

Spectral theory of dynamical systems

Adam Kanigowski* and Mariusz Lemańczyk†

Outline

Glossary and notation	2
1 Definition of the subject	4
2 Introduction	4
2.1 General unitary representations	4
2.2 Koopman representations	6
2.3 Markov operators, joinings and Koopman representations, disjointness and spectral disjointness, entropy	7
3 Maximal spectral type of a Koopman representation, Alexeyev’s Theorem	9
4 Spectral theory of weighted operators	10
4.1 Maximal spectral type of weighted operators over rotations	11
4.2 The multiplicity problem for weighted operators over rotations	13
4.3 Remarks on the Banach problem	13
4.4 Lifting mixing properties	14
5 The multiplicity function	15
5.1 Cocycle approach	15
5.2 Multiplicity for Gaussian and Poissonian automorphisms	16
5.3 Rokhlin’s uniform multiplicity problem	17
6 Rokhlin cocycles	18
7 Rank-1 and related systems	19
8 Spectral theory of dynamical systems of probabilistic origin	22
9 Inducing and spectral theory	25
10 Rigid sequences	26
11 Spectral theory of parabolic dynamical systems	28
11.1 Time-changes of algebraic systems	28
11.2 Special flows, flows on surfaces, interval exchange transformations	30
11.2.1 Interval exchange transformations	30
11.2.2 Smooth flows on surfaces and their special representations	31
11.3 Special flows over rotations and interval exchange transformations	33
12 Spectral theory for locally compact groups of type I	35
12.1 Groups of type I	35
12.2 Spectral properties of Heisenberg group actions	37
12.3 Heisenberg odometers	38
12.4 On the “finitely dimensional” part of the spectrum	39

*Department of Mathematics, University of Maryland at College Park, College Park, MD 20740, USA, adkanigowski@gmail.com

†Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, 87-100 Toruń, Chopin street 12/18, Poland, mlem@mat.umk.pl

Glossary and notation

Spectral decomposition of a unitary representation. If $\mathcal{U} = (U_a)_{a \in \mathbb{A}}$ is a continuous unitary representation of a locally compact second countable (l.c.s.c.) Abelian group \mathbb{A} in a separable Hilbert space H then a decomposition $H = \bigoplus_{i=1}^{\infty} \mathbb{A}(x_i)$ is called *spectral* if $\sigma_{x_1} \gg \sigma_{x_2} \gg \dots$ (such a sequence of measures is also called *spectral*); here $\mathbb{A}(x) := \overline{\text{span}}\{U_a x : a \in \mathbb{A}\}$ is called the *cyclic space* generated by $x \in H$ and σ_x stands for the spectral measure of x .

Maximal spectral type and the multiplicity function of \mathcal{U} . The *maximal spectral type* $\sigma_{\mathcal{U}}$ of \mathcal{U} is the type of σ_{x_1} in any spectral decomposition of H ; the *multiplicity function* $M_{\mathcal{U}} : \widehat{\mathbb{A}} \rightarrow \{1, 2, \dots\} \cup \{+\infty\}$ is defined $\sigma_{\mathcal{U}}$ -a.e. and $M_{\mathcal{U}}(\chi) = \sum_{i=1}^{\infty} 1_{Y_i}(\chi)$, where $Y_1 = \widehat{\mathbb{A}}$ and $Y_i = \text{supp } \frac{d\sigma_{x_i}}{d\sigma_{x_1}}$ for $i \geq 2$.

A representation \mathcal{U} is said to have *simple spectrum* if H is reduced to a single cyclic space. The multiplicity is *uniform* if there is only one essential value of $M_{\mathcal{U}}$. The essential supremum of $M_{\mathcal{U}}$ is called the *maximal spectral multiplicity*. \mathcal{U} is said to have *discrete spectrum* if H has an orthonormal basis consisting of eigenvectors of \mathcal{U} ; \mathcal{U} has *singular (Haar, absolutely continuous) spectrum* if the maximal spectral type of \mathcal{U} is singular with respect to (equivalent to, absolutely continuous with) a Haar measure of $\widehat{\mathbb{A}}$.

Koopman representation of a dynamical system \mathcal{T} . Let \mathbb{A} be a l.c.s.c. (and not compact) Abelian group and $\mathcal{T} : a \mapsto T_a$ a representation of \mathbb{A} in the group $\text{Aut}(X, \mathcal{B}, \mu)$ of (measure-preserving) automorphisms of a standard probability Borel space (X, \mathcal{B}, μ) . The *Koopman representation* $\mathcal{U} = \mathcal{U}_{\mathcal{T}}$ of \mathcal{T} in $L^2(X, \mathcal{B}, \mu)$ is defined as the unitary representation $a \mapsto U_{T_a} \in U(L^2(X, \mathcal{B}, \mu))$, where $U_{T_a}(f) = f \circ T_a$.

Ergodicity, weak mixing, mild mixing, mixing and rigidity of \mathcal{T} . A measure-preserving \mathbb{A} -action $\mathcal{T} = (T_a)_{a \in \mathbb{A}}$ is called *ergodic* if $\chi_0 \equiv 1 \in \widehat{\mathbb{A}}$ is a simple eigenvalue of $\mathcal{U}_{\mathcal{T}}$. It is *weakly mixing* if $\mathcal{U}_{\mathcal{T}}$ has a continuous spectrum on the subspace $L_0^2(X, \mathcal{B}, \mu)$ of zero mean functions. \mathcal{T} is said to be *rigid* if there is a sequence (a_n) going to infinity in \mathbb{A} such that the sequence $(U_{T_{a_n}})$ goes to the identity in the strong (or weak) operator topology; \mathcal{T} is said to be *mildly mixing* if it has no non-trivial rigid factors. We say that \mathcal{T} is *mixing* if the operator equal to zero is the only limit point of $\{U_{T_a}|_{L_0^2(X, \mathcal{B}, \mu)} : a \in \mathbb{A}\}$ in the weak operator topology.

Spectral disjointness. Two \mathbb{A} -actions \mathcal{S} and \mathcal{T} are called *spectrally disjoint* if the maximal spectral types of their Koopman representations $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\mathcal{S}}$ on the corresponding L_0^2 -spaces are mutually singular.

SCS property. We say that a Borel measure σ on $\widehat{\mathbb{A}}$ satisfies the *strong convolution singularity property* (SCS property) if, for each $n \geq 1$, in the disintegration (given by the map $(\chi_1, \dots, \chi_n) \mapsto \chi_1 \cdot \dots \cdot \chi_n$) $\sigma^{\otimes n} = \int_{\widehat{\mathbb{A}}} \nu_{\chi} d\sigma^{(n)}(\chi)$

the conditional measures ν_x are atomic with exactly $n!$ atoms ($\sigma^{(n)}$ stands for the n -th convolution $\sigma * \dots * \sigma$). An \mathbb{A} -action \mathcal{T} satisfies the SCS property if the maximal spectral type of $\mathcal{U}_{\mathcal{T}}$ on L^2_0 is a type of an SCS measure.

Kolmogorov group property. An \mathbb{A} -action \mathcal{T} satisfies the *Kolmogorov group property* if $\sigma_{\mathcal{U}_{\mathcal{T}}} * \sigma_{\mathcal{U}_{\mathcal{T}}} \ll \sigma_{\mathcal{U}_{\mathcal{T}}}$.

Weighted operator. Let T be an ergodic automorphism of (X, \mathcal{B}, μ) and $\xi : X \rightarrow \mathbb{T}$ be a measurable function. The (unitary) operator $V = V_{\xi, T}$ acting on $L^2(X, \mathcal{B}, \mu)$ by the formula $V_{\xi, T}(f)(x) = \xi(x)f(Tx)$ is called a *weighted operator*.

Induced automorphism. Assume that T is an automorphism of a standard probability Borel space (X, \mathcal{B}, μ) . Let $A \in \mathcal{B}$, $\mu(A) > 0$. The *induced automorphism* T_A is defined on the conditional space $(A, \mathcal{B}_A, \mu_A)$, where \mathcal{B}_A is the trace of \mathcal{B} on A , $\mu_A(B) = \mu(B)/\mu(A)$ for $B \in \mathcal{B}_A$ and $T_A(x) = T^{k_A(x)}x$, where $k_A(x)$ is the smallest $k \geq 1$ for which $T^k x \in A$.

AT property of an automorphism. An automorphism T of a standard probability Borel space (X, \mathcal{B}, μ) is called *approximatively transitive* (AT for short) if for every $\varepsilon > 0$ and every finite set f_1, \dots, f_n of non-negative L^1 -functions on (X, \mathcal{B}, μ) we can find $f \in L^1(X, \mathcal{B}, \mu)$ also non-negative such that $\|f_i - \sum_j \alpha_{ij} f \circ T^{nj}\|_{L^1} < \varepsilon$ for all $i = 1, \dots, n$ (for some $\alpha_{ij} \geq 0$, $n_j \in \mathbb{N}$).

Cocycles and group extensions. If T is an ergodic automorphism, G is a l.c.s.c. Abelian group and $\varphi : X \rightarrow G$ is measurable then the pair (T, φ) generates a *cocycle* $\varphi^{(\cdot)}(\cdot) : \mathbb{Z} \times X \rightarrow G$, where

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \dots + \varphi(T^{n-1}x) & \text{for } n > 0, \\ 0 & \text{for } n = 0 \\ -(\varphi(T^n x) + \dots + \varphi(T^{-1}x)) & \text{for } n < 0. \end{cases}$$

(That is, $(\varphi^{(n)})$ is a standard 1-cocycle in the algebraic sense for the \mathbb{Z} -action $n(f) = f \circ T^n$ on the group of measurable functions on X with values in G ; hence the function $\varphi : X \rightarrow G$ itself is often called a *cocycle*.)

Assume additionally that G is compact. Using the cocycle φ , we define a *group extension* T_φ on $(X \times G, \mathcal{B} \otimes \mathcal{B}(G), \mu \otimes \lambda_G)$ (λ_G stands for Haar measure of G), where $T_\varphi(x, g) = (Tx, \varphi(x) + g)$.

Special flow. Given an ergodic automorphism T on a standard probability Borel space (X, \mathcal{B}, μ) and a positive integrable function $f : X \rightarrow \mathbb{R}^+$ we put

$$X^f = \{(x, t) \in X \times \mathbb{R} : 0 \leq t < f(x)\}, \quad \mathcal{B}^f = \mathcal{B} \otimes \mathcal{B}(\mathbb{R})|_{X^f},$$

and define μ^f as normalized $\mu \otimes \lambda_{\mathbb{R}}|_{X^f}$. By a *special flow* we mean the \mathbb{R} -action $T^f = (T_t^f)_{t \in \mathbb{R}}$ under which a point $(x, s) \in X^f$ moves vertically with the unit speed, and once it reaches the graph of f , it is identified with $(Tx, 0)$.

Time change. Let $\mathcal{R} = (R_t)_{t \in \mathbb{R}}$ be a flow on (X, \mathcal{B}, μ) and let $v \in L^1(X, \mathcal{B}, \mu)$ be a positive function. The function v determines a cocycle over \mathcal{R} given by the formula

$$v(t, x) := \int_0^t v(R_s x) ds.$$

Then for a.e. $x \in X$ and all $t \in \mathbb{R}$, there exists a unique $u = u(t, x)$ such that

$$\int_0^u v(R_s x) ds = t.$$

Now, we can define the flow $\tilde{R}_t(x) := R_{u(t,x)}(x)$. The new flow $\tilde{\mathcal{R}} = (\tilde{R}_t)_{t \in \mathbb{R}}$ has the same orbits as the original flow, and it preserves the measure $\tilde{\mu} \ll \mu$ (hence it is ergodic if \mathcal{R} was), where $\frac{d\tilde{\mu}}{d\mu} = v / \int_X v d\mu$.

Markov operator. A linear operator $J : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(Y, \mathcal{C}, \nu)$ is called *Markov* if it sends non-negative functions to non-negative functions and $J1 = J^*1 = 1$.

Unitary actions on Fock spaces. If H is a separable Hilbert space then by $H^{\odot n}$ we denote the subspace of n -tensors of $H^{\otimes n}$ symmetric under all permutations of coordinates, $n \geq 1$; then the Hilbert space $F(H) := \bigoplus_{n=0}^{\infty} H^{\odot n}$ is called a *symmetric Fock space*. If $V \in U(H)$ then $F(V) := \bigoplus_{n=0}^{\infty} V^{\odot n} \in U(F(H))$, where $V^{\odot n} = V^{\otimes n}|_{H^{\odot n}}$.

1 Definition of the subject

Spectral theory of dynamical systems is a study of special unitary representations, called Koopman representations (see the glossary). Invariants of such representations are called spectral invariants of measure-preserving systems. Together with the entropy, they constitute the most important invariants used in the study of measure-theoretic intrinsic properties and classification problems of dynamical systems as well as in applications. Spectral theory was originated by von Neumann, Halmos and Koopman in the 1930s. In this article we will focus on recent progresses in the spectral theory of finite measure-preserving dynamical systems.

2 Introduction

Throughout \mathbb{A} denotes a non-compact l.c.s.c. Abelian group (\mathbb{A} will be most often \mathbb{Z} or \mathbb{R}). The assumption of second countability implies that \mathbb{A} is metrizable, σ -compact and the space $C_0(\mathbb{A})$ is separable. Moreover the dual group $\widehat{\mathbb{A}}$ is also l.c.s.c. Abelian.

2.1 General unitary representations

We are interested in *unitary*, that is with values in the unitary group $U(H)$ of a Hilbert space H , (weakly) continuous representations $V : \mathbb{A} \ni a \mapsto V_a \in U(H)$ of such groups (the scalar valued maps $a \mapsto \langle V_a x, y \rangle$ are continuous for each $x, y \in H$).

Let $H = L^2(\widehat{\mathbb{A}}, \mathcal{B}(\widehat{\mathbb{A}}), \mu)$, where $\mathcal{B}(\widehat{\mathbb{A}})$ stands for the σ -algebra of Borel sets of $\widehat{\mathbb{A}}$ and $\mu \in M^+(\widehat{\mathbb{A}})$ (whenever X is a l.c.s.c. space, by $M(X)$ we denote the

set of complex Borel measures on X , while $M^+(X)$ stands for the subset of positive (finite) measures). Given $a \in \mathbb{A}$, for $f \in L^2(\widehat{\mathbb{A}}, \mathcal{B}(\widehat{\mathbb{A}}), \mu)$ put

$$V_a^\mu(f)(\chi) = i(a)(\chi) \cdot f(\chi) = \chi(a) \cdot f(\chi) \quad (\chi \in \widehat{\mathbb{A}}),$$

where $i : \mathbb{A} \rightarrow \widehat{\mathbb{A}}$ is the canonical Pontriagin isomorphism of \mathbb{A} with its second dual. Then $V^\mu = (V_a^\mu)_{a \in \mathbb{A}}$ is a unitary representation of \mathbb{A} . Since $C_0(\widehat{\mathbb{A}})$ is dense in $L^2(\widehat{\mathbb{A}}, \mu)$, the latter space is separable. Therefore also direct sums $\bigoplus_{i=1}^\infty V^{\mu_i}$ of such type representations will be unitary representations of \mathbb{A} in separable Hilbert spaces.

Lemma 1 (Wiener Lemma) *If $F \subset L^2(\widehat{\mathbb{A}}, \mu)$ is a closed V_a^μ -invariant subspace for all $a \in \mathbb{A}$ then $F = 1_Y L^2(\widehat{\mathbb{A}}, \mathcal{B}(\widehat{\mathbb{A}}), \mu)$ for some Borel subset $Y \subset \widehat{\mathbb{A}}$.*

Notice however that since \mathbb{A} is not compact (equivalently, $\widehat{\mathbb{A}}$ is not discrete), we can find μ continuous and therefore V^μ has no irreducible (non-zero) subrepresentation. From now on only unitary representations of \mathbb{A} in separable Hilbert spaces will be considered and we will show how to classify them.

A function $r : \mathbb{A} \rightarrow \mathbb{C}$ is called *positive definite* if

$$(1) \quad \sum_{n,m=0}^N r(a_n - a_m) z_n \overline{z_m} \geq 0$$

for each $N > 0$, $(a_n) \subset \mathbb{A}$ and $(z_n) \subset \mathbb{C}$. The central result about positive definite functions is the following theorem (see e.g. [242]).

Theorem 1 (Bochner-Herglotz) *Let $r : \mathbb{A} \rightarrow \mathbb{C}$ be continuous. Then r is positive definite if and only if there exists (a unique) $\sigma \in M^+(\widehat{\mathbb{A}})$ such that*

$$r(a) = \int_{\widehat{\mathbb{A}}} \chi(a) d\sigma(\chi) \quad \text{for each } a \in \mathbb{A}.$$

If now $\mathcal{U} = (U_a)_{a \in \mathbb{A}}$ is a representation of \mathbb{A} in H then for each $x \in H$ the function $r(a) := \langle U_a x, x \rangle$ is continuous and satisfies (1), so it is positive definite. By the Bochner-Herglotz Theorem there exists a unique measure $\sigma_{\mathcal{U},x} = \sigma_x \in M^+(\widehat{\mathbb{A}})$ (called the *spectral measure* of x) such that

$$\widehat{\sigma}_x(a) := \int_{\widehat{\mathbb{A}}} i(a)(\chi) d\sigma_x(\chi) = \langle U_a x, x \rangle$$

for each $a \in \mathbb{A}$. Since the partial map $U_a x \mapsto i(a) \in L^2(\widehat{\mathbb{A}}, \sigma_x)$ is isometric and equivariant, there exists a unique extension of it to a unitary operator $W : \mathbb{A}(x) \rightarrow L^2(\widehat{\mathbb{A}}, \sigma_x)$ giving rise to an isomorphism of $\mathcal{U}|_{\mathbb{A}(x)}$ and V^{σ_x} . Then the existence of a spectral decomposition is proved by making use of separability and a choice of maximal cyclic spaces at every step of an induction procedure. Moreover, a spectral decomposition is unique in the following sense.

Theorem 2 (Spectral Theorem) *If $H = \bigoplus_{i=1}^{\infty} \mathbb{A}(x_i) = \bigoplus_{i=1}^{\infty} \mathbb{A}(x'_i)$ are two spectral decompositions of H then $\sigma_{x_i} \equiv \sigma_{x'_i}$ for each $i \geq 1$.*

It follows that the representation \mathcal{U} is entirely determined by the types (the sets of equivalent measures to a given one) of a decreasing sequence of measures or, equivalently, \mathcal{U} is determined by its maximal spectral type $\sigma_{\mathcal{U}}$ and its multiplicity function $M_{\mathcal{U}}$.

Notice that eigenvalues of \mathcal{U} correspond to Dirac measures: $\chi \in \widehat{\mathbb{A}}$ is an eigenvalue (i.e. for some $\|x\| = 1$, $U_a(x) = \chi(a)x$ for each $a \in \mathbb{A}$) if and only if $\sigma_{\mathcal{U},x} = \delta_{\chi}$. Therefore \mathcal{U} has a discrete spectrum if and only if the maximal spectral type of \mathcal{U} is a discrete measure.

We refer the reader to [106], [159], [185], [210], [219] for presentations of spectral theory needed in the theory of dynamical systems – such presentations are usually given for $\mathbb{A} = \mathbb{Z}$ but once we have the Bochner-Herglotz Theorem and the Wiener Lemma, their extensions to the general case are straightforward.

2.2 Koopman representations

We will consider *measure-preserving* representations of \mathbb{A} . It means that we fix a probability standard Borel space (X, \mathcal{B}, μ) and by $Aut(X, \mathcal{B}, \mu)$ we denote the group of automorphisms of this space, that is, $T \in Aut(X, \mathcal{B}, \mu)$ if $T : X \rightarrow X$ is a bimeasurable (a.e.) bijection satisfying $\mu(A) = \mu(TA) = \mu(T^{-1}A)$ for each $A \in \mathcal{B}$. Consider then a representation of \mathbb{A} in $Aut(X, \mathcal{B}, \mu)$, that is, a group homomorphism $a \mapsto T_a \in Aut(X, \mathcal{B}, \mu)$; we write $\mathcal{T} = (T_a)_{a \in \mathbb{A}}$. Moreover, we require that the associated Koopman representation $\mathcal{U}_{\mathcal{T}}$ is continuous. Unless explicitly stated, \mathbb{A} -actions are assumed to be *free*, that is, we assume that for μ -a.e. $x \in X$ the map $a \mapsto T_a x$ is 1-1. In fact, since constant functions are obviously invariant for U_{T_a} , equivalently, the trivial character 1 is always an eigenvalue of $\mathcal{U}_{\mathcal{T}}$, the Koopman representation is considered only on the subspace $L_0^2(X, \mathcal{B}, \mu)$ of zero mean functions. We will restrict our attention only to *ergodic* dynamical systems (see the glossary). It is easy to see that \mathcal{T} is ergodic if and only if whenever $A \in \mathcal{B}$ and $A = T_a(A)$ (μ -a.e.) for all $a \in A$ then $\mu(A)$ equals 0 or 1. In case of ergodic Koopman representations, all eigenvalues are simple. In particular, (ergodic) Koopman representations with discrete spectra have simple spectra. The reader is referred to monographs mentioned above as well as to [46], [221], [246], [272], [300] for basic facts on the theory of dynamical systems. See also survey articles [187] and [51].

The passage $\mathcal{T} \mapsto \mathcal{U}_{\mathcal{T}}$ can be seen as functorial (contravariant). In particular, a measure-theoretic isomorphism of \mathbb{A} -systems \mathcal{T} and \mathcal{T}' implies spectral isomorphism of the corresponding Koopman representations; hence spectral properties are measure-theoretic invariants. Since unitary representations are completely classified, one of the main questions in the spectral theory of dynamical systems is to decide which pairs $([\sigma], M)$ can be realized by Koopman representations. The most spectacular is the Banach problem concerning a realization, for $\mathbb{A} = \mathbb{Z}$, of $([\lambda_{\mathbb{T}}], M \equiv 1)$, see Section 4.3. Another important problem is to give a complete spectral classification in some classes of dynamical systems

(classically, it was done in the theory of Kolmogorov and Gaussian dynamical systems). We will also see how spectral properties of dynamical systems can determine their statistical (ergodic) properties; a culmination given by the fact that a spectral isomorphism may imply measure-theoretic similitude (discrete spectrum case, Gaussian-Kronecker case). An old conjecture is that a dynamical system \mathcal{T} either is spectrally determined or there are uncountably many pairwise non-isomorphic systems spectrally isomorphic to \mathcal{T} .

We could also consider Koopman representations in L^p for $1 \leq p \neq 2$. However, whenever $W : L^p(X, \mathcal{B}, \mu) \rightarrow L^p(Y, \mathcal{C}, \nu)$ is a surjective isometry and $W \circ U_{T_a} = U_{S_a} \circ W$ for each $a \in \mathbb{A}$ then by the Lamperti Theorem (e.g. [241]) the isometry W has to come from a non-singular pointwise map $R : Y \rightarrow X$ and, by ergodicity, R “preserves” the measure ν and hence establishes a measure-theoretic isomorphism [144] (see also [185]). Thus spectral classification of such Koopman representations goes back to the measure-theoretic classification of dynamical systems, so it looks hardly interesting. This does not mean that there are no interesting dynamical questions for $p \neq 2$. Let us mention still open Thouvenot’s question (formulated in the 1980s) for \mathbb{Z} -actions: *For every ergodic T acting on (X, \mathcal{B}, μ) , does there exist $f \in L^1(X, \mathcal{B}, \mu)$ such that the closed linear span of $f \circ T^n$, $n \in \mathbb{Z}$, equals $L^1(X, \mathcal{B}, \mu)$?*

Iwanik [125], [126] proved that if T is a system with positive entropy then its L^p -multiplicity is ∞ for all $p > 1$. Moreover, Iwanik and de Sam Lazaro [131] proved that for Gaussian systems (they will be considered in Section 8) the L^p -multiplicities are the same for all $p > 1$ (see also [197]).

2.3 Markov operators, joinings and Koopman representations, disjointness and spectral disjointness, entropy

We would like to emphasize that spectral theory is closely related to the theory of joinings (see de la Rue’s article [254] for needed definitions). The elements ρ of the set $J(\mathcal{S}, \mathcal{T})$ of joinings of two \mathbb{A} -actions \mathcal{S} and \mathcal{T} are in a 1-1 correspondence with Markov operators $J = J_\rho$ between the L^2 -spaces equivariant with the corresponding Koopman representations (see the glossary and [254]). The set of ergodic self-joinings of an ergodic \mathbb{A} -action \mathcal{T} is denoted by $J_2^e(\mathcal{T})$.

Each Koopman representation $\mathcal{U}_\mathcal{T}$ consists of Markov operators (indeed, U_{T_a} is clearly a Markov operator). In fact, if $U \in U(L^2(X, \mathcal{B}, \mu))$ is Markov then it is of the form U_R , where $R \in \text{Aut}(X, \mathcal{B}, \mu)$ ([192]). This allows us to see Koopman representations as unitary Markov representations, but since a spectral isomorphism does not “preserve” the set of Markov operators, spectrally isomorphic systems can have drastically different sets of self-joinings.

We will touch here only some aspects of interactions (clearly, far from completeness) between the spectral theory and the theory of joinings.

In order to see however an example of such interactions let us recall that the simplicity of eigenvalues for ergodic systems yields a short “joining” proof of the classical isomorphism theorem of Halmos-von Neumann in the discrete spectrum case: *Assume that $\mathcal{S} = (S_a)_{a \in \mathbb{A}}$ and $\mathcal{T} = (T_a)_{a \in \mathbb{A}}$ are ergodic \mathbb{A} -actions on (X, \mathcal{B}, μ) and (Y, \mathcal{C}, ν) respectively. Assume that both Koopman representations*

have purely discrete spectrum and that their group of eigenvalues are the same. Then \mathcal{S} and \mathcal{T} are measure-theoretically isomorphic. Indeed, each ergodic joining of \mathcal{T} and \mathcal{S} is the graph of an isomorphism of these two systems (see [185]; see also Goodson's proof in [109]). Another example of such interactions appear in the study Rokhlin's multiple mixing problem and its relation with the *pairwise independence property* (PID) for joinings of higher order. We will not deal with this subject here, referring the reader to [254] (see however Section 4.4).

Following [102], two \mathbb{A} -actions \mathcal{S} and \mathcal{T} are called *disjoint* provided the product measure is the only element in $J(\mathcal{S}, \mathcal{T})$ (if they are disjoint, one of these actions has to be ergodic). It was already noticed in [117] that spectrally disjoint systems are disjoint in the Furstenberg sense; indeed, $\text{Im}(J_\rho|_{L_0^2}) = \{0\}$ since $\sigma_{\mathcal{T}, J_\rho f} \ll \sigma_{\mathcal{S}, f}$.

Notice that whenever we take $\rho \in J_2^e(\mathcal{T})$ we obtain a new ergodic \mathbb{A} -action $(T_a \times T_a)_{a \in \mathbb{A}}$ defined on the probability space $(X \times X, \rho)$. One can now ask how close spectrally to \mathcal{T} is this new action? It turns out that except of the obvious fact that the marginal σ -algebras are factors, $(\mathcal{T} \times \mathcal{T}, \rho)$ can have other factors spectrally disjoint from \mathcal{T} : the most striking phenomenon here is a result of Smorodinsky and Thouvenot [274] (see also [56]) saying that each zero entropy system is a factor of an ergodic self-joining system of a fixed Bernoulli system (Bernoulli systems themselves have countable Haar spectrum). The situation changes if $\rho = \mu \otimes \mu$. In this case for $f, g \in L^2(X, \mathcal{B}, \mu)$ the spectral measure of $f \otimes g$ is equal to $\sigma_{\mathcal{T}, f} * \sigma_{\mathcal{T}, g}$. A consequence of this observation is that an ergodic action $\mathcal{T} = (T_a)_{a \in \mathbb{A}}$ is weakly mixing (see the glossary) if and only if the product measure $\mu \otimes \mu$ is an ergodic self-joining of \mathcal{T} .

The entropy which is a basic measure-theoretic invariant does not appear when we deal with spectral properties. We will not give here any formal definition of entropy for amenable group actions referring the reader to [216]. Assume that \mathbb{A} is countable and discrete. We always assume that \mathbb{A} is Abelian, hence it is amenable. For each dynamical system $\mathcal{T} = (T_a)_{a \in \mathbb{A}}$ acting on (X, \mathcal{B}, μ) , we can find a largest invariant sub- σ field $\mathcal{P} \subset \mathcal{B}$, called the *Pinsker σ -algebra*, such that the entropy of the corresponding quotient system is zero. Generalizing the classical Rokhlin-Sinai Theorem (see also [147] for \mathbb{Z}^d -actions), Thouvenot (unpublished) and independently Dooley and Golodets [61] proved this theorem for groups even more general than those considered here: *The spectrum of $\mathcal{U}_{\mathcal{T}}$ on $L^2(X, \mathcal{B}, \mu) \ominus L^2(\mathcal{P})$ is Haar with uniform infinite multiplicity*. This general result is quite intricate and based on methods introduced to entropy theory by Rudolph and Weiss [248] with a very surprising use of Dye's Theorem on orbital equivalence of all ergodic systems. For \mathbb{A} which is not countable the same result was proved in [24] in case of unimodular amenable groups which are not increasing union of compact subgroups. It follows that spectral theory of dynamical systems essentially reduces to the zero entropy case.

3 Maximal spectral type of a Koopman representation, Alexeyev's Theorem

Only few general properties of maximal spectral types of Koopman representations are known. The fact that a Koopman representation preserves the space of real functions implies that its maximal spectral type is the type of a symmetric (invariant under the map $\chi \mapsto \bar{\chi}$) measure.

Recall that the *Gelfand spectrum* $\sigma(\mathcal{U})$ of a representation $\mathcal{U} = (U_a)_{a \in \mathbb{A}}$ is defined as the set of *approximative eigenvalues* of \mathcal{U} , that is, $\sigma(\mathcal{U}) \ni \chi \in \widehat{\mathbb{A}}$ if for a sequence (x_n) bounded and bounded away from zero, $\|U_a x_n - \chi(a)x_n\| \rightarrow 0$ for each $a \in \mathbb{A}$. The spectrum is a closed subset in the topology of pointwise convergence, hence, in the compact-open topology of $\widehat{\mathbb{A}}$. In case of $\mathbb{A} = \mathbb{Z}$, the above set $\sigma(\mathcal{U})$ is equal to $\{z \in \mathbb{C} : U - z \cdot Id \text{ is not invertible}\}$.

Assume now that \mathbb{A} is countable and discrete (and Abelian). Then there exists a Følner sequence $(B_n)_{n \geq 1}$ whose elements tile \mathbb{A} [216]. Take a free and ergodic action $\mathcal{T} = (T_a)_{a \in \mathbb{A}}$ on (X, \mathcal{B}, μ) . By [216] for each $\varepsilon > 0$ we can find a set $Y_n \in \mathcal{B}$ such that the sets $T_b Y_n$ are pairwise disjoint for $b \in B_n$ and $\mu(\bigcup_{b \in B_n} T_b Y_n) > 1 - \varepsilon$. For each $\chi \in \widehat{\mathbb{A}}$, by considering functions of the form $f_n = \sum_{b \in B_n} \chi(b) 1_{T_b Y_n}$, we obtain that $\chi \in \sigma(\mathcal{U}_{\mathcal{T}})$. It follows that the topological support of the maximal spectral type of the Koopman representation of a free and ergodic action is full ([159], [185], [210]). The theory of Gaussian systems shows in particular that there are symmetric measures on the circle whose topological support is the whole circle but which cannot be maximal spectral types of Koopman representations.

An open well-known question remains of whether an absolutely continuous measure ρ is the maximal spectral type of a Koopman representation if and only if ρ is equivalent to a Haar measure of $\widehat{\mathbb{A}}$ (this is unknown for $\mathbb{A} = \mathbb{Z}$).

Another interesting question was raised by A. Katok (see [157]): *Is it true that the topological supports of all measures in a spectral sequence of a Koopman representation are full?* If the answer to this question is positive then for example the essential supremum of $M_{\mathcal{U}_{\mathcal{T}}}$ is the same on all balls of $\widehat{\mathbb{A}}$.

Notice that the fact that all spectral measures in a spectral sequence are symmetric means that $\mathcal{U}_{\mathcal{T}}$ is isomorphic to $\mathcal{U}_{\mathcal{T}^{-1}}$. A. del Junco [139] showed that generically for \mathbb{Z} -actions, T and its inverse are not measure-theoretically isomorphic (in fact, he proved disjointness).

Let \mathcal{T} be an \mathbb{A} -action on (X, \mathcal{B}, μ) . One can ask whether a “good” function can realize the maximal spectral type of $\mathcal{U}_{\mathcal{T}}$. In particular, can we find a function $f \in L^\infty(X, \mathcal{B}, \mu)$ that realizes the maximal spectral type? The answer is given in the following general theorem (see [199]).

Theorem 3 (Alexeyev's Theorem) *Assume that $\mathcal{U} = (U_a)_{a \in \mathbb{A}}$ is a unitary representation of \mathbb{A} in a separable Hilbert space H . Assume that $F \subset H$ is a dense linear subspace. Assume moreover that with some F -norm $\|\cdot\|$ – stronger than the norm $\|\cdot\|$ given by the scalar product – F becomes a Fréchet space. Then, for each spectral measure $\sigma \ll \sigma_{\mathcal{U}}$ there exists $y \in F$ such that $\sigma_y \gg \sigma$. In particular, there exists $y \in F$ that realizes the maximal spectral type.*

By taking $H = L^2(X, \mathcal{B}, \mu)$ and $F = L^\infty(X, \mathcal{B}, \mu)$ we obtain the positive answer to the original question. Alexeyev [20] proved the above theorem for $\mathbb{A} = \mathbb{Z}$ using analytic functions. Refining Alexeyev's original proof, Frączek [93] showed the existence of a sufficiently regular function realizing the maximal spectral type depending only on the “regularity” of the underlying probability space, e.g. when X is a compact metric space (compact smooth manifold) then one can find a continuous (smooth) function realizing the maximal spectral type.

By the theory of systems of probabilistic origin (see Section 8), in case of simplicity of the spectrum, one can easily point out spectral measures whose types are not realized by (essentially) bounded functions. However, it is still an open question whether for each Koopman representation $\mathcal{U}_{\mathcal{T}}$ there exists a sequence $(f_i)_{i \geq 1} \subset L^\infty(X, \mathcal{B}, \mu)$ such that the sequence $(\sigma_{f_i})_{i \geq 1}$ is a spectral sequence for $\mathcal{U}_{\mathcal{T}}$. For $\mathbb{A} = \mathbb{Z}$ it is unknown whether the maximal spectral type of a Koopman representation can be realized by a characteristic function.

The group $\text{Aut}(X, \mathcal{B}, \mu)$ considered with the weak operator topology is closed in $U(L^2(X, \mathcal{B}, \mu))$,¹ hence becomes a Polish group. One can then ask what are “typical” (largeness is understood as a residual subset) properties of an automorphism of (X, \mathcal{B}, μ) . It is classical (Halmos) that typically an automorphism is weakly mixing, rigid and has simple spectrum. Some other typical properties will be discussed later. While Halmos already noticed that in the weak operator topology mixing automorphisms form a meager set, in [286], S. Tikhonov considers a special (Polish) topology on the set of mixing automorphisms. In fact, this topology was introduced by Alpern [22] in 1985 and Tikhonov disproves a conjecture by Alpern by showing that a generic mixing transformation has simple singular spectrum and is mixing of arbitrary order; moreover, all its powers are disjoint. In [287], the topology is extended to mixing actions of infinite countable groups H , it is given by the metric d_m , where for two H -actions \mathcal{T}_i and $H \ni h \mapsto |h| \in \mathbb{N}$ so that $\sum_{h \in H} 1/2^{|h|} < +\infty$, we have

$$d_m(\mathcal{T}_1, \mathcal{T}_2) := \sum_{h \in H} \frac{1}{2^{|h|}} d(T_{1,h}, T_{2,h}) + \sup_{h \in H} \sum_{i,j \geq 1} \frac{1}{2^{i+j}} |\mu((T_{1,h}A_i \cap A_j) - \mu(T_{2,h}A_i \cap A_j))|.$$

Bashtanov [31], [32] proved that the conjugacy classes (of mixing automorphisms) are dense in this topology. Hence, properties like to have trivial centralizer and no (non-trivial) factors are typical in this topology.

4 Spectral theory of weighted operators

We now pass to the problem of possible essential values for the multiplicity function of a Koopman representation. However, one of known techniques is a use of cocycles, so before we tackle the multiplicity problem, we will go through some

¹If we choose $\{A_i : i \geq 1\}$ a dense subset in \mathcal{B} (considered modulo null sets), then the weak operator topology is metrizable with the metric $d(T_1, T_2) := \sum_{i \geq 1} \frac{1}{2^i} (\mu(T_1(A_i) \Delta T_2(A_i)) + \mu(T_1^{-1}(A_i) \Delta T_2^{-1}(A_i)))$.

results concerning spectral theory of compact group extensions automorphisms which in turn entail a study of weighted operators (see the glossary).

Assume that T is an ergodic automorphism of a standard Borel probability space (X, \mathcal{B}, μ) . Let $\xi : X \rightarrow \mathbb{T}$ be a measurable function and let $V = V_{\xi, T}$ be the corresponding weighted operator. To see a connection of weighted operators with Koopman representations of compact group extensions consider a compact (metric) Abelian group G and a cocycle $\varphi : X \rightarrow G$. Then U_{T_φ} (see the glossary) acts on $L^2(X \times G, \mu \otimes \lambda_G)$. But

$$L^2(X \times G, \mu \otimes \lambda_G) = \bigoplus_{\chi \in \widehat{G}} L_\chi, \quad \text{where } L_\chi = L^2(X, \mu) \otimes \chi,$$

where L_χ is a U_{T_φ} -invariant (clearly, closed) subspace. Moreover, the map $f \otimes \chi \mapsto f$ settles a unitary isomorphism of $U_{T_\varphi}|_{L_\chi}$ with the operator $V_{\chi \circ \varphi, T}$. Therefore, spectral analysis of such Koopman representations reduces to the spectral analysis of weighted operators $V_{\chi \circ \varphi, T}$ for all $\chi \in \widehat{G}$.

4.1 Maximal spectral type of weighted operators over rotations

The spectral analysis of weighted operators $V_{\xi, T}$ is especially well developed in case of rotations, i.e. when we additionally assume that T is an ergodic rotation on a compact monothetic group X : $Tx = x + x_0$, where x_0 is a topologically cyclic element of X (and μ will stand for Haar measure λ_X of X). In this case, Helson's analysis [119] applies (see also [112], [128], [185], [224]) leading to the following conclusions:

- The maximal spectral type $\sigma_{V_{\xi, T}}$ is either discrete or continuous.
- When $\sigma_{V_{\xi, T}}$ is continuous it is either singular or Lebesgue.
- The spectral multiplicity of $V_{\xi, T}$ is uniform.

We now pass to a description of some results in case when $Tx = x + \alpha$ is an irrational rotation on the additive circle $X = [0, 1)$. It was already noticed in the original paper by Anzai [23] that when $\xi : X \rightarrow \mathbb{T}$ is an affine cocycle ($\xi(x) = \exp(nx + c)$, $0 \neq n \in \mathbb{Z}$) then $V_{\xi, T}$ has a Lebesgue spectrum. It was then considered by several authors (originated by [180], see also [44], [46]) to which extent this property is stable when we perturb our cocycle. Since the topological degree of affine cocycles is different from zero, when perturbing them we consider smooth perturbations by cocycles of degree zero.

Theorem 4 ([128]) *Assume that $Tx = x + \alpha$ is an irrational rotation. If $\xi : [0, 1) \rightarrow \mathbb{T}$ is of non-zero degree, absolutely continuous, with the derivative of bounded variation then $V_{\xi, T}$ has a Lebesgue spectrum.*

In the same paper, it is noticed that if we drop the assumption on the derivative then the maximal spectral type of $V_{\xi, T}$ is a Rajchman measure (i.e. its Fourier

transform vanishes at infinity). It is still an open question, whether one can find ξ absolutely continuous with non-zero degree and such that $V_{\xi,T}$ has singular spectrum. „Below” absolute continuity, topological properties of the cocycle seem to stop playing any role – in [128] a continuous, degree 1 cocycle ξ of bounded variation is constructed such that $\xi(x) = \eta(x)/\eta(Tx)$ for a measurable $\eta : [0, 1) \rightarrow \mathbb{T}$ (that is ξ is a *coboundary*) and therefore $V_{\xi,T}$ has purely discrete spectrum (it is isomorphic to U_T). For other results about Lebesgue spectrum for Anzai skew products see also [44], [94], [127] (in [94] \mathbb{Z}^d -actions of rotations and so called winding numbers instead of topological degree are considered). For recent generalizations, see [39], [281].

When the cocycle is still smooth but its degree is zero the situation drastically changes. Given an absolutely continuous function $f : [0, 1) \rightarrow \mathbb{R}$ M. Herman [121], using the Denjoy-Koksma inequality (see e.g. [177]), showed that $f_0^{(q_n)} \rightarrow 0$ uniformly (here $f_0 = f - \int_0^1 f d\lambda_{[0,1]}$ and (q_n) stands for the sequence of denominators of α). It follows that $T_{e^{2\pi i f}}$ is rigid and hence has a singular spectrum. B. Fayad [68] shows that this result is no longer true if one dimensional rotation is replaced by a multi-dimensional rotation (his counterexample is in the analytic class). See also [189] for the singularity of spectrum for functions f whose Fourier transform satisfies $o(\frac{1}{|n|})$ condition or to [130], where it is shown that sufficiently small variation implies singularity of the spectrum.

A natural class of weighted operators arises when we consider Koopman operators of rotations on nil-manifolds. We only look at the particular example of such a rotation on a quotient of the Heisenberg group $(\mathbb{R}^3, *)$ (a general spectral theory of nil-actions was mainly developed by W. Parry [220] – these actions have countable Lebesgue spectrum in the orthocomplement of the subspace of eigenfunctions) that is take the nil-manifold $\mathbb{R}^3/\mathbb{Z}^3$ on which we define the nil-rotation $S((x, y, z) * \mathbb{Z}^3) = (\alpha, \beta, 0) * (x, y, z) * \mathbb{Z}^3$, where α, β and 1 are rationally independent. It can be shown that S is isomorphic to the skew product defined on $[0, 1)^2 \times \mathbb{T}$ by

$$T_\varphi : (x, y, z) \mapsto (x + \alpha, y + \beta, z \cdot e^{2\pi i \varphi(x, y)}) = (x + \alpha, y + \beta, z + \alpha y) * \mathbb{Z}^3,$$

where $\varphi(x, y) = \alpha y - \psi(x + \alpha, y + \beta) + \psi(x, y)$ with $\psi(x, y) = x[y]$. Since nil-cocycles can be considered as a certain analog of affine cocycles for one-dimensional rotations, it would be nice to explain to what kind of perturbations the Lebesgue spectrum property is stable.

Yet another interesting problem which is related to the spectral theory of extensions given by so called *Rokhlin cocycles* (see Section 6) arises, when given $f : [0, 1) \rightarrow \mathbb{R}$, we want to describe spectrally the one-parameter set of weighted operators $W_c := V_{e^{2\pi i c f}, T}$; here T is a fixed irrational rotation by α . As proved by quite sophisticated arguments in [130], if we take $f(x) = x$ then for all non-integer $c \in \mathbb{R}$ the spectrum of W_c is continuous and singular (see also [112] and [208] where some special α 's are considered). It has been open for some time if at all one can find $f : [0, 1) \rightarrow \mathbb{R}$ such that for each $c \neq 0$, the operator W_c has a Lebesgue spectrum. The positive answer is given in [301]: for example if $f(x) = x^{-(2+\varepsilon)}$ ($\varepsilon > 0$) and α has bounded partial quotients then W_c has a

Lebesgue spectrum for all $c \neq 0$. All functions with such a property considered in [301] are non-integrable. It would be interesting to find an integrable f with the above property.

We refer to [109] and the references therein for further results especially for transformations of the form $(x, y) \mapsto (x + \alpha, 1_{[0, \beta)}(x) + y)$ on $[0, 1) \times \mathbb{Z}/2\mathbb{Z}$. Recall however that earlier Katok and Stepin [158] used this kind of transformations to give a first counterexample to the Kolmogorov group property (see the glossary) for the spectrum.

4.2 The multiplicity problem for weighted operators over rotations

In case of perturbations of affine cocycles, this problem was already raised by Kushnirenko [180]. Some significant results were obtained by M. Guenais. Before we state her results let us recall a useful criterion to find an upper bound for the multiplicity: *If there exist $c > 0$ and a sequence $(F_n)_{n \geq 1}$ of cyclic subspaces of H such that for each $y \in H$, $\|y\| = 1$ we have $\liminf_{n \rightarrow \infty} \|\text{proj}_{F_n} y\|^2 \geq c$, then $\text{esssup}(M_U) \leq 1/c$ which follows from a well-known lemma of Chacon ([40], [46], [165], [185]).* Using this and a technique which is close to the idea of local rank one (see [80], [165]) M. Guenais [113] proved a series of results on multiplicity which we now list.

Theorem 5 *Assume that $Tx = x + \alpha$ and let $\xi : [0, 1) \rightarrow \mathbb{T}$ be a cocycle.*

- (i) *If $\xi(x) = e^{2\pi icx}$ then $M_{V_{\xi, T}}$ is bounded by $|c| + 1$.*
- (ii) *If ξ is absolutely continuous and ξ is of topological degree zero, then $V_{\xi, T}$ has a simple spectrum.*
- (iii) *if ξ is of bounded variation, then $M_{V_{\xi, T}} \leq \max(2, 2\pi \text{Var}(\xi)/3)$.*

4.3 Remarks on the Banach problem

We already mentioned in Introduction the Banach problem in ergodic theory, which is simply the question whether there exists a Koopman representation for $\mathbb{A} = \mathbb{Z}$ with simple Lebesgue spectrum. In fact no example of a Koopman representation with Lebesgue spectrum of finite multiplicity is known. Helson and Parry [120] asked for the validity of a still weaker version: *Can one construct T such that U_T has a Lebesgue component in its spectrum whose multiplicity is finite?* Quite surprisingly in [120] they give a general construction yielding for each ergodic T a cocycle $\varphi : X \rightarrow \mathbb{Z}/2\mathbb{Z}$ such that the unitary operator U_{T_φ} has a Lebesgue spectrum in the orthocomplement of functions depending only on the X -coordinate. Then Mathew and Nadkarni [207] gave examples of cocycles over so called dyadic adding machine for which the multiplicity of the Lebesgue component was equal to 2. In [184] this was generalized to so called *Toeplitz $\mathbb{Z}/2\mathbb{Z}$ -extensions* of adding machines: for each even number k we can find a two-point extension of an adding machine so that the multiplicity of the Lebesgue component is k . Moreover, it was shown that Mathew and Nadkarni's constructions from [207] in fact are close to systems arising from

number theory (like the famous Rudin-Shapiro sequence, e.g. [224]), relating the result about multiplicity of the Lebesgue component to results by Kamae [146] and Queffelec [224]. Independently of [184], Ageev [14] showed that one can construct 2-point extensions of the Chacon transformation realizing (by taking powers of the extension) each even number as the multiplicity of the Lebesgue component. Contrary to all previous examples, Ageev's constructions are weakly mixing.

Still an open question remains whether for $\mathbb{A} = \mathbb{Z}$ one can find a Koopman representation with the Lebesgue component of multiplicity 1 (or even odd).

In [114], M. Guenais studies the problem of Lebesgue spectrum in the classical case of Morse sequences (see [161] as well as [181], where the problem of spectral classification in this class is studied). All dynamical systems arising from Morse sequences have simple spectra [181]. It follows that if a Lebesgue component appears in a Morse dynamical system, it has multiplicity one. Guenais [114] using a Riesz product technique relates the Lebesgue spectrum problem with the still open problem (a variation of the classical Littlewood problem) of whether a construction of so called L^1 -ultraflat trigonometric polynomials with coefficients ± 1 is possible (in the very recent preprint [30], the Littlewood problem of existence of uniformly flat trigonometric polynomials has been solved, but it is unclear whether it yields the ultraflatness condition). However, it is proved in [114] that such a construction can be carried out on some compact Abelian groups and it leads, for an Abelian countable torsion group \mathbb{A} , to a construction of an ergodic action of \mathbb{A} with simple spectrum and a Haar component in its spectrum.

In [222], A. Prikhodko published a construction of a rank one flow (see Section 7) having Lebesgue spectrum. As rank one implies simple spectrum, the result yields solution of Banach problem for $\mathbb{A} = \mathbb{R}$. To carry out the construction, Prikhodko proved the following L^1 - ultraflat version of the Littlewood conjecture: *For all $0 < a < b$ and $n \geq 1$, there are polynomials $P_n(t) = \sum_{j=0}^{n-1} e^{2\pi i w_j^{(n)} t}$ for some real numbers $w_j^{(n)}$, so that $\|P_n\|_{L^1([a,b])} / \sqrt{n} \rightarrow 1$ when $n \rightarrow \infty$.* It seems however that some of the arguments in the paper are written too briefly and no further clarifying presentation of methods/results/ideas from [222] has appeared so far.²

4.4 Lifting mixing properties

We now give one more example of interactions between spectral theory and joinings (see Introduction) that gives rise to a quick proof of the fact that r -fold mixing property of T ($r \geq 2$) lifts to a weakly mixing compact group extension T_φ (the original proof of this fact is due to D. Rudolph [244]). Indeed, to prove r -fold mixing for a mixing(=2-mixing) transformation S (acting on (Y, \mathcal{C}, ν)) one has to prove that each weak limit of off-diagonal self-joinings (given by powers

²It would also be extremely nice to explain the status of [8] by H. El Abdalaoui, first posted on arXiv in 2015, which claims the solution of the original Banach problem (i.e. in the conservative infinite measure-preserving category).

of S , see [254]) of order r is simply the product measure $\nu^{\otimes r}$. We need also a Furstenberg's lemma ([103]) about relative unique ergodicity (RUE) of compact group extensions T_φ : *If $\mu \otimes \lambda_G$ is an ergodic measure for T_φ then it is the only (ergodic) invariant measure for T_φ whose projection on the first coordinate is μ .* Now, the result about lifting r -fold mixing to compact group extensions follows directly from the fact that whenever T_φ is weakly mixing, $(\mu \otimes \lambda_G)^{\otimes r}$ is an ergodic measure (this approach was shown to the second author by A. del Junco). In particular, if T is mixing and T_φ is weakly mixing then for each $\chi \in \widehat{G} \setminus \{1\}$, the maximal spectral type of $V_{\chi \circ \varphi, T}$ is Rajchman.

See Section 6 for a generalization of the lifting result to Rokhlin cocycle extensions.

5 The multiplicity function

In this section only $\mathbb{A} = \mathbb{Z}$ is considered. For other groups, even for \mathbb{R} , much less is known. Clearly, given an automorphism T , by inducing its Koopman \mathbb{Z} -representation, we obtain a one-parameter group $(V_t)_{t \in \mathbb{R}}$ of unitary operators, which has precisely the same properties as the original one, except that we added the eigenvalues $n \in \mathbb{R}$. Moreover, classically, the induced Koopman representation is given by the suspension of T , i.e. by the special flow T^f (see the glossary), where $f = 1$, whence it is also Koopman but is never weakly mixing. See [54] and [59], where some of the results below proved for $\mathbb{A} = \mathbb{Z}$ have been extended to (weakly mixing) flows. See also the case of so called *product \mathbb{Z}^d -actions* [86] and [275] for general countable Abelian group actions.

Contrary to the case of maximal spectral type, it is rather commonly believed that there are no restrictions for the set of essential values of Koopman representations. In fact, if we drop the assumption that we consider the finite measure-preserving case and let ourselves consider μ σ -finite and infinite, Danilenko and Ryzhikov [57], [58] proved that all subsets of $\{1, 2, \dots\} \cup \{\infty\}$ are Koopman realizable (in the weak mixing and mixing class, respectively).

5.1 Cocycle approach

We will only concentrate on some results of the last four decades. In 1983, refining an earlier idea of Oseledets from the 1960th, E.A. Robinson [233] proved that for each $n \geq 1$ there exists an ergodic transformation whose maximal spectral multiplicity is n . Another important result was proved in [234] (see also [152]), where it is shown that given a finite set $M \subset \mathbb{N}$ containing 1 and closed under the least common multiple one can find (even a weakly mixing) T so that the set of essential values of the multiplicity function equals M . This result was then extended in [110] to infinite sets and finally in [183] (see also [17]) to all subsets $M \subset \mathbb{N}$ containing 1. In fact, as we have already noticed in the previous section the spectral theory for compact Abelian group extensions is reduced to a study of weighted operators and then to comparing maximal

spectral types for such operators. This leads to sets of the form

$$M(G, v, H) = \#\{(\{\chi \circ v^i : i \in \mathbb{Z}\} \cap \text{anih}(H)) : \chi \in \text{anih}(H)\}$$

($H \subset G$ is a closed subgroup and v is a continuous group automorphism of G). Then an algebraic lemma has been proved in [183] saying that each set M containing 1 is of the form $M(G, v, H)$ and the techniques to construct “good” cocycles and a passage to “natural factors” yielded the following: *For each $M \subset \{1, 2, \dots\} \cup \{\infty\}$ containing 1 there exists an ergodic automorphism such that the set of essential values for its Koopman representation equals M .* See also [235] for the case of non-Abelian group extensions.

A similar in spirit approach (that means, a passage to a family of factors) is present in the paper of Ageev [19] in which he first applies Katok’s analysis (see [152], [156]) for spectral multiplicities of the Koopman representation associated with Cartesian products $T^{\times k}$ for a generic transformation T . In a natural way this approach leads to study multiplicities of tensor products of unitary operators. Roughly, the multiplicity is computed as the number of atoms (counted modulo obvious symmetries) for conditional measures (see [152]) of a product measure over its convolution. Ageev [19] proved that for a typical automorphism T the set of the values of the multiplicity function for $U_{T^{\times k}}$ equals $\{k, k(k-1), \dots, k!\}$ and then he just passes to “natural” factors for the Cartesian products by taking sets invariant under a fixed subgroup of permutations of coordinates. In particular, he obtains all sets of the form $\{2, 3, \dots, n\}$ on L_0^2 . He also shows that such sets of multiplicities are realizable in the category of mixing transformations. See also Ryzhikov [263] for a realization of sets of the form $\{k, l, kl\}$, $\{k, l, m, kl, km, lm, klm\}$, etc.

A further progress was done in 2009-2012, when first Katok and Lemańczyk [157] proved that each finite subset $M \subset \{1, 2, \dots\} \cup \{\infty\}$ containing 2 can be realized as the set of essential values of an ergodic automorphism which was then, by overcoming some algebraic difficulties, extended by Danilenko [49], [50] (in the mixing category) to all subsets containing 2.

5.2 Multiplicity for Gaussian and Poissonian automorphisms

We refer the reader to Section 8 for the definition and basic properties of Gaussian and Poissonian automorphisms. Recall that given a Poissonian automorphism, there is a Gaussian automorphism spectrally isomorphic to it (whether the converse holds, is unknown). In Gaussian case, the classical Girsanov’s theorem from the 1950th asserts that the maximal spectral multiplicity in this case is either one or infinity (with a possibility that ∞ is not an essential value), see [179] for an elegant proof of this theorem. What is the family of subsets appearing as sets of essential values of the multiplicity functions of Koopman operators given by Gaussian (and also Poissonian) automorphisms was studied by Danilenko and Ryzhikov in [58]. They prove the following remarkable results:

- this family contains all multiplicative sub-semigroups of \mathbb{N} ;

- this family contains other sets than multiplicative sub-semigroups of \mathbb{N} .

The latter shows that Proposition 6.4.4 (multiplicative nature of M_T in the Gaussian case) claimed in the book [159] (and also in [241]) is not true.

In the unpublished preprint [264], Ryzhikov shows that all subsets containing ∞ are Gaussian “realizable” (even in the mixing category).

5.3 Rokhlin’s uniform multiplicity problem

The Rokhlin multiplicity problem (see the recent book by Anosov [21]) was, given $n \geq 2$, to construct an ergodic transformation with uniform multiplicity n on L_0^2 . A solution for $n = 2$ was independently given by Ageev [15] and Ryzhikov [260] (see also [21] and [109]) and in fact it consists in showing that for some T (actually, any T with simple spectrum for which $\frac{1}{2}(Id + U_T)$ is in the weak operator closure of the powers of U_T will do) the multiplicity of $T \times T$ is uniform and equal to 2 (see also Section 13). In [285] ([266]), the case $n = 2$ is solved in case of mixing automorphisms (flows).

In [18], Ageev proposed a new approach which consists in considering actions of „slightly non-Abelian” groups and showing that for a “typical” action of such a group a fixed “direction” automorphism has a uniform multiplicity. Shortly after publication of [18], Danilenko [48], following Ageev’s approach, considerably simplified the original proof. We will present Danilenko’s arguments.

Fix $n \geq 1$. Denote $\bar{e}_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$, $i = 1, \dots, n$. We define an automorphism L of \mathbb{Z}^n setting $L(\bar{e}_i) = \bar{e}_{i+1}$, $i = 1, \dots, n - 1$ and $L(\bar{e}_n) = \bar{e}_1$. Using L we define a semi-direct product $G := \mathbb{Z}^n \rtimes \mathbb{Z}$ defining multiplication as $(u, k) \cdot (w, l) = (u + L^k w, k + l)$. Put $e_0 = (0, 1)$, $e_i = (\bar{e}_i, 0)$, $i = 1, \dots, n$ (and $Le_i = (L\bar{e}_i, 0)$). Moreover, denote $e_{n+1} = e_0^n = (0, n)$. Notice that $e_0 \cdot e_i \cdot e_0^{-1} = Le_i$ for $i = 1, \dots, n$ ($L(e_{n+1}) = e_{n+1}$).

Theorem 6 (Ageev, Danilenko) *For every unitary representation \mathcal{U} of G in a separable Hilbert space H , for which $U_{e_1 - L^r e_1}$ has no non-trivial fixed points for $1 \leq r < n$, the essential values of the multiplicity function for $U_{e_{n+1}}$ are contained in the set of multiples of n . If, in addition, U_{e_0} has a simple spectrum, then $U_{e_{n+1}}$ has uniform multiplicity n .*

It is then a certain work to show that the assumption of the second part of the theorem is satisfied for a typical action of the group G . Using a special (C, F) -construction with all the cut-and-stack parameters explicit, Danilenko [48] was also able to show that each set of the form $k \cdot M$, where $k \geq 1$ and M is an arbitrary subset of natural numbers containing 1, is realizable as the set of essential values of a Koopman representation.

Tikhonov [285] proved the existence of a mixing automorphism of uniform multiplicity n on L_0^2 for all $n \geq 1$.

6 Rokhlin cocycles

We consider now a certain class of extensions which should be viewed as a generalization of the concept of compact group extensions. We will focus on \mathbb{Z} -actions only.

Assume that T is an ergodic automorphism of (X, \mathcal{B}, μ) . Let G be a l.c.s.c. Abelian group. Assume that this group acts on (Y, \mathcal{C}, ν) , that is we have a G -action $\mathcal{S} = (S_g)_{g \in G}$ on (Y, \mathcal{C}, ν) . Let $\varphi : X \rightarrow G$ be a cocycle. We then define an automorphism $T_{\varphi, \mathcal{S}}$ of the space $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ by

$$T_{\varphi, \mathcal{S}}(x, y) = (Tx, S_{\varphi(x)}(y)).$$

Such an extension is called a *Rokhlin cocycle extension* (the map $x \mapsto S_{\varphi(x)}$ is called a *Rokhlin cocycle*). Such an operation generalizes the case of compact group extensions; indeed, when G is compact the action of G on itself by rotations preserves Haar measure. (It is quite surprising, that when only we admit G non-Abelian, then, as shown in [53], **each** ergodic extension of T has a form of a Rokhlin cocycle extension.) Ergodic and spectral properties of such extensions are examined in several papers: [105], [107], [188], [190], [191], [192], [236], [245]. Since in these papers rather joining aspects are studied (among other things in [188] Furstenberg's RUE lemma is generalized to this new context), we will mention here only few results, mainly spectral, following [188] and [192]. We will constantly assume that G is non-compact. As $\varphi : X \rightarrow G$ is then a cocycle with values in a non-compact group, the theory of such cocycles is much more complicated (see e.g. [268]), and in fact the theory of Rokhlin cocycle extensions leads to interesting interactions between classical ergodic theory, the theory of cocycles and the theory of non-singular actions arising from cocycles taking values in non-compact groups – especially, the Mackey action associated to φ plays a crucial role here (see the problem of invariant measures for $T_{\varphi, \mathcal{S}}$ in [191] and [53]); see also monographs [2], [152], [155], [268]. Especially, two Borel subgroups of \widehat{G} are important here:

$$\Sigma_\varphi = \{\chi \in \widehat{G} : \chi \circ \varphi = c \cdot \xi / \xi \circ T \text{ for a measurable } \xi : X \rightarrow \mathbb{T} \text{ and } c \in \mathbb{T}\}.$$

and its subgroup Λ_φ given by $c = 1$. Λ_φ turns out to be the group of L^∞ -eigenvalues of the Mackey action (of G) associated to the cocycle φ . This action is the quotient action of the natural action of G (by translations on the second coordinate) on the space of ergodic components of the skew product T_φ – the Mackey action is (in general) not measure-preserving, it is however non-singular. We refer the reader to [4], [123] and [210] for other properties of those subgroups.

Theorem 7 ([191],[192]) (i) $\sigma_{T_{\varphi, \mathcal{S}}} |_{L^2(X \times Y, \mu \otimes \nu) \ominus L^2(X, \mu)} = \int_{\widehat{G}} \sigma_{V_{\chi \circ \varphi, T}} d\sigma_{\mathcal{S}}$.

(ii) $T_{\varphi, \mathcal{S}}$ is ergodic if and only if T is ergodic and $\sigma_{\mathcal{S}}(\Lambda_\varphi) = 0$.

(iii) $T_{\varphi, \mathcal{S}}$ is weakly mixing if and only if T is weakly mixing and \mathcal{S} has no eigenvalues in Σ_φ .

(iv) if T is mixing, \mathcal{S} is mildly mixing, φ is recurrent and not cohomologous to a cocycle with values in a compact subgroup of G then $T_{\varphi, \mathcal{S}}$ remains mixing.

(v) If T is r -fold mixing, φ is recurrent and $T_{\varphi, \mathcal{S}}$ is mildly mixing then $T_{\varphi, \mathcal{S}}$ is also r -fold mixing.

(vi) If T and R are disjoint, the cocycle φ is ergodic and \mathcal{S} is mildly mixing then $T_{\varphi, \mathcal{S}}$ remains disjoint from R .

Let us recall ([104], [270]) that an \mathbb{A} -action $\mathcal{S} = (S_a)_{a \in \mathbb{A}}$ is mildly mixing (see the glossary) if and only if the \mathbb{A} -action $(S_a \times \tau_a)_{a \in \mathbb{A}}$ remains ergodic for every properly ergodic non-singular \mathbb{A} -action $\tau = (\tau_a)_{a \in \mathbb{A}}$.

Coming back to Smorodinsky-Thouvenot's result about factors of ergodic self-joinings of a Bernoulli automorphism we would like to emphasize here that the disjointness result (vi) above was used in [191] to give an example of an automorphism which is disjoint from all weakly mixing transformations but which has an ergodic self-joining whose associated automorphism has a non-trivial weakly mixing factor. In a sense this is opposed to Smorodinsky-Thouvenot's result as here from self-joinings we produced a "more complicated" system (namely the weakly mixing factor) than the original system.

It would be interesting to develop the theory of spectral multiplicity for Rokhlin cocycle extensions as it was done in the case of compact group extensions. However a difficulty is that in the compact group extension case we deal with a countable direct sum of representations of the form $V_{\chi \circ \varphi, T}$ while in the non-compact case we have to consider an integral of such representations.

7 Rank-1 and related systems

Although properties like mixing, weak (and mild) mixing as well as ergodicity, are clearly spectral properties, they have "good" measure-theoretic formulations (expressed by a certain behaviour on sets). Simple spectrum property is another example of a spectral property, and it was a popular question in the 1980s whether simple spectrum property of a Koopman representation can be expressed in a more "measure-theoretic" way. We now recall rank-1 concept which can be seen as a notion close to Katok's and Stepin's theory of cyclic approximation [158] (see also [46]).

Assume that T is an automorphism of a standard probability Borel space (X, \mathcal{B}, μ) . T is said to have the *rank one* property if there exists an increasing sequence of Rokhlin towers tending to the partition into points (a *Rokhlin tower* is a family $\{F_n, TF_n, \dots, T^{h_n-1}F_n\}$ of pairwise disjoint sets, while "tending to the partition into points" means that we can approximate every set in \mathcal{B} by unions of levels of towers in the sequence). Hence, basically, rank one automorphism is given by two sequences of parameters: $r_n, n \geq 1$, which is the number of subcolumns on which we divide the n th tower given by F_n , and $s_{n,j}, n \geq 1, j = 0, 1, \dots, r_n - 1$, the sequence of spacers put over consecutive subcolumns. A "typical" automorphism of a standard probability Borel space has the rank one property. The "typicality" of rank one is still true in the Alpern-Tikhonov topology we mentioned in Section 3 by [31].

Baxter [33] showed that the maximal spectral type of such a T is realized by a characteristic function. Since the cyclic space generated by the characteristic

function of the base contains characteristic functions of all levels of the tower, by the definition of rank one, the increasing sequence of cyclic spaces tends to the whole L^2 -space, therefore rank one property implies simplicity of the spectrum for the Koopman representation. It was a question for some time whether rank-1 is just a characterization of simplicity of the spectrum, disproved by del Junco [138]. We refer the reader to [82] as a good source for basic properties of rank-1 transformations.

Similarly to the rank one property, one can define *finite rank* automorphisms (simply by requiring that an approximation is given by a sequence of a fixed number of towers) – see e.g. [215], or even, a more general property, namely the *local rank one* property can be defined, just by requiring that the approximating sequence of single towers fills up a fixed fraction of the space (see [80], [165]). Local rank one property implies finite multiplicity [165] and the maximal spectral multiplicity is always bounded by rank. Mentzen [209] showed that for each $n \geq 1$ one can construct an automorphism with simple spectrum and having rank n and later Kwiatkowski and Lacroix [182] showed that for each pair (m, r) with $m \leq r$, one can construct a rank r automorphism whose maximal spectral multiplicity is m . In [198] there is an example of a simple spectrum automorphism which is not of local rank one. Ferenczi [81] deals with the notion of funny rank one (approximating towers are Rokhlin towers with “holes”) - the concept that has been introduced by Thouvenot. Funny rank one also implies simplicity of the spectrum. An example is given in [81] which is even not loosely Bernoulli (see Section 9, we recall that local rank one property implies loose Bernoullicity [80]).

The notion of AT (see the glossary) has been introduced by Connes and Woods [45]. They proved that AT property implies zero entropy. They also proved that funny rank one automorphisms are AT. In [62] it is proved that the system induced by the classical Morse-Thue system is AT (it is an open question by S. Ferenczi whether this system has funny rank one property). A question by Dooley and Quas is whether AT implies funny rank one property. AT property implies “simplicity of the spectrum in L^1 ” which we already considered in Introduction (a “generic” proof of this fact is due to J.-P. Thouvenot).

A persistent question was formulated in the 1980s whether rank one itself is a spectral property. In [85], the authors maintained that this is not the case, based on an unpublished preprint of the first named author of [85] in which there was a construction of a Gaussian-Kronecker automorphism (see Section 8) having rank-1 property. This latter construction turned out to be false. In fact, de la Rue [250] proved that no Gaussian automorphism can be of local rank one. Therefore the question whether: *Rank one is a spectral property* remains one of interesting open questions in that theory. Downarowicz and Kwiatkowski [63] proved that rank-1 is a spectral property in the class of systems generated by generalized Morse sequences.

One of the most beautiful theorems about rank-1 automorphisms is the following result of J. King [164] (for a different proof see [256]).

Theorem 8 (WCT) *If T is of rank one then for each element S of the cen-*

tralizer $C(T)$ of T there exists a sequence (n_k) such that $U_T^{n_k} \rightarrow U_S$ strongly.

A conjecture of J. King is that in fact for rank-1 automorphisms each indecomposable Markov operator $J = J_\rho$ ($\rho \in J_2^c(T)$) is a weak limit of powers of U_T (see [166], also [256]). To which extent the WCT remains true for actions of other groups is not clear. In [303] the WCT is proved in case of rank one flows, however the main argument seems to be based on the fact that a rank one flow has a non-zero time automorphism T_{t_0} which is of rank one, which is not true. After the proof of the WCT by Ryzhikov in [256] there is a remark that the rank one flow version of the theorem can be proved by a word for word repetition of the arguments. He also proves that if the flow $(T_t)_{t \in \mathbb{R}}$ is mixing, then T_1 does not have finite rank. On the other hand, for $\mathbb{A} = \mathbb{Z}^2$, Downarowicz and Kwiatkowski [64] gave a counterexample to the WCT. But see also [136].

Even though it looks as if rank one construction is not complicated, mixing in this class is possible; historically the first mixing constructions were given by D. Ornstein [214] in 1970, using probability type arguments for a choice of spacers. Once mixing was shown, the question arose whether absolutely continuous spectrum is also possible, as this would give automatically the positive answer to the Banach problem. However Bourgain [36], relating spectral measures of rank one automorphisms with some classical constructions of Riesz product measures, proved that a certain subclass of Ornstein's class consists of automorphisms with singular spectrum (see also [7] and [11]). Since in Ornstein's class spacers are chosen in a certain "non-constructive" way, quite a lot of attention was devoted to the rank one automorphism defined by cutting a tower at the n -th step into $r_n = n$ subcolumns of equal "width" and placing i spacers over the i -th subcolumn. The mixing property conjectured by M. Smorodinsky, was proved by Adams [12] (in fact Adams proved a general result on mixing of a class of staircase transformations). Spectral properties of rank-1 transformations are also studied in [169], where the authors proved that whenever $\sum_{n=1}^{\infty} r_n^{-2} = +\infty$ then the spectrum is automatically singular, see also more recent [47]. H. Abdalaoui [7] gives a criterion for singularity of the spectrum of a rank one transformation; his proof uses a central limit theorem. It seems that still the question whether rank one implies singularity of the spectrum remains the most important question of this theory.

We have already seen in Section 4 that for a special class of rank one systems, namely those with discrete spectra ([137]), we have a nice theory for weighted operators. It would be extremely interesting to find a rank one automorphism with continuous spectrum for which a substitute of Helson's analysis exists.

B. Fayad [70] constructs a rank one differentiable flow, as a special flow over a two-dimensional rotation. In [71] he gives new constructions of smooth flows with singular spectra which are mixing (with a new criterion for a Rajchman measure to be singular). In [66] a certain smooth change of time for an irrational flows on the 3-torus is given, so that the corresponding flow is partially mixing and has the local rank one property.

Motivated by Sarnak's conjecture on Möbius disjointness, see [178], a certain recent activity was to study spectral disjointness of powers for rank one

automorphisms. Let σ be a probability measure on the additive circle $[0, 1)$. Given a real number $a > 0$, we denote by σ^a the image of σ under the map $x \mapsto ax \bmod 1$. If $r \geq 1$ is an integer, then by σ_r , we will denote the measure which is obtained first by taking the image of σ under the map $x \mapsto \frac{1}{r}x$, i.e. the measure $\sigma^{1/r}$, and then repeating this new measure periodically in intervals $[\frac{j}{r}, \frac{j+1}{r})$. The following holds:³

$$(2) \quad \text{if } (r, s) = 1 \text{ then } \sigma^r \perp \eta^s \text{ if and only if } \sigma_s \perp \eta_r.$$

In [37], Bourgain used Riesz product technique to show that for the class of so called rank one automorphisms with bounded parameters (both (r_n) and $(s_{n,j})$ are bounded and no spacer over the last column) we have $\sigma_r \perp \sigma_s$ for $r \neq s$ prime. In view of (2) it follows that different prime powers are spectrally disjoint. In [10], a much larger class of rank one automorphisms is considered. No boundedness assumption on (r_n) is made but a certain bounded recurrence is required on the sequence of spacers. Spectral disjointness of different powers (for the continuous part of the maximal spectral type) is derived from the existence, in the weak closure of powers, of sufficiently many analytic functions of the Koopman operator U_T .

For a spectral disjointness of the continuous part of the maximal spectral type for powers of automorphisms like the substitutional system given by the Thue-Morse sequence and related (rank two systems), see [9]. Weak closure of powers for Chacon automorphism is described in [134].

8 Spectral theory of dynamical systems of probabilistic origin

Let us just recall that when $(Y_n)_{n=-\infty}^{\infty}$ is a stationary process then its distribution μ on $\mathbb{R}^{\mathbb{Z}}$ is invariant under the shift S on $\mathbb{R}^{\mathbb{Z}}$: $S((x_n)_{n \in \mathbb{Z}}) = (y_n)_{n \in \mathbb{Z}}$, where $y_n = x_{n+1}$, $n \in \mathbb{Z}$. In this way we obtain an automorphism S defined on $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \mu)$. For each automorphism T we can find $f : X \rightarrow \mathbb{R}$ measurable such that the smallest σ -algebra making the stationary process $(f \circ T^n)_{n \in \mathbb{Z}}$ measurable is equal to \mathcal{B} , therefore, for the purpose of this article, by a system of probabilistic origin we will mean (S, μ) obtained from a stationary infinitely divisible process (see e.g. [205], [267]). In particular, the theory of Gaussian dynamical systems is indeed a classical part of ergodic theory (e.g. [212], [213], [296], [297]). If $(X_n)_{n \in \mathbb{Z}}$ is a stationary real centered Gaussian process and σ denotes the *spectral measure of the process*, i.e. $\widehat{\sigma}(n) = E(X_n \cdot X_0)$, $n \in \mathbb{Z}$, then by $S = S_\sigma$ we denote the corresponding Gaussian system on the shift space (recall also that for each symmetric measure σ on \mathbb{T} there is exactly one stationary real centered Gaussian process whose spectral measure is σ). Notice that if σ has an atom, then in the cyclic space generated by X_0 there exists an eigenfunction Y for S_σ – if now S_σ were ergodic, $|Y|$ would be a constant function which is not

³e.g. in the unpublished notes by H. El Abdalaoui, J. Kułaga-Przymus, M. Lemańczyk and T. de la Rue.

possible by the nature of elements in $\mathbb{Z}(X_0)$. In what follows we assume that σ is continuous.

It follows that U_{S_σ} restricted to $\mathbb{Z}(X_0)$ is spectrally the same as $V = V^\sigma$ acting on $L^2(\mathbb{T}, \sigma)$, and we obtain that $(U_{S_\sigma}, L^2(\mathbb{R}^\mathbb{Z}, \mu_\sigma))$ can be represented as the symmetric Fock space built over $H = L^2(\mathbb{T}, \sigma)$ and $U_{S_\sigma} = F(V)$ – see the glossary ($H^{\odot n}$ is called the n -th chaos). In other words the spectral theory of Gaussian dynamical systems is reduced to the spectral theory of special tensor products unitary operators. Classical results (see [46]) which can be obtained from this point of view are for example the following:

- (A) ergodicity implies weak mixing,
- (B) the multiplicity function is either 1 or is unbounded,
- (C) the maximal spectral type of U_{S_σ} is equal to $\exp(\sigma)$, hence Gaussian systems enjoy the Kolmogorov group property.

However, we can also look at a Gaussian system in a different way, simply by noticing that the variables $e^{2\pi i f}$ (f is a real variable), where $f \in \mathbb{Z}(X_0)$ generate $L^2(\mathbb{R}^\mathbb{Z}, \mu_\sigma)$. Now, calculating the spectral measure of $e^{2\pi i f}$ is not difficult and we obtain easily (C). Moreover, integrals of type $\int e^{2\pi i f_0} e^{2\pi i f_1 \circ T^n} e^{2\pi i f_2 \circ T^{n+m}} d\mu_\sigma$ can also be calculated, whence in particular we easily obtain Leonov's theorem on the multiple mixing property of Gaussian systems [201].

One of the most beautiful parts of the theory of Gaussian systems concerns ergodic properties of S_σ when σ is concentrated on a thin Borel set. Recall that a closed subset $K \subset \mathbb{T}$ is said to be a *Kronecker set* if each $f \in C(K)$ is a uniform limit of characters (restricted to K). Each Kronecker set has no rational relations. Gaussian-Kronecker automorphisms are, by definition, those Gaussian systems for which the measure σ (always assumed to be continuous) is concentrated on $K \cup \overline{K}$, K a Kronecker set. The following theorem has been proved in [89] (see also [46]).

Theorem 9 (Foiş-Stratila Theorem) *If T is an ergodic automorphism and f is a real-valued element of L^2_0 such that the spectral measure σ_f is concentrated on $K \cup \overline{K}$, where K is a Kronecker set, then the process $(f \circ T^n)_{n \in \mathbb{Z}}$ is Gaussian.*

This theorem is indeed striking as it gives examples of weakly mixing automorphisms which are spectrally determined (like rotations). A relative version of the Foiş-Stratila Theorem has been proved in [188].

The Foiş-Stratila Theorem implies that whenever a spectral measure σ is Kronecker, it has no realization of the form σ_f with f bounded. We will see however in Section 13 that for some automorphisms T (having the SCS property) the maximal spectral type σ_T has the property that S_{σ_T} has a simple spectrum.

Gaussian-Kronecker automorphisms are examples of automorphisms with simple spectra. In fact, whenever σ is concentrated on a set without rational relations, then S_σ has a simple spectrum (see [46]). Examples of mixing automorphisms with simple spectra are known [212], however it is still unknown (Thouvenot's question) whether the Foiş-Stratila property may hold in the mixing class. F. Parreau [217] using independent Helson sets gave an example of mildly mixing Gaussian system with the Foiş-Stratila property.

Joining theory of a class of Gaussian system, called GAG, is developed in [195]. A Gaussian automorphism S_σ with the Gaussian space $H \subset L_0^2(\mathbb{R}^{\mathbb{Z}}, \mu_\sigma)$ is called a GAG if for each ergodic self-joining $\rho \in J_2^e(S_\sigma)$ and arbitrary $f, g \in H$ the variable

$$(\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}, \rho) \ni (x, y) \mapsto f(x) + g(y)$$

is Gaussian. For GAG systems one can describe the centralizer and factors, they turn out to be objects close to the probability structure of the system. One of the crucial observations in [195] was that all Gaussian systems with simple spectrum are GAG.

It is conjectured (J.P. Thouvenot) that in the class of zero entropy Gaussian systems the PID property holds true.

For the spectral theory of classical factors of a Gaussian system see [197]; also spectrally they share basic spectral properties of Gaussian systems. Recall that historically one of the classical factors namely the σ -algebra of sets invariant for the map

$$(\dots, x_{-1}, x_0, x_1, \dots) \mapsto (\dots, -x_{-1}, -x_0, -x_1, \dots)$$

was the first example with zero entropy and countable Lebesgue spectrum (indeed, we need a singular measure σ such that $\sigma * \sigma$ is equivalent to Lebesgue measure [213]). For factors obtained as functions of a stationary process see [129].

T. de la Rue [250] proved that Gaussian systems are never of local rank-1, however his argument does not apply to classical factors. We conjecture that Gaussian systems are disjoint from rank-1 automorphisms (or even from local rank-1 systems).

We now turn the attention to Poissonian systems (see [46] for more details). Assume that (X, \mathcal{B}, μ) is a standard Borel space, where μ is infinite. Without entering too much into details, the new configuration space \tilde{X} is taken as the set of all countable subsets $\{x_i : i \geq 1\}$ of X . Once a set $A \in \mathcal{B}$, of finite measure is given one can define a map $N_A : \tilde{X} \rightarrow \mathbb{N}(\cup\{\infty\})$ just counting the number of elements belonging to A . The measure-theoretic structure $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ is given so that the maps N_A become random variables with Poisson distribution of parameter $\mu(A)$ and such that whenever $A_1, \dots, A_k \subset X$ are of finite measure and are pairwise disjoint then the variables N_{A_1}, \dots, N_{A_k} are independent.

Assume now that T is an automorphism of (X, \mathcal{B}, μ) . It induces a natural automorphism on the space $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ defined by $\tilde{T}(\{x_i : i \geq 1\}) = \{Tx_i : i \geq 1\}$. The automorphism \tilde{T} is called the *Poisson suspension* of T (see [46]). Such a suspension is ergodic if and only if no set of positive and finite μ -measure is T -invariant. Moreover, the ergodicity of \tilde{T} implies weak mixing. In fact the spectral structure of $U_{\tilde{T}}$ is very similar to the Gaussian one: namely the first chaos equals $L^2(X, \mathcal{B}, \mu)$ (we emphasize that this is about the whole L^2 and not only L_0^2) on which $U_{\tilde{T}}$ acts as U_T and the $L^2(\tilde{X}, \tilde{\mu})$ together with the action of $U_{\tilde{T}}$ has the structure of the symmetric Fock space $F(L^2(X, \mathcal{B}, \mu))$ (see the glossary).

We refer to [38], [132], [237], [238] for ergodic properties of systems given by symmetric α -stable stationary processes, or more generally infinitely divisible processes. Again, they share spectral properties similar to the Gaussian case: ergodicity implies weak mixing, while mixing implies mixing of all orders.

In [240], E. Roy clarifies the dynamical “status” of such systems. He uses Poisson suspension automorphisms and the Maruyama representation of an infinitely divisible process mixed with basic properties of automorphisms preserving infinite measure (see [2]) to prove that as a dynamical system, a stationary infinitely divisible process (without the Gaussian part), is a factor of the Poisson suspension over the Lévy measure of this process. In [239] a theory of ID-joinings is developed (which should be viewed as an analog of the GAG theory in the Gaussian class). Parreau and Roy [218] study Poisson suspensions without non-trivial factors.

Many natural problems still remain open here, for example (assuming always zero entropy of the dynamical system under consideration): Are Poisson suspensions disjoint from Gaussian systems? In [135] there are examples of Poissonian systems which are disjoint from all Gaussian systems. What is the spectral structure for dynamical systems generated by symmetric α -stable processes? Are such systems disjoint whenever $\alpha_1 \neq \alpha_2$? Are Poissonian systems disjoint from local rank one automorphisms (cf. [250])? In [141] it is proved that Gaussian systems are disjoint from so called simple systems (see [292], [143] and [254]); we will come back to an extension of this result in Section 13. It seems that flows of probabilistic origin satisfy the Kolmogorov group property for the spectrum. One can therefore ask how different are systems satisfying the Kolmogorov group property from systems for which the convolutions of the maximal spectral type are pairwise disjoint (see also Section 13 and the SCS property).

We also mention here another problem which will be taken up in Section 11.2 – *Is it true that flows of probabilistic origin are disjoint from smooth flows on surfaces?*

Yet one more (joining) property seems to be characteristic in the class of systems of probabilistic origin, namely they satisfy so called ELF property (see [60] and de la Rue’s article [254]). Vershik asked whether the ELF property is spectral – however the example of a cocycle from [301] together with Theorem 7 (i) yields a certain Rokhlin extension of a rotation which is ELF and has countable Lebesgue spectrum in the orthocomplement of the eigenfunctions (see [302]); on the other hand any affine extension of that rotation is spectrally the same, while it cannot have the ELF property.

Prikhodko and Thouvenot (private communication) have constructed weakly mixing and non-mixing rank one automorphisms which enjoy the ELF property.

9 Inducing and spectral theory

Assume that T is an ergodic automorphism of a standard probability Borel space (X, \mathcal{B}, μ) . Can „all” dynamics be obtained by inducing (see the glossary)

from one fixed automorphism was a natural question from the very beginning of ergodic theory. Because of Abramov's formula for entropy $h(T_A) = h(T)/\mu(A)$ it is clear that positive entropy transformations cannot be obtained from inducing on a zero entropy automorphism. However here we are interested in spectral questions and thus we ask how many spectral types we obtain when we induce. It is proved in [101] that the family of $A \in \mathcal{B}$ for which T_A is mixing is dense for the (pseudo) metric $d(A_1, A_2) = \mu(A_1 \Delta A_2)$. De la Rue [251] proves the following result: *For each ergodic transformation T of a standard probability space (X, \mathcal{B}, μ) the set of $A \in \mathcal{B}$ for which the maximal spectral type of U_{T_A} is Lebesgue is dense in \mathcal{B} .* The multiplicity function is not determined in that paper. Recall (without giving a formal definition, see [215]) that a zero entropy automorphism is *loosely Bernoulli* (LB for short) if and only if it can be induced from an irrational rotation (see also [79], [153]). The LB theory shows that not all dynamical systems can be obtained by inducing from an ergodic rotation. However an open question remained whether LB systems exhaust spectrally all Koopman representations. An interesting question of M. Ratner [225] is whether from every ergodic automorphism T one can induce an automorphism which has countable Lebesgue spectrum (Ratner in [225] shows that this can be done if T is an irrational rotation).

In a deep paper [249], de la Rue studies LB property in the class of Gaussian-Kronecker automorphisms, in particular he constructs S which is not LB. Suppose now that T is LB and for some $A \in \mathcal{B}$, U_{T_A} is isomorphic to U_S . Then by the Foias-Stratila Theorem, T_A is isomorphic to S , and hence T_A is not LB. However, an induced automorphism from an LB automorphism is LB, a contradiction.

Another fruitful source of non LB systems comes from taking Cartesian products of some natural LB systems. In [215], it is proved that there exists a rank one (and hence LB) system whose Cartesian square is not LB. Moreover, in [226], it was shown that the square of the horocycle flow is not LB (the horocycle flow itself being LB, [225]). Recently, [151], the authors showed that there are staircase rank one transformations whose Cartesian product is not LB.

10 Rigid sequences

Recall (see the glossary) that an automorphism T of a standard probability Borel space (X, \mathcal{B}, μ) is called *rigid* if there exists a strictly increasing sequence $q_n \rightarrow \infty$ such that $\mu(T^{q_n} A \Delta A) \rightarrow 0$ as $n \rightarrow \infty$, for each $A \in \mathcal{B}$.⁴ Equivalently, for each $f \in L^2(X, \mathcal{B}, \mu)$, $U_T^{q_n} f \rightarrow f$ in $L^2(X, \mathcal{B}, \mu)$ (it is not hard to see that the latter is equivalent that $\int_0^1 e^{2\pi i q_n x} d\sigma_f(x) \rightarrow 1$ for any σ_f representing the maximal spectral type of T , $\|f\| = 1$). We call (q_n) a *rigidity sequence* of T . Rigidity is one of (purely spectral) fundamental phenomena in ergodic theory. Assuming that T is aperiodic, it is not hard to see that for any rigidity sequence (q_n) , we must have $q_{n+1} - q_n \rightarrow \infty$. Typical automorphism is rigid and weakly

⁴In fact, to have a global rigidity sequence, as observed by Thouvenot, we only need to know that for each $A \in \mathcal{B}$ there is a sequence $(q_{n,A})$ so that $\mu(T^{q_{n,A}} A \Delta A) \rightarrow 0$.

mixing but since weak mixing implies $U_T^n \rightarrow 0$ weakly on $L_0^2(X, \mathcal{B}, \mu)$ along a sequence of n of full density, there is no much “room” left for rigidity sequences. So positive density sequences cannot be rigid but beyond that, in the class of zero density sequences there can be other, for example algebraic in nature, obstructions for rigidity. For example, as noticed in [34], [65], if $P \in \mathbb{Q}[x]$ is any non-zero polynomial taking integer values on \mathbb{Z} then the sequence $(P(n))$ cannot be rigid for any ergodic automorphism. It is also easy to see that (2^n) is a rigidity sequence while $(2^n + 1)$ is not.

A systematic study of sequences which can be rigidity sequences was originated in [34] and [65]. Both papers use harmonic analysis approach to construct rigid sequences (via the standard Gaussian functor). Other constructions are also presented in [34]: rank one constructions, weighted operators, Poisson suspensions, while [65] rather concentrates on so called linear dynamical systems and studies rigidity for weakly mixing automorphisms. One of the results in [34] and [65] states that if either $q_{n+1}/q_n \rightarrow \infty$ or q_{n+1}/q_n is an integer then (q_n) is a rigidity sequence (for a weakly mixing automorphism). On the other hand, Eisner and Grivaux in [65] give an example of a rigid sequence (q_n) , for a weakly mixing automorphism, such that $q_{n+1}/q_n \rightarrow 1$. As a matter of fact, both [34] and [65] deal with the case of denominators (q_n) of an irrational $\alpha \in [0, 1)$ (which are obviously rigidity sequences for the corresponding irrational rotations $Tx = x + \alpha$ on the additive circle) to show that such sequences are rigid for some weakly mixing automorphisms. Let us also mention Aaronson’s result [1]: *Given any sequence (r_n) of density 0, there is a sequence (q_n) such that $q_n < r_n$, $n \geq 1$, and (q_n) is rigid for some weakly mixing automorphisms.*

Moreover, in [34] two basic questions have been formulated: *Given any sequence rigid for some T with discrete spectrum, must it be rigid for some weakly mixing automorphism? What about the converse?*

The positive answer to the first question was given by Adams [13] and Fayad and Thouvenot [77]. On the other hand, surprisingly, Fayad and Kanigowski [73] answered negatively the second question: there are rigidity sequences (for weakly mixing automorphisms) which are not rigidity sequences for any rotation. For a strengthening of this result (the existence of a rigidity sequence which, as a subset, is dense in the Bohr topology on \mathbb{Z}), see [112].

One can also consider a notion stronger than rigidity, called IP-rigidity (see e.g. [34]): (q_n) is an IP-rigidity sequence for an automorphism T acting on (X, \mathcal{B}, μ) if $T^\xi \rightarrow Id$ (in the strong topology of $L^2(X, \mathcal{B}, \mu)$) in the IP sense, that is, when $\xi \rightarrow \infty$, where $\xi = q_{m_1} + \dots + q_{m_k}$ and we require that the smallest element q_{m_1} is going to ∞ . This notion is studied in [3] relating it to non-singular ergodic theory (more precisely, to groups of so called L^∞ -eigenvalues of non-singular automorphisms). As proved in [3], in this category, the answer to the second question (above) from [34] turns out to be positive. Moreover, the paper provides an example of a super lacunary sequence (which must be a rigidity sequence by [34] and [65]) which is not an IP-rigid.

In the recent preprint [29], rigidity sequences are compared to other classical notions in harmonic analysis. It is proved that rigidity sequences (q_n) are nullpotent, i.e. there exists a topology τ on \mathbb{Z} making it a topological group

such that $q_n \rightarrow 0$ but they are never Kazhdan.⁵ We find also there a rather surprising result that the family of all rigidity sequences considered as a subset in $\mathbb{Z}^{\mathbb{N}}$ is Borel.

11 Spectral theory of parabolic dynamical systems

We say a system is *algebraic* if it is a \mathbb{Z} (or \mathbb{R}) translation on a quotient of a Lie group by a lattice. Spectral theory (and mixing properties) of *algebraic systems* is by now well understood. The two main classes are actions on quotients of semi-simple and nilpotent Lie groups. In the first case, the two main examples are horocycle and geodesic flows on quotients of $SL(2, \mathbb{R})$. More generally, one can talk about quasi-unipotent and partially hyperbolic actions. Recall that in the setting of algebraic actions, being quasi-unipotent is equivalent to *zero entropy*, while being partially hyperbolic is equivalent to *positive entropy*.

It is known that in both cases the spectrum is countable Lebesgue (we refer the reader to [159] for a nice description of spectral theory of horocycle and geodesic flows). Actions on quotients of nilpotent Lie groups are also known to have countable Lebesgue spectrum in the orthocomplement of the eigenspace (we refer to [220] for details). Quantitative mixing (and higher order mixing) of algebraic systems is also well understood (we refer the reader to a recent paper, [35], for general results on decay of correlations for algebraic systems on semi-simple Lie groups).

Much less is known in spectral theory of parabolic systems beyond algebraic world. There is no strict definition for a system to be parabolic. However, characteristic features of parabolic systems are: zero-entropy, polynomial orbit growth, strong mixing and equidistribution properties. We describe some classes of (non-algebraic) parabolic systems below. One of the main difficulty in studying non-algebraic systems is a lack of many tools from representation theory, which is available in the algebraic setting. Below, we focus on known results and questions in spectral theory of non-algebraic parabolic systems.

11.1 Time-changes of algebraic systems

Perhaps the simplest class of non-algebraic parabolic dynamical systems is given by time-changes (or reparametrizations) of algebraic systems. As for algebraic systems, it is natural to consider separately the cases of time-changes of unipotent systems and nilpotent systems. We do it in two paragraphs below.

Time-changes of unipotent systems In recent years we witnessed substantial development in understanding the theory of time-changes of unipotent

⁵A subset $B \subset \mathbb{Z}$ is called Kazhdan if there exists $\varepsilon > 0$ such that each unitary operator U on a separable Hilbert space H having a unit vector x with $\sup_{n \in B} \|U^n x - x\| < \varepsilon$ has a non-zero fixed point.

flows. The first (and most studied) case is that of smooth time changes of horocycle flows. Recall first that M. Ratner, [228], established measures and joinings rigidity phenomena for time-changes of horocycle flows that are analogous to the Ratner’s theory in algebraic setting. In particular in [229] Ratner proved the H -property for all (sufficiently smooth time-changes). It is not known if Ratner’s joinings and measure rigidity also holds for time-changes of general unipotent flows.

Mixing for smooth time-changes of horocycle flows was established by Marcus [204], generalizing earlier work of Kushnirenko [180] who required additionally small derivative in the geodesic direction. A crucial result for the theory is by L. Flaminio and G. Forni [87], where the authors classify all invariant distributions and as a consequence show that a typical time-change is not a (measurable) quasi-coboundary (and hence the time-change is not trivially isomorphic to the original flow). A. Katok and J.-P. Thouvenot conjectured, [159], that every sufficiently smooth time change of the horocycle flow has countable Lebesgue spectrum. A partial answer to this conjecture was given by G. Forni and C. Ulcigrai, [92], where the authors show that the maximal spectral type of the time-changed flow remains Lebesgue (see also a result of R. Tiedra de Aldecoa, [279], where absolute continuity of the spectrum is proven, and [282], [283] for further applications of the commutator method in ergodic theory). A full solution of the Katok-Thouvenot conjecture (i.e. countable Lebesgue spectrum) was recently given by G. Forni, B. Fayad and A. Kanigowski, [72]. Generalizing the approach from [92], L. Simonelli, [271], showed that the spectrum of smooth time-changes of general unipotent flows remains Lebesgue. It is not known if the multiplicity is infinite, but it seems that the approach from [72] has the potential of being applicable in this setting.

Recall that Ratner’s work [227] allows one to classify joinings between horocycle flows. Recently, there was progress in understanding joinings for time-changes of horocycle flows. In [150] (see also a result in [88]) the authors show that there is a strong dichotomy for two smooth time-changes of horocycle flows: either the time-change functions are cohomologous or the resulting time-changed flows are disjoint.

Even though quantitative mixing for time-changes of unipotent flows is now well understood, not much is known for quantitative higher order correlations:

Question 1 *Is the decay of higher correlations for non-trivial time-changes of horocycle flows (or more generally, unipotent flows) polynomial?*

For trivial time-changes, i.e. for the horocycle flow, decay of higher correlations is indeed polynomial by a recent result of M. Björklund, M. Einsiedler and A. Gorodnik, [35] (in fact this applies to general unipotent flows).

Time-changes of nilpotent systems Recall that nilpotent flows are never weakly mixing since they always have a non-trivial Kronecker factor. An interesting question is therefore whether one can improve mixing properties of the system by a time-change. The first result in this direction by A. Avila, G. Forni

and C. Ulcigrai [27] is that there exists a dense set of smooth functions on the *Heisenberg nilmanifold* such that the resulting time-changed Heisenberg flow is *mixing*. This result was strengthened by D. Ravotti in [231] to quasi-abelian nilflows, and recently to all nilflows by Avila, Forni, Ravotti and Ulcigrai [28]. It is important to mention that the mixing mechanism is non-quantitative. Therefore two questions are natural to ask: what are mixing properties of general time-changes of nilflows and can one obtain some quantitative mixing results. The only case in which some progress has been recently made is that of time-changes of Heisenberg nilflows. In [90] the authors show stretched polynomial decay of correlations for smooth time-changes of full measure set of Heisenberg nilflows (parametrized by the frequency of the Kronecker factor). In the case the flow is of *bounded type* the authors prove polynomial speed of decay of correlations. Moreover, in [91] the authors show that for time-changes of bounded type Heisenberg nilflows, every non-trivial time-change enjoys the *R*-property and as a consequence is *mildly mixing*. Moreover, in the above setting, it also follows that every mixing time-change is mixing of all orders.

Mixing and spectral properties of time-changes of general nilflows are poorly understood. In particular, the following questions seems interesting:

Question 2 *Are all non-trivial smooth time-changes of general nilflows mixing?*

The mixing mechanism from [27] (see also [231] and [28]) is non-quantitative. Therefore the following question seems to be far more challenging:

Question 3 *Does there exist a smooth time change of a general nilflow with AC (Lebesgue) spectrum?*

11.2 Special flows, flows on surfaces, interval exchange transformations

In this section we will describe spectral results for special flows over interval exchange transformations (IETs) (irrational rotations begin a particular case). As described below, such special flows arise as representation of smooth locally hamiltonian flows on surfaces.

11.2.1 Interval exchange transformations

To define an interval exchange transformation (IET) of m intervals we need a permutation π of $\{1, \dots, m\}$ and a probability vector $\lambda = (\lambda_1, \dots, \lambda_m)$ (with positive entries). Then we define $T = T_{\lambda, \pi}$ of $[0, 1)$ by putting

$$T_{\lambda, \pi}(x) = x + \beta_i^\pi - \beta_i \text{ for } x \in [\beta_i, \beta_{i+1}),$$

where $\beta_i = \sum_{j < i} \lambda_j$, $\beta_i^\pi = \sum_{\pi(j) < \pi(i)} \lambda_j$. Obviously, each IET preserves Lebesgue measure. One of possible approaches to study ergodic properties of IET is an “a.e” approach “seen” in the definition of $T_{\lambda, \pi}$. It is based on the fundamental fact that the induced transformation on a subinterval of $[0, 1)$ is also IET (see

[46]). This leads to a very delicate and deep mathematics based on Rauzy induction, which is a way of inducing on special intervals, considering only irreducible permutations whose set is partitioned into orbits of some maps (any such an orbit is called a *Rauzy class*). If now \mathcal{R} is a Rauzy class of permutations and λ lies in the standard simplex Δ_{m-1} then the Rauzy induction together with a natural renormalization leads to a map $\mathcal{P} : \mathcal{R} \times \Delta_{m-1} \rightarrow \mathcal{R} \times \Delta_{m-1}$. For a better understanding of the dynamics of the Rauzy map Veech [293] introduced the space of *zippered rectangles*. A zippered rectangle associated to the Rauzy class \mathcal{R} is a quadruple (λ, h, a, π) , where $\lambda \in \mathbb{R}_+^m$, $h \in \mathbb{R}_+^m$, $a \in \mathbb{R}_+^m$, $\pi \in \mathcal{R}$ and the vectors h and a satisfy some equations and inequalities. Every zippered rectangle (λ, h, a, π) determines a Riemann structure on a compact connected surface. Denote by $\Omega(\mathcal{R})$ the space of all zippered rectangles, corresponding to a given Rauzy class \mathcal{R} and satisfying the condition $\langle \lambda, h \rangle = 1$. In [293], Veech defined a flow $(P^t)_{t \in \mathbb{R}}$ on the space $\Omega(\mathcal{R})$ putting

$$P^t(\lambda, h, a, \pi) = (e^t \lambda, e^{-t} h, e^{-t} a, \pi)$$

and extended the Rauzy map. On so called *Veech moduli space* of zippered rectangles, the flow (P^t) becomes the *Teichmüller flow* and it preserves a natural Lebesgue measure class; by passing to a transversal and projecting the measure on the space of IETs $\mathcal{R} \times \Delta_{m-1}$ Veech has proved the following fundamental theorem ([293], see also [206]) which is a generalization of the fact that Gauss measure $\frac{1}{\ln 2} \frac{1}{1+x} dx$ is invariant for the Gauss map which sends $t \in (0, 1)$ into the fractional part of its inverse.

Theorem 10 (Veech, Masur, 1982) *Assume that \mathcal{R} is a Rauzy class. There exists a σ -finite measure $\mu_{\mathcal{R}}$ on $\mathcal{R} \times \Delta_{m-1}$ which is \mathcal{P} -invariant, equivalent to “Lebesgue” measure, conservative and ergodic.*

In [293] it is proved that a.e. (in the above sense) IET is then of rank one (and hence is ergodic and has a simple spectrum). He also formulated as an open problem whether we can achieve the weak mixing property a.e. This has been recently answered in positive by A. Avila and G. Forni [26] (for π which is not a rotation).

Katok [154] proved that IET have no mixing factors (in fact his proof shows more: the IET class is disjoint from the class of mixing transformations). By their nature, IET transformations are of finite rank (see [46]) so they are of finite multiplicity. They need not be of simple spectrum (see remarks in [159] pp. 88-90). It remains an open question whether an IET can have a non-singular spectrum. Joining properties in the class of exchange of 3 and more intervals are studied in [83], [84]. In [41] the authors show that a.e. 3-IET is not simple. This answers a special case of a question of Veech [292] whether a.e. IET is simple. The case of d -IET’s with $d > 3$ is still widely open.

11.2.2 Smooth flows on surfaces and their special representations

We consider a closed, connected, smooth and orientable surface S of genus $g \geq 1$. Let $X : S \mapsto TS$ be a smooth vector field with finitely many fixed points and

such that the corresponding flow (ϕ_t^X) preserves a smooth area form ω . The flow (ϕ_t^X) is called a *locally Hamiltonian flow*; it is locally given by a smooth hamiltonian H (up to an additive constant), so that (ϕ_t^X) is a solution to

$$(3) \quad \frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = \frac{-\partial H}{\partial x}.$$

There has recently been some progress in understanding ergodic and spectral properties of locally Hamiltonian flows. It follows by [25] that (ϕ_t^X) can be decomposed into finite number of topological discs filled with periodic orbits and finite number of connected components on which the flow is minimal. We are always interested in properties of (ϕ_t^X) on the minimal components. A classical way of studying locally hamiltonian flows is through their *special representations*. On each minimal component one chooses a smooth transversal and represents (ϕ_t^X) by the first return map (the base transformation T) and the first return time (the roof function f). We will denote the special representation of (ϕ_t^X) by T^f . It is well known that T is either a rotation (in genus 1 case) or an interval exchange transformation (in higher genus) and the roof function is smooth except finitely many points at which it explodes according to the nature of the fixed points. Notice first that if (ϕ_t^X) has no fixed points then S is a two-dimensional torus (by Gauss-Bonet). In this case (ϕ_t^X) is a smooth time-change of a linear flow on the two-torus. Therefore it is never mixing [170], disjoint from mixing flows [95] and in particular has purely singular spectrum. From now on we will therefore assume that the set of fixed points is non-empty. If all fixed points are non-degenerated (i.e. hessian of H is non-zero at every critical point) the roof function f has *logarithmic singularities*, i.e. it blows up logarithmically at finitely many points. Logarithmic singularities can be either *symmetric* (for instance, if there are no saddle connections) or *asymmetric* (in case there is a saddle loop). Ergodic properties of (ϕ_t^X) depend strongly on whether the roof has symmetric or asymmetric singularities. If there are *degenerated* fixed points, Kochergin, [171], constructed smooth hamiltonian for which the roof function at singularities blows up power-like (like x^γ , $\gamma \in (-1, 0)$). By changing a speed as it is done in [98] so that critical points of the vector field in (3) become singular points, Arnold's special representation is transformed to a special flow over the same irrational rotation however under a piecewise smooth function. If the sum of jumps is not zero then in fact we come back to von Neumann's class of special flows considered in [211]. Similar classes of special flows (when the roof function is of bounded variation) are obtained from ergodic components of flows associated to billiards in convex polygons with rational angles [160].

It is convenient to express ergodic and spectral properties of locally hamiltonian flows in terms of their special representation. We will do it in the next subsection.

11.3 Special flows over rotations and interval exchange transformations

Special flows were introduced to ergodic theory by von Neumann in his fundamental work [211] in 1932. Also in that work he explains how to compute eigenvalues for special flows, namely: $r \in \mathbb{R}$ is an eigenvalue of T^f if and only if the following functional equation

$$e^{2\pi i r f(x)} = \xi(x)/\xi(Tx)$$

has a measurable solution $\xi : X \rightarrow \mathbb{T}$. We recall also that the classical Ambrose-Kakutani theorem asserts that practically each ergodic flow has a special representation ([46], see also Rudolph’s theorem on special representation therein). As described in the previous subsection, we will consider the case when the roof functions is: of bounded variation, has logarithmic singularities (symmetric or asymmetric) or has power singularities.

Bounded variation roof function. Kochergin [170] showed that special flows over irrational rotations and under bounded variation functions are never mixing. This has been recently strengthened in [95] to the following: *If T is an irrational rotation and f is of bounded variation then the special flow T^f is spectrally disjoint from all mixing flows.* In particular all such flows have singular spectra. Moreover, in [95] it is proved that whenever the Fourier transform of the roof function f is of order $O(\frac{1}{n})$ then T^f is disjoint from all mixing flows (see also [96]). In fact in the papers [95] – [98] the authors discuss the problem of disjointness of those special flows with all ELF-flows conjecturing that no flow of probabilistic origin has a smooth realization on a surface. In [200], the analytic case is considered leading to a “generic” result on disjointness with the ELF class generalizing the classical Shklover’s result on the weak mixing property [273]. A. Katok [154] proved the absence of mixing for special flows over IET when the roof function is of bounded variation (see also [257]). Katok’s theorem was strengthened in [97] to the disjointness theorem with the class of mixing flows. A. Avila and G. Forni [26] proved that a.e. translation flow on a surface (of genus at least two) is weakly mixing (which is a drastic difference with the linear flow case of the torus, where the spectrum is always discrete).

One important property in the class of special flows over rotations and IET’s is Ratner’s property (R-property). This property may be viewed as a particular way of divergence of orbits of close points; it was shown to hold for horocycle flows by M. Ratner [227]. We refer the reader to [227] and the survey article [277] for the formal definitions and basic consequences of R-property. In particular, R-property implies “rigidity” of joinings and it also implies the PID property; hence mixing and R-property imply mixing of all orders. In [98], [100] a version of R-property is shown for the class of von Neumann special flows (however α is assumed to have bounded partial quotients). This allowed one to prove there that such flows are even mildly mixing (mixing is excluded by a Kochergin’s result). The eigenvalue problem (mainly how many frequencies can have the

group of eigenvalues) for special flows over irrational rotations is studied in [75], [78], [115].

It follows from [95] that von Neumann flows have singular spectrum. However nothing is known about their multiplicity.

Question 4 *What is the spectral multiplicity of von Neumann flows?*

Symmetric logarithmic singularities. Kochergin [172] proved the absence of mixing for flows where the roof function has finitely many singularities, however some Diophantine restriction is put on α . In [186], where also the absence of mixing is considered for the symmetric logarithmic case, it was conjectured (and proved for arbitrary rotation) that a necessary condition for mixing of a special flow T^f (with arbitrary T and f) is the condition that the sequence of distributions $((f_0^{(n)})_*)_n$ tends to δ_∞ in the space of probability measures on $\overline{\mathbb{R}}$. K. Schmidt [269] proved it using the theory of cocycles and extending a result from [5] on tightness of cocycles. Ulcigrai, [290], showed that for every $d \geq 2$ and a.e. IET of d intervals, the corresponding special flow T^f is *not* mixing. However, Chaika and Wright, [43], proved existence of an IET T such that T^f is mixing. Notice that by [95] it also follows that for a.e. irrational rotation T^f has purely singular spectral type. Recent result, [42], shows that one can also prove singularity of the spectrum for *symmetric IET's* in the base. This in particular shows that minimal flows on genus 2 surfaces (with two isometric saddles) have purely singular spectral type. In [149] the authors showed that if T is an IET of *bounded type* then T^f is mildly mixing (and has the R -property). For symmetric logarithmic singularities the following two questions are open:

Question 5 *What is the maximal spectral type of T^f for a general IET's?*

Nothing is known about multiplicity of the spectrum in this setting.

Asymmetric logarithmic singularities. In this case mixing properties are different. Khanin and Sinai, [163], showed that T^f is mixing for a.e. irrational rotation T . This was strengthened by Ulcigrai to a.e. IET, [289]. Moreover, [232], Ravotti obtained quantitative mixing estimates (with sub-logarithmic speed of decay of correlations). Not much was known about multiple mixing for T^f . This changed recently: in [74] the authors showed that for a.e. irrational rotation, T^f enjoys a variant of the R -property and hence is multiple mixing. This was strengthened to a.e. IET's in [148]. The following two questions seem to be natural (the first one already stated as Questions 34, 35 in [76]):

Question 6 *What is the maximal spectral type of T^f ?*

Moreover, one can ask about quantitative higher order decay:

Question 7 *Is the decay of higher order correlations sub-logarithmic?*

Both of the above question are open even for rotations in the base. As in the symmetric case, nothing is known about multiplicity of the spectrum.

Power singularities. In case f has power singularities, T^f was shown to be mixing by Kochergin, [171] (for any uniquely ergodic irrational rotation and IET's). In [74] the authors show that if T is a rotation of bounded type, then T^f is multiple mixing (it enjoys a variant of the R -property). Polynomial decay for some flows with power singularities was obtained by B. Fayad, [69]. In [72] the authors considered the spectrum of T^f . They showed that if f has sufficiently strong power singularity (of the form $x^{-1+\eta}$ for small $\eta > 0$) then T^f has countable Lebesgue spectrum for a.e. irrational rotation. To the best of our knowledge, this is the only result dealing with multiplicity for smooth surface flows. The following problems are natural (see Question 34 in [76]):

Question 8 *What is the maximal spectral type of T^f when f has power singularities? What is the multiplicity?*

To answer this one needs to consider general IET's in the base, as well as functions with weaker power singularity than in [72]. The following question is still open (Question 38 in [76]):

Question 9 *Are all mixing surface flows mixing of all orders?*

Finally, it may also be useful to show that smooth flows on surfaces are disjoint from flows of probabilistic origin – see [141], [142], [194], [265], [278].

B. Fayad [71] gives a criterion that implies singularity of the maximal spectral type for a dynamical system on a Riemannian manifold. As an application he gives a class of smooth mixing flows (with singular spectra) on \mathbb{T}^3 obtained from linear flows by a time change (again this is a drastic difference with dimension two, where a smooth time change of a linear flow leads to non-mixing flows [46]).

We mention at the end that if we drop here (and in other problems) the assumption of regularity of f then the answers will be always positive because of the LB theory; in particular, there is a section of any horocycle flow (it has the LB property [225]) such that in the corresponding special representation T^f , the map T is an irrational rotation. Using a Kochergin's result [173] on cohomology (see also [152], [245]) the L^1 -function f is cohomologous to a positive function g which is even continuous, thus T^f is isomorphic to T^g .

12 Spectral theory for locally compact groups of type I

This section has been written by A. Danilenko.

12.1 Groups of type I

The spectral theory presented here for Abelian group actions extends potentially to probability preserving actions of non-Abelian locally compact groups of type I . We now provide the definition of type I . Let G be a locally compact second countable group, \mathcal{H} a separable Hilbert space and $\pi : G \ni g \mapsto \pi(g)$ is a (weakly)

continuous unitary representation of G in \mathcal{H} . We say that π is of *type I* if there is a subset $A \subset \{1, 2, \dots, +\infty\}$ such that π is unitarily equivalent to the orthogonal sum $\bigoplus_{k \in A} U_k \otimes I_k$, where U_k is a unitary representation of G with a simple spectrum and I_k is the trivial representation in the Hilbert space of dimension k .

Definition 1 If every unitary representation of G is of type *I* then G is called of *type I*.

Denote by \widehat{G} the *unitary dual* of G , i.e. the set of unitarily equivalent classes of all irreducible unitary representations of G . If G is Abelian then every irreducible representation of G is 1-dimensional. Hence \widehat{G} is identified naturally with the group of characters of G . In the general case, let $\text{Irr}_n(G)$ stand for the set of all irreducible unitary representations of G in the n -dimensional separable Hilbert space \mathcal{K}_n , $1 \leq n \leq +\infty$. Endow it with the natural Borel structure, i.e. the smallest one in which the mapping $\pi \mapsto \langle \pi(g)f, h \rangle$ is Borel for every $g \in G$ and $f, h \in \mathcal{K}_n$. It is standard. Let \widehat{G}_n be the quotient of $\text{Irr}_n(G)$ by the unitary equivalence relation. Endow \widehat{G}_n with the quotient Borel structure. Since $\widehat{G} = \bigsqcup_{n=1}^{\infty} \widehat{G}_n \sqcup \widehat{G}_{\infty}$, we obtain a Borel structure on \widehat{G} . It is called *Mackey Borel structure* on \widehat{G} . By the Glimm theorem, the Mackey Borel structure is standard if and only if G is of type *I* [108].

For $n \in \mathbb{N} \cup \{\infty\}$, denote by I_n the identity operator on \mathcal{K}_n . Then for each unitary representation π of G in \mathcal{H} , there is a measure λ , a measurable field $\widetilde{G} \ni \omega \mapsto \mathcal{H}_{\omega}$ of Hilbert spaces, a measurable field $\widehat{G} \ni \omega \mapsto V_{\omega}$ of irreducible unitary G -representations such that $V_{\omega} \in \omega$ and \mathcal{H}_{ω} is the space of V_{ω} on \widehat{G} and a measurable map $m : \widehat{G} \rightarrow \mathbb{N} \cup \{+\infty\}$ such that

$$\mathcal{H} = \int_{\widehat{G}}^{\oplus} \mathcal{H}_{\omega} \otimes \mathcal{K}_{m(\omega)} d\lambda(\omega) \text{ and}$$

$$\pi(g) = \int_{\widehat{G}}^{\oplus} V_{\omega}(g) \otimes I_{m(\omega)} d\lambda(\omega).$$

It appears that if G is of type *I* then the equivalence class of λ is defined uniquely by π and the function m is defined up to a λ -zero subset. We call the class of λ the *maximal spectral type* of π , and we call m the *spectral multiplicity* of π . If we have a probability preserving action $T = (T_g)_{g \in G}$ of G then we can consider the corresponding Koopman unitary representation π of G . The maximal spectral type of π and the spectral multiplicity of π is called the *maximal spectral type of T* and the *spectral multiplicity of T* respectively. If G is Abelian then these concepts coincide with their classic counterparts considered above in the survey.

All compact groups, Abelian groups, connected semisimple Lie groups, nilpotent Lie groups (or, more generally, exponential Lie groups) are of type *I*. Each subgroup of $GL(n, \mathbb{R})$ determined by a system of algebraic equations is also of type *I*. Solvable Lie groups can be of type *I* or not of type *I*. If G is a countable (discrete) group then G is of type *I* if and only if it is virtually Abelian, i.e. it contains an Abelian subgroup of finite index [280].

Even if we know that a non-Abelian group G is of type I , it is usually not an easy problem to describe \widehat{G} explicitly. Kirillov introduced an *orbit method* for a description of \widehat{G} when G is a connected, simply connected nilpotent Lie group (the method was developed further for solvable groups). He identified \widehat{G} with the space of orbits for the co-adjoint G -action on the dual space \mathfrak{g}^* of its Lie algebra \mathfrak{g} . Though Kirillov's method gives an algorithm how to describe \widehat{G} , not so many groups are known for which the unitary dual is described explicitly.

12.2 Spectral properties of Heisenberg group actions

The 3-dimensional Heisenberg group $H_3(\mathbb{R})$ is perhaps one of the simplest examples of non-Abelian nilpotent Lie groups for which the orbit method leads to a very concrete description of the unitary dual [167]. Recall that

$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

For simplicity, we will denote the matrix $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ by $[a, b, c]$. The unitary

dual $\widehat{H_3(\mathbb{R})}$ is identified with $\mathbb{R}^2 \sqcup \mathbb{R}^*$ endowed with the natural standard Borel structure. Every irreducible unitary representation of $H_3(\mathbb{R})$ is unitarily equivalent to either a one-dimensional $\pi_{\alpha, \beta}$ with $(\alpha, \beta) \in \mathbb{R}^2$ or an infinite dimensional π_γ in $L^2(\mathbb{R}, \text{Leb})$, with $\gamma \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$, such that

$$\pi_{\alpha, \beta}[a, b, c] := e^{2\pi i(\alpha a + \beta b)},$$

$$\pi_\gamma[a, b, c]f(x) := e^{2\pi i(c + bx)}f(x + a), \quad f \in L^2(\mathbb{R}, \text{Leb}).$$

Now, given a measure preserving $H_3(\mathbb{R})$ -action on a standard probability space (X, μ) , let U denote the corresponding Koopman representation in $L^2(X, \mu)$. Then there are a probability measure $\sigma^{1,2}$ on \mathbb{R}^2 , a function $l^{1,2} : \mathbb{R}^2 \rightarrow \mathbb{N}$ a probability measure σ^3 on \mathbb{R}^* , a function $l^3 : \mathbb{R}^* \rightarrow \mathbb{N}$ such that

$$L^2(X, \mu) = \int_{\mathbb{R}^2}^{\oplus} \bigoplus_{j=1}^{l^{1,2}(\alpha, \beta)} \mathbb{C} d\sigma^{1,2}(\alpha, \beta) \oplus \int_{\mathbb{R}^*}^{\oplus} \bigoplus_{j=1}^{l^3(\alpha, \beta)} L^2(\mathbb{R}, \text{Leb}) d\sigma^3(\gamma) \text{ and}$$

$$U = \int_{\mathbb{R}^2}^{\oplus} \bigoplus_{j=1}^{l^{1,2}(\alpha, \beta)} \pi_{\alpha, \beta} d\sigma^{1,2}(\alpha, \beta) \oplus \int_{\mathbb{R}^*}^{\oplus} \bigoplus_{j=1}^{l^3(\alpha, \beta)} \pi_\gamma d\sigma^3(\gamma).$$

We now compare the spectral properties of T with the spectral properties of the restriction T to the center of $H_3(\mathbb{R})$. The center is the subgroup $\{[0, 0, t] : t \in \mathbb{R}\}$.

Proposition 1 ([52]) *The maximal spectral type of the flow $(T_{[0,0,t]})_{t \in \mathbb{R}}$ contains the measure $\sigma^{1,2}(\mathbb{R}^2)\delta_0 + \sigma^3$. The corresponding spectral multiplicity is $\int_{\mathbb{R}^2} l^{1,2} d\sigma^{1,2}$ at the point $t = 0$ and the infinity if $t \in \mathbb{R}^*$.*

Theorem 11 ([52]) *If $(T_{[0,0,t]})_{t \in \mathbb{R}}$ is ergodic then:*

1. $\sigma^{1,2}(\mathbb{R}^2 \setminus \{(0,0)\}) = 0$, i.e. there are no non-trivial one-dimensional representations in the spectral decomposition of U . The maximal spectral type of T equals the maximal spectral type of the restriction of T to the center of $H_3(\mathbb{R})$ (modulo the natural identification);
2. T is mixing (see also [258], [259]);
3. the weak closure of the group $\{T_g : g \in H_3(\mathbb{R})\}$ in $\text{Aut}(X, \mu)$ is the union of $\{T_g : g \in H_3(\mathbb{R})\}$ and the weak closure of $\{T_{[0,0,t]} : t \in \mathbb{R}\}$;
4. if T is rigid then $(T_{[0,0,t]})_{t \in \mathbb{R}}$ is rigid.

In [52], there were constructed explicit examples of mixing of all orders rank-one (and hence zero entropy) actions T of the Heisenberg group.

The concept of simplicity for ergodic $H_3(\mathbb{R})$ -actions is defined in a similar way as for the Abelian actions. A simple $H_3(\mathbb{R})$ -action T has MSJ if the centralizer of the action is $\{T_{[0,0,t]} : t \in \mathbb{R}\}$. It was shown in [52] that the examples of mixing $H_3(\mathbb{R})$ -actions constructed there satisfy also the following:

1. The flow $(T_{[0,0,t]})_{t \in \mathbb{R}}$ is simple and the centralizer of it equals the group $\{T_g : g \in H_3(\mathbb{R})\}$.
2. The transformation $T_{[0,0,1]}$ is simple and the centralizer of it equals $\{T_g : g \in H_3(\mathbb{R})\}$.
3. T has MSJ.

As a corollary, we obtain examples of mixing Poisson and mixing Gaussian (probability preserving) actions of $H_3(\mathbb{R})$.

12.3 Heisenberg odometers

In [55], the authors isolated a special class of ergodic $H_3(\mathbb{R})$ -actions, called *the odometer* actions. Namely, let $\Gamma_1 \supset \Gamma_2 \supset \dots$ be a sequence of lattices in $H_3(\mathbb{R})$. Then, we can associate a sequence of homogeneous $H_3(\mathbb{R})$ -spaces intertwined with $H_3(\mathbb{R})$ -equivariant maps:

$$H_3(\mathbb{R})/\Gamma_1 \longleftarrow H_3(\mathbb{R})/\Gamma_2 \longleftarrow \dots$$

Denote by X the projective limit of this sequence. Then X is a compact Polish G -space. Endow each space $H_3(\mathbb{R})/\Gamma_n$ with the Haar probability measure. The projective limit of the sequence of these measures is a $H_3(\mathbb{R})$ -invariant probability measure μ . Of course, the $H_3(\mathbb{R})$ -action on (X, μ) is ergodic. It is called *the Heisenberg odometer associated with $(\Gamma_n)_{n=1}^\infty$* . We consider the Heisenberg odometers as non-commutative counterparts of the ergodic \mathbb{Z} -actions and \mathbb{R} -actions with pure point rational spectrum.

A complete spectral decomposition of the Heisenberg odometers is found in [55]. Denote by $p : H_3(\mathbb{R}) \rightarrow \mathbb{R}^2$ the homomorphism $[a, b, c] \mapsto (a, b)$. The kernel

of this homomorphism is the center of the Heisenberg group. Given a lattice Γ in $H_3(\mathbb{R})$, we denote by ξ_Γ a positive real such that $\Gamma \cap \text{Ker } p = \{[0, 0, n\xi_\Gamma] \mid n \in \mathbb{Z}\}$. Of course, $p(\Gamma)$ is a lattice in \mathbb{R}^2 . If $p(\Gamma) = A(\mathbb{Z}^2)$ for some matrix $A \in GL(2, \mathbb{R})$ then we denote by $p(\Gamma)^*$ the dual lattice $(A^*)^{-1}\mathbb{Z}^2$ in \mathbb{R}^2 .

Theorem 12 *Let U stand for the Koopman unitary representation of the Heisenberg odometer associated with a sequence of lattices $\Gamma_1 \supset \Gamma_2 \supset \dots$. If $\bigcup_{n=1}^\infty p(\Gamma_n)^*$ is not closed in \mathbb{R}^2 then*

$$U = \bigoplus_{(\alpha, \beta) \in \bigcup_{n=1}^\infty p(\Gamma_n)^*} \pi_{\alpha, \beta} \oplus \bigoplus_{0 \neq \gamma \in \bigcup_{n=1}^\infty \xi_{\Gamma_n}^{-1}\mathbb{Z}} \bigoplus_1^\infty \pi_\gamma.$$

An analogous decomposition is found also for the case where $\bigcup_{n=1}^\infty p(\Gamma_n)^*$ is closed in \mathbb{Z}^2 (see [55] for details). Thus, we see that the maximal spectral type of Heisenberg odometers is purely atomic.

For a decreasing sequence $\Gamma = (\Gamma_n)_{n=1}^\infty$ of lattices in $H_3(\mathbb{R})$ we let $S(\Gamma) := \bigcup_{n=1}^\infty p(\Gamma_n)^*$ and $\xi_\Gamma := \bigcup_{n=1}^\infty \xi_{\Gamma_n}^{-1}\mathbb{Z}$. The following theorem (except for the first claim) and the below remarks demonstrate a drastic difference between $H_3(\mathbb{R})$ -odometers and \mathbb{Z} -odometers.

Theorem 13 *Two Heisenberg odometers T and T' associated with decreasing sequences of lattices Γ and Γ' respectively are unitarily equivalent if and only if $S(\Gamma) = S(\Gamma')$ and $\xi_\Gamma = \xi_{\Gamma'}$. The direct product $T \times T' := (T_g \times T'_g)_{g \in G}$ is not spectrally equivalent to any Heisenberg odometer. $T \times T'$ is ergodic if and only if $S(\Gamma) \cap S(\Gamma') = \{0\}$. $T \times T'$ is ergodic and has discrete maximal spectral type if and only if $S(\Gamma) \cap S(\Gamma') = \{0\}$ and $\xi_\Gamma \cap \xi_{\Gamma'} = \{0\}$.*

It was shown in [55] that Heisenberg odometers are not *isospectral*, i.e. the unitary equivalence, in general, does not imply isomorphism for the underlying $H_3(\mathbb{R})$ -actions. It was also shown in [55] that Heisenberg odometers are not *spectrally determined*: a Heisenberg odometer is constructed which is unitarily equivalent to an $H_3(\mathbb{R})$ -action which is not isomorphic to any Heisenberg odometer.

12.4 On the “finitely dimensional” part of the spectrum

Suppose that G is an arbitrary locally compact second countable group. If G is not of type I then \widehat{G} furnished with the Mackey Borel structure is a “bad” (not standard) Borel space. Then a decomposition of a unitary representation of G into irreducibles can be done in essentially non-unique way (see [167]). Nevertheless, this “badness” is related only to the infinite-dimensional part of the spectrum. Thus, the union $\bigsqcup_{n=1}^\infty \widehat{G}_n = \widehat{G} \setminus \widehat{G}_\infty$ is a “good” standard Borel space. Thus, given a measure preserving G -action, we can study “the finitely dimensional part” of the spectrum of the corresponding Koopman unitary representation of G .

This approach was used by Mackey in [203], where he made an attempt to extend the theory of action with pure discrete spectrum to non-Abelian groups.

For that he isolated a class of ergodic G -actions T for which the Koopman representation U_T decomposes into a (countable) family of finite dimensional irreducible representations. We note that the family is uniquely defined by T . Mackey called such T *an action with pure point spectrum*. He established a structure for these actions. He showed that T has a pure point spectrum if and only if T is isomorphic to a G -action by rotations on a homogeneous space of G by a compact subgroup. However, in general, in contrast with the Abelian case, the G -actions with pure point spectrum are not necessarily isospectral even in the case of finite G (see [288] and [196] Section 6 for counterexamples).

In [202], Lightwood, Şahin, and Ugarcovici considered certain odometer actions of the discrete Heisenberg group $H_3(\mathbb{Z}) := \{[a, b, c] : a, b, c \in \mathbb{Z}\}$. This group is not of type I (in contrast with $H_3(\mathbb{R})$) because each subgroup of finite index in $H_3(\mathbb{Z})$ is non-Abelian. Hence $\widehat{H_3(\mathbb{Z})}$ is a “bad” Borel space. On the other hand, the (standard Borel) subspace $\bigsqcup_{n=1}^{\infty} \widehat{H_3(\mathbb{Z})}_n$ of it is explicitly described in [124]. Consider now an $H_3(\mathbb{Z})$ -odometer T generated by a decreasing sequence $\Gamma_1 \supset \Gamma_2 \supset \dots$ of normal subgroups of finite index in $H_3(\mathbb{Z})$. We call such an odometer *normal*. Then T has a pure point spectrum in the sense of Mackey. Moreover, the full list of the finitely dimensional irreducible unitary representations that occur in U_T is found in [202] in terms of the sequence $(\Gamma_n)_{n=1}^{\infty}$. It was shown later in [55] that the multiplicity of each irreducible component in U_T equals the dimension of this component. It was also proved in [55] that the normal $H_3(\mathbb{Z})$ -odometers are isospectral. Thus, the unitary equivalence of the Koopman representations implies isomorphism of the underlying normal $H_3(\mathbb{Z})$ -odometers.

13 Future directions

We have already seen several cases where spectral properties interact with measure-theoretic properties of a system. Let us mention a few more cases which require further research and deeper understanding.

We recall that the weak mixing property can be understood as a property complementary to discrete spectrum (more precisely to the distality [103]), or similarly mild mixing property is complementary to rigidity. This can be phrased quite precisely by saying that T is not weakly (mildly) mixing if and only if it has a non-trivial factor with discrete spectrum (it has a non-trivial rigid factor). It has been a question for quite a long time if in a sense mixing can be “built” on the same principle. In other words we seek a certain “highly” non-mixing factor. It was quite surprising when in 2005 F. Parreau (private communication) gave the positive answer to this problem.

Theorem 14 (Parreau) *Assume that T is an ergodic automorphism of a standard probability space (X, \mathcal{B}, μ) . Assume moreover that T is not mixing. Then there exists a non-trivial factor (see below) of T which is disjoint from all mixing automorphisms.*

In fact, Parreau proved that each factor of T given by $\mathcal{B}_\infty(\rho)$ (this σ -algebra is described in [195]), where $U_T^{n_k} \rightarrow J_\rho$, is disjoint from all mixing transformations. This proof leads to some other results of the same type, for example: *Assume that T is an ergodic automorphism of a standard probability space. Assume that there exists a non-trivial automorphism S with a singular spectrum which is not disjoint from T . Then T has a non-trivial factor which is disjoint from any automorphism with a Lebesgue spectrum.*

The problem of spectral multiplicity of Cartesian products for “typical” transformation studied by Katok [152] and then its solution in [19] which we already considered in Section 5 lead to a study of those T for which

$$(CS) \quad \sigma^{(m)} \perp \sigma^{(n)} \quad \text{whenever } m \neq n,$$

where $\sigma = \sigma_T$ just stands for the reduced maximal spectral type of U_T (which is constantly assumed to be a continuous measure), see also Stepin’s article [276].

Usefulness of the above property (CS) in ergodic theory was already shown in [140], where a spectral counterexample machinery was presented using the following observation: *If \mathcal{A} is a $T^{\times\infty}$ -invariant sub- σ -algebra such that the maximal spectral type on $L^2(\mathcal{A})$ is absolutely continuous with respect to σ_T then \mathcal{A} is contained in one of the coordinate sub- σ -algebras \mathcal{B} .* Based on that in [140] it is shown how to construct two weakly isomorphic actions which are not isomorphic or how to construct two non-disjoint automorphisms which have no common non-trivial factors (such constructions were previously known for so called minimal self-joining automorphisms [243]). See also [284] for extensions of those results to \mathbb{Z}^d -actions.

Prikhodko and Ryzhikov [223] proved that the classical Chacon transformation enjoys the (CS) property. The SCS property defined in the glossary is stronger than the (CS) condition above; the SCS property implies that the corresponding Gaussian system S_{σ_T} has a simple spectrum. Ageev [16] shows that Chacon’s transformation satisfies the SCS property; moreover in [19] he shows that the SCS property is satisfied generically and he gives a construction of a rank one mixing SCS-system (see also [262]). In [193] it is proved that some special flows considered in Section 11.2 (including the von Neumann class, however with α having unbounded partial quotients) have the SCS property. It is quite plausible that the SCS property is commonly seen for smooth flows on surfaces.

A classical open problem is whether each ergodic automorphism has a smooth model. While this problem stays open even for so called dyadic adding machine (ergodic, discrete spectrum automorphism having roots of unity of degree 2^n , $n \geq 1$, as eigenvalues), A. Katok suggested many years ago one can construct a Kronecker measure so that the corresponding Gaussian system (\mathbb{Z} -action (!)) has a smooth representation on the torus. No “written” proof of this fact yet appeared.

In [295], A.M. Vershik sketches a proof of the fact (claimed by himself for decades) that Pascal adic transformation is weakly mixing. Earlier, X. Méla in his PhD showed that this transformation is ergodic and has zero entropy.

Moreover, in [133], Janvresse and de la Rue proved the LB property of this map. No complete proof of weak mixing of Pascal adic transformation seems to exist in the published literature.

In [298], Vershik proposed to study a new equivalence between measure-preserving automorphisms, called quasi-similarity. Two automorphisms T on (X, \mathcal{B}, μ) and S on (Y, \mathcal{C}, ν) are *quasi-similar* if there are Markov operators $J : L^2(X, \mathcal{B}, \mu) \rightarrow L^2(Y, \mathcal{C}, \nu)$ and $K : L^2(Y, \mathcal{C}, \nu) \rightarrow L^2(X, \mathcal{B}, \mu)$, both with dense ranges, intertwining the corresponding Koopman operators U_T and U_S . This new equivalence is strictly stronger than spectral isomorphism but strictly weaker than (weak) isomorphism as shown in [99] answering one of questions by Vershik. However, possible invariants for this new equivalence are not well understood. For example, in [99] it is shown that an automorphism quasi-similar to a K-automorphism must also be K, but an intriguing question remains whether Bernoulli property is an invariant of quasi-similarity. As weak isomorphism for finite multiplicity automorphisms is in fact isomorphism, another question is whether we can have two non-isomorphic quasi-similar automorphisms with simple spectrum (rank one)? (see Problem 4 in [99]). See also [116] for the problem of Markov quasi-factors in the class of Abramov automorphisms.

Not too many zero entropy classical dynamical systems with purely Lebesgue spectrum are known: non-zero time automorphisms of horocycle flows and their smooth time changes and even factor of special Gaussian systems. Can we produce a Poissonian suspension over a conservative infinite measure-preserving automorphism having purely Lebesgue spectrum?

Katok and Thouvenot (private communication) considered systems called *infinitely divisible* (ID). These are systems T on (X, \mathcal{B}, μ) which have a family of factors \mathcal{B}_ω indexed by $\omega \in \bigcup_{n=0}^{\infty} \{0, 1\}^n$ ($\mathcal{B}_\varepsilon = \mathcal{B}$) such that $\mathcal{B}_{\omega_0} \perp \mathcal{B}_{\omega_1}$, $\mathcal{B}_{\omega_0} \vee \mathcal{B}_{\omega_1} = \mathcal{B}_\omega$ and for each $\eta \in \{0, 1\}^{\mathbb{N}}$, $\bigcap_{n \in \mathbb{N}} \mathcal{B}_{\eta[0, n]} = \{\emptyset, X\}$. They showed (unpublished) that there are discrete spectrum transformations which are ID, and that there are rank one transformations with continuous spectra which are also ID (clearly Gaussian systems are ID). In [194], it is proved that dynamical systems coming from stationary ID processes are factors of ID automorphisms; moreover, ID automorphisms are disjoint from all systems having the SCS property. It would be nice to decide whether Koopman representations associated to ID automorphisms satisfy the Kolmogorov group property.

While some lacunary sequences have realization as rigidity sequences for weakly mixing automorphisms, it would be interesting to determine what happens in the non-lacunary case. An especially interesting case is when we consider the non-lacunary multiplicative semigroup $\{2^i 3^j : i, j \geq 0\}$. Is the corresponding sequence a rigidity sequence? What about $\{2^k + 3^\ell : k, \ell \geq 0\}$ [34]?

References

- [1] J. Aaronson, *Rational ergodicity, bounded rational ergodicity, and some continuous measures on the circle*, Israel J. Math. **33** (3-4) (1979), 181-197.
- [2] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Mathematical Surveys and Monographs, vol. **50**, American Mathematical Society, 1997. J. Aaronson. Rational ergodicity,

- bounded rational ergodicity, and some continuous measures on the circle. *Israel J. Math.* **33**(3–4) (1979), 181–197.
- [3] J. Aaronson, M. Hosseini, M. Lemańczyk, *IP-rigidity and eigenvalue groups*, *Ergodic Theory Dynam. Systems* **34** (2014), 1057–1076.
- [4] J. Aaronson, M.G. Nadkarni, *L^∞ eigenvalues and L^2 spectra of nonsingular transformations*, *Proc. London Math. Soc. (3)* **55** (1987), 538–570.
- [5] J. Aaronson, B. Weiss, *Remarks on the tightness of cocycles*, *Coll. Math.* **84/85** (2000), 363–376.
- [6] H. El Abdalaoui, *On the spectrum of the powers of Ornstein transformations*, *Ergodic theory and harmonic analysis (Mumbai, 1999)*, *Sankhya Ser. A* **62** (2000), 291–306.
- [7] H. El Abdalaoui, *A new class of rank 1 transformations with singular spectrum*, *Ergodic Theory Dynam. Systems* (2007).
- [8] H. El Abdalaoui, *Ergodic Banach problem on simple Lebesgue spectrum and flat polynomials*, arXiv 1508.06439v4.
- [9] H. El Abdalaoui, S. Kasjan, M. Lemańczyk, *0-1 sequences of the Thue-Morse type and Sar-nak’s conjecture*, *Proceedings of the American Mathematical Society* **144** (2016), 161–176.
- [10] H. El Abdalaoui, M. Lemańczyk, T. de la Rue, *On spectral disjointness of powers for rank-one transformations and Möbius orthogonality*, *Journal of Functional Analysis* **266** (2014), 284–317.
- [11] H. El Abdalaoui, F. Parreau, A. Prikhodko, *A new class of Ornstein transformations with singular spectrum*, *Ann. Inst. H. Poincaré Probab. Statist.* **42** (2006), 671–681.
- [12] T. Adams, *Smorodinsky’s conjecture on rank one systems*, *Proc. Amer. Math. Soc.* **126** (1998), 739–744.
- [13] T. Adams, *Tower multiplexing and slow weak mixing*, *Colloq. Math.* **138** (2015), 47–72.
- [14] O.N. Ageev, *Dynamical systems with a Lebesgue component of even multiplicity in the spectrum*, (Russian) *Mat. Sb. (N.S.)* **136(178)** (1988), 307–319, 430; translation in *Math. USSR-Sb.* **64** (1989), 305–317.
- [15] O.N. Ageev, *On ergodic transformations with homogeneous spectrum*, *J. Dynam. Control Systems* **5** (1999), 149–152.
- [16] O.N. Ageev, *On the spectrum of Cartesian powers of classical automorphisms*, (Russian) *Mat. Zametki* **68** (2000), 643–647; translation in *Math. Notes* **68** (2000), 547–551.
- [17] O.N. Ageev, *On the multiplicity function of generic group extensions with continuous spectrum*, *Ergodic Theory Dynam. Systems* **21** (2001), 321–338.
- [18] O.N. Ageev, *The homogeneous spectrum problem in ergodic theory*, *Invent. Math.* **160** (2005), 417–446.
- [19] O.N. Ageev, *Mixing with staircase multiplicity function*, *Ergodic Theory Dynam. Systems* **28** (2008), 1687–1700.
- [20] V.M. Alexeyev, *Existence of bounded function of the maximal spectral type*, *Vestnik Mosc. Univ.* **5** (1958), 13–15 and *Ergodic Theory Dynam. Syst.* **2** (1982), 259–261.
- [21] D.V. Anosov, *On Spectral Multiplicities in Ergodic Theory*, Institut im. V.A. Steklova, Moscow, 2003.
- [22] S. Alpern, *Conjecture: in general a mixing transformation is not two-fold mixing*, *Annals Prob.* **13** (1985), 310–313.
- [23] H. Anzai, *Ergodic skew product transformations on the torus*, *Osaka J. Math.* **3** (1951), 83–99.
- [24] N. Arni, *Spectral and mixing properties of actions of amenable groups*, *Electron. Res. Announc. AMS* **11** (2005), 57–63

- [25] V.I. Arnold, *Topological and ergodic properties of closed 1-forms with incommensurable periods*, (Russian) Funktsional. Anal. Prilozhen. **25** (1991) 1-12.
- [26] A. Avila, G. Forni, *Weak mixing for interval exchange transformations and translation flows*, Annals of Math. **165** (2007), 637-664.
- [27] A. Avila, G. Forni, C. Ulcigrai, *Mixing for the time-changes of Heisenberg nilflows*, J. Diff. Geom. **89** (2011), 369-410.
- [28] A. Avila, G. Forni, D. Ravotti, C. Ulcigrai, *Mixing for smooth time-changes of general nilflows*, preprint, arXiv: 1905.11628.
- [29] C. Badea, S. Grivaux, E. Matheron, *Rigidity sequences, Kazhdan sets and group topologies on the integers*, arXiv:1812.09014.
- [30] P. Balister, B. Bollobas, R. Morris, J. Suhasrabudhe, M. Tiba, *Flat Littlewood polynomials exist*, arXiv 1907.09464.
- [31] A.I. Bashtanov, *Generic mixing transformations are rank 1*, Math. Notes **93** (2013), 209-216.
- [32] A.I. Bashtanov, *Conjugacy classes are dense in the space of mixing Z^d -actions*, Math. Notes **99** (2016), 9-23.
- [33] J.R. Baxter, *A class of ergodic transformations having simple spectrum*, Proc. Amer. Math. Soc. **27** (1971), 275-279.
- [34] V. Bergelson, A. del Junco, M. Lemańczyk, J. Rosenblatt, *Rigidity and non-recurrence along sequences*, Ergodic Theory Dynam. Systems **34** (2014), 1464-1502.
- [35] M. Björklund, M. Einsiedler, A. Gorodnik, *Quantitative multiple mixing*, to appear in J. European Math. Soc.
- [36] J. Bourgain, *On the spectral type of Ornstein's class of transformations*, Israel J. Math. **84** (1993), 53-63.
- [37] J. Bourgain, *On the correlation of the Moebius function with rank-one systems*, Journal d'Analyse Mathématique **120** (2013), 105-130.
- [38] S. Cambanis, K. Podgórski, A. Weron, *Chaotic behaviour of infinitely divisible processes*, Studia Math. **115** (1995), 109-127.
- [39] P.A. Cecchi, R. Tiedra de Aldecoa, *Furstenberg transformations on Cartesian products of infinite-dimensional tori*, Potential Anal. **44** (2016), 43-51.
- [40] R.V. Chacon, *Approximation and spectral multiplicity*, in *Contributions to Ergodic Theory and Probability* (A. Dold and B. Eckmann, eds.) Springer, Berlin, 1970, 18-27.
- [41] J. Chaika, A. Eskin, *Self-joinings for 3-IET's*, arXiv:1805.11167v2.
- [42] J. Chaika, K. Frączek, A. Kanigowski, C. Ulcigrai, *Singularity of the spectrum for smooth area-preserving flows in genus two*, preprint.
- [43] J. Chaika, A. Wright, *A smooth mixing flow on a surface with nondegenerate fixed points*, J. Amer. Math. Soc. **32** (2019), 81-117.
- [44] G.H. Choe, *Products of operators with singular continuous spectra*, Operator theory: operator algebras and applications, Part 2 (Durham, NH, 1988), 65-68, Proc. Sympos. Pure Math., 51, Part 2, Amer. Math. Soc., Providence, RI, 1990.
- [45] A. Connes, E. Woods, *Approximately transitive flows and ITPFI factors*, Ergodic Theory Dynam. Syst. **5** (1985), 203-236.
- [46] I.P. Cornfeld, S.V. Fomin, Y.G. Sinai, *Ergodic Theory*, Springer-Verlag, New York, 1982.
- [47] D. Creutz, C. Silva, *Mixing on rank-one transformations*, Studia Math. **199** (2010), 43-72.
- [48] A.I. Danilenko, *Explicit solution of Rokhlin's problem on homogeneous spectrum and applications*, Ergodic Theory Dynam. Systems **26** (2006), 1467-1490.

- [49] A.I. Danilenko, *On new spectral multiplicities for ergodic maps*, Studia Math. **197** (2010), 57-68.
- [50] A.I. Danilenko, *New spectral multiplicities for mixing transformations*, Ergodic Theory Dynam. Systems **32** (2012), 517-534.
- [51] A.I. Danilenko, *A survey on spectral multiplicities of ergodic actions*, Ergodic Theory Dynam. Systems **33** (2013), 81-117.
- [52] A.I. Danilenko, *Mixing actions of the Heisenberg group*, Ergodic Theory Dynam. Systems **34** (2014), 1142-1167.
- [53] A.I. Danilenko, M. Lemańczyk, *A class of multipliers for \mathcal{W}^\perp* , Israel J. Math. **148** (2005), 137-168.
- [54] A.I. Danilenko, M. Lemańczyk, *Spectral multiplicities for ergodic flows*, Discrete Continuous Dynam. Systems **33** (2013), 4271-4289.
- [55] A.I. Danilenko, M. Lemańczyk, *Odometer actions of the Heisenberg group*, J. d'Analyse Math. **128** (2016), 107-157.
- [56] A.I. Danilenko, K.K. Park, *Generators and Bernouillian factors for amenable actions and cocycles on their orbits*, Ergodic Theory Dynam. Systems **22** (2002), 1715-1745.
- [57] A.I. Danilenko, V.V. Ryzhikov, *Spectral multiplicities for infinite measure preserving transformations*, Funct. Anal. Appl. **44** (2010), 161-170.
- [58] A.I. Danilenko, V.V. Ryzhikov, *Mixing constructions with infinite invariant measure and spectral multiplicities*, Ergodic Theory Dynam. Systems **31** (2011), 853-873.
- [59] A.I. Danilenko, A.V. Solomko, *Ergodic Abelian Actions with Homogenous Spectrum*, (Contemporary Math. 532), American Math. Soc., Providence, RI, 2010, pp. 137-148.
- [60] Y. Derriennic, K. Frączek, M. Lemańczyk, F. Parreau, *Ergodic automorphisms whose weak closure of off-diagonal measures consists of ergodic self-joinings*, Coll. Math. **110** (2008), 81-115.
- [61] A. Dooley, V.Y. Golodets, *The spectrum of completely positive entropy actions of countable amenable groups*, J. Funct. Anal. **196** (2002), 1-18.
- [62] A. Dooley, A. Quas, *Approximate transitivity for zero-entropy systems*, Ergodic Theory Dynam. Systems **25** (2005), 443-453.
- [63] T. Downarowicz, J. Kwiatkowski, *Spectral isomorphism of Morse flows*, Fundamenta Math. **163** (2000), 193-213.
- [64] T. Downarowicz, J. Kwiatkowski, *Weak Cosure Theorem fails for \mathbb{Z}^d -actions*, Studia Math. **153** (2002), 115-125.
- [65] T. Eisner, S. Grivaux, *Hilbertian Jamison sequences and rigid dynamical systems*, J. Funct. Anal. **261** (2011), 2013-2052.
- [66] B. Fayad, *Partially mixing and locally rank 1 smooth transformations and flows on the torus \mathbb{T}^d , $d \geq 3$* , J. London Math. Soc. (2) **64** (2001), 637-654.
- [67] B. Fayad, *Polynomial decay of correlations for a class of smooth flows on the two torus*, Bull. Soc. Math. France **129** (2001), 487-503.
- [68] B. Fayad, *Skew products over translations on T^d , $d \geq 2$* , Proc. Amer. Math. Soc. **130** (2002), 103-109.
- [69] B. Fayad, *Analytic mixing reparametrizations of irrational flows*, Ergodic Theory Dynam. Systems **22** (2002), 437-468.
- [70] B. Fayad, *Rank one and mixing differentiable flows*, Invent. Math. **160** (2005), 305-340.
- [71] B. Fayad, *Smooth mixing flows with purely singular spectra*, Duke Math. J. **132** (2006), 371-391.

- [72] B. Fayad, G. Forni, A. Kanigowski, *Lebesgue spectrum of countable multiplicity for conservative flows on the torus*, submitted.
- [73] B. Fayad, A. Kanigowski, *Rigidity times for a weakly mixing dynamical system which are not rigidity times for any irrational rotation*, Ergodic Theory Dynam. Systems **35** (2015), 2529–2534.
- [74] B. Fayad, A. Kanigowski, *Multiple mixing for a class of conservative surface flows*, Inv. Math. **203** (2) (2016), 555–614.
- [75] B. Fayad, A.B. Katok, A. Windsor, *Mixed spectrum reparameterizations of linear flows on \mathbb{T}^2* , Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary. Mosc. Math. J. **1** (2001), 521–537.
- [76] B. Fayad, R. Krikorian, *Some questions around quasi-periodic dynamics*, arXiv:1809.10375.
- [77] B. Fayad, J.-P. Thouvenot, *On the convergence to 0 of $m_n \xi \bmod 1$* , Acta Arith. **165** (2014), 327–332.
- [78] B. Fayad, A. Windsor, *A dichotomy between discrete and continuous spectrum for a class of special flows over rotations*, J. Mod. Dyn. **1** (2007), 107–122.
- [79] J. Feldman, *New K -automorphisms and a problem of Kakutani*, Israel J. Math. **24** (1976), 16–38.
- [80] S. Ferenczi, *Systèmes localement de rang un*, Ann. Inst. H. Poincaré Probab. Statist. **20** (1984), 35–51.
- [81] S. Ferenczi, *Systèmes de rang un gauche*, Ann. Inst. H. Poincaré Probab. Statist. **21** (1985), 177–186.
- [82] S. Ferenczi, *Systems of finite rank*, Colloq. Math. **73** (1997), 35–65.
- [83] S. Ferenczi, C. Holton, L.Q. Zamboni, *Joinings of three-interval exchange transformations*, Ergodic Theory Dynam. Systems **25** (2005), 483–502.
- [84] S. Ferenczi, C. Holton, L.Q. Zamboni, *Structure of three-interval exchange transformations III: ergodic and spectral properties*, J. Anal. Math. **93** (2004), 103–138.
- [85] S. Ferenczi, M. Lemańczyk, *Rank is not a spectral invariant*, Studia Math. **98** (1991), 227–230.
- [86] I. Filipowicz, *Product \mathbf{Z}^d -actions on a Lebesgue space and their applications*, Studia Math. **122** (1997), 289–298.
- [87] L. Flaminio, G. Forni, *Invariant distributions and time averages for horocycle flows*, Duke Math. J., 119 2003, no. 3, 465–526.
- [88] L. Flaminio, G. Forni, *Orthogonal powers and Möbius conjecture for smooth time changes of horocycle flows*, preprint, arXiv:1811.04652.
- [89] C. Foias, S. Stratila, *Ensembles de Kronecker dans la théorie ergodique*, C.R. Acad. Sci. Paris, sr. A-B **267** (1968), A166–A168.
- [90] G. Forni, A. Kanigowski, *Time-changes of Heisenberg nilflows*, preprint, arXiv:1711.05543.
- [91] G. Forni, A. Kanigowski *Multiple mixing and disjointness for time changes of bounded-type Heisenberg nilflows*, preprint, arXiv:1810.13319
- [92] G. Forni, C. Ulcigrai, *Time-changes of horocycle flows*, J. Mod. Dynam. **6** (2), 2012, 251–273.
- [93] K. Frączek, *On a function that realizes the maximal spectral type*, Studia Math. **124** (1997), 1–7.
- [94] K. Frączek, *Circle extensions of \mathbf{Z}^d -rotations on the d -dimensional torus*, J. London Math. Soc. (2) **61** (2000), 139–162.
- [95] K. Frączek, M. Lemańczyk, *A class of special flows over irrational rotations which is disjoint from mixing flows*, Ergodic Theory Dynam. Systems **24** (2004), 1083–1095.

- [96] K. Frączek, M. Lemańczyk, *On symmetric logarithm and some old examples in smooth ergodic theory*, Fund. Math. **180** (2003), 241–255.
- [97] K. Frączek, M. Lemańczyk, *On disjointness properties of some smooth flows*, Fund. Math. **185** (2005), 117–142.
- [98] K. Frączek, M. Lemańczyk, *On mild mixing of special flows over irrational rotations under piecewise smooth functions*, Ergodic Theory Dynam. Systems **26** (2006), 719–738.
- [99] K. Frączek, M. Lemańczyk, *A note on quasi-similarity of Koopman operators*, J. Lond. Math. Soc. (2) **82** (2010), 361–375.
- [100] K. Frączek, M. Lemańczyk, E. Lesigne, *Mild mixing property for special flows under piecewise constant functions*, Discrete and Continuous Dynamical Systems **19** (2007), 691–710.
- [101] N.A. Friedman, D.S. Ornstein, *Ergodic transformations induce mixing transformations*, Advances in Math. **10** (1973), 147–163.
- [102] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem of Diophantine approximation*, Math. Systems Th. **1** (1967), 1–49.
- [103] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, New Jersey, 1981.
- [104] H. Furstenberg, B. Weiss, *The finite multipliers of infinite ergodic transformations*, Lect. Notes in Math. **668** (1978), 127–132.
- [105] E. Glasner, *On the class of multipliers for \mathcal{W}^\perp* , Ergodic Theory Dynam. Systems **14** (1994), 129–140.
- [106] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs **101**, AMS, Providence, RI, 2003.
- [107] E. Glasner, B. Weiss, *Processes disjoint from weak mixing*, Trans. Amer. Math. Soc. **316** (1989), 689–703.
- [108] J.G. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340.
- [109] G.R. Goodson, *A survey of recent results in the spectral theory of ergodic dynamical systems*, J. Dynam. Control Systems **5** (1999), 173–226.
- [110] G.R. Goodson, J. Kwiatkowski M. Lemańczyk, P. Liardet, *On the multiplicity function of ergodic group extensions of rotations*, Studia Math. **102** (1992), 157–174.
- [111] J. Griesmer, *Recurrence, rigidity, and popular differences*, Ergodic Theory Dynam. Systems, to appear.
- [112] A.L. Gromov, *Spectral classification of some types of unitary weighted shift operators*, (Russian) Algebra i Analiz **3** (1991), 62–87; translation in St. Petersburg Math. J. **3** (1992), 997–1021.
- [113] M. Guenais, *Une majoration de la multiplicité spectrale d’opérateurs associés à des cocycles réguliers*, Israel J. Math. **105** (1998), 263–284.
- [114] M. Guenais, *Morse cocycles and simple Lebesgue spectrum*, Ergodic Theory Dynam. Systems **19** (1999), 437–446.
- [115] M. Guenais, F. Parreau, *Valeurs propres de transformations liées aux rotations irrationnelles et aux fonctions en escalier*, preprint 2005.
- [116] M. Haase, N. Moriakov, *On systems with quasi-discrete spectrum*, Studia Math. **241** (2018), 173–199.
- [117] F. Hahn, W. Parry, *Some characteristic properties of dynamical systems with quasi-discrete spectra*, Math. Systems Theory **2** (1968), 179–190.
- [118] B. Hasselblatt, A.B. Katok, *Principal Structures*, Handbook of dynamical systems, Vol. 1A, 1–203, North-Holland, Amsterdam, 2002.

- [119] H. Helson, *Cocycles on the circle*, J. Operator Theory **16** (1986), 189-199.
- [120] H. Helson, W. Parry, *Cocycles and spectra*, Arkiv Math. **16** (1978), 195-206.
- [121] M. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. IHES **49** (1979), 5-234.
- [122] B. Host, *Mixing of all orders and pairwise independent joinings of systems with singular spectrum*, Israel J. Math. **76** (1991), 289-298.
- [123] B. Host, F. Méla, F. Parreau, *Non-singular transformations and spectral analysis of measures*, Bull. Soc. Math. France **119** (1991), 33-90.
- [124] F.K. Indukaev, *The twisted Burnside theory for the discrete Heisenberg group and for the wreath products of some groups*, Moscow Univ. Math. Bull. **62** (2007), 219-227.
- [125] A. Iwanik, *The problem of L^p -simple spectrum for ergodic group automorphisms*, Bull. Soc. Math. France **119** (1991), 91-96.
- [126] A. Iwanik, *Positive entropy implies infinite L^p -multiplicity for $p > 1$* , Ergodic theory and related topics, III (Güstrow, 1990), 124-127, Lecture Notes in Math. **1514**, Springer, Berlin, 1992.
- [127] A. Iwanik, *Anzai skew products with Lebesgue component of infinite multiplicity*, Bull. London Math. Soc. **29** (1997), 195-199.
- [128] A. Iwanik, M. Lemańczyk, D. Rudolph, *Absolutely continuous cocycles over irrational rotations*, Israel J. Math. **83** (1993), 73-95.
- [129] A. Iwanik, M. Lemańczyk, J. de Sam Lazaro, T. de la Rue, *Quelques remarques sur les facteurs des systèmes dynamiques gaussiens*, Studia Math. **125** (1997), 247-254.
- [130] A. Iwanik, M. Lemańczyk, C. Mauduit, *Piecewise absolutely continuous cocycles over irrational rotations*, J. London Math. Soc. (2) **59** (1999), 171-187.
- [131] A. Iwanik, J. de Sam Lazaro, *Sur la multiplicité L^p d'un automorphisme gaussien*, C.R. Acad. Sci. Paris, Série I, **312** (1991), 875-876.
- [132] A. Janicki, A. Weron, *Simulation and chaotic behavior of α -stable stochastic processes*, Monographs and Textbooks in Pure and Applied Mathematics, 178 Marcel Dekker, Inc., New York, 1994.
- [133] E. Janvresse, T. de la Rue, *The Pascal adic transformation is loosely Bernoulli*, Ann. Inst. H. Poincaré Probab. Statist. **40** (2004), 133-139.
- [134] E. Janvresse, A.A. Prikhod'ko, T. de la Rue, V.V. Ryzhikov, *Weak limits of powers of Chacon's automorphism*, Ergodic Theory Dynam. Systems **35** (2015), 128-141.
- [135] E. Janvresse, E. Roy, T. de la Rue, *Poisson suspensions and Sushis*, Ann. Sci. École Norm. Supérieure. (4) **50** (2017), 1301-1334.
- [136] E. Janvresse, T. de la Rue, V.V. Ryzhikov, *Around King's rank one theorems: flows and \mathbb{Z}^n -actions*, Dynamical systems and group actions, 143-161, Contemporary Math., 567, Amer. Math. Soc., Providence, RI, 2012.
- [137] A. del Junco, *Transformations with discrete spectrum are stacking transformations*, Canad. J. Math. **28** (1976), 836-839.
- [138] A. del Junco, *A transformation with simple spectrum which is not of rank one*, Canad. J. Math. **29** (1977), 655-663.
- [139] A. del Junco, *Disjointness of measure-preserving transformations, minimal self-joinings and category*, Prog. Math. **10** (1981), 81-89.
- [140] A. del Junco, M. Lemańczyk, *Generic spectral properties of measure-preserving maps and applications*, Proc. Amer. Math. Soc. **115** (1992), 725-736.
- [141] A. del Junco, M. Lemańczyk, *Simple systems are disjoint with Gaussian systems*, Studia Math. **133** (1999), 249-256.

- [142] A. del Junco, M. Lemańczyk, *Joinings of distally simple systems*, preprint (2005).
- [143] A. del Junco, D. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergodic Theory Dynam. Systems **7** (1987), 531-557.
- [144] A.G. Kachurovskii, *A property of operators generated by ergodic automorphisms*, (Russian) Optimizatsiya **47** (64) (1990), 122–125.
- [145] S. Kalikow, *Two fold mixing implies three fold mixing for rank one transformations*, Ergodic Theory Dynam. Systems **4** (1984), 237-259.
- [146] T. Kamae, *Spectral properties of automata generating sequences*, unpublished preprint.
- [147] B. Kamiński, *The theory of invariant partitions for \mathbb{Z}^d -actions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **29** (1981), 349–362.
- [148] A. Kanigowski, J. Kułaga-Przymus, C. Ulcigrai, *Multiple mixing and parabolic divergence in smooth area-preserving flows on higher genus surfaces*, preprint, arXiv:1606.09189v3, to appear in J. European Math. Soc.
- [149] A. Kanigowski, J. Kułaga-Przymus, *Ratner’s property and mild mixing for smooth flows on surfaces*, Ergodic Theory Dynam. Systems **36** (2016), 2512–2537.
- [150] A. Kanigowski, M. Lemańczyk, C. Ulcigrai, *On disjointness properties of some parabolic flows*, preprint, arXiv:1810.11576v1.
- [151] A. Kanigowski, T. de la Rue, *Product of two staircase rank one transformations that is not loosely Bernoulli*, accepted in J. d’Analyse Math.
- [152] A.B. Katok, *Constructions in ergodic theory*, unpublished lecture notes.
- [153] A.B. Katok, *Monotone equivalence in ergodic theory*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), 104–157.
- [154] A.B. Katok, *Interval exchange transformations and some special flows are not mixing*, Israel J. Math. **35** (1980), 301–310.
- [155] A.B. Katok, *Cocycles, cohomology and combinatorial constructions in ergodic theory*, In collaboration with E. A. Robinson, Jr. Proc. Sympos. Pure Math., 69, Smooth ergodic theory and its applications (Seattle, WA, 1999), 107–173, Amer. Math. Soc., Providence, RI, 2001.
- [156] A.B. Katok, *Combinatorial constructions in ergodic theory and dynamics*, University Lecture Series, 30. American Mathematical Society, Providence, RI, 2003.
- [157] A. Katok, M. Lemańczyk, *Some new cases of realization of spectral multiplicity function for ergodic transformations*, Fundamenta Math. **206** (2009), 185-215.
- [158] A.B. Katok, A.M. Stepin, *Approximations in ergodic theory*, (Russian) Uspekhi Mat. Nauk **22** (1967) (137), 81–106.
- [159] A. Katok, J.-P. Thouvenot, *Spectral Properties and Combinatorial Constructions in Ergodic Theory*, Handbook of dynamical systems. Vol. 1B, 649–743, Elsevier B. V., Amsterdam, 2006.
- [160] A.B. Katok, A.N. Zemlyakov, *Topological transitivity of billiards in polygons*, (Russian) Mat. Zametki **18** (1975), 291–300.
- [161] M. Keane, *Generalized Morse sequences*, Z. Wahr. Verw. Gebiete **10** (1968), 335–353.
- [162] M. Keane, *Interval exchange transformations* Math. Z. **141** (1975), 25–31.
- [163] K.M. Khanin, Ya.G. Sinai, *Mixing of some classes of special flows over rotations of the circle*, Funct. Anal. Appl. **26** (1992), 155-169.
- [164] J. King, *The commutant is the weak closure of the powers, for rank-1 transformations*, Ergodic Theory Dynam. Systems **6** (1986), 363-384.
- [165] J.L. King, *Joining-rank and the structure of finite rank rank mixing transformations*, J. Analyse Math. **51** (1988), 182-227.

- [166] J.L. King, *Flat stacks, joining-closure and genericity*, preprint (2001).
- [167] A. A. Kirillov, *Lectures on the orbit method*, Providence, RI : American Math. Soc., 2004. Series: Graduate studies in mathematics, 64.
- [168] I. Klemes, *The spectral type of the staircase transformation*, Tohoku Math. J. **48** (1996), 247-248.
- [169] I. Klemes, K. Reinhold, *Rank one transformations with singular spectral type*, Israel J. Math. **98** (1997), 1–14.
- [170] A.V. Kočergin, *On the absence of mixing in special flows over the rotation of a circle and in flows on a two-dimensional torus*, (Russian) Dokl. Akad. Nauk SSSR **205** (1972), 949-952.
- [171] A.V. Kochergin, *Mixing in special flows over rearrangement of segments and in smooth flows on surfaces*, Math-USSR Acad. Sci. **25** (1975), 441-469.
- [172] A.V. Kochergin, *Non-degenerate saddles and absence of mixing*, (Russian) Mat. Zametky **19** (1976), 453-468.
- [173] A. V. Kochergin, *On the homology of function over dynamical systems*, Dokl. Akad. Nauk SSSR **231** (1976), 795–798.
- [174] A.V. Kochergin, *A mixing special flow over a rotation of the circle with an almost Lipschitz function*, Sb. Math. **193** (2002), 359-385.
- [175] A.V. Kochergin, *Nondegenerate fixed points and mixing in flows on a two-dimensional torus. II.*, (Russian) Mat. Sb. **195** (2004), 15–46.
- [176] B.O. Koopman, *Hamiltonian systems and transformations in Hilbert space*, Proc. Nat. Acad. Sci. USA **17** (1931), 315-318.
- [177] L. Kuipers, H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, 1974.
- [178] J. Kułaga-Przymus, M. Lemańczyk, *Sarnak’s conjecture from ergodic theory point of view*, Encyclopedia of Complexity and System Science (2019).
- [179] J. Kułaga, F. Parreau, *Disjointness properties for Cartesian products of weakly mixing systems*, Colloq. Math. **128** (2012), 153-177.
- [180] A.G. Kushnirenko, *Spectral properties of some dynamical systems with polynomial divergence of orbits*, Vestnik Moskovskogo Univ. **1-3** (1974), 101-108.
- [181] J. Kwiatkowski, *Spectral isomorphism of Morse dynamical systems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **29** (1981), 105–114.
- [182] J. Kwiatkowski, Y. Lacroix, *Multiplicity, rank pairs*, J. d’Analyse Math. **71** (1997), 205-235.
- [183] J. Kwiatkowski (jr.), M. Lemańczyk, *On the multiplicity function of ergodic group extensions.II*, Studia Math. **116** (1995), 207-215.
- [184] M. Lemańczyk, *Toeplitz Z_2 -extensions*, Ann. Inst. H. Poincaré **24** (1988), 1-43.
- [185] M. Lemańczyk, *Introduction to Ergodic Theory from the Point of View of the Spectral Theory*, Lecture Notes of the tenth KAIST mathematics workshop, Taejon 1996, 1-153.
- [186] M. Lemańczyk, *Sur l’absence de mélange pour des flots spéciaux au dessus d’une rotation irrationnelle*, Coll. Math. **84/85** (2000), 29-41.
- [187] M. Lemańczyk, *Spectral Theory of Dynamical Systems*, Encyclopedia of Complexity and System Science, Springer-Verlag (2009), 8554-8575.
- [188] M. Lemańczyk, E. Lesigne, *Ergodicity of Rokhlin cocycles*, J. Anal. Math. **85** (2001), 43-86.
- [189] M. Lemańczyk, C. Mauduit, *Ergodicity of a class of cocycles over irrational rotations*, Journal London Math. Soc. **49** (1994), 124-132.
- [190] M. Lemańczyk, M.K. Mentzen, H. Nakada, *Semisimple extensions of irrational rotations*, Studia Math. **156** (2003), 31-57.

- [191] M. Lemańczyk, F. Parreau, *Rokhlin extensions and lifting disjointness*, Ergodic Theory Dynam. Systems **23** (2003), 1525-1550.
- [192] M. Lemańczyk, F. Parreau, *Lifting mixing properties by Rokhlin cocycles*, Ergodic Theory Dynam. Systems **32** (2012), 763-784.
- [193] M. Lemańczyk, F. Parreau, *Special flows over irrational rotation with simple convolution property*, preprint (2007).
- [194] M. Lemańczyk, F. Parreau, E. Roy, *Systems with simple convolutions, distal simplicity and disjointness with infinitely divisible systems*, Proc. Amer. math. Soc. **139** (2011), 185-199.
- [195] M. Lemańczyk, F. Parreau, J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self-joinings are Gaussian*, Fundamenta Math. **164** (2000), 253-293.
- [196] M. Lemańczyk, J.-P. Thouvenot, B. Weiss, *Relative discrete spectrum and joinings*, Monatsh. Math. **137** (2002), 57-75.
- [197] M. Lemańczyk, J. de Sam Lazaro, *Spectral analysis of certain compact factors for Gaussian dynamical systems*, Isr. J. Math. **98** (1997), 307-328.
- [198] M. Lemańczyk, A. Sikorski, *A class of not local rank one automorphisms arising from continuous substitutions*, Prob. Th. Rel. Fields **76** (1987), 421-428.
- [199] M. Lemańczyk, M. Wasieczko, *A new proof of Alexeyev's Theorem*, preprint (2006).
- [200] M. Lemańczyk, M. Wysokińska, *On analytic flows on the torus which are disjoint from systems of probabilistic origin*, Fundamenta Math. **195** (2007), 97-124.
- [201] V.P. Leonov, *The use of the characteristic functional and semi-invariants in the ergodic theory of stationary processes*, Dokl. Akad. Nauk SSSR **133** (1960), 523-526 (Sov. Math. **1** (1960), 878-881).
- [202] S. Lightwood, A. Şahin, I. Ugarcovici, *The structure and the spectrum of Heisenberg odometers*, Proc. Amer. Math. Soc. **142** (2014), 2429-2443.
- [203] G.W. Mackey, *Ergodic transformation groups with a pure point spectrum*, Illinois J. Math. **8** (1964), 593-600.
- [204] B. Marcus, *Ergodic properties of horocycle flows for surfaces of negative curvature*, Annals Math. (2) **105** (1977), 81-105.
- [205] G. Maruyama, *Infinitely divisible processes*, Theory Prob. Appl. **15** (1) (1970), 1-22.
- [206] H. Masur, *Interval exchange transformations and measured foliations*, Annals Math. **115** (1982), 169-200.
- [207] J. Mathew, M.G. Nadkarni, *Measure-preserving transformation whose spectrum has Lebesgue component of multiplicity two*, Bull. London Math. Soc. **16** (1984), 402-406.
- [208] H. Medina, *Spectral types of unitary operators arising from irrational rotations on the circle group*, Michigan Math. J. **41** (1994), 39-49.
- [209] M.K. Mentzen, *Some examples of automorphisms with rank r and simple spectrum*, Bull. Pol. Ac. Sc. **7-8** (1988), 417-424.
- [210] M.G. Nadkarni, *Spectral Theory of Dynamical Systems*, Hindustan Book Agency, New Delhi, 1998.
- [211] J. von Neumann, *Zur Operatorenmethode in der Klassischen Mechanik*, Annals Math. **33** (1932), 587-642.
- [212] D. Newton, *On Gaussian processes with simple spectrum*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete **5** (1966), 207-209.
- [213] D. Newton, W. Parry, *On a factor automorphism of a normal dynamical system*, Ann. Math. Statist **37** (1966), 1528-1533.

- [214] D. Ornstein, *On the root problem in ergodic theory*, In: Proc. 6th Berkeley Symp. Math. Stats. Prob., University California Press, Berkeley, 1970, 348-356.
- [215] D. Ornstein, D. Rudolph, B. Weiss, *Equivalence of measure preserving transformations*, Mem. Amer. Math. Soc. **37** (1982), no. 262.
- [216] D. Ornstein, B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. Analyse Math. **48** (1987), 1–141.
- [217] F. Parreau, *On the Foias and Stratila Theorem*, Proceedings of the conference on Ergodic Theory, Toruń 2000.
- [218] F. Parreau, E. Roy, *Prime Poisson suspensions*, Ergodic Theory Dynam. Systems **35** (2015), 2216-2230.
- [219] W. Parry, *Topics in Ergodic Theory*, Cambridge Tracts in Mathematics, 75 Cambridge University Press, Cambridge-New York, 1981.
- [220] W. Parry, *Spectral analysis of G -extensions of dynamical systems*, Topology **9** (1970), 217–224.
- [221] K. Petersen, *Ergodic Theory*, Cambridge Univ. Press, Cambridge, 1983.
- [222] A.A. Prikhodko, *Littlewood polynomials and their applications to the spectral theory of dynamical systems*, Mat. Sb. **204** (2013), 135-160, translation in Sb. Math. 204 (2013).
- [223] A.A. Prikhodko, V.V. Ryzhikov, *Disjointness of the convolutions for Chacon's automorphism*, Dedicated to the memory of Anzelm Iwanik, Colloq. Math. **84/85** (2000), 67–74.
- [224] M. Quefelec, *Substitution Dynamical Systems – Spectral Analysis*, Lecture Notes in Math. **1294**, Springer-Verlag, 1988.
- [225] M. Ratner, *Horocycle flows are loosely Bernoulli*, Israel J. Math. **31** (1978), 122–132.
- [226] M. Ratner, *The Cartesian square of the horocycle flow is not loosely Bernoulli*, Israel J. Math. 34 no. 1-2, (1979), 72–96.
- [227] M. Ratner, *Horocycle flows, joinings and rigidity of products*, Annals Math. **118** (1983), 277-313.
- [228] M. Ratner, *Rigid reparametrizations and cohomology for horocycle flows*, Invent. Math., Vol. 88, (2)987,341–374.
- [229] M. Ratner, *Rigidity of time changes for horocycle flows*, Acta Math. —bf 156 (1986), 1–32.
- [230] G. Rauzy, *Echanges d'intervalles et transformations induites*, Acta Arith. **34** (1979), 315–328.
- [231] D. Ravotti, *Mixing for suspension flows over skew-translations and time-changes of quasi-abelian filiform nilflows*, Erg. Th. Dynam. Sys., published online at <https://doi.org/10.1017/etds.2018.19>.
- [232] D. Ravotti, *Quantitative mixing for locally Hamiltonian flows with saddle loops on compact surfaces*, Ann. H. Poincaré **18** (12) (2017), 3815–3861.
- [233] E.A. Robinson (jr.), *Ergodic measure preserving transformations with arbitrary finite spectral multiplicities*, Invent. Math. **72** (1983), 299–314.
- [234] E.A. Robinson (jr.), *Transformations with highly nonhomogeneous spectrum of finite multiplicity*, Israel J. Math. **56** (1986), 75–88.
- [235] E.A. Robinson (jr.), *Nonabelian extensions have nonsimple spectrum*, Compositio Math. **65** (1988), 155–170.
- [236] E.A. Robinson (jr.), *A general condition for lifting theorems*, Trans. Amer. Math. Soc. **330** (1992), 725–755.
- [237] J. Rosiński, T. Żak, *Simple condition for mixing of infinitely divisible processes*, Stochastic Processes Appl. **61** (1996), 277-288.

- [238] J. Rosiński, T. Żak, *The equivalence of ergodicity and weak mixing for infinitely divisible processes*, J. Theoret. Probab. **10** (1997), 73-86.
- [239] E. Roy, *Mesures de Poisson, infinie divisibilité et propriétés ergodiques*, Thèse de doctorat de l'Université Paris 6 (2005).
- [240] E. Roy, *Ergodic properties of Poissonian ID processes*, Annals Probab. **35** (2007), 551-576.
- [241] H.L. Royden, *Real Analysis*, McMillan, New York, 1968.
- [242] W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Applied Mathematics, No. 12 Interscience Publishers (a division of John Wiley and Sons), New York-London 1962.
- [243] D.J. Rudolph, *An example of a measure-preserving map with minimal self-joinings and applications*, J. Anal. Math. **35** (1979), 97-122.
- [244] D. Rudolph, *k-fold mixing lifts to weakly mixing isometric extensions*, Ergodic Theory Dynam. Systems **5** (1985), 445-447.
- [245] D. Rudolph, \mathbb{Z}^n and \mathbb{R}^n cocycle extension and complementary algebras, Ergodic Theory Dynam. Systems **6** (1986), 583-599.
- [246] D. Rudolph, *Fundamentals of Measurable Dynamics*, Oxford Sc. Publ. 1990.
- [247] D. Rudolph, *Pointwise and L^1 mixing relative to a sub-sigma algebra*, Illinois J. Math. **48** (2004), 505-517.
- [248] D. Rudolph, B. Weiss, *Entropy and mixing for amenable group actions*, Annals Math. (2) **151** (2000), 1119-1150.
- [249] T. de la Rue, *Systèmes dynamiques gaussiens d'entropie nulle, lâchement et non lâchement Bernoulli*, Ergodic Theory Dynam. Systems **16** (1996), 379-404.
- [250] T. de la Rue, *Rang des systèmes dynamiques gaussiens*, Israel J. Math. **104** (1998), 261-283.
- [251] T. de la Rue, *L'ergodicité induit un type spectral maximal équivalent à la mesure de Lebesgue*, Ann. Inst. H. Poincaré Probab. Statist. **34** (1998), 249-263.
- [252] T. de la Rue, *L'induction ne donne pas toutes les mesures spectrales*, Ergodic Theory Dynam. Syst. **18** (1998), 1447-1466.
- [253] T. de la Rue, *An extension which is relatively twofold mixing but not threefold mixing*, Colloq. Math. **101** (2004), 271-277.
- [254] T. de la Rue, *Joinings in ergodic theory*, Encyclopedia of Complexity and System Science
- [255] V.V. Ryzhikov, *Joinings of dynamical systems. Approximations and mixing*, (Russian) Uspekhi Mat. Nauk **46** (1991), no. 5(281), 177-178.
- [256] V.V. Ryzhikov, *Mixing, rank and minimal self-joining of actions with invariant measure*, (Russian) Mat. Sb. **183** (1992), 133-160.
- [257] V.V. Ryzhikov, *The absence of mixing in special flows over rearrangements of segments*, (Russian) Mat. Zametki **55** (1994), 146-149.
- [258] V.V. Ryzhikov, *Skew products and multiple mixing of dynamical systems*, Russ. Math. Surv. **49** (1994), 170-171.
- [259] V.V. Ryzhikov, *Stochastic intertwining and multiple mixing of dynamical systems*, J. Dynam. Control Systems **2** (1996), 1-19.
- [260] V.V. Ryzhikov, *Transformations having homogeneous spectra*, J. Dynam. Control Systems **5** (1999), 145-148.
- [261] V.V. Ryzhikov, *The Rokhlin problem on multiple mixing in the class of actions of positive local rank*, (Russian) Funktsional. Anal. i Prilozhen. **34** (2000), 90-93.
- [262] V.V. Ryzhikov, *Weak limits of powers, simple spectrum symmetric products and mixing rank one constructions*, Math. Sb. **198** (2007), 733-754.

- [263] V.V. Ryzhikov, *Spectral multiplicities and asymptotic operator properties of actions with invariant measure*, Sb. Math. **200** (2009), 1833-1845.
- [264] V.V. Ryzhikov, *On spectral multiplicities of Gaussian actions*, arXiv 1406.3321.
- [265] V.V. Ryzhikov, J.-P. Thouvenot, *Disjointness, divisibility, and quasi-simplicity of measure-preserving actions*, (Russian) Funktsional. Anal. i Prilozhen. **40** (2006), 85–89.
- [266] V.V. Ryzhikov, A.E. Troitskaya, *Mixing flows with a homogeneous spectrum of multiplicity 2*, (Russian) Fundam. Prikl. Mat. **21** (2016), 191–197.
- [267] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, 1999.
- [268] K. Schmidt, *Cocycles of Ergodic Transformation Groups*, Lecture Notes in Math. 1, Mac Millan of India, 1977.
- [269] K. Schmidt, *Dispersing cocycles and mixing flows under functions*, Fund. Math. **173** (2002), 191–199.
- [270] K. Schmidt, P. Walters, *Mildly mixing actions of locally compact groups*, Proc. London Math. Soc. (3) **45** (1982), 506-518.
- [271] L. D. Simonelli, *Absolutely continuous spectrum for parabolic flows/maps*, Disc. Cont. Dyn. Syst. (1) **38** (2018), 263–292.
- [272] Ya.G. Sinai, *Topics in Ergodic Theory*, Princeton University Press, Princeton, 1994.
- [273] M. Shklover, *Classical dynamical systems on the torus with continuous spectrum*, (Russian) Izv. Vys. Uceb. Zaved. Matematika **10** (65) (1967), 113-124.
- [274] M. Smorodinsky, J.-P. Thouvenot, *Bernoulli factors that span a transformation*, Isr. J. Math. **32** (1979), 39-43.
- [275] A.V. Solomko, *New spectral multiplicities for ergodic actions*, Studia Math. **208** (2012), 229-247.
- [276] A.M. Stepin, *Spectral properties of generic dynamical systems*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), 801–834.
- [277] J.-P. Thouvenot, *Some properties and applications of joinings in ergodic theory*, Ergodic Th. and its Connections with Harmonic Anal., London Math. Soc. 1995, 207-235.
- [278] J.-P. Thouvenot, *Les systèmes simples sont disjoints de ceux qui sont infiniment divisibles et plongeables dans un flot*, Coll. Math. **84/85** (2000), 481-483.
- [279] R. Tiedra de Aldecoa, *Spectral Analysis of Time-Changes of the Horocycle Flow*, J. Mod. Dynam. **6** (2) (2012), 275–285.
- [280] E. Thoma, *Über unitäre Darstellungen abzählbarer, diskreter Gruppen*, Math. Ann. **153** (1964), 111–138.
- [281] R. Tiedra de Aldecoa, *The absolute continuous spectrum of skew products of compact Lie groups*, Israel J. Math. **208** (2015), 323-350.
- [282] R. Tiedra de Aldecoa, *Commutator methods for the spectral analysis of uniquely ergodic dynamical systems*, Ergodic Theory Dynam. Systems **35** (2015), 944–967.
- [283] R. Tiedra de Aldecoa, *Commutator criteria for strong mixing*, Ergodic Theory Dynam. Systems **37** (2017), 308–323.
- [284] S.V. Tikhonov, *On a relation between the metric and spectral properties of \mathbb{Z}^d -actions*, Fundam. Prikl. Mat. **8** (2002), 1179–1192.
- [285] S.V. Tikhonov, *Mixing transformations with homogeneous spectrum*, Mat. Sb. **202**:8 (2011), 139–160; English transl. in Sb Math. 202:8 (2011), 1231–1252.
- [286] S.V. Tikhonov, *Genericity of a multiple mixing*, (Russian) Uspekhi Mat. Nauk **67** (2012), no. 4(406), 187–188; translation in Russian Math. Surveys 67 (2012), no. 4, 779–780.

- [287] S.V. Tikhonov, *Complete metric on mixing actions of general groups*, J. Dyn. Control Syst. **19** (2013), 17–31.
- [288] J.A. Todd, *On a conjecture in group theory*, J. London Math. Soc. **25** (1950), 246.
- [289] C. Ulcigrai, *Mixing of asymmetric logarithmic suspension flows over interval exchange transformations*, Ergodic Theory Dynam. Syst. **27** (2007), 991–1035.
- [290] C. Ulcigrai, *Absence of mixing in area-preserving flows on surfaces*, Annals Math. **173** (3) (2011), 1743–1778.
- [291] W. Veech, *Interval exchange transformations*, J. Analyse Math. **33** (1978), 222–272.
- [292] W.A. Veech, *A criterion for a process to be prime*, Monatshefte Math. **94** (1982), 335–341.
- [293] W. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Annals Math. (2) **115** (1982), 201–242.
- [294] W. Veech, *The metric theory of interval exchange transformations. I. Generic spectral properties*, Amer. J. Math. **106** (1984), 1331–1359.
- [295] A.M. Vershik, *The Pascal automorphism has a continuous spectrum*, (Russian) Funktsional. Anal. i Prilozhen. **45** (2011), 16–33; translation in Funct. Anal. Appl. **45** (2011), 173–186.
- [296] A.M. Vershik, *On the theory of normal dynamic systems*, Math. Sov. Dokl. **144** (1962), 625–628.
- [297] A.M. Vershik, *Spectral and metric isomorphism of some normal dynamical systems*, Math. Sov. Dokl. **144** (1962), 693–696.
- [298] A.M. Vershik, *Polymorphisms, Markov processes, and quasi-similarity*, Discrete Contin. Dyn. Syst. **13** (2005), 1305–1324.
- [299] R. Yassawi, *Multiple mixing and local rank group actions*, Ergodic Theory Dynam. Systems **23** (2003), 1275–1304.
- [300] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, 1982.
- [301] M. Wysokińska, *A class of real cocycles over an irrational rotation for which Rokhlin cocycle extensions have Lebesgue component in the spectrum*, Topol. Methods Nonlinear Anal. **24** (2004), 387–407.
- [302] M. Wysokińska, *Ergodic properties of skew products and analytic special flows over rotations*, PhD thesis, Toruń, 2007.
- [303] P. Zeitz, *The centralizer of a rank one flow*, Israel J. Math. **84** (1993), 129–145.