The Möbius function and continuous extensions of rotations

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Abstract

Let $f\colon \mathbb{T}\to\mathbb{R}$ be of class $C^{1+\delta}$ for some $\delta>0$ and let $c\in\mathbb{Z}$. We show that for a generic $\alpha\in\mathbb{R}$, the extension $T_{c,f}\colon\mathbb{T}^2\to\mathbb{T}^2$ of the irrational rotation $Tx=x+\alpha$, given by $T_{c,f}(x,u)=(x+\alpha,u+cx+f(x)) \pmod 1$ satisfies Sarnak's conjecture.

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1 Introduction

Recall that the Möbius function $\boldsymbol{\mu}\colon\mathbb{N}\to\{-1,0,1\}$ is a multiplicative function[1] defined as $\boldsymbol{\mu}(p_1\cdot\ldots\cdot p_k)=(-1)^k$ for distinct prime numbers p_j, $\boldsymbol{\mu}(1)=1$ and 0 otherwise. Its importance is reflected in the fact that the prime number theorem[2] is equivalent to the condition $\sum_{k\leq x}\boldsymbol{\mu}(k)=\mathrm{o}(x)$ and the Riemann hypothesis is equivalent to the condition $\sum_{k\leq x}\boldsymbol{\mu}(k)=\mathrm{O}_\varepsilon(x^{\frac12+\varepsilon})$ for any $\varepsilon>0$

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[1]Recall that $\nu\colon\mathbb{N}\to\mathbb{Z}$ is said to be multiplicative if $\nu(mn)=\nu(m)\nu(n)$ for n,m relatively prime.

[2]Recall that the prime number theorem states that $\pi(x)=\frac{x}{\ln x}+\mathrm{o}(\frac{x}{\ln x})$, where $\pi(x)$ is the number of primes less than x.

(when $x \to \infty$). The Möbius function appears to behave rather randomly and this statement was formalized in the following conjecture of Sarnak:

Conjecture 1 ([26])**.** *Let X be a compact metric space and let $T\colon X \to X$ be a homeomorphism of zero topological entropy. Let $x \in X$, $g \in C(X)$. Then*

$$\sum_{n \le N} g(T^n x)\boldsymbol{\mu}(n) = \mathrm{o}(N). \tag{1}$$

Whenever condition (1) is true for some T for each $x \in X$ and each $g \in C(X)$, we say that *Sarnak's conjecture holds* for T or that T is *disjoint from the Möbius function.*[3]

Sarnak's conjecture is known to hold in several situations, including rotations [9], nilsystems [15], horocycle flows [7], large class of rank one maps [6, 2] and certain subclasses of dynamical systems generated by generalized Morse sequences [19], including the classical Thue-Morse system: [1, 5, 8, 14, 16, 24] and the dynamical system generated by the Rudin-Shapiro sequence [25].

One of the most fruitful tools used for proving disjointness with the Möbius function turns out to be the following orthogonality criterion of Katai-Bourgain-Sarnak-Ziegler (we will refer to it as KBSZ criterion).

Theorem 1.0.1 ([17, 7])**.** *Let $F\colon \mathbb{N} \to \mathbb{C}$ be a bounded sequence. Suppose that*

$$\sum_{n \le N} F(rn)\overline{F(sn)} = \mathrm{o}(N) \tag{2}$$

for any pair of sufficiently large primes $r \neq s$. Then

$$\sum_{n \le N} F(n)\nu(n) = \mathrm{o}(N), \tag{3}$$

for any multiplicative function ν with $|\nu| \le 1$.

In order to use this theorem for proving Conjecture 1 for a given homeomorphism $T\colon X \to X$, one takes

$$F(n) := g(T^n x) \text{ for } n \in \mathbb{Z}, x \in X \text{ and } g \in C(X). \tag{4}$$

Notice that the expression (2) (for $F(n)$ given by (4)) takes the form

$$\frac{1}{N}\sum_{n \le N} g \otimes \overline{g}\left((T^r \times T^s)^n(x,x)\right) = \int_{X^2} g \otimes \overline{g}\ d\left(\frac{1}{N}\sum_{n \le N} \delta_{(T^r \times T^s)^n(x,x)}\right),$$

where $x \in X$. It follows that we are interested in the limit measures $\rho = \lim_{k\to\infty} \frac{1}{N_k}\sum_{n \le N_k} \delta_{(T^r \times T^s)^n(x,x)}$ which are clearly $T^r \times T^s$-invariant. Therefore, to prove disjointness of T with the Möbius function, it suffices to show that the following holds for T, for r, s relatively prime:

[3] Notice that it suffices to show that (1) holds for a linearly dense set of continuous functions to obtain disjointness with the Möbius function.

(a) The ergodic components of $T^r \times T^s$ are pairwise disjoint closed sets filling up the whole space.[4]

(b) The ergodic components are uniquely ergodic (this implies that all points are generic for $T^r \times T^s$ for relevant invariant measures).

(c) There exists a linearly dense set[5] of continuous functions $\mathcal{F} \subset C(X)$ such that for each $g \in \mathcal{F}$, $g \circ T \neq g$, we have $\int_{X^2} g \otimes \overline{g}\, d\rho = 0$ for any $T^r \times T^s$-ergodic measure ρ, whenever r, s are sufficiently large.

We will use the strategy (a), (b), (c) to study disjointness with the Möbius function in the following setting. Denote by $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1)$ the additive circle and consider

$$\mathbb{T}^2 \ni (x,y) \mapsto T_{c,f}(x,y) := (x+\alpha, y+cx+f(x)) \in \mathbb{T}^2. \tag{5}$$

where $c \in \mathbb{Z}$ and $cx + f(x)$ is a lift of a continuous circle map, i.e. $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, periodic of period 1, and c is the degree of the map in question. In other words, we consider continuous case of the classical Anzai skew product extensions of a rotation on the circle [3].

Liu and Sarnak in their recent paper [23] proved the following.

Theorem 1.0.2 ([23])**.** *Assume that in* (5)*, $f\colon \mathbb{R} \to \mathbb{R}$ is an analytic periodic function of period* 1*. Assume additionally that $|\widehat{f}(m)| \gg e^{-\tau|m|}$ for some $\tau > 0$. Then $T_{c,f}$ satisfies Conjecture 1.*

The technical condition on the Fourier transform, namely $|\widehat{f}(m)| \gg e^{-\tau|m|}$, seems to be necessary for the methods of [23] to work. On the other hand, there is no condition on α. Moreover, for some αs, the result is obtained using Theorem 1.0.1. Under some additional assumptions, also a quantitative version (i.e. concerning the speed of convergence to zero in (1)) of Theorem 1.0.2 is proved in [23]. This is achieved by treating the problem in a more direct way than applying Theorem 1.0.1.

A natural question arises whether the strong assumptions on f in Theorem 1.0.2 can be relaxed. We do so in the main result of the paper (Theorem 1.0.3 below) to obtain Sarnak's conjecture for each sufficiently smooth f at the cost of reducing "for each α" in Theorem 1.0.2 to "for a generic α". Hence, our result can be viewed as a complementary to Liu-Sarnak's result.

Theorem 1.0.3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function of class $C^{1+\delta}$ for some $\delta > 0$, periodic of period* 1*. Let $c \in \mathbb{Z}$. Then for a generic set[6] of α, the automorphism $T_{c,f}$ of $\mathbb{T}^2$ given by* (5) *satisfies Conjecture 1.*

[4] The proof of the main result of the paper (Theorem 1.0.3 below) says also that the ergodic decomposition will be the same as the decomposition into minimal components which seems to be a fact of independent interest. In case of continuous compact group extensions the existence of the decomposition into minimal components is guaranteed by a result of Auslander [4] and Ellis [10] on distal systems.

[5] Notice that we do not aim to prove (2) (for $F(n)$ given by (4)) for each $g \in C(X)$ (and each $x \in X$) – as a matter of fact (2) (for $F(n)$ given by (4)) is not satisfied for some continuous functions already for an irrational rotation; we provide examples in Appendix, see Proposition 3.3.1. Our aim is to prove (2) for a linearly dense set of $g \in C(X)$ (and each $x \in X$), as it implies that (3) holds for each $g \in C(X)$.

[6] The question of whether an analogous result to Theorem 1.0.3 is true for f which is only assumed to be continuous, remains open. We recall that under the continuity assumptions on f, even, it is open whether f is not a quasi-coboundary for a generic set of α.

Before we give the proof of Theorem 1.0.3, we will first show that Conjecture 1 holds in the following two natural cases: the case of an arbitrary continuous extension of a rational rotation (see Proposition 2.3.3 below)[7] and to get an independent proof of the purely affine case (i.e. when $f = 0$) for each α, first proved in a larger context in [23].

Theorem 1.0.4 ([23]). *For any $\alpha, \gamma \in \mathbb{R}$ and for any $c \in \mathbb{Z}$, the automorphism $(x, y) \mapsto (x + \alpha, cx + y + \gamma)$ satisfies Conjecture 1.*

Let us now describe how we check conditions (a), (b) and (c). Let $\alpha \notin \mathbb{Q}$ and let $Tx = x+\alpha$. In either setting (purely affine or with a non-trivial perturbation) the base rotation $T^r \times T^s$ is the same. Its ergodic components are obtained by taking the partition of $\mathbb{T}^2$ into closed invariant sets $A_{c_1} = A_{c_1}^{r,s} := \{(x, y + c_1) \in \mathbb{T}^2 \colon sx = ry\}$, $c_1 \in [0, \frac{1}{r})$. In fact, these sets are at the same time the minimal components of $T^r \times T^s$ and they are uniquely ergodic. Thus, we are interested in the action of $(T_{c,f})^r \times (T_{c,f})^s$ on the sets $I_{c_1} = I_{c_1}^{r,s} := A_{c_1} \times \mathbb{T}^2$. It turns out that this is equivalent to dealing with extensions of T by the following $\mathbb{T}^2$-valued cocycles:

$$\psi_{c_1}(x) = (\psi^{(r)}(rx), \psi^{(s)}(sx + c_1)),$$

where $\psi(x) = f(x) + cx$ and $c_1 \in [0, \frac{1}{r})$. The cocycle ψ_{c_1} is ergodic if and only if

$$e^{2\pi i(A\psi^{(r)}(rx)+B\psi^{(s)}(sx+c_1))} \text{ is not a multiplicative coboundary}^{8}$$

for $A, B \in \mathbb{Z}$, $A^2+B^2 \neq 0$ (see Remark 3.2.2). This is the situation we aim for in course of the proof of Theorem 1.0.3. For a generic α we indeed obtain ergodicity of ψ_{c_1} for all c_1 – for the details see Corollary 2.5.8. Statement (b) follows from the fact that we deal with compact group extensions of rotations. Finally, we show that also (c) holds: given a non-trivial character $\chi \in \widehat{\mathbb{T}}^2$, we prove that for r and s relatively prime and large enough, the corresponding integrals of $\chi \otimes \overline{\chi}$ are zero. In case of Theorem 1.0.4, the problem is in a sense more delicate – some of the sets I_{c_1} are too large to be the ergodic components and they need to be partitioned further into smaller subsets. This refined partition will be however satisfying (a). Condition (b) is proved in the same way as in Theorem 1.0.3. To prove that also (c) holds, we take again $\mathcal{F} = \widehat{\mathbb{T}}^2$.

For reader's convenience, we added Appendix collecting some necessary facts concerning cocycles.

2 Results

2.1 On the strategy of the proofs

Our approach to proving disjointness with the Möbius function was described in (a), (b) and (c). We will now make some more comments on this method. Recall that in view of Theorem 1.0.1, for a linearly dense set $\mathcal{F}$ of $g \in C(X)$

[7] Liu and Sarnak [23] prove such a result for f smooth.

[8] We identify $\mathbb{T}$ with the multiplicative circle $\mathbb{S}^1 = \{z \in \mathbb{C} \colon |z| = 1\}$. We will use both, the additive and the multiplicative notation, whichever is more convenient for us at the moment. In particular, $e^{2\pi ix}$ is to be understood multiplicatively and x in the exponent is to be understood additively.

and each $x \in X$, what we want to prove is

$$\frac{1}{N}\sum_{n\leq N} g(T^{rn}x)\overline{g(T^{sn}x)} \to 0 \text{ as } N\to\infty \tag{6}$$

for distinct, sufficiently large prime numbers r, s. When this is realized through (a), (b) and (c), we prove more. Namely, for each $g \in \mathcal{F}$ and for sufficiently large primes $r \neq s$, the following holds for all $x, y \in X$:

$$\frac{1}{N}\sum_{n\leq N} g(T^{rn}x)\overline{g(T^{sn}y)} \to 0 \text{ as } N\to\infty \tag{7}$$

(the condition on r, s is independent of the choice of x and y).

Remark 2.1.1.

(i) In view of the above discussion, in order to prove that Sarnak conjecture holds for T, it suffices to check conditions (a), (b) and (c) for T^k for some $k \geq 1$. Indeed, by applying (7) to T^k, we obtain

$$\sum_{n\leq N} g(T^{r(kn+j)}x)\overline{g(T^{s(kn+j)}x)} =$$
$$= \sum_{n\leq N} g((T^k)^{rn}T^{rj}x)\overline{g((T^k)^{sn}T^{sj}x)} = \mathrm{o}(N) \text{ for } 0 \leq j < k,$$

which implies that (6) holds for T.

(ii) Notice that whenever the conditions (a), (b) and (c) are satisfied for some homeomorphism T then they are also satisfied for T^{-1}. However,

(iii) It is unclear how to prove directly that if T^k for some $k \in \mathbb{Z}\setminus\{0\}$ is disjoint from $\boldsymbol{\mu}$ then also T is disjoint from $\boldsymbol{\mu}$, or even to show that if the assumptions of the KBSZ criterion are satisfied for T^k for some $k \in \mathbb{Z}\setminus\{0\}$ then they are satisfied for T.

Let now $T_\varphi \colon X\times\mathbb{T} \to X\times\mathbb{T}$ be a continuous circle group extension of a homeomorphism $T\colon X\to X$ by $\varphi\colon X\to\mathbb{T}$.

Remark 2.1.2. Suppose that T_φ satisfies Sarnak's conjecture. Then for any $k \geq 1$ also $T_{k\varphi}$ satisfies Sarnak's conjecture as it is a topological factor of T_φ.

In the case of affine extensions of rotations, Remark 2.1.1 (i) and Remark 2.1.2 are complementary in the following sense. Let $T^{(\alpha)}$ stand for the rotation $T^{(\alpha)}x = x+\alpha$ and let $\psi(x) = x$. Then $(T_\psi^{(\frac{\alpha}{k})})^k = T^{(\alpha)}_{k\psi+\frac{k-1}{2}\alpha}$ and, by Remark 2.1.1 (i), the following implication holds:

$$\text{(a), (b) and (c) hold for } T^{(\alpha)}_{k\psi+\frac{k-1}{2}\alpha} \Rightarrow T_\psi^{(\frac{\alpha}{k})} \text{ satisfies Conjecture 1.}$$

2.2 General remarks

From now on, our assumption will be that

$$r, s \geq 3 \text{ are odd and relatively prime.}$$

Let $\alpha \notin \mathbb{Q}$ and denote by $T\colon \mathbb{T} \to \mathbb{T}$ the irrational rotation $Tx = x + \alpha$. For $c_1 \in [0, \frac{1}{r})$ let

$$I_{c_1} = I_{c_1}^{r,s} := A_{c_1} \times \mathbb{T}^2, \text{ where } A_{c_1} = A_{c_1}^{r,s} := \{(x, y + c_1) \in \mathbb{T}^2 \colon sx = ry\}.^9 \quad (8)$$

Lemma 2.2.1. *The decomposition of $\mathbb{T}^2$ into minimal components of $T^r \times T^s$ consists of sets A_{c_1}, $c_1 \in [0, \frac{1}{r})$. It is the same as the ergodic decomposition. Moreover, $(T^r \times T^s)|_{A_{c_1}}$ is topologically isomorphic to T. The isomorphism is given by*

$$W = W_{c_1} \colon A_{c_1} \to \mathbb{T},\ W(x, y + c_1) = ax + by, \tag{9}$$

where $a, b \in \mathbb{Z}$ are such that $ar + bs = 1$.

Proof. Notice first that the sets A_{c_1} are closed and invariant under $T^r \times T^s$ and $\cup_{c_1 \in [0, \frac{1}{r})} A_{c_1} = \mathbb{T}^2$. Fix $c_1 \in [0, \frac{1}{r})$, let $a, b \in \mathbb{Z}$ be such that $ar + bs = 1$ and let W be given by (9). Then $W \circ (T^r \times T^s)|_{A_{c_1}} = T \circ W$. For $(x, y + c_1) \in A_{c_1}$, we have $r(ax + by) = x$ and $s(ax + by) = y$. Therefore, W is bijective. Moreover, W preserves the measure, as rotations are uniquely ergodic. □

For a measurable function $\psi\colon \mathbb{T} \to \mathbb{T}$, let $\Psi\colon \mathbb{T}^2 \to \mathbb{T}^2$ be given by

$$\Psi(x, y) = (\psi^{(r)}(x), \psi^{(s)}(y)). \tag{10}$$

Then clearly the automorphism $(T_\psi)^r \times (T_\psi)^s$ is topologically isomorphic to $(T^r \times T^s)_\Psi$.

Lemma 2.2.2. *For $c_1 \in [0, \frac{1}{r})$ the sets I_{c_1} are invariant under $(T^r \times T^s)_\Psi$. Moreover, $(T^r \times T^s)_\Psi|_{I_{c_1}}$ is topologically isomorphic to $T_{\psi_{c_1}}$, where $\psi_{c_1}(x) = (\psi^{(r)}(rx), \psi^{(s)}(sx + c_1))$. The isomorphism is given by*

$$V = V_{c_1} \colon I_{c_1} \to \mathbb{T}^3,\ V(x, y + c_1, u, v) = (ax + by, u, v), \tag{11}$$

with $a, b \in \mathbb{Z}$ such that $ar + bs = 1$.

Proof. Fix $c_1 \in [0, \frac{1}{r})$, let $a, b \in \mathbb{Z}$ be such that $ar + bs = 1$ and let V be given by (11). Then

$$\begin{aligned} V \circ (T^r \times T^s)_\Psi &(x, y + c_1, u, v) \\ &= V(x + r\alpha, y + s\alpha + c_1, u + \psi^{(r)}(x), v + \psi^{(s)}(y + c_1)) \\ &= (ax + by + \alpha, u + \psi^{(r)}(x), v + \psi^{(s)}(y + c_1)) \end{aligned}$$

and

$$\begin{aligned} T_{\psi_{c_1}} \circ V(x, y + c_1, u, v) &= T_{\psi_{c_1}}(ax + by, u, v) \\ &= (ax + by + \alpha, u + \psi^{(r)}(r(ax + by)), v + \psi^{(s)}(s(ax + by) + c_1)). \end{aligned}$$

Moreover, $r(ax + by) = x$ and $s(ax + by) = y$, which completes the proof. □

[9] Whenever r, s are fixed, we will write I_{c_1} and A_{c_1}. For r, s varying, we will use $I_{c_1}^{r,s}$ and $A_{c_1}^{r,s}$.

Remark 2.2.3. The inverse of $V_{c_1}: I_{c_1} \to \mathbb{T}^3$ is given by

$$V_{c_1}^{-1}(t, u, v) = (rt, st + c_1, u, v).$$

Moreover, $art + bst = t$ for $a, b \in \mathbb{Z}$ such that $ar + bs = 1$.

2.3 Continuous extensions of rational rotations

We will now show that all continuous extensions of rational rotations satisfy Sarnak's conjecture.

Lemma 2.3.1. *Assume that $S(x, y) = (x, f(x) + y)$ with $f : \mathbb{T} \to \mathbb{T}$ continuous. Then for each $(x_i, y_i) \in \mathbb{T}^2$, $i = 1, 2$, $\chi \in \widehat{\mathbb{T}^2}$ we have*

$$\frac{1}{N} \sum_{n \leq N} \chi(S^{rn}(x_1, y_1)) \overline{\chi(S^{sn}(x_2, y_2))} \to 0 \quad \text{as} \quad N \to \infty \tag{12}$$

for sufficiently large prime numbers $r \neq s$, whenever $\chi \neq \eta \otimes \mathbb{1}_{\mathbb{T}}$ ($\eta \in \widehat{\mathbb{T}}$) and $f(x_1) \notin \mathbb{Q}$ or $f(x_2) \notin \mathbb{Q}$.

Proof. Note that for each $m \geq 1$,

$$S^m(x, y) = (x, mf(x) + y).$$

Let $\chi(x, y) = e^{2\pi i(ax+by)}$ for some $a, b \in \mathbb{Z}$, with $b \neq 0$ by assumption. Hence

$$\frac{1}{N} \sum_{n \leq N} \chi(S^{rn}(x_1, y_1)) \overline{\chi(S^{sn}(x_2, y_2))}$$
$$= e^{2\pi i(a(x_1 - x_2) + b(y_1 - y_2))} \frac{1}{N} \sum_{n \leq N} e^{2\pi i b(rf(x_1) - sf(x_2))n}.$$

If exactly one of the numbers $f(x_1)$ and $f(x_2)$ is irrational then the result follows from Weyl's criterion. If both $f(x_1)$ and $f(x_2)$ are irrational then there is at most one pair (r, s) of relatively prime numbers such that $rf(x_1) - sf(x_2) \in \mathbb{Q}$ and we can again make use of Weyl's criterion, this time for r, s sufficiently large. □

Remark 2.3.2. Notice that the above proof says more. Namely, the convergence in (12) does not depend on y_1, y_2.

Proposition 2.3.3. *Each continuous extension $R(x, y) = (x + \frac{p}{q}, f(x) + y)$ of a rational rotation $x \mapsto x + p/q$ satisfies Sarnak's conjecture.*

Proof. We need to check (1) only for $F = \chi \in \widehat{\mathbb{T}^2}$. First, note that

$$R^q(x, y) = (x, f_q(x) + y), \tag{13}$$

where $f_q(x) = f(x) + f(x + \frac{1}{q}) + \ldots + f(x + \frac{q-1}{q})$. Given $n \geq 1$, we take n' such that $n = qn' + j$ with $0 \leq j < q$. Then, for each $\chi \in \widehat{\mathbb{T}^2}$, we have

$$\chi(R^{rn}(x, y)) \overline{\chi(R^{sn}(x, y))} = \chi(R^{qrn'}(R^{rj}(x, y))) \overline{\chi(R^{qsn'}(R^{sj}(x, y)))},$$

where the first coordinates of the points $R^{rj}(x,y), R^{rj}(x,y)$ belong to the finite set $\{x, x+\frac{1}{q}, \dots, x+\frac{q-1}{q}\}$ (hence do not depend on r,s). Hence, to show

$$\frac{1}{N}\sum_{n\le N}\chi(R^{rn}(x,y))\overline{\chi(R^{sn}(x,y))}\to 0$$

for sufficiently large prime numbers $r\neq s$, we need to show that

$$\frac{1}{N}\sum_{n'\le N/q}\chi(R^{qrn'}(x_1,*))\overline{\chi(R^{qsn'}(x_2,*))}\to 0,$$

for $x_1,x_2\in\{x,x+1/q,\dots,x+\frac{q-1}{q}\}$. This is the case by Lemma 2.3.1, Remark 2.3.2 and (13), provided that $\chi\neq\eta\otimes\mathbb{1}_{\mathbb{T}}$ and $f_q(x_1)\notin\mathbb{Q}$ or $f_q(x_2)\notin\mathbb{Q}$.

If $\chi=\eta\otimes\mathbb{1}_{\mathbb{T}}$, (1) follows from Sarnak's conjecture for finite systems. Suppose now that $f_q(x+jrp/q), f_q(x+jsp/q)\in\mathbb{Q}$. This is possible only if $f_q(x)\in\mathbb{Q}$ since $f_q(\cdot)$ is constant on the orbit of x under the rotation $x\mapsto x+\frac{p}{q}$. Moreover, if $n=qn'+j$ with $0\le j<q$ then

$$f^{(n)}(x):=\sum_{i=0}^{n-1}f(x+ip/q)=f^{(j)}(x)+f^{(qn')}(x+jp/q)=f^{(j)}(x)+n'f_q(x). \quad (14)$$

It follows that

$$\begin{aligned}
&\frac{1}{N}\sum_{n\le N}\chi(R^n(x,y))\boldsymbol{\mu}(n)\\
&=\sum_{j=0}^{q-1}\frac{1}{N}\sum_{n'\le N/q}\chi(x+(qn'+j)p/q, f^{(j)}(x)+n'f_q(x)+y)\boldsymbol{\mu}(qn'+j)\\
&=\sum_{j=0}^{q-1}\frac{1}{N}\sum_{n'\le N/q}\chi(x+jp/q, f^{(j)}(x)+n'f_q(x)+y))\boldsymbol{\mu}(qn'+j)\\
&=\sum_{j=0}^{q-1}e^{2\pi i(a(x+jp/q)+bf^{(j)}(x))}\frac{1}{N}\sum_{n'\le N/q}e^{2\pi i b f_q(x)n'}\boldsymbol{\mu}(qn'+j)\\
&=\sum_{j=0}^{q-1}e^{2\pi i(a(x+jp/q)+bf^{(j)}(x))}\frac{1}{N}\sum_{n'\le N/q}e^{2\pi i b\frac{c}{d}n'}\boldsymbol{\mu}(qn'+j),
\end{aligned}$$

where $\chi(x,y)=e^{2\pi i(ax+by)}$, $f_q(x)=\frac{c}{d}$ with $a,b,c,d\in\mathbb{Z}$, $b\neq 0$ and $d>0$. By seting $n'=dn''+k$ with $0\le k<d$, we obtain $qn'+j=qdn''+(qk+j)$ and rewriting the sum to the form

$$\sum_{j=0}^{q-1}\sum_{k=0}^{d-1}\frac{1}{N}\sum_{n''\le N/(dq)}A_{j,k}\boldsymbol{\mu}(qdn''+kq+j),$$

the result again follows from Sarnak's conjecture for finite systems. □

2.4 Affine case

Recall that we are interested in the disjointness with the Möbius function of $(x,y)\mapsto(x+\alpha, cx+y+\gamma)$, where $c\in\mathbb{Z}$ and $\alpha,\gamma\in\mathbb{R}$.

Remark 2.4.1. If $c \neq 0$, it follows immediately from Remark 2.1.2 that instead of $(x,y) \mapsto (x+\alpha, y+cx+\gamma)$, we can consider $(x,y) \mapsto (x+\alpha, y+x+\frac{\gamma}{c})$. By Proposition 2.3.3, this shows that it suffices to consider only the cases $c=1$ and $c=0$, with $\alpha \notin \mathbb{Q}$.

2.4.1 Case $c=1$, $\alpha \notin \mathbb{Q}$

We will now deal with $(x,y) \mapsto (x+\alpha, y+x+\gamma)$, i.e. with T_ψ, where $\psi(x) = x+\gamma$. Let Ψ be given by (10), i.e.

$$\Psi(x,y) = \left(rx + \frac{(r-1)r}{2}\alpha + r\gamma, sy + \frac{(s-1)s}{2}\alpha + s\gamma \right).$$

Recall that given $r, s \in \mathbb{N}$ (odd and relatively prime) and $c_1 \in [0, \frac{1}{r})$, we have

$$I_{c_1} = I_{c_1}^{r,s} = \left\{ (x, y+c_1, u, v) \in \mathbb{T}^4 \colon sx = ry \right\}.$$

For c_1 such that $rs((s-r)\gamma - rc_1) \in \alpha\mathbb{Q} + \mathbb{Q}$ and $c_2 \in [0, \frac{1}{r^2})$, define

$$\begin{aligned} J_{c_1,c_2} = J_{c_1,c_2}^{r,s} :=& \Big\{ (x, y+c_1, u, v+c_2) \in \mathbb{T}^4 \colon sx = ry \text{ and} \\ & l_0 s^2 u = l_0 r^2 v + \left(l_0 rs\frac{r-s}{2} - k_0 \right)(ax+by) \Big\}, \end{aligned}$$

where $l_0 = l_0^{r,s,c_1}$ is the smallest positive integer such that

$$l_0 rs((s-r)\gamma - rc_1) \in \alpha\mathbb{Z} + \mathbb{Z} \tag{15}$$

and $k_0 = k_0^{r,s,c_1} \in \mathbb{Z}$ is such that

$$l_0 rs((s-r)\gamma - rc_1) - k_0\alpha \in \mathbb{Z}. \tag{16}$$

Lemma 2.4.2. *For $c_1 \in [0, \frac{1}{r})$ the homeomorphism $T_{\psi_{c_1}} \colon \mathbb{T}^3 \to \mathbb{T}^3$, where $\psi_{c_1}(x) = (\psi^{(r)}(rx), \psi^{(s)}(sx+c_1))$, is topologically isomorphic to $T_{\varphi_{c_1}} \colon \mathbb{T}^3 \to \mathbb{T}^3$, where $\varphi_{c_1} \colon \mathbb{T} \to \mathbb{T}^2$ is given by $\varphi_{c_1}(x) = (r^2x + r\gamma, s^2x + sc_1 + s\gamma)$. The isomorphism is given by*

$$U = U_{c_1} \colon \mathbb{T}^3 \to \mathbb{T}^3,\ U(x,u,v) = \left(x, u - \frac{r(r-1)}{2}x, v - \frac{s(s-1)}{2}x \right). \tag{17}$$

Proof. We have

$$\begin{aligned} \psi_{c_1}(x) &= \left(\psi^{(r)}(rx), \psi^{(s)}(sx+c_1) \right) \\ &= \left(r^2x + \frac{r(r-1)}{2}\alpha + r\gamma, s(sx+c_1) + \frac{s(s-1)}{2}\alpha + s\gamma \right). \end{aligned}$$

Since for $\theta(x) = \left(-\frac{r(r-1)}{2}x, -\frac{s(s-1)}{2}x \right)$ we have

$$\left(\frac{r(r-1)}{2}\alpha, \frac{s(s-1)}{2}\alpha \right) = \theta(x) - \theta(x+\alpha),$$

it follows that U given by (17) is indeed the required isomorphism. Notice that θ is continuous, whence U is a homeomorphism. $\square$

Proposition 2.4.3. *The decomposition of $\mathbb{T}^4$ into minimal components for $(T^r \times T^s)_\Psi$ is the same as the decomposition into ergodic components. It consists of sets of the form I_{c_1} for $c_1 \notin \alpha\mathbb{Q} + \mathbb{Q}$ and J_{c_1,c_2} for $c_1 \in \alpha\mathbb{Q} + \mathbb{Q}$, where $c_1 \in [0, \frac{1}{r})$, $c_2 \in [0, \frac{1}{r^2})$. Moreover, on each such component, $(T^r \times T^s)_\Psi$ is uniquely ergodic. In particular, each point in $\mathbb{T}^4$ is generic (for a relevant invariant measure).*

Proof. In view of Lemma 2.4.2, instead of $(T^r \times T^s)_\Psi$, we may consider $T_{\varphi_{c_1}}$, with $\varphi_{c_1}(x) = (r^2x + r\gamma, s^2 + sc_1 + s\gamma)$. By Remark 3.2.7, Proposition 3.2.1 and Proposition 3.2.4, for the first part of the assertion, we need to show that the equation

$$\chi \circ \varphi_{c_1} = \xi - \xi \circ T. \tag{18}$$

has a measurable solution $\xi\colon \mathbb{T} \to \mathbb{T}$ for $\chi(u, v) = Au + Bv$ for $A, B \in \mathbb{Z}$ with $A^2 + B^2 \neq 0$ if and only if it has a continuous one. We have

$$\chi \circ \varphi_{c_1}(x) = \chi(r^2x + r\gamma, s^2x + sc_1 + s\gamma) = (Ar^2 + Bs^2)x + (Ar + Bs)\gamma + Bsc_1.$$

By [3], $\chi \circ \varphi_{c_1}$ can be a measurable coboundary only if

$$Ar^2 + Bs^2 = 0 \text{ and } (Ar + Bs)\gamma + Bsc_1 \in \alpha\mathbb{Z} + \mathbb{Z}. \tag{19}$$

Then the solution to (18) is given by $\xi(x) = -kx$, where $k \in \mathbb{Z}$ is such that $(Ar + Bs)\gamma + Bsc_1 - k\alpha \in \mathbb{Z}$. In particular, all measurable solutions to (18) are continuous. It follows by Remark 3.2.7 that the decomposition into minimal components for $(T^r \times T^s)_\Psi$ is the same as the decomposition into ergodic components.

We will now describe the ergodic (i.e. minimal) components of each $T_{\varphi_{c_1}}$ and show that they are uniquely ergodic. Suppose that $\chi \circ \varphi_{c_1}$ is a coboundary. It follows from (19) (recall that $r \neq s$ are coprime) that $r^2|B$ and $s^2|A$, and therefore $A = A's^2$ and $B = -A'r^2$ for some $A' \in \mathbb{Z}$. Hence, the second part of condition (19) takes the form

$$A'rs((s - r)\gamma - rc_1) \in \alpha\mathbb{Z} + \mathbb{Z}. \tag{20}$$

Having this in mind, we consider two cases:

(i) $(s - r)\gamma - rc_1 \notin \mathbb{Q}\alpha + \mathbb{Q}$,

(ii) $(s - r)\gamma - rc_1 \in \mathbb{Q}\alpha + \mathbb{Q}$.

In case (i), it follows immediately from the first part of the proof that $(T^r \times T^s)_\Psi|_{I_{c_1}}$ is ergodic and minimal. Unique ergodicity follows from Proposition 3.2.8.

We consider now case (ii). We will describe characters χ, such that (18) has a (measurable and continuous) solution. In view of and (15) and (20), $A' = al_0$ for some $a \in \mathbb{Z}$. Therefore, a measurable solution $\xi\colon \mathbb{T} \to \mathbb{T}$ to (18) exists precisely for the characters χ of $\mathbb{T}^2$ of the form

$$\chi(u, v) = al_0s^2u - al_0r^2v \text{ for } a \in \mathbb{Z}. \tag{21}$$

Denote the set of such characters by $\Gamma = \Gamma_{top}$. It is easy to see that

$$\text{ann } \Gamma = \left\{(u, v) \in \mathbb{T}^2 \colon l_0s^2u = l_0r^2v\right\}.$$

We claim that

$$\widetilde{J}^{r,s}_{c_1,c_2} := \left\{(t,u,v+c_2)\in\mathbb{T}^3 \colon l_0 s^2 u = l_0 r^2 v - k_0 t\right\} \text{ for } c_2\in[0,1/r^2)$$

are minimal components of $T_{\varphi_{c_1}}$. Indeed:

- each $\widetilde{J}^{r,s}_{c_1,c_2}$ is closed,
- by (16), each $\widetilde{J}^{r,s}_{c_1,c_2}$ is $T_{\varphi_{c_1}}$-invariant,
- $\bigcup_{c_2}\widetilde{J}^{r,s}_{c_1,c_2} = \mathbb{T}^3$ (notice that the projection of $\widetilde{J}^{r,s}_{c_1,c_2}$ onto the first two coordinates is equal to $\mathbb{T}^2$),
- $(u,v)\in \operatorname{ann}\Gamma$ if and only if $\widetilde{J}^{r,s}_{c_1,c_2}+(0,u,v)=\widetilde{J}^{r,s}_{c_1,c_2}$

(see Proposition 3.2.4). Unique ergodicity follows, as in the previous case, from Proposition 3.2.8.

To find the minimal component corresponding to each of the sets $\widetilde{J}^{r,s}_{c_1,c_2}$ described above, we need to find the preimage of $\widetilde{J}^{r,s}_{c_1,c_2}$ via $U_{c_1}\circ V_{c_1}$. We have

$$\begin{aligned}
&U_{c_1}^{-1}(\widetilde{J}^{r,s}_{c_1,c_2}) =\\
&\quad=\left\{(t,u,v+c_2)\colon l_0 s^2\left(u-\frac{r(r-1)}{2}t\right) = l_0 r^2\left(v-\frac{s(s-1)}{2}t\right)-k_0 t\right\}\\
&\quad=\left\{(t,u,v+c_2)\colon l_0 s^2 u = l_0 r^2 v + \left(l_0 rs\frac{r-s}{2}-k_0\right)t\right\},
\end{aligned}$$

whence, by Remark 2.2.3,

$$\begin{aligned}
(U_{c_1}V_{c_1})^{-1}(\widetilde{J}^{r,s}_{c_1,c_2}) =& \{(rt, st+c_1, u, v+c_2)\colon\\
& l_0 s^2 u = l_0 r^2 v + \left(l_0 rs\frac{r-s}{2}-k_0\right)t\Big\}\\
=&\Big\{(x, y+c_1, u, v+c_2)\colon sx = ry \text{ and}\\
& l_0 s^2 u = l_0 r^2 v + \left(l_0 rs\frac{r-s}{2}-k_0\right)(ax+by)\Big\}.
\end{aligned}$$

Therefore $(U_{c_1}V_{c_1})^{-1}\left(\widetilde{J}^{r,s}_{c_1,c_2}\right) = J^{r,s}_{c_1,c_2}$, which completes the proof. □

Remark 2.4.4. The sets $I^{r,s}_{c_1}$ are translates of $I^{r,s}_0$, which is a subgroup of $\mathbb{T}^4$. Therefore, the conditional measures given by the ergodic decomposition of $(T^r\times T^s)_\Psi$, which are supported on the sets I_{c_1} for c_1 such that $(s-r)\gamma - rc_1\notin \mathbb{Q}\alpha+\mathbb{Q}$, are translates of Haar measure on $I^{r,s}_0$.

For c_1 with $(s-r)\gamma - rc_1\in\mathbb{Q}\alpha+\mathbb{Q}$, the sets $\widetilde{J}^{r,s}_{c_1,c_2}$ are translates of $\widetilde{J}^{r,s}_{c_1,0}$, which is a subgroup of $\mathbb{T}^3$. As before, the conditional measures given by the ergodic decomposition of $T_{\varphi_{c_1}}$, are translates of Haar measure on $\widetilde{J}^{r,s}_{c_1,0}$. Notice that V_{c_1} carries a translate of Haar measure to Haar measure. Therefore, also the conditional measures given by the ergodic decomposition of $(T^r\times T^s)_\Psi$, which are supported on the sets $J^{r,s}_{c_1,c_2}$, are translates of the corresponding Haar measures.

Remark 2.4.5. Let G be a compact Abelian group. Let $\chi \in \widehat{G}$ and let $H = \ker \chi$. Then χ is constant on each coset of H. Moreover, if $x + H \neq y + H$ then $\chi(x + H) \neq \chi(y + H)$.

Remark 2.4.6. Let $\chi \in \widehat{G}$ and let $H \subset G$ be a subgroup. Then the integral of χ with respect to Haar measure on each coset of H is zero if and only if $\chi|_H$ is not constant.

Lemma 2.4.7. *Let G be a compact metric Abelian group. Let $\chi \in \widehat{G}$. Then there exists $\delta > 0$ such that whenever $\{y_1, \dots, y_n\} \subset G$ is a δ-net and $\chi(y_i) = 1$ for $i = 1, \dots, n$ then $\chi \equiv 1$.*

Proof. Let d stand for the metric on G. Since χ is uniformly continuous, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|\chi(x) - \chi(y)| < \frac{1}{4}$. Since $\{y_1, \dots, y_n\}$ is a δ-net, it follows that $|\chi(G) - 1| < \frac{1}{2}$. This is however possible only when $\chi \equiv 1$, as $\chi(G)$ is a closed subgroup of $\mathbb{S}^1$. $\square$

Lemma 2.4.8. *Let $\chi \in \widehat{\mathbb{T}^4}$ be a non-trivial character. If $r, s \in \mathbb{N}$ are large enough then $\chi|_{I_{c_1}}$ is not constant for $c_1 \in [0, \frac{1}{r})$.*

Proof. Fix $1 \not\equiv \chi \in \widehat{\mathbb{T}^4}$ and let $\delta > 0$ be as in Lemma 2.4.7. In view of Lemma 2.4.7, we need to show that whenever r, s are large enough then there exists a δ-net of $\mathbb{T}^4$ contained in $I_{c_1}^{r,s}$ for $c_1 \in [0, \frac{1}{r})$. Since $I_{c_1}^{r,s}$ is a translate of $I_0^{r,s}$ and the third and fourth coordinate in I_0 is arbitrary, it suffices to prove that for r, s sufficiently large we can always find a δ-net of $\mathbb{T}^2$ contained in the set $A_0^{r,s}$. To complete the proof it suffices to consider sets of the form $\left\{ \left(\frac{i}{s}, \frac{j}{r} \right) : 0 \leq i < s, 0 \leq j < r \right\}$. $\square$

Lemma 2.4.9. *Let $\chi \in \widehat{\mathbb{T}^4}$ be a non-trivial character. If $r, s \in \mathbb{N}$ are large enough then $\chi|_{J_{c_1,c_2}^{r,s}}$ is not constant for $c_1 \in [0, \frac{1}{r})$ such that $c_1 \in \alpha\mathbb{Q} + \mathbb{Q}$ and $c_2 \in [0, \frac{1}{r^2})$.*

Proof. Fix $1 \not\equiv \chi \in \widehat{\mathbb{T}^4}$ and let $\delta > 0$ be as in Lemma 2.4.7. Since $J_{c_1,c_2}^{r,s}$ is a translation of $J_{c_1,0}^{r,s}$, by Lemma 2.4.7, it suffices to prove that for r, s sufficiently large there exists a δ-net of $\mathbb{T}^4$ in $J_{c_1,0}^{r,s}$.

Notice first that the projection of $J_{c_1,0}^{r,s}$ onto the first two coordinates is equal to $A_{c_1}^{r,s}$. Indeed, for any $z \in \mathbb{T}$ the equation $l_0 s^2 u = l_0 r^2 v + z$ has a solution (u, v) as $\mathbb{T}$ is an infinitely divisible group. By the proof of the previous lemma, we can find a δ-net of $\mathbb{T}^2$

$$\{(x_i, y_j) \colon 0 \leq i, j < n\} \subset A_{c_1}^{r,s}.$$

Moreover, by the infinite divisibility of $\mathbb{T}$, for each (i, j) there exist $u_{i,j}, v_{i,j}$ such that $(x_i, y_j, u_{i,j}, v_{i,j}) \in J_{c_1,0}^{r,s}$. Moreover, for $0 \leq a, b < l_0 s^2$

$$\left(x_i, y_j, u_{i,j} + \frac{a}{l_0 s^2}, v_{i,j} + \frac{b}{l_0 s^2} \right) \in J_{c_1,0}^{r,s}.$$

Therefore, whenever r, s are sufficiently large, the set

$$\left\{ \left(x_i, y_j, u_{i,j} + \frac{a}{l_0 s^2}, v_{i,j} + \frac{b}{l_0 s^2} \right) : 0 \leq i, j < n, 0 \leq a, b < l_0 s^2 \right\}$$

is the required δ-net.[10] This completes the proof. $\square$

[10] Notice that since $l_0 \geq 1$, the condition on r, s is independent of c_1.

Proposition 2.4.10. *For any $1 \not\equiv \chi \in \widehat{\mathbb{T}}^4$ whenever $r, s \in \mathbb{N}$ are odd, relatively prime and large enough then $\int_I \chi \, d\lambda_I = 0$ for each ergodic component of $(T^r \times T^s)_\Psi$, where λ_I is the relevant invariant measure.*

Proof. The assertion follows immediately from Proposition 2.4.3, Lemma 2.4.8 and Lemma 2.4.9. $\square$

Proof of Theorem 1.0.4 in case $c = 1$, $\alpha \notin \mathbb{Q}$. We only need to check that (a), (b), (c) hold. Conditions (a) and (b) follows from Proposition 2.4.3. It follows by Proposition 2.4.10 that also condition (c) is satisfied (we take $\mathcal{F} = \widehat{\mathbb{T}}^2$). $\square$

2.4.2 Case $c = 0$, $\alpha \notin \mathbb{Q}$

Proof of Theorem 1.0.4 in case $c = 0$, $\alpha \notin \mathbb{Q}$. We have $T_\psi(x, u) = (x+\alpha, u+\gamma)$, i.e. T_ψ is a rotation on $\mathbb{T}^2$. The decomposition into minimal components of $(T^r \times T^s)_\Psi$ consists of the cosets of $I_0 = I_0^{r,s} = \{(x, y, 0, 0) \in \mathbb{T}^4 \colon sx = ry\}$. They are at the same the ergodic components which are moreover uniquely ergodic, and we conclude as previously. $\square$

2.5 Generic case – compact group extensions

Let $f \colon \mathbb{R} \to \mathbb{R}$ (periodic of period 1) be in $L^2(\mathbb{T})$. Denote by

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}$$

the Fourier expansion of f. Recall that our goal is to prove disjointness of

$$\mathbb{T}^2 \ni (x, y) \mapsto (x + \alpha, y + cx + f(x)) \in \mathbb{T}^2,$$

with the Möbius function $\boldsymbol{\mu}$ for a generic set of α (under some additional assumptions on f).

Recall the following result.

Theorem 2.5.1 ([18], [21][11])**.** *Suppose that $f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}$ is in $C^{1+\delta}(\mathbb{T})$ for some $\delta > 0$, and has zero mean. Denote by $T \colon \mathbb{T} \to \mathbb{T}$ the irrational rotation $x \mapsto x + \alpha$. Assume that for a sequence $(p_n/q_n)_{n \in \mathbb{N}}$ of rational numbers we have*

$$\frac{|\widehat{f}(q_n)|}{\sum_{k \geq 1} |\widehat{f}(kq_n)|} > c > 0 \tag{22}$$

and

$$\frac{\left|\alpha - \frac{p_n}{q_n}\right| q_n}{|\widehat{f}(q_n)|} \to 0. \tag{23}$$

Then for each $\lambda \in \mathbb{S}^1$ the cocycle $\lambda e^{2\pi i f(\cdot)}$ is not a T-coboundary.

We will now prove a modified version of the above theorem. It will be our main tool in course of the proof of Theorem 1.0.3.

[11]In [18] f is assumed to be of class C^2 and the proof in fact requires that $\sum n|\widehat{f}(n)| < \infty$. See [21] for the proof of Theorem 2.5.1 with $f \in C^{1+\delta}(\mathbb{T})$ for some $\delta > 0$.

Theorem 2.5.2. *Suppose that $f(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi inx}$ is in $C^{1+\delta}(\mathbb{T})$ for some $\delta > 0$, and has zero mean. Denote by $T\colon \mathbb{T} \to \mathbb{T}$ the irrational rotation $x \mapsto x+\alpha$. Let $r, s \in \mathbb{N}$ ($r > s$) be relatively prime. Assume that $(p_n/q_n)_{n\in\mathbb{N}}$ is a subsequence of convergents of α in its continued fraction expansion such that*

$$\frac{|\widehat{f}(q_n)|}{\sum_{k\geq 1} |\widehat{f}(kq_n)|} > c_0 > 0, \tag{24}$$

$$\frac{|\widehat{f}(q_n)|}{|\widehat{f}(q_n)| + \sum_{k\geq 1} |\widehat{f}(k\frac{r}{s}q_n)|} > c_0 > 0, \text{ whenever } s|q_n, \tag{25}$$

and

$$\frac{\left|\alpha - \frac{p_n}{q_n}\right| q_n}{|\widehat{f}(q_n)|} \to 0. \tag{26}$$

Then for each $\lambda \in \mathbb{S}^1$, $h \in \mathbb{R}$ and $A, B \in \mathbb{R}$ with $A^2 + B^2 \neq 0$ the cocycle $\lambda e^{2\pi i(Af^{(r)}(rx)+Bf^{(s)}(sx+h))}$ is not a T-coboundary.

Proof. Fix $A, B \in \mathbb{R}$ with $A^2 + B^2 \neq 0$, relatively prime numbers $r > s$ and $h \in \mathbb{R}$. Set

$$F(x) := Af^{(r)}(rx) + Bf^{(s)}(sx+h).$$

For any $k \in \mathbb{N}$ we have

$$f^{(k)}(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\frac{1-e^{2\pi inka}}{1-e^{2\pi in\alpha}}e^{2\pi inx}.$$

Therefore,

$$F(x) = A\sum_{n\in\mathbb{Z}} \widehat{f}(n)\frac{1-e^{2\pi inr\alpha}}{1-e^{2\pi in\alpha}}e^{2\pi inrx} + B\sum_{n\in\mathbb{Z}} \widehat{f}(n)\frac{1-e^{2\pi ins\alpha}}{1-e^{2\pi in\alpha}}e^{2\pi in(sx+h)}. \tag{27}$$

Suppose first that $A \cdot B = 0$. We may assume without loss of generality that $A = 0$ and $B \neq 0$. Then, for all $n \in \mathbb{N}$, by (27), we have

$$\widehat{F}(sn) = B\widehat{f}(n)\frac{1-e^{2\pi ins\alpha}}{1-e^{2\pi in\alpha}}e^{2\pi inh}.$$

Therefore, since

$$\left|\frac{1-e^{2\pi iq_ns\alpha}}{1-e^{2\pi iq_n\alpha}}\right| \to s \text{ as } n \to \infty, \tag{28}$$

$$|\widehat{F}(sq_n)| \geq |B| \cdot \frac{s}{2} \cdot |\widehat{f}(q_n)| \text{ for } n \text{ sufficiently large} \tag{29}$$

and

$$|\widehat{F}(msq_n)| = |B| \cdot |\widehat{f}(mq_n)| \cdot \left|\frac{1-e^{2\pi imsq_n\alpha}}{1-e^{2\pi imq_n\alpha}}\right| \leq |B| \cdot s \cdot |\widehat{f}(mq_n)| \tag{30}$$

for all $n \in \mathbb{N}$, $m \in \mathbb{Z}$. It follows immediately by (29) and (30) that for n sufficiently large

$$\frac{|\widehat{F}(sq_n)|}{\sum_{m\geq 1} |\widehat{F}(msq_n)|} \geq \frac{|B| \cdot \frac{s}{2} \cdot |\widehat{f}(q_n)|}{\sum_{m\geq 1} |B| \cdot s \cdot |\widehat{f}(mq_n)|} = \frac{|\widehat{f}(q_n)|}{2\sum_{m\geq 1} |\widehat{f}(mq_n)|} \tag{31}$$

$$\frac{\left|\alpha - \frac{sp_n}{sq_n}\right| sq_n}{|\widehat{F}(sq_n)|} \leq \frac{2}{|B|} \cdot \frac{\left|\alpha - \frac{p_n}{q_n}\right| q_n}{|\widehat{f}(q_n)|}. \tag{32}$$

Now, (31) and (32) and the assumptions (24) and (26) imply that for n sufficiently large

$$\frac{|\widehat{F}(sq_n)|}{\sum_{m\geq 1} |\widehat{F}(msq_n)|} > \frac{c_0}{2} > 0 \text{ and } \frac{\left|\alpha - \frac{sp_n}{sq_n}\right| sq_n}{|\widehat{F}(sq_n)|} \to 0.$$

In view of Theorem 2.5.1, this completes the proof in case $A \cdot B = 0$.

Suppose now that $A \cdot B \neq 0$. Applying the fact that for any absolutely summable sequence $(y_n)_{n\in\mathbb{Z}}$

$$\sum_{n\in\mathbb{Z}} y_n = \sum_{r|n} y_n + \sum_{r\nmid n} y_n = \sum_{n\in\mathbb{Z}} y_{nr} + \sum_{r\nmid n} y_n = \sum_{s|n} y_{n\frac{r}{s}} + \sum_{r\nmid n} y_n$$ [12]

to the second summand in formula (27), we obtain

$$\begin{aligned} F(x) = \sum_{s|n} & \underbrace{\left(A\widehat{f}(n)\frac{1-e^{2\pi inr\alpha}}{1-e^{2\pi in\alpha}} + B\widehat{f}(n\frac{r}{s})\frac{1-e^{2\pi inr\alpha}}{1-e^{2\pi in\frac{r}{s}\alpha}}e^{2\pi in\frac{r}{s}h}\right)}_{a_{rn}} e^{2\pi inrx} \\ & + A\sum_{s\nmid n} \widehat{f}(n)\frac{1-e^{2\pi irn\alpha}}{1-e^{2\pi in\alpha}}e^{2\pi inrx} + B\sum_{r\nmid n} \widehat{f}(n)\frac{1-e^{2\pi ins\alpha}}{1-e^{2\pi in\alpha}}e^{2\pi inh}e^{2\pi insx}. \end{aligned} \tag{33}$$

For $m \in \mathbb{N}$, let $B_m := \{n \in \mathbb{N} \colon m \nmid q_n\}$. We will consider the following cases:

(i) B_r or B_s is infinite,

(ii) both sets B_r and B_s are finite.

We will cover first case (i). Without loss of generality we may assume that B_r is infinite. Suppose that

$$\lambda e^{2\pi i(Af^{(r)}(rx)+Bf^{(s)}(sx+h))} \text{ is a coboundary.} \tag{34}$$

Since $f^{(r)}(rx)$ is $\frac{1}{r}$-periodic, it follows immediately that

$$\lambda e^{2\pi i(Af^{(r)}(rx)+Bf^{(s)}(sx+\frac{s}{r}+h))} \text{ is also a coboundary.} \tag{35}$$

Hence, dividing the expression from (34) by the one from (35), we conclude that

$$e^{2\pi iB(f^{(s)}(sx)-f^{(s)}(sx+\frac{s}{r}))} \text{ is a coboundary as well.} \tag{36}$$

We will show that this is impossible in view of Theorem 2.5.1. Indeed, we have

$$g(x) := f^{(s)}(sx) - f^{(s)}\left(sx + \frac{s}{r}\right) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\frac{1-e^{2\pi ins\alpha}}{1-e^{2\pi in\alpha}}(1-e^{2\pi in\frac{s}{r}})e^{2\pi insx}.$$

[12] These equalities hold for all $r, s \in \mathbb{N}$.

We claim that for $n \in B_r$,

$$\frac{|\widehat{g}(sq_n)|}{\sum_{k\geq 1}|\widehat{g}(ksq_n)|} > c_1 \text{ for some } c_1 > 0 \tag{37}$$

and

$$\frac{\left|\alpha - \frac{sp_n}{sq_n}\right| sq_n}{|\widehat{g}(sq_n)|} \to 0. \tag{38}$$

Indeed, for $n \in B_r$, we have

$$|\widehat{g}(sq_n)| = |\widehat{f}(q_n)| \cdot \left|\frac{1 - e^{2\pi i s q_n \alpha}}{1 - e^{2\pi i q_n \alpha}}\right| \cdot |e^{2\pi i q_n \frac{s}{r}} - 1|. \tag{39}$$

Since $r \nmid q_n$ and r and s are relatively prime,

$$\left|e^{2\pi i q_n \frac{s}{r}} - 1\right| \geq \left|e^{2\pi i \frac{1}{r}} - 1\right| =: c_2 > 0. \tag{40}$$

Using again (28), we obtain from (39) and (40) that for n sufficiently large,

$$|\widehat{g}(sq_n)| \geq \frac{c_2 s}{2} \cdot |\widehat{f}(q_n)|. \tag{41}$$

On the other hand, since $\left|\frac{1-e^{2\pi i k s q_n \alpha}}{1-e^{2\pi i k q_n \alpha}}\right| \leq s$,

$$\begin{aligned}\sum_{k\geq 1}|\widehat{g}(ksq_n)| &= \sum_{k\geq 1}|\widehat{f}(kq_n)| \cdot \left|\frac{1 - e^{2\pi i k s q_n \alpha}}{1 - e^{2\pi i k q_n \alpha}}\right| \cdot |e^{2\pi i q_n \frac{s}{r}} - 1| \\ &\leq \sum_{k\geq 1}|\widehat{f}(kq_n)| \cdot s \cdot |e^{2\pi i q_n \frac{s}{r}} - 1| \leq 2s \sum_{k\geq 1}|\widehat{f}(kq_n)|.\end{aligned} \tag{42}$$

Thus, using (41), (42) and (24) we obtain, for n large enough,

$$\frac{|\widehat{g}(sq_n)|}{\sum_{k\geq 1}|\widehat{g}(ksq_n)|} \geq \frac{c_2 s|\widehat{f}(q_n)|}{4s\sum_{k\geq 1}|\widehat{f}(kq_n)|} > \frac{c_0 c_2}{4},$$

which shows that (37) holds. Using (41) and (26), we obtain that (38) also holds:

$$\frac{\left|\alpha - \frac{sp_n}{sq_n}\right| sq_n}{|\widehat{g}(sq_n)|} \leq \frac{2\left|\alpha - \frac{p_n}{q_n}\right| sq_n}{c_2 s|\widehat{f}(q_n)|} = \frac{2}{c_2} \cdot \frac{\left|\alpha - \frac{p_n}{q_n}\right| q_n}{|\widehat{f}(q_n)|} \to 0.$$

It follows from Theorem 2.5.1 that $\lambda e^{2\pi i B(f^{(s)}(sx) - f^{(s)}(sx + \frac{s}{r}))}$ cannot be a coboundary, contrary to (36). This completes the proof in case (i).

We cover now case (ii). We claim that for $n \notin B_s$ (i.e. n such that $s|q_n$), for some $c_3 > 0$, we have

$$\frac{|a_{rq_n}|}{\sum_{m\geq 1}|a_{mrq_n}|} > c_3 > 0 \tag{43}$$

and

$$\frac{|\alpha - \frac{rp_n}{rq_n}| rq_n}{|a_{rq_n}|} \to 0. \tag{44}$$

To prove (43), we will estimate $|a_{rq_n}|$ from below and $\sum_{m\geq 1}|a_{mrq_n}|$ from above in an appropriate way. We begin by estimating $|a_{rq_n}|$. We have

$$\begin{aligned}|a_{rq_n}| &\geq |A|\cdot|\widehat{f}(q_n)|\cdot\left|\frac{1-e^{2\pi irq_n\alpha}}{1-e^{2\pi iq_n\alpha}}\right| - |B|\cdot\left|\widehat{f}\left(q_n\frac{r}{s}\right)\right|\cdot\left|\frac{1-e^{2\pi iq_nr\alpha}}{1-e^{2\pi iq_n\frac{r}{s}\alpha}}\right| \\ &\geq |A|\cdot|\widehat{f}(q_n)|\cdot\frac{r}{2} - |B|\cdot\left|\widehat{f}\left(q_n\frac{r}{s}\right)\right|\cdot\left|\frac{1-e^{2\pi iq_nr\alpha}}{1-e^{2\pi iq_n\frac{r}{s}\alpha}}\right|, \quad (45)\end{aligned}$$

where the latter inequality follows from (28) and is valid for n sufficiently large. It follows by (25) that for $n\notin B_s$,

$$\left|\widehat{f}\left(q_n\frac{r}{s}\right)\right| < \frac{1}{c_0}|\widehat{f}(q_n)|. \quad (46)$$

We will now estimate $|1-e^{2\pi iq_n\frac{r}{s}\alpha}|$. Notice that

$$\frac{2}{\pi} < \frac{|e^{2\pi ix}-e^{2\pi iy}|}{|x-y|} < 1 \quad (47)$$

for $x,y\in\mathbb{R}$ such that $0<|x-y|<1$. Since $|1-e^{2\pi iq_n\alpha}|\to 0$, for n sufficiently large,

$$\left|1-e^{2\pi iq_n\alpha}\right| < \frac{s}{\pi r}\left|1-e^{2\pi i\frac{1}{s}}\right|.$$

Therefore and by (47), for such n, we have

$$\begin{aligned}\left|e^{2\pi i\frac{[q_n\alpha]}{s}} - e^{2\pi iq_n\frac{1}{s}\alpha}\right| &< \left|\frac{[q_n\alpha]}{s} - q_n\frac{1}{s}\alpha\right| = \frac{1}{s}\cdot|[q_n\alpha]-q_n\alpha| \\ &< \frac{\pi}{2s}\cdot|e^{2\pi i[q_n\alpha]}-e^{2\pi iq_n\alpha}| = \frac{\pi}{2s}\cdot|1-e^{2\pi iq_n\alpha}| \quad (48) \\ &< \frac{\pi}{2s}\cdot\frac{s}{\pi r}\cdot|1-e^{2\pi i\frac{1}{s}}| = \frac{1}{2r}\cdot\left|1-e^{2\pi i\frac{1}{s}}\right|.\end{aligned}$$

Since for all $x,y\in\mathbb{R}$ and all $k\in\mathbb{N}$

$$|e^{2\pi ikx}-e^{2\pi iky}| \leq k|e^{2\pi ix}-e^{2\pi iy}|,$$

it follows from (48) that

$$|e^{2\pi i\frac{[q_n\alpha]r}{s}} - e^{2\pi iq_n\frac{r}{s}\alpha}| < \frac{1}{2}|1-e^{2\pi i\frac{1}{s}}|.$$

Since q_n is a denominator of α and $s|q_n$, we have $|c-\frac{q_n}{s}\alpha| > |p_n-q_n\alpha|$ for all $0\leq c\leq\frac{q_n}{s}$ (see(52) in Section 3.1). Using this inequality and (47), we obtain

$$\frac{\pi}{2}|1-e^{2\pi i\frac{q_n}{s}\alpha}| > |c-\frac{q_n}{s}\alpha| > |p_n-q_n\alpha| > |1-e^{2\pi iq_n\alpha}|. \quad (49)$$

By the first two lines of (48) and (49), we have

$$\left|e^{2\pi i\frac{[q_n\alpha]}{s}} - e^{2\pi iq_n\frac{1}{s}\alpha}\right| < \frac{\pi}{2s}|1-e^{2\pi iq_n\alpha}| < \frac{\pi}{2s}\cdot\frac{\pi}{2}|1-e^{2\pi i\frac{q_n}{s}\alpha}|. \quad (50)$$

Suppose that $s|[q_n\alpha]$. Then (50) implies $1<\frac{\pi}{2s}\cdot\frac{\pi}{2}$, i.e. $s<(\pi/2)^2$. This is however impossible, as $s\geq 3$. Therefore $s\nmid[q_n\alpha]$, which implies $s\nmid[q_n\alpha]r$, i.e.

$$\left|1-e^{2\pi i\frac{[q_n\alpha]r}{s}}\right| \geq |1-e^{2\pi i\frac{1}{s}}|.$$

Therefore

$$\begin{aligned}\left|1-e^{2\pi i q_n \frac{r}{s}\alpha}\right| &\geq \left|1-e^{2\pi i \frac{[q_n\alpha]r}{s}}\right| - \left|e^{2\pi i \frac{[q_n\alpha]r}{s}} - e^{2\pi i q_n \frac{r}{s}\alpha}\right| \\ &\geq \left|1-e^{2\pi i \frac{[q_n\alpha]r}{s}}\right| - \frac{1}{2}\left|1-e^{2\pi i \frac{1}{s}}\right| \geq \frac{1}{2}\left|1-e^{2\pi i \frac{1}{s}}\right| =: c_4 > 0.\end{aligned}$$

It follows that

$$\left|\frac{1-e^{2\pi i r q_n \alpha}}{1-e^{2\pi i q_n \frac{r}{s}\alpha}}\right| \leq \frac{1}{c_4}|1-e^{2\pi i r q_n\alpha}| \leq \frac{r}{c_4}|1-e^{2\pi i q_n \alpha}| \to 0.$$

This and (45), (46) imply that for n sufficiently large

$$|a_{rq_n}| \geq c_5|\widehat{f}(q_n)| \text{ for some } c_5 > 0. \tag{51}$$

We will estimate now $\sum_{m\geq 1}|a_{mrq_n}|$. By (24) and (25) we have

$$\sum_{m\geq 1}|\widehat{f}(mq_n)| < \frac{1}{c_0}|\widehat{f}(q_n)| \text{ and } \sum_{m\geq 1}\left|\widehat{f}\left(mq_n\frac{r}{s}\right)\right| < \frac{1}{c_0}|\widehat{f}(q_n)|.$$

Hence

$$\sum_{m\geq 1}|a_{mrq_n}| \leq \frac{|A|r+|B|s}{c_0}\cdot|\widehat{f}(q_n)|.$$

Using this estimate and (51), we obtain

$$\frac{|a_{rq_n}|}{\sum_{m\geq 1}|a_{mrq_n}|} \geq \frac{c_5|\widehat{f}(q_n)|}{\frac{|A|r+|B|s}{c_0}|\widehat{f}(q_n)|} = \frac{c_0c_5}{|A|r+|B|s} > 0$$

and (43) follows. Notice that (51) together with (26) implies that also (44) is true. By Theorem 2.5.1, we conclude that $\lambda e^{2\pi i F(x)}$ cannot be a coboundary, which completes the proof in case (ii). $\square$

Remark 2.5.3. Recall that if $f \in C^{1+\delta}(\mathbb{T})$ for some $\delta > 0$ then $\widehat{f}(n) = \mathrm{o}(1/n^{1+\delta'})$ for $0 < \delta' < \delta$ (the speed of convergence to zero depends on δ').

Lemma 2.5.4 (cf. Lemma 4 in [21]). *Let $g\colon \mathbb{N} \to (0,\infty)$ be a non-increasing positive function such that $g(mn) \leq g(m)g(n)$ for all $m, n \in \mathbb{N}$ and $\sum_{m\geq 1} g(m) = C < \infty$. Let $(x_n)_{n\in\mathbb{N}} \subset [0,\infty)$ be a summable sequence such that $x_n = o(g(n))$, $n \in \mathbb{N}$. Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} > 0$. Let $b \geq 1$ and let*

$$\varepsilon_k = \frac{x_{n_k}}{x_{n_k} + \sum_{m\geq 1} x_{m[bn_k]}}.$$

Then $\varepsilon_k \not\to 0$.

Proof. We will choose a subsequence of $(\varepsilon_k)_{k\in\mathbb{N}}$ recursively. Let $k_1 \geq 1$ and $\delta_1 > 0$ be such that

$$\frac{x_{n_{k_1}}}{g(n_{k_1})} > \delta_1 \text{ and } \frac{x_n}{g(n)} \leq \delta_1 \text{ for } n > n_{k_1}.$$

Suppose first that $[bn_{k_1}] = n_{k_1}$. Then

$$\begin{aligned} x_{n_{k_1}} + \sum_{m\geq 1} x_{m[bn_{k_1}]} &= 2x_{n_{k_1}} + \sum_{m\geq 2} x_{m[bn_{k_1}]} \leq 2x_{n_{k_1}} + \sum_{m\geq 2} g(m[bn_{k_1}])\delta_1 \\ &\leq 2x_{n_{k_1}} + g([bn_{k_1}])\delta_1 \sum_{m\geq 2} g(m) \leq 2x_{n_{k_1}} + g(n_{k_1})\delta_1 C < (C+2)x_{n_{k_1}}. \end{aligned}$$

Suppose now that $[bn_{k_1}] > n_{k_1}$. In a similar way as before, we obtain

$$x_{n_{k_1}} + \sum_{m\geq 1} x_{m[bn_{k_1}]} \leq x_{n_{k_1}} + g(n_{k_1})\delta_1 \sum_{m\geq 1} g(m) < (C+1)x_{n_{k_1}}.$$

Once we have chosen $k_1, \dots, k_j$, we pick $k_{j+1} > k_j$ and $\delta_{j+1} > 0$ such that

$$\frac{x_{n_{k_{j+1}}}}{g(n_{k_{j+1}})} > \delta_{j+1} \text{ and } \frac{x_n}{g(n)} \leq \delta_{j+1} \text{ for } n > n_{k_{j+1}}.$$

As before, we obtain

$$\sum_{m\geq 1} x_{m[bn_{k_{j+1}}]} < (C+2)x_{n_{k_{j+1}}}$$

which completes the proof. □

Remark 2.5.5. Given $b_1, \dots, b_l \geq 1$, under the assumptions of the above lemma, by a diagonalizing procedure we can find an increasing sequence $(n_k) \subset \mathbb{N}$ such that

$$\varepsilon_{n_k}(b_i) > c > 0 \text{ for } 1 \leq i \leq l.$$

Corollary 2.5.6. *Suppose that $f \in C^{1+\delta}(\mathbb{T})$ for some $\delta > 0$ and it is not a trigonometric polynomial. Then for a generic α and any $A, B \in \mathbb{R}^2$ with $A^2 + B^2 \neq 0$, any relatively prime numbers r and s, any $h \in \mathbb{R}$ and any $\lambda \in \mathbb{T}$, the cocycle $\lambda e^{2\pi i(Af^{(r)}(rx)+Bf^{(s)}(sx+h))}$ is not a T-coboundary for $Tx = x + \alpha$.*

Proof. Let $(q_n)_{n\in\mathbb{N}}$ be such that $\widehat{f}(q_n) \neq 0$. Then by (53), for a residual set of irrationals α, we have

$$\frac{\left|\alpha - \frac{p_n}{q_n}\right| q_n}{|\widehat{f}(q_n)|} \to 0$$

along some subsequence of $(q_n)_{n\in\mathbb{N}}$ which, for convenience, we will still denote by $(q_n)_{n\in\mathbb{N}}$. In other words, (26) holds. It follows from Remark 2.5.3, Lemma 2.5.4 and Remark 2.5.5 (for $l = 2$, $b_1 = 1$, $b_2 = \frac{r}{s}$) that (24) and (25) also hold. Therefore, we can apply Theorem 2.5.2 to complete the proof. □

Remark 2.5.7. Let $c \neq 0$, $r, s, A, B \in \mathbb{Z}$ be such that $Ar^2 + Bs^2 \neq 0$. Let $\varphi(x) = cx + f(x)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is of class $C^{1+\delta}$ for some $\delta > 0$ and periodic of period 1. Since the topological degree of $e^{2\pi i(A\varphi^{(r)}(r\cdot)+B\varphi^{(s)}(s\cdot+h))}$ is equal to $(Ar^2 + Bs^2)c$, an immediate consequence of (56) is that the cocycle $e^{2\pi i(A\varphi^{(r)}(r\cdot)+B\varphi^{(s)}(s\cdot+h))}$ is not a coboundary for all $h \in \mathbb{R}$. On the other hand, if $Ar^2 + Bs^2 = 0$, then for some λ of modulus 1, we have

$$e^{2\pi i(A\varphi^{(r)}(r\cdot)+B\varphi^{(s)}(s\cdot+h))} = \lambda \cdot e^{2\pi i(Af^{(r)}(r\cdot)+Bf^{(s)}(s\cdot+h))}.$$

Corollary 2.5.8. *Let $c \in \mathbb{Z}$ and suppose that $f \in C^{1+\delta}(\mathbb{T})$ for some $\delta > 0$ and it is not a trigonometric polynomial. Let $\varphi(x) = cx + f(x)$. Then for a generic α the cocycle $(e^{2\pi i\varphi^{(r)}(r\cdot)}, e^{2\pi i\varphi^{(s)}(s\cdot+h)})$ is ergodic (as a cocycle over $Tx = x+\alpha$) for all relatively prime numbers $r \neq s$ and any $h \in \mathbb{R}$.*

Proof. In view of Remark 3.2.2, the assertion follows immediately from Corollary 2.5.6 and Remark 2.5.7. □

Proof of Theorem 1.0.3. Recall that $T_{c,f}(x,y) = (x+\alpha, y+cx+f(x))$. We can assume that f is not a trigonometric polynomial. Indeed, otherwise f is a coboundary with the transfer function also being a trigonometric polynomial and the problem is reduced to the affine case.

It follows from Lemma 2.2.2 and Corollary 2.5.8 that for a generic α the decomposition of $\mathbb{T}^4$ into minimal components of $(T_{c,f})^r \times (T_{c,f})^s$ is the same as the decomposition into ergodic components: it consists of sets I_{c_1}, $c_1 \in [0, \frac{1}{r})$ (see (8) on page 6). Moreover, each ergodic component is uniquely ergodic. It follows that all points are generic for $(T_{c,f})^r \times (T_{c,f})^s$ (for the relevant invariant measures). Thus, conditions (a) and (b) are satisfied.

To complete the proof, it suffices to show that the set $\widehat{\mathbb{T}}^2$ satisfies condition (c). This is however true by Lemma 2.4.8 and the result follows. □

3 Appendix

3.1 Notation and basic facts

Automorphisms of standard Borel spaces Let $(X, \mathcal{B}, \mu)$ be a probability standard Borel space. By $\mathrm{Aut}(X, \mathcal{B}, \mu)$ we will denote the space of all bi-measurable measure-preserving bijections of X which we will call automorphisms.

Irrational rotation We will identify the multiplicative circle $\mathbb{S}^1 = \{z \in \mathbb{C} \colon |z| = 1\}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with $X = [0,1)$ with addition mod 1. Therefore, real functions defined on the circle will be identified with one-periodic functions defined on $\mathbb{R}$. Let $\lambda_{\mathbb{T}}$ denote Lebesgue measure on X.

Assume that $T \colon X \to X$ is an irrational rotation, $Tx = x + \alpha \pmod 1$, $x \in X$. Clearly $T \in \mathrm{Aut}(X, \mathcal{B}(X), \lambda_{\mathbb{T}})$. Let $\alpha = [0; a_1, a_2, \dots]$ be the continued fraction expansion of α. Let

$$q_0 = 1, q_1 = a_1, q_{n+1} = a_{n+1}q_n + q_{n-1},$$
$$p_0 = 0, p_1 = 1, p_{n+1} = a_{n+1}p_n + p_{n-1}$$

for $n \geq 1$. The rationals p_n/q_n are called the *convergents* of α.

Recall (see e.g. [20]) that every convergent p_n/q_n is a best approximation of α in the following sense:

$$\text{if } \frac{c}{d} \neq \frac{p_n}{q_n} \text{ and } 0 < d \leq q_n \text{ then } |c - d\alpha| > |p_n - q_n\alpha|. \tag{52}$$

Recall also (see e.g. [21]) that given an infinite set $\{q_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ and a positive real valued function $R = R(q_n)$ the set

$$\mathcal{A} = \Big\{\alpha \in [0,1)\colon \text{ for infinitely many } n \text{ we have } \left|\alpha - \frac{p_n}{q_n}\right| < R(q_n), \\ \text{where } \frac{p_n}{q_n} \text{ are convergents of } \alpha\Big\} \text{ is residual in } \mathbb{T}. \quad (53)$$

Cocycles and group extensions Let $T \in \mathrm{Aut}(X,\mathcal{B},\mu)$. For a locally compact second countable Abelian group G[13] with a Haar measure λ_G, a measurable map $\varphi\colon \mathbb{Z}\times X \to G$ is called a *cocycle* if

$$\varphi^{(n+m)}(x) = \varphi^{(n)}(x)\varphi^{(m)}(T^n x) \text{ for each } n,m \in \mathbb{Z}.$$

The generator $\varphi(x) = \varphi^{(1)}(x)$ determines $\varphi^{(n)}(x)$ for any $(n,x)\in\mathbb{Z}\times X$:

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x)\cdot\varphi(Tx)\cdot\ldots\cdot\varphi(T^{n-1}x), & \text{if } n>0,\\ 1, & \text{if } n=0,\\ (\varphi(T^n x)\cdot\ldots\cdot\varphi(T^{-1}x))^{-1}, & \text{if } n<0.\end{cases}$$

Thus, we will call a *cocycle* any measurable function $\varphi\colon X\to G$ as well. Given φ, we consider a G-extension of T, acting on $(X\times G, \mu\otimes\lambda_G)$, defined by the formula

$$T_\varphi(x,g) = (Tx, \varphi(x)g).$$

A cocycle φ is called a *T-coboundary* (or simply a *coboundary*) if it is of the form

$$\varphi(x) = \xi(x)(\xi(Tx))^{-1}$$

for some measurable function $\xi\colon X\to G$ (called a *transfer function*). Two cocycles $\phi,\psi\colon X\to G$ are *cohomologous* if for some measurable function $f\colon X\to G$ we have

$$(f(Tx))^{-1}\phi(x)f(x) = \psi(x). \quad (54)$$

Analogous notions to the above ones are also present in topological dynamics. Let $T\colon X\to X$ be a minimal homeomorphism of a compact metric space. Let $\varphi\colon X\to G$ be a continuous function. We say that φ is a *topological T-coboundary* if it is a measurable T-coboundary with a continuous transfer function.

3.2 Compact group extensions: ergodicity and minimality

Let G be a compact Abelian metrizable group and assume that $T\in\mathrm{Aut}(X,\mathcal{B},\mu)$ is ergodic. Let $\varphi : X\to G$ be a cocycle. Then

$$\begin{array}{l} T_\varphi \text{ is ergodic if and only if the equation} \\ \chi\circ\varphi = \xi/\xi\circ T \text{ has only trivial solution } \chi = \mathbb{1}, \xi = \text{const} \\ \text{in } \chi\in\widehat{G} \text{ and a measurable } \xi : X\to\mathbb{S}^1. \end{array} \quad (55)$$

For example, see [11],

$$\begin{array}{l} \text{if } Tx = x+\alpha,\ \varphi:\mathbb{T}\to\mathbb{S}^1 \text{ is Lipschitz with non-zero degree} \\ \text{then } T_\varphi \text{ is ergodic.} \end{array} \quad (56)$$

[13]We use multiplicative notation in G.

If T_φ is ergodic, we will also say that φ is ergodic (for T).

Recall that G acts on $X \times G$ via $g \mapsto R_g$, where:

$$R_g(x,h) := (x, hg^{-1}) \text{ for } (x,h) \in X \times G,\ g \in G.$$

Ergodic components Let $\varphi\colon X \to G$ be a cocycle. Let $\mathcal{P}(T_\varphi, \mathcal{B}, \mu)$ stand for the set of T_φ-invariant Borel measures whose projection on X is μ. Fix an ergodic measure $\lambda \in \mathcal{P}(T_\varphi, \mathcal{B}, \mu)$. Denote by H be the stabilizer of λ in G, i.e.

$$H := \{g \in G\colon \lambda \circ R_g = \lambda\}.$$

Notice that

$$\Gamma := \{\chi \in \widehat{G}\colon \chi \circ \varphi \text{ is a coboundary}\}$$

is a (closed) subgroup of $\widehat{G}$ and let

$$F = \operatorname{ann} \Gamma := \{g \in G\colon \text{ for each } \chi \in \Gamma,\ \chi(g) = 1\}.$$

Proposition 3.2.1 (see e.g. [22])**.** *The system $(X \times G, T_\varphi, \lambda)$ is isomorphic to $(X \times H, T_\psi, \mu \otimes \lambda_H)$ for some ergodic $\psi\colon X \to H$. Moreover, $F = H$.*

Remark 3.2.2. In view of (55), T_φ is ergodic if and only if $\Gamma = \{\mathbb{1}\}$.

Minimal components Let $T\colon X \to X$ be a minimal homeomorphism of a compact metric space. Let $\varphi\colon X \to G$ be a continuous function and let M be a minimal component of T_φ, i.e. $M \subset X \times G$ is closed and invariant with no proper subsets having the same properties. Let H_{top} be the stabilizer of M in G, i.e.

$$H_{top} := \{g \in G\colon R_g(M) = M\}$$

(this definition is independent of the initial choice of M). Notice that

$$\Gamma_{top} := \{\chi \in \widehat{G}\colon \chi \circ \varphi \text{ is a topological coboundary}\}$$

is a closed subgroup of $\widehat{G}$ and let

$$F_{top} := \operatorname{ann} \Gamma_{top} = \{g \in G\colon \text{ for each } \chi \in \Gamma_{top},\ \chi(g) = 1\}.$$

Lemma 3.2.3 ([27])**.** *There exists a continuous map $\tau\colon X \to G/H_{top}$ such that*

$$\tau(Tx) = \varphi(x)\tau(x). \tag{57}$$

Moreover, $M := \cup_{x \in X}\{x\} \times \tau(x)$ is a minimal set.

The proof of the following result is analogous to the one in the measure-theoretical case. We include it here for the sake of completeness.

Proposition 3.2.4. *$H_{top} = F_{top}$.*

Proof. Let $g_0 \in H_{top}$ and let $\chi \in \Gamma_{top}$, i.e.

$$\chi \circ \varphi(x) = h(Tx)/h(x)$$

for some continuous function $h\colon X \to \mathbb{S}^1$. We define $w\colon X \times G \to \mathbb{S}^1$:

$$w(x,g) = h(x)^{-1}\chi(g).$$

This function is continuous and T_φ-invariant, whence it is constant on each minimal component. It follows that $h(x)^{-1}\chi(g)\chi(g_0) = w(x, gg_0) = w(x, g) = h(x)^{-1}\chi(g)$, which implies $\chi(g_0) = 1$. Therefore $g_0 \in F_{top}$.

Suppose now that there exists $g_0 \in F_{top} \setminus H_{top}$. Then there exists $\chi \in \widehat{G}$ such that $\chi(g_0) \neq 1$ and $\chi(H_{top}) = \{1\}$. Let $\tau \colon X \to G/H_{top}$ satisfy (57). It follows that $\chi \circ \tau$ is well-defined and we obtain

$$\chi \circ \tau(Tx) = \chi \circ \varphi(x) \cdot \chi \circ \tau(x)$$

which implies

$$\chi \circ \varphi(x) = \frac{\chi \circ \tau(Tx)}{\chi \circ \tau(x)},$$

i.e. $\chi \in \Gamma_{top}$ and consequently $\chi(g_0) = 1$ which is a contradiction. □

Relation between the ergodic and the minimal components Let T be a minimal homeomorphism of a compact metric space X and let μ be a T-invariant probability Borel measure, ergodic with respect to T. Let $\varphi \colon X \to G$ be continuous. Then T_φ is a homeomorphism of $X \times G$ and $T_\varphi \in \mathrm{Aut}(X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes \lambda_G)$. Let $\lambda \in \mathcal{P}(T_\varphi, \mathcal{B}, \mu)$. There are two natural partitions associated to T_φ:

- $\mathcal{P}_{erg}$ – the partition into the ergodic components of T_φ,
- $\mathcal{P}_{min}$ – the partition into the minimal components of T_φ.

Partition $\mathcal{P}_{erg}$ is clearly measurable.

Proposition 3.2.5. *Partition $\mathcal{P}_{min}$ is measurable.*

Proof. Let $M \subset X \times G$ be as in Lemma 3.2.3, in particular, M is a minimal component of T_φ. Let $s \colon G/H_{top} \to G$ be a Borel selector of the canonical projection $\pi \colon G \to G/H_{top}$. Consider $\eta \colon X \to G$ given by $\eta = s \circ \tau$. We obtain

$$\varphi'(x) := \varphi(x)\eta(x)(\eta(Tx))^{-1} \in H_{top}$$

and

$$M = \bigcup_{x \in X} \{x\} \times (\eta(x) H_{top}),$$

so the map

$$(x, h) \mapsto (x, \eta(x)h) \tag{58}$$

settles an equivariant Borel isomorphism of $X \times H$ (considered with $T_{\varphi'}$) and M (considered with T_φ). Moreover, (58) can be naturally extended to a Borel isomorphism of $X \times H_{top} \times G/H_{top}$ and $X \times G$. Clearly the partition of $X \times H_{top} \times G/H_{top}$ given by relevant translations of $X \times H_{top} \times \{1\}$ (indexed by G/H_{top}) is measurable for the product measure $\mu \otimes \lambda_{H_{top}} \otimes \lambda_{G/H_{top}}$. Hence its image by the Borel extensions of (58) is also a measurable partition for the image of the measure $\mu \otimes \lambda_{H_{top}} \otimes \lambda_{G/H_{top}}$. This image is equal to $\mu \otimes \lambda_G$, so the partition into minimal components is indeed measurable. □

Remark 3.2.6. Since the partition into the ergodic components can be defined as the finest measurable partition whose atoms are invariant under the action of the homeomorphism in question, $\mathcal{P}_{erg}$ is finer than $\mathcal{P}_{min}$.

As a direct consequence of the above remark we obtain the following:

Remark 3.2.7. We have $H \supset H_{top}$. The condition $H = H_{top}$ is necessary and sufficient for the ergodic components of T_φ to be the same as its minimal components. Moreover, $H = H_{top}$ if and only if $\Gamma = \Gamma_{top}$.

Unique ergodicity

Proposition 3.2.8 (Furstenberg, see the proof of Theorem I.4 in [12] and Proposition 3.10 in [13]). *Let $T\colon X \to X$ be uniquely ergodic and let $\varphi\colon X \to G$ be a continuous cocycle with values in a compact Abelian group. If T_φ is ergodic with respect to $\mu \otimes \lambda_G$ then it is uniquely ergodic.*

3.3 A remark on the KBSZ criterion for $Tx = x + \alpha$

Let $Tx = x + \alpha$ be an irrational rotation on $\mathbb{T}$.

Proposition 3.3.1. *Let $A \subset \mathbb{Z}$.*

(i) If A is such that

$$rA \cap sA = \emptyset \text{ for sufficiently large prime numbers } r \neq s, \tag{59}$$

then (6) *holds true for every $f \in C(\mathbb{T})$ such that* $\operatorname{supp} \widehat{f} := \{n \in \mathbb{Z} : \widehat{f}(n) \neq 0\} = A$.

(ii) Suppose that A contains infinitely many primes. Let $f \in C(\mathbb{T})$ be such that $\operatorname{supp} \widehat{f} = A$ *and all nonzero Fourier coefficients are positive. Then* (6) *fails for f.*

Remark 3.3.2. Note that every finite set satisfies (59), so all trigonometric polynomials satisfy (6); the set $\{2^n : n \geq 1\}$ is an example of an infinite set satisfying 59.

Proof of Proposition 3.3.1. We will consider the behavior of the sums in (6) at $(0, 0)$. Given r, s, two different prime numbers, we set $I_{r,s} := \{(x, y) : sx = ry\}$, which is a closed subgroup of $\mathbb{T}^2$ (of course $(0, 0) \in I_{r,s}$), invariant under $T^r \times T^s$. It is not hard to see that

$$\frac{1}{N} \sum_{n \leq N} \delta_{rn\alpha, sn\alpha} \to \lambda_{I_{r,s}} \tag{60}$$

and that

$$\begin{array}{l} W\colon I_{r,s} \to \mathbb{T},\ W(x, y) = ax + by\ (ar + bs = 1) \\ \text{is a continuous group isomorphism,} \end{array} \tag{61}$$

in particular, it sends $\lambda_{I_{r,s}}$ to $\lambda_{\mathbb{T}}$. In view of (60),

$$\frac{1}{N} \sum_{n \leq N} f(T^{rn}0)\overline{f(T^{sn}0)} \to \int_{I_{r,s}} f \otimes \overline{f}\, d\lambda_{I_{r,s}}.$$

Since $W^{-1}(t) = (rt, st)$, it follows that

$$\int_{I_{r,s}} f \otimes \overline{f}\, d\lambda_{I_{r,s}} = \int_{\mathbb{T}} (f \otimes \overline{f}) \circ W^{-1}(t)\, dt = \int_{\mathbb{T}} f(rt)\overline{f(st)}\, dt. \tag{62}$$

Now, if $f(t) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n t}$ then

$$f(rt) = \sum_{n \in \mathbb{Z}} c_n n e^{2\pi i r n t}, \quad f(st) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i s n t} \tag{63}$$

and we can compute $\int_{I_{r,s}} f \otimes \overline{f}\, d\lambda_{I_{r,s}}$ using (62), (63) and Parseval's formula.

(ii) Fix infinitely many pairs $(r_j, s_j) \in A \times A$ of distinct prime numbers, $r_j, s_j \to \infty$. Take $j \geq 1$. It is enough to show that $\int_{I_{r_j,s_j}} f \otimes \overline{f}\, d\lambda_{I_{r_j,s_j}} \neq 0$. By Parseval's formula, and the fact that all Fourier coefficients of f are positive, we have $\int_{I_{r_j,s_j}} f \otimes \overline{f}\, d\lambda_{I_{r_j,s_j}} \geq 0$. Since $r_j s_j = s_j r_j$, the $s_j r_j$th Fourier coefficient of $f(r_j \cdot)$ is $c_{s_j} > 0$, while the $r_j s_j$th Fourier coefficient of $f(s_j \cdot)$ is $c_{r_j} > 0$. Hence, by Parseval's formula, $\int_{I_{r_j,s_j}} f \otimes \overline{f}\, d\lambda_{I_{r_j,s_j}} > 0$.

(i) Each $(x, y) \in \mathbb{T}^2$ belongs to a coset of $I_{r,s}$. The proof for an arbitrary coset of $I_{r,s}$ goes along the same lines as for $I_{r,s}$ itself (on the cosets of $I_{r,s}$, we consider translations of Haar measure $\lambda_{I_{r,s}}$ and W is practically the same). Since $rA \cap sA = \emptyset$, the supports of the Fourier transforms of $f(r\cdot)$ and $f(s\cdot)$ are disjoint, whence $\int_{\mathbb{T}} f(rt)\overline{f(st)}\, dt = 0$. $\square$

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