_k < +\infty.$$

We then obtain two more free systems:

$$\widetilde{\mathscr{B}} = \{b'_1, c_1, b'_2, c_2, \ldots\} \text{ and } \mathscr{B}' = \{b'_1, b'_2, \ldots\}.$$

Using (2.21), (2.22) and Remark 2.2.11, we obtain

$$X_{\widetilde{\mathscr{B}}} \subset X_{\mathscr{B}'} \subset \bigcap_{k \geq 1} Y_{k, s_k}(X_{\mathscr{B}}) \subset X_{\mathscr{B}},$$

where $s_k \geq 2$ for $k \geq 1$. But the point spectra of $(S, X_{\widetilde{\mathscr{B}}}, \nu_{\widetilde{\mathscr{B}}})$ and $(S, X_{\mathscr{B}}, \nu_{\mathscr{B}})$ are the same. Finally, $\nu_{\widetilde{\mathscr{B}}}(Y) = 0$.

Now, we give a condition on κ which implies that the corresponding product type measure $\nu_{\mathscr{B}} * \kappa$ is such that (T, Ω) is a factor of $(S, X_\eta, \nu_{\mathscr{B}} * \kappa)$. It is unclear, whether this condition implies that $(\nu_{\mathscr{B}} * \kappa)(Y) = 1$.

Proposition 2.2.13. *If $\kappa \in \mathcal{P}^e(S, \{0,1\}^{\mathbb{Z}})$ yields a totally ergodic system then $(S, X_\eta, \nu_{\mathscr{B}} * \kappa)$ has full rational spectrum, i.e. $(T, \Omega, \mathbb{P})$ is its factor.*

Proof. We will proceed as in Remark 2.2.6, detailing more on C and the towers for the odometer $(T, \Omega, \mathbb{P})$ (which allows us to bypass the existence of r in Remark 2.2.6).

Assume that $(S, \{0,1\}^{\mathbb{Z}}, \kappa)$ is totally ergodic and let $J := C_0^1$. It follows from (2.14) that $C \times J \in \mathcal{G} = (M \circ (\varphi \times Id))^{-1}(\mathcal{B}(X_\eta))$. We will show that also $C \times \{0,1\}^{\mathbb{Z}} \in \mathcal{G}$. For this aim, consider

$$\bigcup_{j=0}^{R-1} T^{jM_k} C \times S^{jM_k} J, \tag{2.27}$$

where $M_k := b_1 \cdot \ldots \cdot b_k$ and $R \geq 1$. Consider a tower for T of height M_k, with the set $\{\omega \in \Omega : \omega(1) = \ldots = \omega(k) = 0\}$ as the base. The levels of this tower are sets of the form $\{\omega \in \Omega : \omega(1) = i_1, \ldots, \omega(k) = i_k\}$, i.e. they are indexed by $(i_1, \ldots, i_k) \in \mathbb{Z}/b_1\mathbb{Z} \times \ldots \times \mathbb{Z}/b_k\mathbb{Z}$. Whenever the level $(i_1, \ldots, i_k)$ contains $i_s = 0$, it is disjoint with C. If at all positions $(i_1, \ldots, i_k)$ we see non-zero values then C is contained in such a level. More than that, we can compute the fraction of the level it occupies (which is smaller than $\sum_{t=k+1} 1/b_t$).

If R is large enough then $\kappa\left(\bigcup_{j=0}^{R-1} S^{jM_k} J\right)$ is as large as we want (since S^{M_k} is ergodic). We need to show that the set (2.27) is close to $C \times \{0,1\}^{\mathbb{Z}}$. Indeed, we have[14]

$$\Big(\bigcup_{j=0}^{R-1} T^{jM_k}C \times S^{jM_k}J\Big)\triangle\Big(C\times\{0,1\}^{\mathbb{Z}}\Big)$$
$$\subset \Big(\Big(\bigcup_{j=0}^{R-1} T^{jM_k}C\setminus C\Big)\times\{0,1\}^{\mathbb{Z}}\Big)\cup\Big(\Omega\times\Big(\{0,1\}^{\mathbb{Z}}\setminus\bigcup_{j=0}^{R-1}S^{jM_k}J\Big)\Big),$$

whence

$$\mathbb{P}\otimes\kappa\Big(\Big(\bigcup_{j=0}^{R-1} T^{jM_k}C \times S^{jM_k}J\Big)\triangle\Big(C\times\{0,1\}^{\mathbb{Z}}\Big)\Big)$$
$$\leq \mathbb{P}\Big(\Big(\bigcup_{j=0}^{R-1} T^{jM_k}C\Big)\setminus C\Big)+\varepsilon.$$

Moreover, since each T^{jM_k} sends the level of the tower into itself, the levels that were disjoint with C remain disjoint and the first summand above is not larger than the approximation of C given by the union of levels containing C. □

2.2.4 Ergodic invariant measures on X_η are of joining type

In Section 2.1.1, we have proved that each measure $\nu\in\mathcal{P}^e(S,Y)$ is of joining type, more precisely, $\nu=M_*(\widetilde{\rho})$, where $\widetilde{\rho}\in\mathcal{P}^e(S\times S,X_\eta\times\{0,1\}^{\mathbb{Z}})$ satisfies $\widetilde{\rho}|_{X_\eta}=\nu_{\mathscr{B}}$. One could now expect that the converse also holds. That is, whenever we have $\widetilde{\rho}\in\mathcal{P}^e(S\times S,X_\eta\times\{0,1\}^{\mathbb{Z}})$ which is an ergodic joining of $\nu_{\mathscr{B}}$ and $\kappa:=\widetilde{\rho}|_{\{0,1\}^{\mathbb{Z}}}$ then $M_*(\widetilde{\rho})\in\mathcal{P}(S,Y)$ (in particular, the corresponding dynamical system has "full" rational discrete spectrum). This is however not true:

Example 5. Consider the situation, where $\mathscr{B}'=\{b_i':i\geq 1\}$ yields a free systems, with $1<b_i'|b_i$, $i\geq 1$. Let

$$\pi:\Omega\to\Omega':=\Pi_{i=1}\mathbb{Z}/b_i'\mathbb{Z},\;\;\pi((\omega(k))_{k\geq 1})=(\omega'(k))_{k\geq 1},$$

$\omega'(k)=\omega(k) \bmod b_k'$, $k\geq 1$. Then π is equivariant and $\pi_*(\mathbb{P})=\mathbb{P}'$. The measure $M_*(\lambda)$, where $\lambda=\nu_{\mathscr{B}}\vee\nu_{\mathscr{B}'}$ stands for the diagonal embedding of $(X_{\eta'},\nu_{\mathscr{B}'})$ in $(X_\eta,\nu_{\mathscr{B}})$, is concentrated on the set

$$W:=\{\varphi(\omega)\cdot\varphi'(\omega'):(\omega,\omega')\in\Omega\times\Omega',\pi(\omega)=\omega'\}.$$

However, whenever $\pi(\omega)=\omega'$, we have $\varphi'(\omega')\leq\varphi(\omega)$. It follows that for each $n\in\mathbb{Z}$, $\varphi(\omega)(n)\cdot\varphi'(\omega')(n)=\varphi'(\omega')(n)$ and therefore

$$M_*(\nu_{\mathscr{B}}\vee\nu_{\mathscr{B}'})=\nu_{\mathscr{B}'}.$$

We will show now, how to extend Theorem 2.1.1 to obtain Theorem 0.0.2, thus providing a description of all invariant measures for $\mathscr{B}$-free systems. As a

[14] We use here the following: whenever $C,A_i\subset X$, $D,B_i\subset Y$, we have $\bigcup_{i=1}^R(A_i\times B_i)\triangle(C\times D)\subset\Big(\Big(\bigcup_{i=1}^R A_i\triangle C\Big)\times Y\Big)\cup\Big(X\times\Big(\bigcup_{i=1}B_i\triangle D\Big)\Big)$.

matter of fact, the proof goes along the same lines as the proof of Theorem 2.1.1. However, to see that similar arguments are indeed valid, we need to define several objects. For $\underline{s}=(s_k)_{k\geq 1}$, $\underline{a}=(\underline{a}^k)_{k\geq 1}$, $\underline{a}^k=\{a_1^k,\dots,a_{s_k}^k\}$, cf. (2.23) and (2.24), let

$$Y_{\underline{s},\underline{a}} := \bigcap_{k\geq 1}\Big(\bigcup_{j=0}^{b_k'-1} S^j Y_{k,s_k;a_1^k,\dots,a_{s_k}^k}\Big).$$

and

$$\overline{Y}_{\underline{s},\underline{a}} := \{x\in\{0,1\}^{\mathbb{Z}} : \text{ any block on } x \text{ occurs on } Y_{\underline{s},\underline{a}}\}.$$

Notice that we can assume without loss of generality that $a_1^k=0$ for each $k\geq 1$. Moreover, since the sets $Y_{\underline{s},\underline{a}}$ are Borel and shift-invariant, for each measure $\nu\in\mathcal{P}^e(S,X_\eta)$, there exist $\underline{s},\underline{a}$ such that $\nu(Y_{\underline{s},\underline{a}})=1$.

Remark 2.2.14. Notice that there exists $\underline{s}$ such that $Y_{\underline{s}}=\emptyset$. Indeed, fix $k_0\geq 1$ and let

$$s_{k_0}:=b_{k_0}-2,\ s_k:=b_k-1 \text{ for } k\neq k_0.$$

Suppose that $Y_{\underline{s}}\neq\emptyset$ and take $x\in Y_{\underline{s}}$. Then there exist $n,m\in\operatorname{supp}(x)$ such that $n-m\not\equiv 0 \bmod b_{k_0}$. This is however impossible since $n-m\equiv 0 \bmod b_k$ for $k\geq 0$, i.e. $n=m$.

From now on, we will assume that $Y_{\underline{s},\underline{a}}\neq\emptyset$. Recall the following result.

Proposition 2.2.15 (see [12], discussion before Lemma 3.3)**.** *We have*

$$h_{top}(S,\overline{Y}_{\underline{s},\underline{a}})=\log 2\cdot\prod_{k\geq 1}\left(1-\frac{s_k}{b_k}\right).$$

Let $x\in\overline{Y}_{\underline{s},\underline{a}}$, fix $K\geq 1$, $n\in\mathbb{Z}$ and consider $x[n,n+b_1\cdot\ldots\cdot b_K-1]$. Since, by Chinese Remainder Theorem, the map

$$W\colon\{n,n+1,\dots,n+b_1\cdot\ldots\cdot b_K-1\}\to\prod_{k=1}^K\mathbb{Z}/b_k\mathbb{Z}$$

given by $W(m)=(m \bmod b_1,\dots,m \bmod b_K)$ is a bijection, therefore

$$\begin{aligned}|\operatorname{supp}(x)\cap&\{n,n+1,\dots,n+b_1\cdot\ldots\cdot b_K-1\}|\\ &=|W(\operatorname{supp}(x)\cap\{n,n+1,\dots,n+b_1\cdot\ldots\cdot b_K-1\})|\\ &\leq\prod_{k=1}^K|\operatorname{supp}(x)\cap\{n,n+1,\dots,n+b_1\cdot\ldots\cdot b_K-1\} \bmod b_k|\\ &\leq\prod_{k=1}^K|\operatorname{supp}(x) \bmod b_k|\leq\prod_{k=1}^K(b_k-s_k).\end{aligned}$$

It follows that

$$\frac{|\operatorname{supp}(x)\cap\{n,n+1,\dots,n+b_1\cdot\ldots\cdot b_K-1\}|}{b_1\cdot\ldots\cdot b_K}\leq\prod_{k=1}^K(1-\frac{s_k}{b_k}). \tag{2.28}$$

We will also need the following simple lemma.

Lemma 2.2.16. *Let $X \subset \{0,1\}^{\mathbb{Z}}$ be closed and shift invariant, and let $\widetilde{X} \subset \{0,1\}^{\mathbb{Z}}$ be the smallest hereditary system containing X. Suppose additionally that for some $d, d' \geq 0$, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $B \in \{0,1\}^n$ which occur on X*

$$\frac{|\{i : B(i) = 1\}|}{n} \in (d - \varepsilon, d' + \varepsilon).$$

Then $d \log 2 \leq h_{top}(S, \widetilde{X}) \leq h_{top}(S, X) + d' \log 2$.

Proof. Let $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ be as in the assumptions of the lemma. Given $n \geq 1$, denote by $p_n(\widetilde{X})$ and $p_n(X)$ the number of n-blocks occurring on X and X, respectively. Notice that the following procedure yields all n-blocks occurring on $\widetilde{X}$:

(i) pick an n-block occurring on X,

(ii) replace some of the 1s with 0s.

Therefore,

$$p_n(\widetilde{X}) \leq p_n(X) \cdot 2^{n(d'+\varepsilon)}$$

for $n \geq n_0$. On the other hand, by fixing one particular n-block occurring on X and exhausting all possibilities given by (ii) of the above procedure, we obtain

$$2^{n(d-\varepsilon)} \leq p_n(\widetilde{X}) \text{ for } n \geq n_0.$$

This implies

$$(d - \varepsilon) \log 2 \leq h_{top}(S, \widetilde{X}) \leq h_{top}(S, X) + (d' + \varepsilon) \log 2$$

and the result follows. □

As an immediate consequence Proposition 2.2.15, (2.28) and Lemma 2.2.16 (with $d = d' = 0$), we obtain:

Corollary 2.2.17. *If $h_{top}(S, \overline{Y}_{\underline{s},\underline{a}}) = 0$ then also the hereditary subshift determined by $\overline{Y}_{\underline{s},\underline{a}}$ is of zero topological entropy.*

Recall (see [9]) that hereditary subshifts of zero topological entropy are uniquely ergodic with $\delta_{(\dots,0,0,0,\dots)}$ being the only invariant measure. Thus, we have shown the following:

Corollary 2.2.18. *If $h_{top}(S, \overline{Y}_{\underline{s},\underline{a}}) = 0$ then $\mathcal{P}(S, \overline{Y}_{\underline{s},\underline{a}}) = \{\delta_{(\dots,0,0,0,\dots)}\}$. In particular, $\mathcal{P}(S, Y_{\underline{s},\underline{a}}) = \emptyset$.*

Remark 2.2.19. Notice that it is possible that $h_{top}(S, \overline{Y}_{\underline{s},\underline{a}}) = 0$ and $Y_{\underline{s},\underline{a}} \neq \emptyset$. Indeed, if $s_k = b_k - 1$ for all $k \geq 1$ then

$$Y_{\underline{s},\underline{a}} = \{S^n(\dots, 0, 0, 1, 0, 0, \dots) : n \in \mathbb{Z}\},$$
$$\overline{Y}_{\underline{s},\underline{a}} = \{S^n(\dots, 0, 0, 1, 0, 0, \dots) : n \in \mathbb{Z}\} \cup \{(\dots, 0, 0, 0, \dots)\}.$$

Fix $\underline{s}, \underline{a}$ and suppose that $h_{top}(\overline{Y}_{\underline{s},\underline{a}}) > 0$. Let

$$\Omega_{\underline{s},\underline{a}} = \prod_{k\geq 1} \mathbb{Z}/b'_k\mathbb{Z}, \text{ where } b'_k \text{ are as in (2.24)}.$$

We define $\varphi_{\underline{s},\underline{a}}\colon \Omega_{\underline{s},\underline{a}} \to \{0,1\}^{\mathbb{Z}}$ by

$$\varphi_{\underline{s},\underline{a}}(\omega)(n) = \begin{cases} 1 & \text{if } \omega(k) - a_i^k + n \neq 0 \bmod b_k \text{ for all } k \geq 1, 1 \leq i \leq s_k, \\ 0 & \text{otherwise.} \end{cases}$$

Fix $k \geq 1$, $z \in \mathbb{Z}/b_k\mathbb{Z}$ and let

$$E_{k,z}^{\underline{s},\underline{a}} := \{\omega \in \Omega_{\underline{s},\underline{a}} : \varphi_{\underline{s},\underline{a}}(\omega)(-z + sb'_k) = 0 \text{ for all } s \geq 1\}.$$

Next, we define

$$(\Omega_{\underline{s},\underline{a}})'_0 := \Omega_{\underline{s},\underline{a}} \setminus \bigcup_{k\geq 1} \bigcup_{z\in\mathbb{Z}/b'_k\mathbb{Z}} \left(\left(\bigcap_{i=1}^{s_k} E_{k,z-a_i^k}^{\underline{s},\underline{a}} \right) \setminus \{\omega \in \Omega_{\underline{s},\underline{a}} : \omega(k) = z\} \right)$$

and we put

$$(\Omega_{\underline{s},\underline{a}})_0 := \bigcap_{k\in\mathbb{Z}} T^k(\Omega_{\underline{s},\underline{a}})'_0$$

We claim that $\varphi_{\underline{s},\underline{a}}$ is 1-1 on $(\Omega_{\underline{s},\underline{s}})_0$ (cf. Lemma 1.2.1) Moreover,

$$\mathbb{P}_{\underline{s},\underline{a}} \left(\left(\bigcap_{i=1}^{s_k} E_{k,z-a_i^k}^{\underline{s},\underline{a}} \right) \setminus \{\omega \in \Omega_{\underline{s},\underline{a}} : \omega(k) = z\} \right) = 0, \tag{2.29}$$

where $\mathbb{P}_{\underline{s},\underline{a}}$ is the normalized Haar measure on $\Omega_{\underline{s},\underline{a}}$. This shows that

$$\mathbb{P}_{\underline{s},\underline{a}}((\Omega_{\underline{s},\underline{a}})_0) = 1.$$

The proof of (2.29) is essentially the same as the proof of Proposition 3.2 in [1]. One of the important steps in this proof is to show that

$$\begin{aligned} (2.30)\quad & \mathbb{P}_{\underline{s},\underline{a}}(\{\omega \in \Omega_{\underline{s},\underline{a}} : \varphi_{\underline{s},\underline{a}}(\omega)(n) = 0\}) \\ &= \mathbb{P}_{\underline{s},\underline{a}} \left(\bigcap_{k\geq 1} \{\omega \in \Omega_{\underline{s},\underline{a}} : w(k) - a_i^k + n \neq 0 \bmod b_k \text{ for all } 1 \leq i \leq s_k\} \right) \end{aligned}$$

is strictly positive. To see that this is indeed the case, notice first that for any set $A \subset \mathbb{Z}/b_k\mathbb{Z}$ such that $A + b'_k = A \bmod b'_k$, we have

$$\begin{aligned} (2.31)\quad & \{\omega \in \mathbb{Z}/b'_k\mathbb{Z} : \omega(k) \neq a \bmod b_k \text{ for all } a \in A\} \\ &= \{\omega \in \mathbb{Z}/b'_k\mathbb{Z} : \omega(k) \neq a \bmod b'_k \text{ for all } a \in A \cap \mathbb{Z}/b'_k\mathbb{Z}\}. \end{aligned}$$

Moreover, since $A = \bigcup_{j=0}^{b_k/b'_k-1} A \cap \{\{0,\ldots,b'_k - 1\} + j\}$ and $|A \cap \{\{0,\ldots,b'_k - 1\} + j\}|$ does not depend on j, we obtain $|A| = b'_k \cdot |A \cap \mathbb{Z}/b'_k\mathbb{Z}|$. Applying this to $A = \{a_i^k - n : 1 \leq i \leq s_k\}$ we conclude that

$$|\{a_i^k - n : 1 \leq i \leq s_k\} \cap \mathbb{Z}/b'_k\mathbb{Z}| = \frac{s_k \cdot b'_k}{b_k}. \tag{2.32}$$

Using (2.31) and (2.32), we obtain

$$|\left\{\omega \in \mathbb{Z}/b'_k\mathbb{Z} : w(k) - a_i^k + n \neq 0 \bmod b_k \text{ for all } 1 \le i \le s_k\right\}| = b'_k - \frac{s_k \cdot b'_k}{b_k}.$$

This, in view of (2.30), gives indeed

$$\mathbb{P}_{\underline{s},\underline{a}}(\{\omega \in \Omega_{\underline{s},\underline{a}} : \varphi_{\underline{s},\underline{a}}(\omega)(n) = 0\}) = \prod_{k\ge 1}\left(1 - \frac{s_k}{b_k}\right) = h_{top}(\overline{Y}_{\underline{s},\underline{a}}) > 0.$$

Remark 2.2.20. Notice that the above calculation shows in particular that

$$\prod_{k\ge 1}\left(1 - \frac{1}{b'_k}\right) \ge \prod_{k\ge 1}\left(1 - \frac{s_k \cdot \frac{b'_k}{b_k}}{b'_k}\right) > 0,$$

i.e. $\{b'_k : k \ge 1\}$ yields a free system.

We also define $\theta_{\underline{s},\underline{a}} \colon Y_{\underline{s},\underline{a}} \to \Omega_{\underline{s},\underline{a}}$ in the following way:

$$\theta_{\underline{s},\underline{a}}(y) = \omega \iff -\omega(k) + a_i^k \notin \operatorname{supp}(y) \bmod b_k \text{ for all } 1 \le i \le s_k.$$

Moreover, denote by $T_{\underline{s},\underline{a}} \colon \Omega_{\underline{s},\underline{a}} \to \Omega_{\underline{s},\underline{a}}$ the map given by

$$T_{\underline{s},\underline{a}}\omega = \omega + (1,1,\ldots) = (\omega(1)+1, \omega(2)+1, \ldots),$$

where $\omega = (\omega(1), \omega(2), \ldots)$.

Lemma 2.2.21 (cf. Lemma 1.2.2)**.** *We have:*

(i) $\theta_{\underline{s},\underline{a}}$ *is equivariant, i.e.* $T_{\underline{s},\underline{a}} \circ \theta_{\underline{s},\underline{a}} = \theta_{\underline{s},\underline{a}} \circ S$.

(ii) For each $\omega \in \Omega_{\underline{s},\underline{a}}$ *and* $y \in Y_{\underline{s},\underline{a}}$ *such that* $\theta(y) = \omega$, *we have* $y \le \varphi_{\underline{s},\underline{a}}(\omega)$.

(iii) $\varphi_{\underline{s},\underline{a}}((\Omega_{\underline{s},\underline{a}})_0) \subset Y_{\underline{s},\underline{a}}$ *(in particular,* $\theta_{\underline{s},\underline{a}} \circ \varphi_{\underline{s},\underline{a}}|_{\Omega_0} = id_{\Omega_0}$*).*

For $n \in \mathbb{N}$ let $M^{(n)} \colon (\{0,1\}^{\mathbb{Z}})^{\times n} \to \{0,1\}^{\mathbb{Z}}$ be given by

$$M^{(n)}((x_i^{(1)})_{i\in\mathbb{Z}}, \ldots, (x_i^{(n)})_{i\in\mathbb{Z}}) = (x_i^{(1)} \cdot \ldots \cdot x_i^{(n)})_{i\in\mathbb{Z}}.$$

Moreover, we define $M^{\infty} \colon (\{0,1\}^{\mathbb{Z}})^{\mathbb{N}} \to \{0,1\}^{\mathbb{Z}}$ as

$$M^{(\infty)}((x_i^{(1)})_{i\in\mathbb{Z}}, (x_i^{(2)})_{i\in\mathbb{Z}}, \ldots) = (x_i^{(1)} \cdot x_i^{(2)} \cdot \ldots)_{i\in\mathbb{Z}}.$$

Lemma 2.2.22. *We have* $(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}}) = M_*^{(\infty)}(\rho)$, *where* ρ *is a joining of a countable number of copies of* $(S, \{0,1\}^{\mathbb{Z}}, \nu_{\mathscr{B}'})$*.*

Proof. For $i \ge 1$ define $R^{(i)} \colon \Omega_{\underline{s},\underline{a}} \to \Omega_{\underline{s},\underline{a}}$ by $R^{(i)}(\omega) = (\omega(k) - \widetilde{a}_i^k)_{k\ge 1}$, where

$$\widetilde{a}_i^k = \begin{cases} a_i^k & \text{if } 1 \le i \le s_k, \\ a_{s_k}^k & \text{if } i > s_k. \end{cases}$$

It follows from (2.31) applied to $A = \{a_i^k - n : 1 \le i \le s_k\}$ that

$$\varphi_{\underline{s},\underline{a}}(\omega) = M^{(\infty)}(\varphi' \circ R^{(1)}(\omega), \varphi' \circ R^{(2)}(\omega), \ldots),$$

where $\varphi'\colon \Omega_{\underline{s},\underline{a}} \to \{0,1\}^{\mathbb{Z}}$ is given by (1.2) with $\mathscr{B}$ replaced with $\mathscr{B}'$. Thus,

$$(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}}) = M_*^{(\infty)} \circ (\varphi' \circ R^{(1)} \times \varphi' \circ R^{(2)} \times \dots)_*(\mathbb{P}_{\underline{s},\underline{a}}).$$

Since $R_*^{(i)}(\mathbb{P}_{\underline{s},\underline{a}}) = \mathbb{P}_{\underline{s},\underline{a}}$ for each $i \geq 1$, it follows that

$$\rho := (\varphi' \circ R^{(1)} \times \varphi' \circ R^{(2)} \times \dots)_*(\mathbb{P}_{\underline{s},\underline{a}})$$

is indeed a joining of a countable number of copies of $\varphi'_*(\mathbb{P}_{\underline{s},\underline{a}}) = \nu_{\mathscr{B}'}$. □

Lemma 2.2.23. *Let $\nu_1, \dots, \nu_n, \nu_{n+1} \in \mathcal{P}(S, \{0,1\}^{\mathbb{Z}})$. Then for any joinings*

- $\rho_{1,n} \in J((S, \{0,1\}^{\mathbb{Z}}, \nu_1), \dots, (S, \{0,1\}^{\mathbb{Z}}, \nu_n))$,
- $\rho_{(1,n),n+1} \in J((S, \{0,1\}^{\mathbb{Z}}, M_*^{(n)}(\rho_{1,n})), (S, \{0,1\}^{\mathbb{Z}}, \nu_{n+1}))$

there exist:

- $\rho_{2,n+1} \in J((S, \{0,1\}^{\mathbb{Z}}, \nu_2), \dots, (S, \{0,1\}^{\mathbb{Z}}, \nu_n), (S, \{0,1\}^{\mathbb{Z}}, n+1))$,
- $\rho_{1,(2,n+1)} \in J((S, \{0,1\}^{\mathbb{Z}}, \nu_1), (S, \{0,1\}^{\mathbb{Z}}, M_*^{(n)}(\rho_{2,n+1})))$

such that $M_*(\rho_{(1,n),n+1}) = M_*(\rho_{1,(2,n+1)})$.[15]

Proof. Clearly, $(S, \{0,1\}^{\mathbb{Z}}, M_*^{(n)}(\rho_{1,n}))$ is a factor of $(S^{\times n}, (\{0,1\}^{\mathbb{Z}})^{\times n}, \rho_{1,n})$. Let $\widehat{\rho}_{(1,n),n+1}$ be the relatively independent extension of $\rho_{(1,n),n+1}$ to a joining of $(S^{\times n}, (\{0,1\}^{\mathbb{Z}})^{\times n}, \rho_{1,n})$ and $(S, \{0,1\}^{\mathbb{Z}}, \nu_{n+1})$. Then

$$(M^{(n)} \times Id)_*(\widehat{\rho}_{(1,n),n+1}) = \rho_{(1,n),n+1}$$

and

$$\widehat{\rho}_{(1,n),n+1} \in J((S, \{0,1\}^{\mathbb{Z}}, \nu_1), (S^{\times n}, (\{0,1\}^{\mathbb{Z}})^{\times n}, \rho_{2,n+1})),$$

where $\rho_{2,n+1}$ is a projection of $\widehat{\rho}_{(1,n),n+1}$ onto the last n coordinates. Let

$$\rho_{1,(2,n+1)} := (Id \times M^{(n)})_*(\widehat{\rho}_{(1,n),n+1}).$$

Clearly, $\rho_{1,(2,n+1)} \in J((S, \{0,1\}^{\mathbb{Z}}, \nu_1), (S, \{0,1\}^{\mathbb{Z}}, M_*^{(n)}(\rho_{2,n+1})))$. Moreover,

$$\begin{aligned} M_*(\rho_{1,(2,n+1)}) &= M_* \circ (Id \times M^{(n)})_*(\widehat{\rho}_{(1,n),n+1}) \\ &= M_* \circ (M^{(n)} \times Id)_*(\widehat{\rho}_{(1,n),n+1}) = M_*(\rho_{(1,n),n+1}) \end{aligned}$$

and the assertion follows. □

Remark 2.2.24. The above lemma remains true when we consider infinite joinings, i.e. instead of $\nu_1, \dots, \nu_n$ we have $\nu_1, \nu_2, \dots$, and instead of $M^{(n)}$ we consider $M^{(\infty)}$.

[15] We could write this property as $M_*(M_*^{(n)}(\nu_1 \vee \dots \vee \nu_n) \vee \nu_{n+1}) = M_*(\nu_1 \vee M_*^{(n)}(\nu_2 \vee \dots \vee \nu_n \vee \nu_{n+1}))$. However, until we say which joining we mean by each symbol $\vee$, this expression has no concrete meaning.

Proof of Theorem 0.0.2. Fix $\nu \in \mathcal{P}^e(S, X_\eta)$ and let $\underline{s}, \underline{a}$ be such that $\nu(Y_{\underline{s},\underline{a}}) = 1$. In view of Lemma 2.2.22, Lemma 2.2.23 and Remark 2.2.24, what we need to show is that there exists $\widetilde{\rho} \in \mathcal{P}(S \times S, \{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}})$ such that the projection of $\widetilde{\rho}$ onto the first coordinate equals $(\varphi_{\underline{s},\underline{a}})_*(\mathbb{P}_{\underline{s},\underline{a}})$ and $M_*(\widetilde{\rho}) = \nu$.

The remaining part of the proof goes exactly along the same lines as the proof of Theorem 2.1.1, with the following modification: we need to replace some objects related to Y by their counterparts related to $Y_{\underline{s},\underline{a}}$. Namely, instead of $\Omega,\ \Theta,\ Y_\infty,\ \widetilde{T},\ \Omega_0,\ \Phi,\ Y_0$ and Ψ, we use

$$\Omega_{\underline{s},\underline{a}},\ \Theta_{\underline{s},\underline{a}},\ (Y_{\underline{s},\underline{a}})_\infty,\ \widetilde{T}_{\underline{s},\underline{a}},\ (\Omega_{\underline{s},\underline{a}})_0,\ \Phi_{\underline{s},\underline{a}},\ (Y_{\underline{s},\underline{a}})_0 \text{ and } \Psi_{\underline{s},\underline{a}},$$

where

- $\Theta_{\underline{s},\underline{a}} \colon Y_{\underline{s},\underline{a}} \to \Omega_{\underline{s},\underline{a}} \times \{0,1\}^{\mathbb{Z}}$ is given by $\Theta_{\underline{s},\underline{a}}(y) := (\theta_{\underline{s},\underline{a}}(y), \widehat{y}_{\varphi_{\underline{s},\underline{a}}(\theta_{\underline{s},\underline{a}}y)})$,
- $(Y_{\underline{s},\underline{a}})_\infty = \{y \in Y_{\underline{s},\underline{a}} : |\text{supp } \varphi_{\underline{s},\underline{a}}(\theta_{\underline{s},\underline{a}}(y)) \cap (-\infty, 0)| = |\text{supp } \varphi_{\underline{s},\underline{a}}(\theta_{\underline{s},\underline{a}}(y)) \cap (0,\infty)| = \infty\}$,
- $\widetilde{T}_{\underline{s},\underline{a}} \colon \Omega_{\underline{s},\underline{a}} \times \{0,1\}^{\mathbb{Z}} \to \Omega_{\underline{s},\underline{a}} \times \{0,1\}^{\mathbb{Z}}$ given by

$$\widetilde{T}_{\underline{s},\underline{a}}(\omega, x) = \begin{cases} (T_{\underline{s},\underline{a}}\omega, x) & \text{if } \varphi_{\underline{s},\underline{a}}(\omega)(0) = 0 \\ (T_{\underline{s},\underline{a}}\omega, Sx) & \text{if } \varphi_{\underline{s},\underline{a}}(\omega)(0) = 1, \end{cases}$$

- $\Phi_{\underline{s},\underline{a}}(\omega, x)$ is the unique element in X_η such that
  - (i) $\Phi_{\underline{s},\underline{a}}(\omega, x) \leq \varphi_{\underline{s},\underline{a}}(\omega)$,
  - (ii) $(\Phi_{\underline{s},\underline{a}}(\omega, x))\widehat{\ }_{\varphi_{\underline{s},\underline{a}}(\omega)} = x$, i.e. the consecutive coordinates of x can be found in $\Phi_{\underline{s},\underline{a}}(\omega, x)$ along $\varphi_{\underline{s},\underline{a}}(\omega)$,
- $(Y_{\underline{s},\underline{a}})_0 = \theta^{-1}_{\underline{s},\underline{a}}((\Omega_{\underline{s},\underline{a}})_0)$,
- $\Psi_{\underline{s},\underline{a}}(\omega, x) = (\omega, \widehat{x}_{\varphi_{\underline{s},\underline{a}}(\omega)})$.

□

We may also extend Theorem 0.0.1 in the following way:

Theorem 2.2.25. *Each of the subshifts $\overline{Y}_{\underline{s},\underline{a}}$ is intrinsically ergodic.*

The proof is very similar to the one of Theorem 0.0.1 presented at the end of the Section 2.1.1. The only difference is that instead of $\Omega, \varphi, \widetilde{T}, C$ we use $\Omega_{\underline{s},\underline{a}}, \varphi_{\underline{s},\underline{a}}, \widetilde{T}_{\underline{s},\underline{a}}, C_{\underline{s},\underline{a}}$, where $C_{\underline{s},\underline{a}} := \{\omega \in \Omega_{\underline{s},\underline{a}} : \varphi_{\underline{s},\underline{a}}(\omega)(0) = 1\}$.

Moreover, using Remark 2.2.20 and Lemma 2.2.21, we obtain the following:

Corollary 2.2.26. *For each $\nu \in \mathcal{P}^e(S, X_\eta)$ the discrete rational part of the spectrum of the corresponding dynamical system (S, X_η, ν) contains all $b'_1 \cdot \ldots \cdot b'_k$-roots of unity, $k \geq 1$, where $\mathscr{B}' = \{b'_k : k \geq 1\}$ is such that* (0.3) *and* (0.4) *are satisfied.*

Corollary 2.2.27. *Let $1 < b'_k | b_k$ for $k \geq 1$. The following are equivalent:*

(a) there exists a measure $\nu \in \mathcal{P}^e(S, X_\eta)$ such that the rational discrete spectrum of (S, X_η, ν) is equal to all $b'_1 \cdot \ldots \cdot b'_k$-roots of unity

(b) $\sum_{k \geq 1} 1/b'_k < +\infty$.

In particular, no ergodic measure for the square-free subshift yields the dynamical system whose spectrum consists of all $p_1 \cdot \ldots \cdot p_k$-roots of unity, $k \geq 1$.

Proof. To see that (b) implies (a), it suffices to take $\nu = \nu_{\mathscr{B}'}$. Suppose now that $\nu \in \mathcal{P}^e(S, X_\eta)$ satisfies (a). It follows by Corollary 2.2.26 that there exists $1 < b''_k | b_k$, $k \geq 1$ such that $\sum_{k\geq 1} 1/b''_k < +\infty$ and the discrete part of the spectrum of (S, X_η, ν) contains all $b''_1 \cdot \ldots \cdot b''_k$-roots of unity, $k \geq 1$. In particular, it contains all b''_k-roots of unity, $k \geq 1$. Therefore, for each $k \geq 1$ there exists ℓ_k such that $b''_k | b'_1 \cdot \ldots \cdot b'_\ell$. Using (0.4), we obtain immediately that $\ell \geq k$ and $b''_k | b'_k$, which yields (b). □

2.3 Combinatorics

Proposition 2.3.1. *Assume that $\mathscr{B} = \{b_k : k \geq 1\}$ and $\mathscr{B}' = \{b'_k : k \geq 1\}$ satisfy* (0.1). *If $X_{\mathscr{B}} = X_{\mathscr{B}'}$ then $\mathscr{B} = \mathscr{B}'$.*

Proof. We can additionally assume that $b_1 < b_2 < \ldots$ and also $b'_1 < b'_2 < \ldots$ Denote by $(T, \Omega, \mathbb{P})$ and $(T', \Omega', \mathbb{P}')$ the corresponding odometers and by $\varphi \colon \Omega_0 \to X_{\mathscr{B}}$, $\varphi' \colon \Omega'_0 \to X_{\mathscr{B}'}$ the relevant genuine embeddings, see (1.2) and Lemma 1.2.1.

We claim now that $\nu_{\mathscr{B}'}(Y) = 1$. Indeed, notice first that $\nu_{\mathscr{B}} * B(1/2, 1/2) = \nu_{\mathscr{B}'} * B(1/2, 1/2)$ since both measures are of maximal entropy on $X_\eta = X_{\eta'}$ and (S, X_η) is intrinsically ergodic. Suppose that $\nu_{\mathscr{B}'}(Y) = 0$. Then $\nu_{\mathscr{B}'}(\bigcap_{k\geq 1} Y_{k,s_k}) = 1$ with at least one $s_k \geq 2$, see Remark 2.2.11. But the set

$$\bigcap_{k\geq 1} \bigcup_{r_k \geq s_k} Y_{k,r_k}$$

is hereditary and clearly $\nu_{\mathscr{B}'} * B(1/2, 1/2)$ is concentrated on it. On the other hand, by Lemma 1.1.3, the measure of maximal entropy is concentrated on Y, a contradiction and our claim follows. Since $\theta_*(\nu_{\mathscr{B}'}) = \mathbb{P}$, we also have $\nu_{\mathscr{B}'}(Y_0) = 1$. In other words, we may assume without loss of generality that $\varphi'(\Omega') \subset Y_0 = \varphi(\Omega_0)$.

Now, for $\omega' \in \Omega'_0$, there exists $\omega \in \Omega_0$ such that $\varphi'(\omega') \leq \varphi(\omega)$ (in fact, $\omega = \theta(\varphi'(\omega'))$, see Lemma 1.2.2 (ii)). Fixing now ω and reversing the roles, we find $\omega''' \in \Omega'_0$ such that $\varphi(\omega) \leq \varphi'(\omega''')$. Thus,

$$\varphi'(\omega') \leq \varphi'(\omega''')$$

which, by (1.16) used for φ', implies that $\omega''' = \omega'$. It follows that $\varphi(\Omega_0) = \varphi'(\Omega'_0)$. Now, $\theta_*(\nu_{\mathscr{B}}) = \mathbb{P} = \theta_*(\nu_{\mathscr{B}'})$ and $\theta|_{\varphi(\Omega_0)}$ is 1-1. It follows that

$$\nu_{\mathscr{B}} = \nu_{\mathscr{B}'}. \tag{2.33}$$

Furthermore,

$$X_{\mathscr{B}} = X_{\mathscr{B}'} \Rightarrow b_1 = b'_1. \tag{2.34}$$

Indeed, suppose $b_1 < b'_1$. Then to obtain (2.34), it is enough to notice that the block $C^1_{\{1,\ldots,b'_1-1\}} \cap C^0_{\{b'_k\}}$ is $\mathscr{B}'$-admissible, while clearly it is not $\mathscr{B}$-admissible.

Set $\widetilde{\mathscr{B}} = \mathscr{B} \setminus \{b_1\}$, $\widetilde{\mathscr{B}}' = \mathscr{B}' \setminus \{b'_1\}$. In view of (1.4), for any finite subset $A \subset \mathbb{N}$, we have

$$\nu_{\mathscr{B}}(C^1_A) = \left(1 - \frac{|A \bmod b_1|}{b_1}\right) \nu_{\widetilde{\mathscr{B}}}(C^1_A)$$

with an analogous formula for $\nu_{\mathscr{B}'}$. In view of (2.34) and (2.33), we deduce $\nu_{\widetilde{\mathscr{B}}} = \nu_{\widetilde{\mathscr{B}'}}$ whence $X_{\widetilde{\mathscr{B}}} = X_{\widetilde{\mathscr{B}'}}$ (the Mirsky measure has full topological support). Using again (2.34), we obtain $b_2 = b'_2$, and by continuing, we conclude $\mathscr{B} = \mathscr{B}'$. □

Remark 2.3.2. Given a subset $A \subset \mathbb{N}$ denote

$$\widetilde{A} := \{C \subset \mathbb{Z} : (\forall C \supset E,\ E \text{ is finite})(\exists k \in \mathbb{Z})\ \ E + k \subset A\}.$$

The result obtained in Proposition 2.3.1 can be reformulated as follows. Assume that $\mathscr{B} = \{b_k : k \geq 1\}$ and $\mathscr{B}' = \{b'_k : k \geq 1\}$ satisfy (0.1) and let $F_{\mathscr{B}}$, $F_{\mathscr{B}'}$ stand for the sets of $\mathscr{B}$- and $\mathscr{B}'$-free numbers, respectively. Then

$$\widetilde{F}_{\mathscr{B}} = \widetilde{F}_{\mathscr{B}'} \text{ if and only if } \mathscr{B} = \mathscr{B}'. \tag{2.35}$$

The proof of Proposition 2.3.1, although short, uses however some non-trivial facts, like intrinsic ergodicity of $\mathscr{B}$-free systems. We will now present an elementary proof, due to Stanisław Kasjan, which has an advantage that it also gives a sufficient and necessary condition for $X_\eta \subset X_{\eta'}$.

Let $\mathscr{B} = \{b_k : k \geq 1\} \subseteq \mathbb{N}$ satisfies (0.1) and assume that $b_1 < b_2 < \ldots$

Lemma 2.3.3. *Let $c \in \mathbb{N}$ be relatively prime to b_k for any $k \geq 1$. Then, for any natural number r the density of the set $\{s \in \mathbb{N} : sc + r \neq 0 \bmod b_k$ for each $k \geq 1\}$ equals $\prod_{k\geq 1}\left(1 - \frac{1}{b_k}\right)$.*[16]

Proof. Fix $m \geq 1$ and consider the finite probability space $\mathbb{Z}/(b_1 \cdot \ldots \cdot b_m)\mathbb{Z}$, that is, the integers mod $M := b_1 \cdot \ldots \cdot b_m$. By the Chinese Remainder Theorem the random variables $X_{b_i}(s)$ which equals 0 if s is divisible by b_i and 1 otherwise, are independent. It follows that the event A such that $n \in A$ if and only if n is not divisible by any of the b_i has probability $p = \prod_{i=1}^m (1 - 1/b_i)$. Now, if c is relatively prime to M, then the addition by c modulo M is transitive (the jc mod M for $j = 0, \ldots, M-1$ are all distinct and yield all residues mod M), the time average of the $\mathbb{1}_A(lc + r)$ equals its space average which is p. Moreover, note that the number of elements in the interval $[1, M]$ which are divisible by some b_{m+i} $(i \geq 1)$ is no more than $M \cdot \sum_{i\geq 1} \frac{1}{b_{m+i}}$. This gives the density result along the subsequence $M_k := b_1 \cdot \ldots \cdot b_k$, $k \geq 1$ and the general result easily follows. □

Note also that a simple induction on finite products shows that

$$\prod_{k\geq 1}\left(1 - \frac{1}{b_k}\right) \geq 1 - \sum_{k\geq 1} \frac{1}{b_k},$$

so for each $\varepsilon > 0$ there exists $N \geq 1$ such that

$$\prod_{k\geq N}\left(1 - \frac{1}{b_k}\right) > 1 - \varepsilon.^{17} \tag{2.36}$$

[16] If $\sum_{k\geq 1} 1/b_k = +\infty$ then this density equals 0.
[17] To obtain (2.36) we need only that $\sum_{k\geq 1} 1/b_k < +\infty$.

Lemma 2.3.4. *Fix $m \geq 1$ and assume that $b' \in \mathbb{N}$ is not divisible by any $b_1, \ldots, b_m$. Then there exists a set $A \subset \mathbb{N}$ containing b' elements which is $\{b_1, \ldots, b_m\}$-admissible and such that A is not $\{b'\}$-admissible.*

Proof. For $i = 1, \ldots, b'$ let e_i denote the product of those numbers from the set $\{b_1, \ldots, b_m\}$ which do not divide i. If every b_j divides i, we set $e_i = 1$. We define

$$A = \{i + e_i b' : i = 1, \ldots, b'\}.$$

By construction, A is not $\{b'\}$-admissible. We will show that A is $\{b_1, \ldots, b_m\}$-admissible by showing that $0 \notin A \bmod b_i$ for each $i = 1, \ldots, m$. Indeed, let $i \in \{1, \ldots, b'\}$, $j \in \{1, \ldots, m\}$. If b_j divides i, then it is relatively prime to e_i (by the choice of e_i and the assumption that the numbers b_i are pairwise relatively prime). Thus b_j does not divide $i + e_i b'$. In the other case, when b_j does not divide i, then it divides e_i and again b_j does not divide $i + e_i b'$. □

Proposition 2.3.5. *Assume that $\mathscr{B} = \{b_k : k \geq 1\}$ and $\mathscr{B}' = \{b'_k : k \geq 1\}$ satisfy* (0.1). *Then:*

(a) $X_{\mathscr{B}} \subset X_{\mathscr{B}'}$ if and only if for any $b' \in \mathscr{B}'$ there exists $b \in \mathscr{B}$ such that b divides b'.

(b) If $X_{\mathscr{B}} = X_{\mathscr{B}'}$ if and only if $\mathscr{B} = \mathscr{B}'$.

Proof. (a) We only need to show if $X_{\mathscr{B}} \subset X_{\mathscr{B}'}$ and $b' \in \mathscr{B}'$ then for some $b \in \mathscr{B}$, b divides b'.

Fix $b' \in \mathscr{B}'$ and assume that b' is not divisible by any $b \in \mathscr{B}$. Using (2.36), we select $m \geq 1$ so that

$$\prod_{k>m} \left(1 - \frac{1}{b_k}\right) > 1 - \frac{1}{b'}.$$

In view of Lemma 2.3.4, we can find a set $A \subset \mathbb{N}$, $|A| = b'$, which is $\{b_1, \ldots, b_m\}$-admissible and is not $\{b'\}$-admissible, hence is not $\mathscr{B}'$-admissible. Denote $c = b_1 \ldots b_m$. It follows that for each $\ell \in \mathbb{N}$, $A + \ell c$ is $\{b_1, \ldots, b_m\}$-admissible and is not $\mathscr{B}'$-admissible. To complete the proof, it is enough to show that for some ℓ_0, $A + \ell_0 c$ is $\mathscr{B}$-admissible. For this aim, we will show that for some $\ell_0 \geq 1$

$$(A + \ell_0 c) \cap \bigcup_{k>m} b_k \mathbb{Z} = \emptyset.$$

Indeed, if not then each $K \geq 1$

$$\frac{1}{K} \big|\{1 \leq \ell \leq K : (A + \ell c) \cap (\bigcup_{i>m} b_i \mathbb{Z}) \neq \emptyset\}\big| = 1.$$

Since $|A| = b'$, it follows that there exists $a \in A$ such that

$$\limsup_{K \to \infty} \frac{1}{K} \left| \left\{ 1 \leq \ell \leq K : \ell c + a \in \bigcup_{i>m} b_i \mathbb{Z} \right\} \right| \geq \frac{1}{b'}$$

and we get a contradiction with the choice of m and Lemma 2.3.3 (applied to $\{b_k : k > m\}$).

(b) Since the elements of $\mathscr{B}$ (resp. $\mathscr{B}'$) are pairwise relatively prime, (b) follows from (a). □

3 Hereditary systems of Sturmian origin

3.1 Intrinsic ergodicity

In this section, we will indicate that our method to prove intrinsic ergodicity for $\mathscr{B}$-free systems can be applied to other hereditary systems. We will detail the case of Sturmian hereditary systems but the method applies to many others.

Consider an irrational rotation $T\colon \mathbb{T} \to \mathbb{T}$, $Tx = x + \alpha$. Fix an interval $J = [a, b) \subset \mathbb{T}$ assuming that the numbers $|J|$ and α are independent over $\mathbb{Q}$. Let

$$X := \overline{\{(\mathbb{1}_J(T^n w))_{n\in\mathbb{Z}} : w \in \mathbb{T}\}}.$$

This is an almost one-one extension of the circle and the only points that have two representatives are the orbits of the endpoints of J (see, e.g., [2]). We will denote the corresponding factor map from X to $\mathbb{T}$ by π.

Remark 3.1.1. Notice that if $x, x' \in \pi^{-1}(a + s\alpha)$, then they differ only at one place, namely, $x[-s] \neq x'[-s]$. It follows that either $x \leq x'$ or $x' \leq x$. The same reasoning applies to the points from the orbit of b.

It follows that we have a continuous map $\pi\colon X \to \mathbb{T}$ intertwining the rotation T with the shift on X, for which $|\pi^{-1}(w)| = 1$, except for countably many points $w \in \mathbb{T}$. This allows us to define “Haar” measure λ on X (which is the lift of Haar measure $\lambda_{\mathbb{T}}$ via the map $\pi : X \to \mathbb{T}$). Moreover, X is uniquely ergodic. Let

$$X_0 := X \setminus \{\pi^{-1}(a + s\alpha), \pi^{-1}(b + s\alpha) : s \in \mathbb{Z}\}$$

and let $\widetilde{X}$ be the hereditary subshift generated by X:

$$\widetilde{X} := \{z \in \{0, 1\}^{\mathbb{Z}} : (\exists x \in X)\ \ z \leq x\}.$$

In view of Lemma 2.2.16 (with X of zero topological entropy), we have

$$h_{top}(S, \widetilde{X}) = \log 2 \cdot \lambda(C_0^1).$$

Finally, let

$$Y := \{y \in \widetilde{X} :\ \text{all blocks from } X \text{ occur on } y\},$$
$$Y_0 := \{y \in Y : (\exists x_0 \in X_0)\ \ y \leq x_0\}.$$

Lemma 3.1.2. *Fix $y \in Y$ and let $x, x' \in X$ be such that $y \leq x$, $y \leq x'$. Then $\pi(x) = \pi(x')$. In particular, if $y \in Y_0$ then $x = x'$, i.e. there is exactly one $x = x(y) \in X$ (in fact, $x \in X_0$) such that $y \leq x$.*

Proof. Assume that $y \leq x$, $y \leq x'$ for some $x, x' \in X$. Fix $N \geq 1$ and choose any maximal block C of length N that occurs on X (i.e. if a block C' of length N occurs on X and $C \leq C'$ then $C = C'$). Now, since $y \leq x$ and C occurs on y, we can find $\ell \in \mathbb{Z}$ such that $C = y[\ell, \ell + N - 1] \leq x[\ell, \ell + N - 1]$ and by maximality, $y[\ell, \ell + N - 1] = x[\ell, \ell + N - 1]$. Hence

$$x[\ell, \ell + N - 1] = x'[\ell, \ell + N - 1].$$

By uniform continuity of π, if N is large enough, then $\pi(S^{\lfloor 3\ell/2\rfloor} x)$ is ε-close to $\pi(S^{\lfloor 3\ell/2\rfloor} x')$. By equivariance and the fact that T is an isometry, $\pi(x) = \pi(x')$. Finally, if $y \in Y_0$ then $x = x'$. $\square$

In view of Lemma 3.1.2 and Remark 3.1.1, we can define a Borel map $\theta\colon Y \to X$ by setting

$$\theta(y) = x \iff x \in X \text{ is the maximal element which dominates } y.$$

This map is equivariant. Each block at positions $[-k,k]$ that occur on the fiber $\theta^{-1}(x)$ is smaller than $x[-k,k]$ and each block smaller than $x[-k,k]$ does occur on the fiber (as on $y \in Y$ we can see all blocks occurring on x). Let $\nu \in \mathcal{P}^e(S, Y_0)$. Then $\theta_*(\nu) = \lambda$ since (S,X) is uniquely ergodic. It is not hard to see that a measure which maximizes the entropy is obtained by independently changing 1 to 0 with probability 1/2, cf. the definition of μ in Section 1.3.1.

Lemma 3.1.3. *Each measure of maximal entropy is supported on Y.*

Proof. Fix a word w that occurs in X and contains a 1. We will estimate the number of dictinct N-blocks that can occur in the set

$$Y(w) := \{y \in \widetilde{X} : (\exists x \in X)\ \ y \leq x \text{ and } w \text{ does not occur on } y\}.$$

Note that if $\nu \in \mathcal{P}^e(S, \widetilde{X})$ then either $\nu(Y(w)) = 0$ or 1 since $Y(w)$ is Borel and invariant under the shift. Hence, if in our estimate we get a bound strictly smaller than the full entropy of $\widetilde{X}$, this will show that a measure of maximal entropy must be supported on Y. Indeed, $Y = \bigcap_w Y(w)^c$.

Define K to be the number of ones that occur in w. Now, we let N be large enough so that:

(a) the exponential number of N-words in $\widetilde{X}$ is very close to $h_{top}(\widetilde{X})$;

(b) in every word u of length N in X there are a fixed fraction of disjoint occurrences of w, say $f > 0$;[18]

(c) The total number of N-words in X is exponentially very small.[19]

Note that f depends on w and the "very close" in (a) and the "very small" in (c) are chosen after we know f.

Now, when we calculate the number of blocks in $\widetilde{X}$ that are dominated by a fixed u_0 of length N in X, we get 2^{aN}, where a is the frequency of ones in u_0. Now, $aN = fKN + (a - fK)N$ (here fKN corresponds to the consecutive disjoint occurrences of w in u_0). However, if w does not occur below u_0 then the number of possibilities is only $(2^K - 1)^{fN} \cdot 2^{(a-fK)N}$, so that the ratio between them is R^N, where $R := (2^K/(2^K - 1))^f > 1$. In view of (c), the entropy of $Y(w)$ will be exponentially comparable with $2^{\varepsilon N} \cdot (2^K - 1)^{fN} \cdot 2^{a-fK}N$, while by (a), the entropy of $\widetilde{X}$ must be exponentially comparable with $2^{\varepsilon N} \cdot 2^{aN}$ (as the frequency of ones in each u_0 is comparable with a), and therefore on $Y(w)$, we have a definite drop in the entropy. □

We have now described a full analogy with the $\mathscr{B}$-free systems. Indeed, X_η corresponds to $\widetilde{X}$, $\varphi(\Omega)$ (a symbolic model of the odometer) corresponds to X (see (1.15) in Remark 1.2.4), Y and Y_0 play the same roles in both cases, and θ in the Sturmian case is immediately with values in X (it is simpler than in

[18]This follows from the fact that (S,X) is uniquely ergodic. We apply (uniformly) the ergodic theorem to the cylinder C corresponding to w with $SC \cup \ldots \cup S^{|w|-1}C$ removed.

[19]This follows from $h_{top}(S,X) = 0$.

the $\mathscr{B}$-free case as we do not need the map φ to get a symbolic model of the odometer embedded in X_η). Now, by repeating the proof of Theorem 0.0.1, we obtain the following result.

Proposition 3.1.4. *Let $(S, \widetilde{X})$ be a Sturmian hereditary system described above. Then it is intrinsically ergodic.*

3.2 Absence of intrinsic ergodicity

3.2.1 Tools

Given a block $C \in \{0,1\}^n$, let x_C be the infinite concatenation of C and let $X_C := \mathcal{O}(x_C) \subset \{0,1\}^{\mathbb{Z}}$. Finally, let $\widetilde{X}_C \subset \{0,1\}^{\mathbb{Z}}$ be the smallest hereditary system containing X_C. We may assume without loss of generality that the smallest period of x_C is equal to $|C|$. It follows directly from Lemma 2.2.16 that

$$h_{top}(S, \widetilde{X}_C) = d \log 2, \;\text{ where }\; d = |\operatorname{supp} C|/|C|. \tag{3.1}$$

Let ν_C be the Haar measure on X_C and let

$$\mu_C^p := \nu_C * \kappa, \text{ where } \kappa = B(p, 1-p), p \in (0,1).$$

Then $\mu_C \in \mathcal{P}^e(S, \widetilde{X}_C)$. Moreover, similar arguments as in Section 2.1.4 yield

$$h(\mu_C^p) = -d(p \log p + (1-p)\log(1-p)). \tag{3.2}$$

In particular, $h(\mu_C^{1/2}) = h_{top}(\widetilde{X}_C)$.[20]

Lemma 3.2.1. *Let $\widetilde{X} \subset \{0,1\}^{\mathbb{Z}}$ be a hereditary subshift. Then there exists $x \in \{0,1\}^{\mathbb{Z}}$ such that $\widetilde{X} \subset \overline{\mathcal{O}(x)}$, $\overline{\mathcal{O}(x)}$ is hereditary and $h_{top}(x) = h_{top}(\widetilde{X})$.*

Proof. For $n \in \mathbb{N}$ let $\mathcal{C}_n := \{B_1^n, \dots, B_{l_n}^n\}$ be the family of all n-blocks occurring on $\widetilde{X}$. Let $L := \lceil \frac{\log 2}{h_{top}(\widetilde{X})} \rceil$ and denote by Z_n the Ln-block consisting of Ln zeroes. Let $n_1 \in \mathbb{N}$ and define $x[1, m_1]$, where $m_1 = l_{n_1} n_1 + l_{n_1} L n_1 = l_{n_1}(L+1)n_1$ by concatenating:

$$B_1^{n_1}, Z_{n_1}, B_2^{n_1}, Z_{n_1}, \dots, B_{l_{n_1}}^{n_1}, Z_{n_1}.$$

Now we begin the inductive procedure. Suppose that $n_1 < \dots < n_{k-1}$ are chosen and m_{k-1} is the largest integer such that $x[1, m_{k-1}]$ is already defined. Let $n_k = m_{k-1}$ and let

$$\mathcal{E}_k := \{E \in \{0,1\}^{n_k} : E \le x[1, n_k]\} = \{E_1^k, \dots, E_{s_k}^k\}.$$

Define $x[m_{k-1}+1, \dots, m_k]$, where

$$m_k = m_{k-1} + (l_{n_k} + s_k)(L+1)n_k$$

by concatenating:

$$Z_{n_k}, B_1^{n_k}, Z_{n_k}, B_2^{n_k}, \dots, Z_{n_k}, B_{l_{n_k}}^{n_k}, Z_{n_k}, E_1^k, Z_{n_k}, E_2^k, \dots, Z_{n_k}, E_{s_k}^k. \tag{3.3}$$

Continuing this procedure, we obtain $x \in \{0,1\}^{\mathbb{N}}$.

Notice that x has the following properties:

[20] From now on, we will simplify entropy notation if no confusion arises.

- for any $B \in x$ and any $C \leq B$ we have $C \in x$, i.e. the orbit closure of x yields a hereditary shift,
- for all $B \in \widetilde{X}$ we have $B \in x$, hence $X \subset \overline{\mathcal{O}(x)}$ and $h_{top}(x) \geq h_{top}(\widetilde{X})$.

We will estimate now from above the number of n_k-blocks occurring on x. Notice that in $x[1, n_k]$ any two consecutive blocks $C, C' \in \mathcal{C}_{n_{k-1}} \cup \mathcal{E}_{k-1}$ are separated by $Z_{n_{k-1}}$ (cf. (3.3) with $k-1$ instead of k). Therefore,

$$d_k := \frac{|\{1 \leq i \leq n_k : x(i) = 1\}|}{n_k} \leq \frac{n_{k-1}}{(L+1)n_{k-1}} = \frac{1}{\lceil \frac{\log 2}{h_{top}(\widetilde{X})} \rceil + 1} \leq \frac{h_{top}(\widetilde{X})}{\log 2}.$$

This implies that

$$|\mathcal{E}_k| = 2^{d_k n_k} \leq 2^{\frac{h_{top}(\widetilde{X})}{\log 2} n_k}. \tag{3.4}$$

Moreover, notice that any n_k-block B occurring on x satisfies (at least) one of the following:

- $B = B'Z$, where Z is a (possibly empty) block consisting of zeroes and for some B'' we have $B''B' \in \mathcal{C}_{n_k} \cup \mathcal{E}_k$,
- $B = ZB'$, where Z is a (possibly empty) block consisting of zeroes and for some B'' we have $B'B'' \in \mathcal{C}_{n_k} \cup \mathcal{E}_k$.

It follows from (3.4) that the number of such blocks with $B''B' \in \mathcal{E}_k$ or $B'B'' \in \mathcal{E}_k$ is bounded from above by

$$p_{n_k}^{\mathcal{E}} := (2n_k + 1) 2^{\frac{h_{top}(\widetilde{X})}{\log 2} n_k}.$$

Moreover, the number of such blocks with $B''B' \in \mathcal{C}_{n_k}$ or $B'B'' \in \mathcal{C}_{n_k}$ is bounded from above by

$$p_{n_k}^{\mathcal{C}} := (2n_k + 1) p_{n_k}(\widetilde{X}),$$

where $p_{n_k}(\widetilde{X})$ stands for the number of n_k-blocks occurring on $\widetilde{X}$. Therefore

$$h_{top}(x) \leq \max \left\{ \lim_{k \to \infty} \frac{1}{n_k} \log p_{n_k}^{\mathcal{E}}, \lim_{k \to \infty} \frac{1}{n_k} \log p_{n_k}^{\mathcal{C}} \right\} = h_{top}(\widetilde{X})$$

and the result follows. □

3.2.2 More than one measure of maximal entropy

For $A := 101001000$, $B := 101000100$ consider $(\widetilde{X}_A, \mu_A)$ and $(\widetilde{X}_B, \mu_B)$, with $\mu_A = \mu_A^{1/2}$, $\mu_B = \mu_B^{1/2}$. Let $\widetilde{X} := \widetilde{X}_A \cup \widetilde{X}_B$.

Proposition 3.2.2. *The measures μ_A and μ_B are ergodic and such that $h_{top}(\widetilde{X}) = h(\mu_A) = h(\mu_B) = \frac{1}{3} \log 2$. Moreover, $\mu_A \neq \mu_B$.*

Proof. The first part follows easily from (3.1) and (3.2). For the second part of the assertion, let $Y_A := \{x \in \widetilde{X} : |\{i : x[i, \ldots, i+8] = A\}| = \infty\}$. To conclude, it suffices to notice that $\mu_A(Y_A) = 1$ (cf. Lemma 3.1.3), whereas $\mu_B(Y_A) = 0$ since for no i, $A \leq (BB)[i, i+8]$. □

Thus we obtain the following corollary which gives the answer to a question raised in [9].

Corollary 3.2.3. *There exists a hereditary shift with more than one ergodic measure of maximal entropy.*

Moreover, as an immediate consequence of Corollary 3.2.3 and of Lemma 3.2.1 we also have:

Corollary 3.2.4. *There exists a transitive hereditary shift with more than one ergodic measure of maximal entropy.*

The above construction also can be modified in such a way that the obtained system has only one minimal subset. Choose a sequence of prime numbers $p_n \to \infty$. We will now define x'_A by "erasing" some positions in x_A. Namely, whenever

$$n = k - 1 \bmod p_1 \cdot \ldots \cdot p_k \text{ for some } k \geq 1 \text{ and } n \neq k - 1,$$

we put $x'_A(9n+i) := 0$ for $0 \leq i \leq 8$ (at all other positions the sequences x_A and x'_A are the same). We also define x'_B adjusting in a similar way x_B. Let X'_A and X'_B be the closure of the orbit under the shift map of x'_A and x'_B, respectively. Notice that arbitrarily long blocks of 0's occur on x'_A and x'_B with bounded gaps. Therefore, the singleton $\{(\ldots, 0, 0, \ldots)\}$ is the only minimal subset of $X := X'_A \cup X'_B$. The same applies to $\widetilde{X}$, i.e. to the minimal hereditary subshift containing X. Notice that $h_{top}(\widetilde{X}) = h_{top}(\widetilde{X}'_A) = h_{top}(\widetilde{X}'_B)$. Similar arguments as the ones used in Proposition 3.2.2 show that the measures of maximal entropy on $\widetilde{X}'_A$ and $\widetilde{X}'_B$ are not the same. Moreover, in view of Lemma 3.2.1, we can enlarge $\widetilde{X}$, so that it becomes transitive, remains hereditary and the topological entropy does not change. Finally, notice that the proof of Lemma 3.2.1 is carried out in such a way that whenever arbitrarily long blocks of 0's occur on $\widetilde{X}$ with bounded gaps, then the same is true for the enlarged system (see (3.3)). Therefore the singleton $\{(\ldots, 0, 0, \ldots)\}$ is the only minimal subset for the enlarged system.

3.2.3 Uncountably many measures of maximal entropy

For $y, a \in \mathbb{R}$ we define a sequence $x^{(y,a)} \in \{0, 1\}^{\mathbb{Z}}$ in the following way:

$$x^{(y,a)}(n) := \mathbb{1}_{[0,1/2)}(\{y + na\}).$$

We will write $x^{(a)}$ for $x^{(0,a)}$. Let $X_a := \overline{\mathcal{O}(x^{(a)})}$. Clearly, for any $a \in \mathbb{R}$, $h_{top}(X_a) = 0$ and if $a \notin \mathbb{Q}$ then $x^{(y,a)} \in X_a$ for any $y \in \mathbb{R}$.

Now, we choose an uncountable set $\mathcal{A} \subset \mathbb{R} \setminus \mathbb{Q}$ satisfying the following conditions:

- any $\alpha \in \mathcal{A}$ has bounded partial quotients with $a_n(\alpha) \leq 2$,
- for any $\alpha, \beta \in \mathcal{A}$, the set $\{1, \alpha, \beta\}$ is rationally independent.

Let now $X := \cup_{\alpha \in \mathcal{A}} X_\alpha$ and let $\widetilde{X}$ be the smallest hereditary subshift containing X.

Remark 3.2.5. It follows from Lemma 2.2.16 that $h_{top}(X_\alpha) = 1/2 \log 2$.

We define μ_α in the following way (cf. Section 1.3.1). X_α is an almost 1-1 extension of a rotation on the circle, i.e. it has only one invariant measure. Now, in each block we erase each 1 with probability 1/2. This is the measure of maximal entropy (cf. Proposition 3.1.4).

Lemma 3.2.6. *If $\alpha \in \mathcal{A}$ and β is such that $|\alpha - \beta| < \frac{1}{48n^2}$ for some n then all n-blocks occurring on $x^{(\beta)}$ occur on $x^{(\alpha)}$.*

Proof. Notice first that by the assumption that $\alpha \in \mathcal{A}$, for each $k \neq 0$ we have

$$\|k\alpha\| \geq \frac{1}{2 \cdot |\sup_{n\in\mathbb{N}} a_n(\alpha)| \cdot k} \geq \frac{1}{6k}.$$

Therefore, for $0 \leq k < k' \leq n-1$ we have

$$\|k\alpha - k'\alpha\| \geq \frac{1}{6|k-k'|} > \frac{1}{6n}$$

and

$$\|k\alpha - k'\alpha + 1/2\| = \inf_{p\in\mathbb{Z}} \frac{|2(k-k')\alpha - 2p + 1|}{2} \geq \frac{1}{24|k-k'|} \geq \frac{1}{24n}. \tag{3.5}$$

Fix $\ell \in \mathbb{Z}$ and let $m \in \mathbb{Z}$ be such that

$$\|\ell\beta - m\alpha\| + n\|\beta - \alpha\| < \frac{1}{48n}$$

and

$$\mathbb{1}_{[0,1/2)}(\ell\beta + k_0\beta) = \mathbb{1}_{[0,1/2)}(m\alpha + k_0\alpha), \tag{3.6}$$

where $0 \leq k_0 < n-1$ satisfies

$$\min\{\|\ell\beta + k_0\beta\|, \|\ell\beta + k_0\beta - 1/2\|\} = \min_{0\leq k\leq n-1} \min\{\|\ell\beta + k\beta\|, \|\ell\beta + k\beta - 1/2\|\}$$

(such $m \in \mathbb{Z}$ exists since the orbit of 0 by the rotation by α is dense). Then, for $0 \leq k \leq n-1$,

$$\|(\ell\beta + k\beta) - (m\alpha + k\alpha)\| < \frac{1}{48n}. \tag{3.7}$$

We claim that there exists at most one $0 \leq k_1 \leq n-1$ such that

$$\|m\alpha + k_1\alpha\| < \frac{1}{48n} \text{ or } \|m\alpha + k_1\alpha - 1/2\| < \frac{1}{48n}. \tag{3.8}$$

Suppose that (3.8) does not hold. There are several possibilities, all of which can be treated in the same way. We will show how to proceed in the case where

$$\|m\alpha + k\alpha\| < \frac{1}{48n} \text{ and } \|m\alpha + k'\alpha - 1/2\| < \frac{1}{48n}$$

for some $0 \leq k < k' \leq n-1$. It follows by (3.5) that

$$\frac{1}{24n} \leq \|(k-k')\alpha + 1/2\| = \|m\alpha + k\alpha - m\alpha - k'\alpha + 1/2\| < \frac{1}{48n} + \frac{1}{48n},$$

which yields a contradiction. Therefore, using (3.6), (3.7) and (3.8), we obtain

$$\mathbb{1}_{[0,1/2)}(\ell\beta + k\beta) = \mathbb{1}_{[0,1/2)}(m\alpha + k\alpha) \text{ for } 0 \leq k \leq n-1,$$

which completes the proof. □

Lemma 3.2.7. $h_{top}(X) = 0$.

Proof. For $n \geq 1$ fix $\alpha_1^{(n)}, \ldots, \alpha_{49n^2}^{(n)} \in \mathcal{A}$ such that for all $\alpha \in \mathcal{A}$ there exists i such that $|\alpha - \alpha_i^{(n)}| < \frac{1}{48n^2}$. It follows from Lemma 3.2.6 that the number of possible n-blocks is of order n^3 which ends the proof. □

Lemma 3.2.8. *For any $\varepsilon > 0$ there exists n_0 such that for $n \geq n_0$ the density of 1's in all n-blocks in X is ε-close to $1/2$.*

Proof. As the indicator function of the upper semicircle is Riemann integrable, we can approximate it by trigonometric polynomials, so that

$$\sum_{-\tau_0}^{\tau_0} a_k e^{2\pi i k t} \leq \mathbb{1}_{[0,1/2)}(t) \leq \sum_{-\tau_0}^{\tau_0} b_k e^{2\pi i k t}, \tag{3.9}$$

with $|a_0 - 1/2|, |b_0 - 1/2| < \delta$. Since all $\alpha \in \mathcal{A}$ have bounded partial quotients with $\sup_{n\in\mathbb{N}} a_n(\alpha) \leq 2$, there exists $c > 0$ such that for each $-\tau_0 \leq k \leq \tau_0$, $k \neq 0$,

$$\left|\frac{1}{n}\sum_{m=0}^{n-1} e^{2\pi i k(x+m\alpha)}\right| = \left|\frac{1}{n}\sum_{m=0}^{n-1} e^{2\pi i k m\alpha}\right| \leq \frac{1}{n}\frac{2}{|1-e^{2\pi i k\alpha}|} \leq \frac{1}{n}\cdot\frac{k}{c}.$$

This, together with (3.9), completes the proof. □

Lemma 3.2.9. $h_{top}(X) = 1/2 \log 2$.

Proof. It suffices to apply Lemma 3.2.7, Lemma 3.2.8 and Lemma 2.2.16. □

Lemma 3.2.10. *For $\alpha, \beta \in \mathcal{A}$, $\mu_\alpha \neq \mu_\beta$ if $\alpha \neq \beta$.*

Proof. Notice first that the closed support of μ_α contains the minimal system X_α. If N is large enough then the orbit of any point (x, y) will spend approximately $1/4$ of time in the upper left quarter of $[0, 1) \times [0, 1)$ on its orbit of length N. The choice of N is uniform, due to unique ergodicity of the rotation by (α, β) on $\mathbb{T}^2$ (recall that $\{1, \alpha, \beta\}$ are rationally independent). This can be interpreted in the following way: for any block B_α in X_α and any block B_β in X_β at approximately half of the places where we can see a 1 in X_α, we see a 0 in X_β. This however means that B_α cannot be seen on $\widetilde{X}_\beta$ and the claim follows as we have found a block of positive μ_α measure and zero μ_β measure. □

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