ISOMETRIC EXTENSIONS, 2-COCYCLES AND ERGODICITY OF SKEW PRODUCTS

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Abstract. We establish existence and uniqueness of a canonical form for isometric extensions of an ergodic non-singular transformation T. This is applied to describe the structure of commutors of the isometric extensions. Moreover, for a compact group G, we construct a G-valued T-cocycle α which generates the ergodic skew product extension T_α and admits a prescribed subgroup in the centralizer of T_α.

0. Introduction

Let T be an ergodic non-singular transformation of a Lebesgue space $(X, \mathfrak{B}, \mu)$. We consider isometric extensions of T, i.e. transformations S of $X \times G/H$ of the form $S = T_{G,H,\alpha}$, $T_{G,H,\alpha}(x, Hg) = (Tx, Hg\alpha(x))$, where $H \subset G$ is a nested pair of compact groups and $\alpha : X \to G$ a Borel function. We show that every ergodic isometric extension is conjugate to another one $T_{G',H',\alpha'}$ with the pair $H' \subset G'$ irreducible which means that H' contains no proper normal subgroups of G'. More impotently, if this condition is satisfied then the corresponding triplet (G', H', α') is determined uniquely up to cohomology (see §1 and Theorem 1.4 for details). This extend the earlier results of T. Hamachi [Ha], where finite extensions are studied.

Denote by $C(T_{G,H,\alpha})$ the centralizer of $T_{G,H,\alpha}$, i.e. the group of all transformations commuting with it, and by $\widetilde{C}(T_{G,H,\alpha})$ the subgroup of those commutors which can be pushed down to X. We apply the above results to describe the structure of elements from $\widetilde{C}(T_{G,H,\alpha})$. Namely, every such element has the form $S_{l,f}$, $S_{l,f}(x, Hg) = (Sx, Hl(g)f(x))$, where $S \in C(T)$, l is an automorphism of G with $l(H) = H$ and $f : X \to G$ is a measurable map with $l(\alpha(x)) = f(x)\alpha(Sx)f(Tx)^{-1}$ (Proposition 2.1). This extends the well-known theorem from [Ne] (see also [Me], [JLM], [D1]), where the particular case $H = \{1_G\}$ and T measure-preserving was considered.

We also study (in §3) the problem of extending of a T-cocycle to a cocycle of a larger group action which is related to the factor problem of isometric extensions (cf. [Kw] and [Le]).

Let K be a compact (in the weak topology) subgroup of T-commutors, G an Abelian compact group and $\alpha : X \to G$ an ergodic cocycle. For simplicity, we shall write T_α instead of $T_{G,\{1_G\},\alpha}$. Let $\pi : \widetilde{C}(T_\alpha) \to C(T)$ stand for the natural projection. Clearly, G is embedded into $\widetilde{C}(T_\alpha)$ as a closed normal subgroup acting

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on $X \times G$ by left translations along the second coordinate. If $K \subset \pi(\widetilde{C}(T_\alpha))$ then we obtain a short exact sequence of compact groups

$$1 \to G \to \pi^{-1}(K) \xrightarrow{\pi} K \to 1. \tag{0-1}$$

As usual, this sequence determines a structure of K-module on G. In turn, this structure plus a 2-cocycle of K with values in G (arising from a Borel cross-section $K \to \pi^{-1}(K)$) determine completely the group structure on $\pi^{-1}(K)$.

In the final §4 we are concerned with the following question. Suppose that a short exact sequence of compact groups is given:

$$1 \to G \to E \to K \to 1 \tag{0-2}$$

with G and K as above. Is it possible to find an ergodic cocycle $\alpha : X \to G$ such that (0-2) is congruent (i.e. identical) to (0-1)? We show that if such an α exists then it must be a measurable solution of some functional equation (see (4-3)) which, in fact, is determined completely by a 2-cohomology class of the 2-cocycle of K associated to (0-2). More precisely, α appears to be a transfer function for a cocycle of a free measurable action of K (it is well known that every cocycle of a free type I action is a coboundary). There is however, abundance of such solutions even if we shall not distinguish T-cohomologous cocycles (in the dynamical system's sense). Thus our problem is to find out are there ergodic solutions? We consider separately 3 cases.

First, let us assume that (0-2) has no splitting quotient-extensions. This means that there are no K-invariant subgroups N of G, $N \neq G$, such that the N-quotient sequence

$$1 \to G/N \to E/N \to K \to 1$$

splits. In the language of 2-cocycles this can be rephrased as follows: there are no Borel cross-sections $K \to E$ such that the associated 2-cocycle of K takes values in N. One of the simplest examples of such group extensions is

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{T} \to \mathbb{T} \to 1.$$

We show that—rather surprisingly—every measurable solution of (4-3) is an ergodic T-cocycle.

Next, let us consider the opposite situation: (0-2) splits. Then it is easy to find non-ergodic solution of (4-3). Nevertheless, we prove that ergodic ones also exists (provided that the K-quotient of (X, μ) is not finite).

Finally, in the general—mixed—case we combine the arguments of both extremal situations to deduce that our problem always has a positive solutions (provided that the K-quotient of (X, μ) is not finite).

Notice that if the K-quotient of (X, μ) is finite, then T has pure point spectrum. This case is also studied: we record necessary and sufficient conditions for positive solution of our problem (Theorem 4.1).

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1. Canonical group covers

Let T and S be non-singular invertible transformations of standard probability spaces $(X, \mathfrak{B}, \mu)$ and $(Y, \mathfrak{C}, \nu)$ respectively. S is called an *extension* of T (and T is a *factor* of S) if there exists a Borel onto map $p : Y \to X$ such that $\nu \circ p^{-1} \sim \mu$ and $pS = Tp$. Let S and S' be two extensions of T. We write $S' \succ S$ if there is a Borel onto map $t : Y' \to Y$ such that $\nu' \circ t^{-1} \sim \nu$ and the diagram

$$\begin{array}{ccccc} Y' \; - - - & & & & - - - \; Y' \\ | & X & \xrightarrow{T} & X & | \\ Y \; - - - & & & & - - - \; Y \end{array}$$

commutes. If t is invertible, then we say that S and S' are *conjugate* and write $S' \simeq S$.

Let G be a compact metric group and λ_G probability Haar measure on G. Suppose that T is ergodic. A Borel map $\alpha : X \to G$ is called a *cocycle* of T with values in G. (We do not distinguish between two maps if they agree almost everywhere.) Let H be a closed subgroup of G and $\lambda_{H\backslash G}$ the probability G-invariant measure on the homogeneous space $H\backslash G$ (the G-action by right translations is implicit). Define a transformation $T_{G,H,\alpha}$ of $(X \times (H\backslash G), \mu \times \lambda_{H\backslash G})$ by setting

$$T_{G,H,\alpha}(x, Hg) = (Tx, Hg\alpha(x)).$$

Clearly $T_{G,H,\alpha}$ is an extension of T with the factor map $p(x, Hg) = x$. We call $T_{G,H,\alpha}$ an *isometric extension* of T with the *group cover* (G, H, α). $T_{G,1_G,\alpha}$ is said to be a *group extension* of T and denoted by T_α if no confusion arises. Since we study dynamical systems up to conjugation, extensions conjugate to isometric (group) ones are also called isometric (group) extensions. It is well-known that T_α is then conjugate to a group extension, more precisely H-extension, of $T_{G,H,\alpha}$. We say that α has *dense range* in G if T_α is ergodic. Assume that $\beta : X \to G$ is a cocycle *cohomologous* to α, i.e. there exists a Borel function $\phi : X \to G$ with $\beta(x) = \phi(x)^{-1}\alpha(x)\phi(Tx)$. Then $T_{G,H,\beta}$ is conjugate to $T_{G,H,\alpha}$; the canonical conjugacy map $t : X \times H\backslash G \to X \times H\backslash G$ is given by $t(x, Hg) = (x, Hg\phi(x))$. If N is a closed normal subgroup of G with $N \subset H$, we define an isomorphism $t : X \times (H/N)\backslash(G/N) \to X \times (H\backslash G)$ by setting

$$t(x, (H/N) \cdot Ng) = (x, Hg).$$

Clearly, $t \circ T_{G/N,H/N,N\cdot\alpha} = T_{G,H,\alpha} \circ t$, where the cocycle $N \cdot \alpha$ is determined by $(N \cdot \alpha)(x) = N\alpha(x)$. Thus $T_{G/N,H/N,N\cdot\alpha} \simeq T_{G,H,\alpha}$.

An inclusion $H \subset G$ is called *irreducible* if the corresponding G-action on $H\backslash G$ by right translations is faithful or, equivalently, H does not contain any nontrivial normal subgroup of G. It follows from the above observation that every isometric extension is conjugate to some $T_{G,H,\alpha}$ with $H \subset G$ irreducible (indeed, for every pair $H' \subset G'$ there exists a biggest normal subgroup N' of G' with $N' \subset H'$).

Let $T_{G,H,\alpha}$ be ergodic. By [Z1] that there exists a closed subgroup K of G and an α-cohomologous cocycle β of T which takes values and has dense range in K. Since $T_{G,H,\beta}$ is also ergodic, it follows that $Hg_0K = G$ for some $g_0 \in G$. Without loss in generality we may assume that $HK = G$ (otherwise set $K' = g_0Kg_0^{-1}$ and $\beta'(x) = g_0\beta(x)g_0^{-1}$ and consider $T_{G,H,\beta'}$). Then $T_{G,H,\beta} \simeq T_{K,H\cap K,\beta}$ and the conjugacy map $t : X \times (H\backslash G) \to X \times ((H \cap K)\backslash K)$ is defined by

$$t(x, Hk) = (x, (H \cap K)k)$$

for all $k \in K$. Notice that if $H \subset G$ is irreducible, so is $H \cap K \subset K$. Thus we have proved

Proposition 1.1. *Let S be an ergodic isometric extension of T. Then there is a group cover (G, H, α) such that $H \subset G$ is irreducible, α has dense range in G, and $S \simeq T_{G,H,\alpha}$.*

We call (G, H, α) a *canonical group cover* of T, if $H \subset G$ is irreducible (and $T_{G,H,\alpha}$ ergodic).

Remark 1.2. If T is measure-preserving then so is every isometric extension of T. In this case M. Mentzen [Me, Theorem 1] proved Proposition 1.1 without claiming that (G, H, α) is a canonical group cover. (He used techniques connected with joinings of dynamical systems.)

Remark 1.3. Theorem 3.1 of [Ha] states that given an ergodic finite-to-one extension S of T, there is a canonical group cover (G, H, α) of T such that $S \simeq T_{G,H,\alpha}$ and G is finite. We give here a short proof of this claim. It follows from the ergodicity of S that there is $n \in \mathbb{N}$ such that the map $X \ni x \mapsto \mathrm{Card}(\{y \mid p(y) = x\})$ is equal to n a.e. Thus we can assume that $Y = X \times \{0, \dots, n-1\}$ and $S(x, j) = (Tx, \alpha(x)^{-1}[j])$, for a Borel function $\alpha : X \to \Sigma(n)$, where $\Sigma(n)$ stands for the group of all permutations of $J := \{0, \dots, n-1\}$. Without loss of generality we can assume that α viewed as a T-cocycle takes values and has dense range in a subgroup $G \subset \Sigma(n)$ (otherwise replace α with a cohomologous cocycle $\beta : X \to \Sigma(n)$ with this property. This results to a new $S(\beta)$, however $S(\beta) \simeq S$). Since S is assumed ergodic and for each $j \in J$

$$\bigcup_{n\in\mathbb{Z}} S^n(X \times \{j\})$$

is S-invariant and of positive measure, we deduce that G acts ergodically (i.e. transitively) on J. Set $H = \{g \in G \mid g[0] = 0\}$. Clearly $H \subset G$ is irreducible. Observe now that (G, H, α) is canonical and moreover S is isomorphic to $T_{G,H,\alpha}$ via $(x, Hg) \mapsto (x, j)$, where $g[j] = 0$.

Our purpose is now to prove

Theorem 1.4. *Let T be an ergodic non-singular transformation of $(X, \mathfrak{B}, \mu)$. Assume that (G, H, α), (G', H', α') are two group covers of T, the first one canonical, the second one with α' having dense range in G'. If $T_{G',H',\alpha'} \succ T_{G,H,\alpha}$ and $t : X \times (H'\backslash G') \to X \times (H\backslash G)$ is the corresponding factor map then:*

(i) *there are a continuous epimorphism $l : G' \to G$ and a Borel function $f : X \to G$ such that $l(H') \subset H$, $t(x, H'g') = (x, Hl(g')f(x))$ and $l(\alpha'(x)) = f(x)\alpha(x)f(Tx)^{-1}$,*

(ii) *$l^{-1}(H) = H'$ if and only if t is an isomorphism,*
(iii) *if $l \restriction H'$ is one-to-one then $H' \subset G'$ is irreducible,*
(iv) *if $H' \subset G'$ is irreducible and t is an isomorphism then l is one-to-one and $l(H') = H$.*

Proof. (i) Consider the product cocycle $\alpha \times \alpha'$ of T with values in $G \times G'$. By [Z1] there exists a closed subgroup $\Pi \subset G \times G'$ and two Borel functions $\phi : X \to G$ and $\psi : X \to G'$ such that the cocycle $\beta \times \beta'$ takes values and has dense range in Π, where $\beta(x) = \phi(x)^{-1}\alpha(x)\phi(Tx)$ and $\beta'(x) = \psi(x)^{-1}\alpha'(x)\psi(Tx)$. Since α and α' have dense ranges in G and G' respectively, the two coordinate projections $\Pi \to G$, $\Pi \to G'$ are onto. Notice that $t(x, H'g') = (x, t_1(x, H'g'))$ for a Borel map $t_1 : X \times (H'\backslash G') \to H\backslash G$. Since $tT_{G',H',\alpha'} = T_{G,H,\alpha}t$, the map $\rho : X \times (H'\backslash G') \to X \times (H \times G)$ given by

$$\rho(x, H'g') = t_1(x, H'g'\psi(x)^{-1})\phi(x) \tag{1-1}$$

satisfies

$$\rho(Tx, H'g'\beta'(x)) = \rho(x, H'g')\beta(x) \tag{1-2}$$

for $\mu \times \lambda_{H'\backslash G'}$-a.e. $(x, H'g')$. Since $\Pi \to G'$ is onto, $\lambda_{G'}$ is the pullback of the Haar measure λ_Π on Π. It follows that (1-2) holds for $\mu \times \lambda_\Pi$-a.e. $(x, g, g') \in X \times \Pi$. We define a Borel function $F : X \times \Pi \to H\backslash G$ by setting $F(x, g, g') = \rho(x, H'g')g^{-1}$. Then

$$F(Tx, g\beta(x), g'\beta'(x)) = \rho(Tx, H'g'\beta'(x))\beta(x)^{-1}g^{-1} = \rho(x, H'g')g^{-1} = F(x, g, g')$$

for $\mu \times \lambda_\Pi$-a.e. (x, g, g'). Hence there is $g_0 \in G$ such that $F(x, g, g') = Hg_0$ and thus

$$\rho(x, H'g') = Hg_0 g \tag{1-3}$$

for $\mu \times \lambda_\Pi$-a.e. (x, g, g'). Without loss of generality we may assume that $g_0 = 1_G$ (otherwise replace β by the cocycle $\beta''(x) = g_0^{-1}\beta(x)g_0$ and instead of Π consider the group $\Pi' = \{(g_0^{-1}gg_0, g') \in G \times G' \mid (g, g') \in \Pi\}$). Let $N_G = \{g \in G \mid (g, 1_{G'}) \in \Pi\}$. Then N_G is a normal subgroup of G. Moreover, it is easy to deduce from (1-3) that for each $n \in N_G$ we have $Hgn = Hg$ for λ_G-a.e. $g \in G$. Since $H \subset G$ is irreducible, $n = 1_G$. Hence N_G is trivial and $\Pi = \{(l(g'), g') \mid g' \in G'\}$ for a continuous epimorphism $l : G' \to G$. Thus

$$\beta(x) = l(\beta'(x)) \tag{1-4}$$

for a.e. x. Moreover, (1-3) (always with $g_0 = 1_G$) entails $\rho(x, H'g') = Hl(g')$ for $(\mu \times \lambda_{G'})$-a.e. $(x, g') \in X \times G'$. Hence $\rho(x, H'g') = \widehat{\rho}(H'g')$ for a.e. $(x, H'g')$, where $\widehat{\rho} : H'\backslash G' \to H\backslash G$ is a continuous map given by $\widehat{\rho}(H'g') = Hl(g')$. Notice that $l(H') \subset H$ and the diagram

$$\begin{array}{ccc} G' & \xrightarrow{\ l\ } & G \\ \big\downarrow & & \big\downarrow \\ H'\backslash G' & \xrightarrow[\widehat{\rho}]{} & H\backslash G \end{array}$$

commutes. We deduce from (1-1) and (1-4) that $t_1(x, H'g') = Hl(g')f(x)$ and $l(\alpha'(x)) = f(x)\alpha(x)f(Tx)^{-1}$, where $f(x) = l(\psi(x))\phi(x)^{-1}$, as desired.

(ii), (iii) are now obvious and (iv) follows from (i) and (ii). □

Remark 1.5. Consider the category of triplets (G, H, α), where G is a compact group, H a subgroup of G and $\alpha : X \to G$ an ergodic cocycle of T. We say that there is a morphism of (G, H, α) onto (G', H', α') if there are non-singular maps $\Phi : X \times G \to X \times G'$ and $\Psi : X \times H\backslash G \to X \times H'\backslash G'$ such that the following diagram commutes

$X \times G$ — — — — $X \times G$

| $X \times H\backslash G$ — — $X \times H\backslash G$ |

| | $X \xrightarrow{T} X$ | |

| $X \times H'\backslash G'$ — —$X \times H'\backslash G'$ |

$X \times G'$ — — — — $X \times G'$,

where the skew arrows are the natural projections along the second coordinate. It follows from Theorem 1.4 that $T_{G,H,\alpha}$ is a minimal object in this category if and only if it is canonical. In the particular case of finite extensions, i.e. when G and G' are finite (see Remark 1.3), this criterion was proved by T. Hamachi [Ha, Theorem 5.1]. However our argument is different. Notice also that the existence of the minimal objects (without the criterion) was established in [JLM, Proposition 1.1].

2. Centralizers of isometric extensions

Throughout this section T is an ergodic non-singular invertible transformation of $(X, \mathfrak{B}, \mu)$ and (G, H, α) a canonical group cover of T. The *centralizer* $C(T)$ of T is the monoid of all μ-non-singular transformations commuting with T. The subgroup of invertible ones will be denoted by $C_*(T)$. T is called *coalescent* if $C(T) = C_*(T)$.

We say that $S \in C_*(T)$ can be *lifted* to $C(T_{G,H,\alpha})$ if there is a $\mu \times \lambda_{H\backslash G}$-non-singular transformation $\widetilde{S}$ of $X \times H\backslash G$ with $\widetilde{S} \in C(T_{G,H,\alpha})$ and $p\widetilde{S} = Sp$, where $p : X \times H\backslash G \to X$ is the first coordinate projection. An easy modification of the argument used in the proof of Theorem 1.4 implies

Proposition 2.1. *Every lift $\widetilde{S}$ of $S \in C_*(T)$ to $C(T_{G,H,\alpha})$ has the form*

$$\widetilde{S}(x, Hg) = S_{l,f}(x, Hg) := (Sx, Hl(g)f(x)),$$

where $f : X \to G$ is a measurable map and $l : G \to G$ is a continuous group epimorphism with $l(H) \subset H$ and

$$l(\alpha(x)) = f(x)\alpha(Sx)f(Tx)^{-1}. \tag{2-1}$$

Moreover, $S_{l,f}$ is invertible if and only if so is l. If this is the case then $l(H) = H$.

Corollary 2.2. *If S can be lifted to $C(T_{G,H,\alpha})$ then it can also be lifted to $C(T_\alpha)$.*

Proposition 2.3. *Let T have pure point spectrum. Then*

(i) *Every element of $C(T_{G,H,\alpha})$ is a lift of some transformation from $C_*(T)$.*
(ii) *If T_α is coalescent then so is $T_{G,H,\alpha}$.*
(iii) *More generally, if $T_{N\cdot\alpha}$ is coalescent for each closed normal subgroup $N \subset G$ then $T_{G,K,\alpha}$ is coalescent for each closed subgroup $K \subset G$ (cf.* [JLM, Corollary 8.1]).

Proof. (i) One should slightly modify the argument of [D1, Proposition 6.1] and apply Proposition 2.1.

(ii) follows directly from (i) while (iii) from (ii). □

Remark 2.4. Note that Propositions 2.1 and 2.3(i) extend the earlier results of [Ne] and [Me, Theorem 4], where the case of group extensions (i.e. $H = \{1_G\}$) of measure-preserving transformations was studied.

Given $k \in G$, we denote by Ad_k the inner automorphism of G defined by k, i.e. $\mathrm{Ad}_k(g) = kgk^{-1}$. Let $N_G(H)$ stand for the normalizer of H in G, i.e. $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$.

Proposition 2.5. *Every lift $\widetilde{\mathrm{Id}}$ of the identity Id of X to $C(T_{G,H,\alpha})$ is of the form*

$$\widetilde{\mathrm{Id}}(x, Hg) = (x, Hkg) = \mathrm{Id}_{\mathrm{Ad}_k,k}(x, Hg)$$

for some $k \in N_G(H)$, where k is regarded as a constant function from X to G. Moreover, $\mathrm{Id}_{\mathrm{Ad}_k,k}$ is the identity of $X \times H\backslash G$ if and only if $k \in H$.

Proof. The result follows from Proposition 2.1, Corollary 2.2 and [D1, Lemma 5.2]. □

This proposition implies that all lifts of the identity are invertible. Moreover, if $S \in C_*(T)$ and S, S^{-1} can both be lifted to $C(T_{G,H,\alpha})$ then every lift of S is invertible. Put

$$L_H(T,\alpha) = \{S \in C_*(T) \mid S, S^{-1} \text{ can be lifted to } C(T_{G,H,\alpha})\},$$
$$\widetilde{C}(T_{G,H,\alpha}) = \{\widetilde{S} \in C(T_{G,H,\alpha}) \mid \widetilde{S} \text{ is a lift of some } S \in L_H(T,\alpha)\}.$$

The two sets are groups. We also notice that

$$\sigma : N_G(H)/H \ni kH \mapsto \sigma(kH) := \mathrm{Id}_{\mathrm{Ad}_k,k} \in \widetilde{C}(T_{G,H,\alpha})$$

is a well defined one-to-one group homomorphism.

For $S \in C_*(T)$, consider a unitary operator U_S in $L^2(X,\mu)$ given by $U_S f(x) = f(Sx)\sqrt{\dfrac{d\mu\circ S}{d\mu}}(x)$. The *weak* topology on $C_*(T)$ is inherited from the strong operator topology on the unitary group $\mathcal{U}(L^2(X,\mu))$ via the embedding $S \mapsto U_S$. It is well known that $C_*(T)$ endowed with the weak topology is a Polish group. A sequence $S_n \in C_*(T)$ weakly converges to $S \in C_*(T)$ if and only if $\dfrac{d\mu\circ S_n}{d\mu} \to \dfrac{d\mu\circ S}{d\mu}$ in the $L^1(X,\mu)$-norm and $\mu(S_n^{-1}A \triangle S^{-1}A) \to 0$ as $n \to \infty$ for each $A \in \mathfrak{B}$. It is easy to see that $\widetilde{C}(T_{G,H,\alpha})$ is a closed subgroup of $C_*(T_{G,H,\alpha})$ and thus Polish.

Moreover, the map $\pi_H : \widetilde{C}(T_{G,H,\alpha}) \to C_*(T)$ given by $\pi_H(S_{l,f}) = S$ is a continuous group homomorphism. By Proposition 2.5 the range of σ is equal to $\pi_H^{-1}(\{\mathrm{Id}\})$ and hence closed. Furnish $N_G(H)/H$ with the quotient topology. Then it is easy to verify that σ is continuous and hence bicontinuous by the open mapping theorem for Polish groups. We endow $L_H(T,\alpha) = \widetilde{C}(T_{G,H,\alpha})/\pi_H^{-1}(\{\mathrm{Id}\})$ with the quotient topology, say the *L_H-topology.* By [Br], $L_H(T,\alpha)$ is a Polish group and

$$1 \to N_G(H)/H \xrightarrow{\sigma} \widetilde{C}(T_{G,H,\alpha}) \xrightarrow{\pi_H} L_H(T,\alpha) \to 1 \tag{2-2}$$

is a short exact sequence of Polish groups. Remark that the L_H- topology is stronger than the weak one restricted to $L_H(T,\alpha)$. Hence $L_H(T,\alpha)$ is a Borel subset of $C_*(T)$.

Denote by $\mathrm{Aut}\, G$ the group of all continuous automorphisms of G. It is Polish when equipped with the topology of uniform convergence. Note that $\mathrm{Aut}_H G := \{l \in \mathrm{Aut}\, G \mid l(H) = H\}$ is a closed subgroup of $\mathrm{Aut}\, G$; $\mathrm{Inn}_H G := \{\mathrm{Ad}_k \mid k \in N_G(H)\}$ and $\widetilde{\mathrm{Inn}}_H G := \{\mathrm{Ad}_k \mid k \in H\}$ are closed normal subgroups of $\mathrm{Aut}_H G$. We put $\mathrm{Out}_H G := \mathrm{Aut}_H G/\mathrm{Inn}_H G$, $\widetilde{\mathrm{Out}}_H G := \mathrm{Aut}_H G/\widetilde{\mathrm{Inn}}_H G$ and endow them with the (Polish by [Br]) quotient topologies. We also put

$$\widetilde{\tau}_H : \widetilde{C}(T_{G,H,\alpha}) \ni S_{l,f} \mapsto l \cdot \widetilde{\mathrm{Inn}}_H G \in \widetilde{\mathrm{Out}}_H G,$$
$$\tau_H : L_H(T,\alpha) \ni S \mapsto l \cdot \mathrm{Inn}_H G \in \mathrm{Out}_H G,$$

where the latter l is determined by (2-1). Proposition 2.5 implies that τ_H is well defined. Notice also that $\widetilde{\tau}_H$ and τ_H are group homomorphisms. It is rather standard to show that they are continuous (see [GLS, §4], [D1, §§5,6]).

We write $WC(T)$ for the weak closure of $\{T^n \mid n \in \mathbb{Z}\}$ in $C_*(T)$. T is said to satisfy the *weak closure property* (WCP) if $C_*(T) = WC(T)$. All ergodic rotations, rank 1 transformations [Ki], Gaussian-Kronecker transformations [FL] satisfy the WCP.

Corollary 2.6 (Weak Closure Theorem for Cocycles, cf. [GLS, Corollary 5.5])**.** *Let T satisfy the WCP and $L_H(T,\alpha) = C_*(T)$. Then $\widetilde{C}_*(T_{G,H,\alpha})$ is equal to the weak closure of $\{T^n_{G,H,\alpha}\sigma(kH) \mid n \in \mathbb{Z},\ k \in N_G(H)\}$.*

Proof. Since $L_H(T,\alpha) = C_*(T)$ and the L_H-topology is stronger than the weak one, it follows that they are equal. Use (2-2) to complete the proof. ☐

By [GLS, Proposition 5.7] if T has pure point spectrum and $L_{\{1_G\}}(T,\alpha) = C_*(T)$ then T_α has pure point spectrum and G is Abelian. Since by Corollary 2.2 $L_{\{1_G\}}(T,\alpha) \subset L_H(T,\alpha)$, we deduce

Corollary 2.7. *If T has pure point spectrum and $H \neq \{1_G\}$ then $L_H(T,\alpha) \neq C(T)$.*

3. Factors of isometric extensions

Let (G,H,α) be a canonical group cover of T. A compact subgroup $\widetilde{K} \subset C_*(T_{G,H,\alpha})$ is called *diagonal* if $\sigma(N_G(H)/H) \cap \widetilde{K} = \{\mathrm{Id}\}$. Such groups were studied in [Kw] and [Le] in connection with the factor problem of group extensions (a diagonal subgroup determines the factor of sets which are fixed by all its elements and under some additional assumptions, all factors arise in this way). In particular, an important question is whether given a compact subgroup $K \subset L_H(T,\alpha)$, there is a diagonal subgroup $\widetilde{K} \subset \widetilde{C}(T_{G,H,\alpha})$ with $\pi_H(\widetilde{K}) = K$? A partial answer to this question was given in [Le] in the case of Abelian G and trivial H (see also an earlier paper [Kw]). In this section we extend their results to arbitrary isometric extensions.

Lemma 3.1. *Let K be a weakly closed subgroup of $L_H(T,\alpha)$. Then the restriction of the L_H-topology and the weak topology to K are the same.*

Proof. One should repeat the argument used in the proof of Corollary 2.6. □

It follows that a subgroup of $L_H(T,\alpha)$ is L_H-compact if and only if it is weakly compact. Given such a subgroup K, we consider the associated short exact sequence of compact groups (cf. with (2-2))

$$1 \to N_G(H)/H \xrightarrow{\sigma} \pi_H^{-1}(K) \xrightarrow{\pi_H} K \to 1. \tag{3-1}$$

Actually, $\pi_H^{-1}(K)$ is compact as a topological group extension of a compact group by another compact one. Remind some concepts of the cohomology groups theory. Let $s : L_H(T,\alpha) \to \widetilde{C}(T_{G,H,\alpha})$ be a Borel normalized cross-section of π_H (see (2-2)), i.e. $\pi_H \circ s = \mathrm{Id}$ and $s(\mathrm{Id}) = \mathrm{Id}$. Define a Borel function $c : L_H(T,\alpha) \times L_H(T,\alpha) \to N_G(H)/H$ by setting $c(S_1,S_2) := s(S_1)s(S_2)s(S_1S_2)^{-1}$. Then c is a *"noncommutative" 2-cocycle*, i.e.

$$c(S_1,S_2)c(S_1S_2,S_3) = \mathrm{Ad}_{s(S_1)}[c(S_2,S_3)]c(S_1,S_2S_3) \text{ and}$$
$$c(S,\mathrm{Id}) = c(\mathrm{Id},S) = 1_{N_G(H)/H}$$

for all $S_1,S_2,S_3,S \in L_H(T,\alpha)$. Notice that given another Borel cross-section $s' : L_H(T,\alpha) \to \widetilde{C}(T_{G,H,\alpha})$, we have $s'(S) = d(S)s(S)$ for a Borel map $d : L_H(T,\alpha) \to N_G(H)/H$ and

$$c'(S_1,S_2) = d(S_1)\mathrm{Ad}_{s(S_1)}[d(S_2)]c(S_1,S_2)d(S_1S_2)^{-1},$$

i.e. the associated 2-cocycles c and c' are 2-*cohomologous.* By a 2-*coboundary* we mean a 2-cocycle 2-cohomologous to the trivial one.

Proposition 3.2. *The following are equivalent*:

(i) *there exists a diagonal subgroup $\widetilde{K} \subset \widetilde{C}(T_{G,H,\alpha})$ with $\pi_H(\widetilde{K}) = K$,*
(ii) (3-1) *splits,*
(iii) *the restriction of c to $K \times K$ is a 2-coboundary (this property does not depend on the particular choice of a cross-section),*
(iv) *there is a Borel cross-section s whose restriction to K is a continuous homomorphism into $\widetilde{C}(T_{G,H,\alpha})$.*

Proof. (i)⟺(ii), (iii)⟹(ii) are well known. (iv)⟹(iii) is obvious. It suffices to show that (i)⟹(iv). For $S \in K$, we set $s(S) := \widetilde{S} \in \widetilde{K}$ if $\pi_H(\widetilde{S}) = S$. Since $\widetilde{K}$ is diagonal, s is well defined. Moreover, s is continuous. Now we extend it to the entire $L_H(T,\alpha)$ in an arbitrary Borel way such that $\pi_H \circ s = \mathrm{Id}$. □

Now we will show that the properties (i)–(ii) are closely related to the problem of extending of $\mathbb{Z}$-cocycles to cocycles of larger actions (studied in [DG], [GLS], [D1]). For simplicity we shall assume that H is trivial.

Let a locally compact group A act (on the left) on $(X, \mathfrak{B}, \mu)$ via non-singular automorphisms and $v : A \ni a \mapsto v_a \in \operatorname{Aut} G$ be a continuous homomorphism. A Borel map $\beta : A \times X \to G$ is an (A, v)-*cocycle* if

$$\beta(ab, x) = \beta(a, bx)v_a(\beta(b, x)) \qquad \text{a.e.}$$

for every $a, b \in A$. If v is trivial we call β an A-*cocycle*. Two (A, v)-cocycles β and γ are *cohomologous* if there exists a Borel function $f : X \to G$ such that

$$\beta(a, x) = f(ax)\gamma(a, x)v_a(f(x))^{-1} \qquad \text{a.e.}$$

for each $a \in A$. β is called an (A, v)-*coboundary* if it is cohomologous to the trivial (A, v)-cocycle. Given a non-singular transformation T and a Borel function $\alpha : X \to G$, we define a $\mathbb{Z}$-cocycle (or simply T-cocycle) $\widehat{\alpha}$ by setting

$$\widehat{\alpha}(1, x) := \alpha(x)^{-1}, \qquad x \in X.$$

Let K be a subgroup of $C_*(T)$. Assume that $T^n \notin K$ for every $n \neq 0$. We say that α *can be extended to* K if there is a homomorphism $l : K \ni S \mapsto l_S \in \operatorname{Aut} G$ and a (K, l)-cocycle β such that

$$\alpha(Sx)^{-1}\beta(S, x) = \beta(S, Tx)l_S(\alpha(x)^{-1}) \qquad \text{a.e.} \tag{3-2}$$

for each $S \in K$. Actually, let $A := \mathbb{Z} \times K$ act on X as $(n, S)x := T^nSx$. Define a homomorphism $v : A \to \operatorname{Aut} G$ by setting $v_{(n,S)} := l_S$ and put

$$\delta((n, S), x) := \widehat{\alpha}(n, Sx)\beta(S, x) \qquad \text{for all } (n, S) \in A,\ x \in X.$$

It is easy to verify that δ is a well defined (A, v)-cocycle and $\delta((1_{\mathbb{Z}}, \mathrm{Id}), x) = \alpha(x)^{-1}$. A similar definition can be given in case of nonfree T-commuting actions of compact groups.

Corollary 3.3. *Let K be a compact subgroup of $L_{\{1_G\}}(T, \alpha)$ such that $T^n \notin K$ for every $n \neq 0$. Then α can be extended to K if and only if one of* (i)–(iv) *from Proposition* 3.2 *is satisfied. If this is the case then α is cohomologous to a T-cocycle α' with $\alpha' \circ S = l_S \circ \alpha'$ for all $S \in K$.*

Proof. Let $\mathcal{M}(X, G)$ stand for the group of G-valued measurable functions on X endowed with the (Polish) topology of convergence in measure. It is easy to deduce from Proposition 2.1 that Proposition 3.2(iv) is equivalent to the following fact: there exist two maps

$$l : K \ni S \mapsto l_S \in \mathrm{Aut}G,$$
$$f : K \ni S \mapsto f_S \in \mathcal{M}(X, G)$$

such that $K \ni S \mapsto S_{l_S, f_S} \in \widetilde{C}(T_\alpha)$ is a continuous group homomorphism. This entails that l is a continuous group homomorphism and f a continuous map satisfying

$$f_{SS'} = l_S \circ f_{S'} \cdot f_S \circ S' \qquad \text{for all } S, S' \in K. \tag{3-3}$$

We set $\beta(S,x) := f_S(x)^{-1}$, $x \in X$, $S \in K$. It follows from (3-3) that β is a (K,l)-cocycle. Then α can be extended to K if and only if (3-2) is fulfilled, which according to (2-1) is equivalent to saying that $S \in L_{\{1_G\}}(T,\alpha)$.

It is a well known fact that every cocycle of a free type I action is a coboundary (see [Sc] and [Z2]). Remind that every measurable action of a compact group is of type I. Hence there exists a function $f : X \to G$ with $\beta(S,x) = f(Sx)^{-1} l_S(f(x))$ a.e. for each $S \in K$. In view of (3-2)

$$\alpha(Sx)^{-1} f(Sx)^{-1} l_S(f(x)) = f(STx)^{-1} l_S(f(Tx)) l_S(\alpha(x)^{-1}) \tag{3-4}$$

a.e. for each $S \in K$. Put $\alpha'(x) := f(x)\alpha(x)f(Tx)^{-1}$, $x \in X$. It follows from (3-4) that α' is as desired. $\square$

Remark 3.4. Let K be as above. We define an action of $\pi^{-1}_{\{1_G\}}(K)$ on X by $S_{l,f}x := Sx$. It is not free and the stability group at each $x \in X$ equals $\pi^{-1}_{\{1_G\}}(\mathrm{Id})$, i.e. G. We set $\beta(S_{l,f},x) := f(x)^{-1}$ for all $x \in X$, $S_{l,f} \in \pi^{-1}_{\{1_G\}}(K)$. Then β is a $(\pi^{-1}_{\{1_G\}}(K), \widetilde{\tau}_{\{1_G\}})$-cocycle. It is easy to verify that α always can be extended to $\pi^{-1}_{\{1_G\}}(K)$, since this is equivalent to (2-1). Notice that $\beta(\mathrm{Id}_{\mathrm{Ad}_k,k},x) = k^{-1}$ at a.e. x for each $k \in G$.

4. Constructing cocycles with prescribed extensions of lifting groups

Let K be a compact subgroup of the centralizer of an ergodic non-singular automorphism T of $(X,\mathfrak{B},\mu)$. We shall assume that G is Abelian and $H = \{1_G\}$. Consider a topological (compact) group extension of K by G:

$$1 \to G \to E \xrightarrow{p} K \to 1. \tag{4-1}$$

Our purpose here is to find a cocycle $\alpha : X \to G$ with dense range in G such that $K \subset L(T,\alpha)$ and (4-1) is congruent to (3-1), i.e. there exists a continuous isomorphism $E \to \pi^{-1}(K)$ such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & E & \xrightarrow{p} & K & \longrightarrow & 1 \\ & & {\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id}} & & \\ 1 & \longrightarrow & G & \longrightarrow & \pi^{-1}(K) & \longrightarrow & K & \longrightarrow & 1 \end{array} \tag{4-2}$$

commutes. Assume that such an α exists. To simplify our notation we shall write $L(T,\alpha), \pi, \tau$ without the lower index $\{1_G\}$ (see §2). Notice that (4-1) determines a continuous group homomorphism $l : K \to \mathrm{Aut}\,\mathrm{G}$ (indeed, E acts on G by inner automorphisms and since G is Abelian, this gives rise to a representation of E/G, i.e. a representation of K). Since $\mathrm{Aut}\, G = \mathrm{Out}\, G$, we deduce from (4-2) that l equals τ (restricted to K). Recall that every transformation $R \in \pi^{-1}(K)$ is of the form $R = S_{l_S,f}$ where $S \in L(T,\alpha)$ and the function $f \in \mathcal{M}(X,G)$ satisfies (2-1). Moreover,

$$\pi^{-1}(S) = \{S_{l_S,\widetilde{f}} \mid \widetilde{f} - f = \text{const a.e.}\}.$$

Hence any Borel cross-section r of the projection $\pi : \pi^{-1}(K) \to K$ can be written as $r(S) = S_{l_S, F(S,.)}$, where $F : K \times X \to G$ is a Borel map satisfying

$$F_\beta(S,x) - F_\beta(S,Tx) = \alpha(Sx) - l_S(\alpha(x)) \tag{4-3}$$

at a.e. x for every $S \in K$. Next, r generates a 2-cocycle $c : K \times K \to G$ by the standard formula

$$r(S_1)r(S_2) = \mathrm{Id}_{1,c(S_1,S_2)} r(S_1S_2),$$

which is equivalent to

$$F(S_1S_2, x) + c(S_1,S_2) = F(S_1, S_2x) + l_{S_1}(F(S_2,x)) \tag{4-4}$$

at a.e. x for all $S_1, S_2 \in K$. It is well known that the 2-cohomology class of c is independent of the particular choice of the cross-section r and determined by (4-1) completely. To summarize we see that the main problem of this section can be restated in equivalent terms as follows: are there an ergodic T-cocycle $\alpha : X \to G$ and a Borel function $F : K \times X \to G$ such that (4-3) and (4-4) hold?

Let us first solve (4-4). Denote by $(Y, \mathfrak{C}, \nu, R)$ the factor of $(X, \mathfrak{B}, \mu, T)$ determined by K. Without loss in generality we may assume that:

(i) $X = Y \times K$, $\mu = \nu \times \lambda_K$,
(ii) K acts on X as $S(y, S') = (y, SS')$ for $(y, S') \in X$, $S \in K$,
(iii) $T = R_\phi$ for a cocycle $\phi : Y \to K$ with dense range in K (the last condition only for ν continuous).

It is easy to verify that the map $F : K \times (Y \times K) \to G$ given by $F(S,(y,S')) = c(S,S')$ satisfies (4-4).

Next, since $K \subset C_*(T)$, it follows that the map

$$K \times X \ni (S,x) \mapsto F(S,Tx) \in G$$

also satisfies (4-4). Then the difference

$$K \times X \ni (S,x) \mapsto F(S,x) - F(S,Tx) \in G$$

is a (K,l)-cocycle. But every (K,l)-cocycle is a coboundary since K is compact and acts freely. Hence there exists a Borel function $\alpha : X \to G$ such that (4-3) holds.

Thus our problem reduces to the following: is it possible to find such an α which has dense range in G as a T-cocycle? (Remark that the set of T-cohomology classes of the solutions of (4-3) does not change if one chooses another solution of (4-4)).

Theorem 4.1. *Let be given an ergodic non-singular automorphism T, a compact subgroup $K \subset C(T)$ and an extension* (4-1) *of K by a compact Abelian group G. If K is of infinite index in $C(T)$ then there exists an ergodic T-cocycle $\alpha : X \to G$ such that $K \subset L(T,\alpha)$ and* (4-2) *commutes. If K is of finite index in $C(T)$ then such an α exists if and only if G/N is monothetic and there is an ergodic rotation $T' : G/N \to G/N$ with $Sp\,T^m \cap Sp\,T' = \{1\}$, where $m = \#(C(T)/K)$ and N is a minimal E-normal subgroup N of G such that* $1 \to G/N \to E/N \to K \to 1$ *splits. In the second case both T and T_α have pure point spectrum.*

Proof. We proceed in several steps. Let us say that (4-1) *has a splitting quotient-extension* if there is a proper closed subgroup $N \subsetneqq G$ which is normal in E and such that the quotient extension of (4-1), namely $1 \to G/N \to E/N \to K \to 1$, splits. Equivalently, c is 2-cohomologous to a 2-cocycle with values in N.

(I) Suppose first that (4-1) has no splitting quotient-extensions. We claim that α viewed as a T-cocycle has dense range in G. Suppose the contrary: there is a proper subgroup $G' \subset G$ and a Borel function $f : X \to G$ such that the T-cocycle $\gamma(x) := f(Tx) + \alpha(x) - f(x)$ takes values and has dense range in G'. Hence, in view of (4-3)

$$(-f(Sx) + F(S,x) + l_S(f(x))) - (-f(STx) + F(S,Tx) + l_S(f(Tx))) = \gamma(Sx) - l_S(\gamma(x)) \tag{4-5}$$

for a.e. $x \in X$, $S \in K$. We need an auxiliary

Lemma 4.2 [Z1]**.** *Let G_1 and G_2 be closed subgroups of G and $\delta_1 : X \to G_1$, $\delta_2 : X \to G_2$ two T-cocycles with dense ranges in G_1 and G_2 respectively. If δ_1 and δ_2 are cohomologous* (*in G*) *then $G_1 = G_2$.*

Continue the proof of Theorem 4.1. It is clear that $\gamma \circ S$ and $l_S \circ \gamma$ are cocycles of T with dense ranges in G' and $l_S(G')$ respectively. It follows from (4-5) that they are cohomologous. By Lemma 4.2, G' is l_S-invariant for all $S \in K$, i.e. G' is a normal subgroup of E. Hence the quotient homomorphism of l, say $\widetilde{l} : K \to \mathrm{Aut}\,(G/G')$, is well defined. Put $\widetilde{f} = q \circ f$, $\widetilde{F} = q \circ F$, $\widetilde{c} = q \circ c$, where $q : G \to G/G'$ is the quotient map. Passing to the quotient group in (4-5) we obtain

$$-\widetilde{f}(Sx) + \widetilde{F}(S,x) + \widetilde{l}_S(\widetilde{f}(x)) = -\widetilde{f}(STx) + \widetilde{F}(S,Tx) + \widetilde{l}_S(\widetilde{f}(Tx))$$

for a.e. $x \in X$ and $S \in K$. Since T is ergodic,

$$\widetilde{F}(S,x) = \widetilde{f}(Sx) - \widetilde{l}_S(\widetilde{f}(x)) + a(S) \tag{4-6}$$

for $\mu \times \lambda_K$-a.e. (x,S), where $a : K \to G/G'$ is a Borel map. It is easy to see that $\widetilde{F}$ satisfies (4-4) with $\widetilde{c}$ being instead of c. Notice that $\widetilde{c}$ is a 2-cocycle associated to the cross-section $q \circ s : K \to G/G'$. It follows from (4-4) and (4-6) that $\widetilde{c}$ is a 2-coboundary. This implies that c is 2-cohomologous to a 2-cocycle with values in G', a contradiction.

(II) We consider here another particular case: (4-1) splits. Then we can assume that c is trivial. It follows that F is trivial and the left hand side of (4-3) is zero. Hence we seek a T-cocycle $\alpha : X \to G$ with dense range in G and such that $\alpha(Sx) - l_S(\alpha(x)) = 0$ a.e. for each $S \in K$. Let $\mathcal{M}_K(X,G) = \{\alpha \in \mathcal{M}(X,G) \mid \alpha \circ S = l_S \circ \alpha \text{ for all } S \in K\}$. Clearly, $\mathcal{M}_K(X,G)$ is a closed subgroup of $\mathcal{M}(X,G)$. It is easy to show that given $\alpha \in \mathcal{M}_K(X,G)$, there is a unique $d \in \mathcal{M}(Y,G)$ such that

$$\alpha((y,S)) = l_S(d(y)) \tag{4-7}$$

for a.e. $(y,S) \in X$. Moreover, the map $\mathcal{M}_K(X,G) \ni \alpha \mapsto d \in \mathcal{M}(Y,G)$ is a homeomorphism. Let $K \ltimes_l G$ stands for the semidirect product of K and G via l. The following statement is a direct corollary of [D2, Theorem 4.4].

Lemma 4.3. *If ν is continuous (i.e. non-atomic) then the subset*

$\mathcal{E} := \{d \in \mathcal{M}(Y,G) \mid$ *the R-cocycle* $\psi : Y \ni y \mapsto (\phi(y), d(y)) \in K \ltimes_l G$ *is ergodic*$\}$

is a dense G_δ in $\mathcal{M}(Y,G)$.

Continue the proof of Theorem 4.1. Take an arbitrary d from $\mathcal{E}$ (which is non-empty by Lemma 4.3) and define a T-cocycle α by (4-7). Then $\alpha \in \mathcal{M}_K(X,G)$. Moreover,

$$(R_\phi)_\alpha((y,S),g) = (Ry, S\phi(y), g + l_S(d(y))) = (Ry, (S,g)\cdot(\phi(y),d(y))) \\ = (Ry, (S,g)\cdot\psi(y)),$$

i.e. $(R_\phi)_\alpha$ is conjugate to R_ψ. Since the latter transformation is ergodic, so is $(R_\phi)_\alpha$. Hence α has dense range in G, as desired.

Now let ν be discrete. Remark that ν is necessary finite, since otherwise R is a free (aperiodic) transitive transformation and hence has no ergodic cocycles. Thus we may assume that there exists $m \in \mathbb{N}$ such that $(Y,\nu) = (\mathbb{Z}/m\mathbb{Z}, \lambda_{\mathbb{Z}/m\mathbb{Z}})$ and $Ry = y+1$ for all $y \in \mathbb{Z}/m\mathbb{Z}$. Now ϕ should be viewed as a cocycle of a nonfree $\mathbb{Z}$-action generated by R. By [Z2] this cocycle is determined by a group homomorphism from the stability group, namely $p\mathbb{Z} \subset \mathbb{Z}$ to K. Since T is ergodic, K is an Abelian monothetic group and T has pure point spectrum. More precisely, we may assume that

(a) $X = \{0,1,\dots,m-1\} \times K$ is an Abelian monothetic group with multiplication law as follows

$$(n,S) \bullet (n',S') = (n \dot{+} n', S + S' + S_0^{[(n+n')/m]})$$

for some $S_0 \in K$ with $\{S_0^n \mid n \in \mathbb{Z}\}$ being dense in K, where $\dot{+}$ stands for addition mod m and [.] for the integer part;

(b) μ is Haar measure on X;

(c) T acts by the formula $T(n,S) = (n,S) \bullet (1,0)$.

Notice that ν is discrete if and only if K is of finite index in $C(T)$ and $m = \#(C(T)/K)$.

Arguing in a similar way, we deduce that a desired T-cocycle α with dense range in G exists if and only if

(i) l is trivial (i.e. $E = K \times G$),

(ii) there is $g_0 \in G$ such that the subgroup of $K \times G$ generated by (S_0, g_0) is dense.

Remark that (ii) is equivalent to

(ii)' there is $g_0 \in G$ such that the g_0-rotation T' on (G, λ_G) is ergodic and $\mathrm{Sp}(T^m) \cap \mathrm{Sp}(T') = \{1\}$, where Sp(.) denotes the point spectrum (i.e. the group of eigenvalues).

Notice that T_α has pure point spectrum.

(III) Now consider the remaining case.

Lemma 4.4. *Let $N_1 \supset N_2 \supset \dots$ be a countable chain of E-normal subgroups of G such that the quotient short exact sequence $1 \to G/N_n \to E/N_n \to K \xrightarrow{p_n} 1$ splits for every $n \in \mathbb{N}$. Then for $N := \bigcap_{n=1}^\infty N_n$, the N-quotient sequence*

$$1 \to G/N \to E/N \to K \to 1$$

also splits.

Proof. Without loss in generality we may assume that $N = \{1_G\}$. Thus we must prove that (4-1) splits. Denote by $q_{n,l} : E/N_n \to E/N_l$ and $q_n : E \to E/N_n$ the natural quotient maps, $n > l$. Clearly, $q_{n,l} \circ q_n = q_l$ for all $n > l$. By the hypothesis of the lemma there are continuous homomorphisms $s_n : K \to E/N_n$ with $p_n \circ s_n = \mathrm{Id}$. Take a countable dense subset K' of K. Passing, if necessary, to a subsequence we may assume that $q_{n,l} \circ s_n(S)$ converges in E/N_l as $n \to \infty$ for every $S \in K'$ and $l \in \mathbb{N}$ (indeed, use a standard diagonal process). Since E (and hence every quotient group of E) is compact, there are continuous homomorphisms $s'_n : K \to E/N_n$ such that $q_{n,l} \circ s_n(S) \to s'_l(S)$ as $n \to \infty$ for every $S \in K$, $l \in \mathbb{N}$. Clearly, $p_l \circ s'_l = \mathrm{Id}$. Moreover, $q_{n,l} \circ s'_n = s'_l$ for all $n > l$. We define a map $s' : K \to E$ by setting $q_n \circ s'(S) = s'_n(S)$ for all $n \in \mathbb{N}$, $S \in K$. Since $E = \operatorname{proj\,lim}_n (E/N_n, q_{n,l})$, it follows that S' is a well defined continuous homomorphism and $p \circ S' = \mathrm{Id}$, i.e. (4-1) splits. $\square$

Continue the proof of Theorem 4.1. Let N be the minimal E-normal subgroup of G determining the (maximal) splitting quotient-extension of (4-1). Such a group exists by Lemma 4.3 and Zorn lemma (indeed, since G is second countable, it is enough to consider only countable chains of G-subgroups). Take any α satisfying (4-3). Like it was done in **(II)**, we perturb α by adding a function $f : X \to G$ such that

(a) $f \circ S = \alpha_S \circ f$ a.e. for each $S \in K$,
(b) the T-cocycle $q \circ \alpha_1 : X \to G/N$ has dense range in G/N, where $\alpha_1 = \alpha + f$ and q stands for the natural projection $G \to G/N$.

We claim that α_1 has dense range in G. Suppose the contrary: there exists a closed subgroup $G' \subset G$ such that α_1 is cohomologous to a cocycle with dense range in G'. As in (I) we see that G' is normal in E and the associated quotient-extension

$$1 \to G/G' \to E/G' \to K \to 1$$

splits. We need the following simple lemma

Lemma 4.5. *Let G be the direct product of two E-normal subgroups G_1 and G_2. Suppose that the top and the bottom lines of the diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_1 & \longrightarrow & E/G_2 & \longrightarrow & K & \longrightarrow & 1 \\
 & & {\scriptstyle p_1}\big\uparrow & & \big\uparrow & & \big\uparrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & G_1 \times G_2 & \longrightarrow & E & \longrightarrow & K & \longrightarrow & 1 \\
 & & {\scriptstyle p_2}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & G_2 & \longrightarrow & E/G_1 & \longrightarrow & K & \longrightarrow & 1
\end{array}$$

split, where $p_i : G_1 \times G_2 \to G_i$ is the natural projection, $i = 1, 2$. Then the middle line also splits.

Complete the proof of Theorem 4.1. It follows from (b) that $q(G_1) = G/N$, i.e. G_1 and N generate the whole G. Since $G/(G_1 \cap N) = G/G_1 \times G/N$, we deduce from Lemma 4.5 that the sequence

$$1 \to G/(G_1 \cap N) \to E/(G_1 \cap N) \to K \to 1$$

splits. By the choice of N we have $G' \cap N = N$, i.e. $G' \subset N$. Let q' stand for the natural projection $G \to G/G'$. By the assumption the T-cocycle $q' \circ \alpha_1$ is a coboundary. But on the other hand $q' \circ \alpha_1$ is a quotient cocycle of $q \circ \alpha_1$ and has dense range in G/G'. It follows that $G = G'$, a contradiction. We leave the case of discrete ν to the reader (combine the above argument with that of **(II)**).

Thus Theorem 4.1 is proved completely. □

Remark 4.6. Notice that the cohomological class of α is not determined uniquely by (4-1). It is interesting to observe that in the case **(I)**—absence of splitting quotient-extensions for (4-1)—every measurable solution α of (4-3) is a T-cocycle with dense range in G.

Remark 4.7. If G is totally disconnected but E (and hence K) is connected, then (4-1) has no splitting quotient-extensions. Actually, otherwise there is a E-normal subgroup $N \subsetneqq G$ such that

$$1 \to G/N \to E/N \to K \to 1$$

splits. Hence E/N is homeomorphic to the direct product G/N and K, a contradiction. We observe, in particular, that

$$1 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{T} \to \mathbb{T} \to 1$$

satisfies the conditions of the remark.

Remark 4.8. Suppose that K acts trivially on G, i.e. $l_S = \mathrm{Id}$ for each $S \in K$. Then G is contained in the center of E and hence each subgroup of G is E-normal. Let $c : K \times K \to G$ denotes the 2-cocycle as above. Then $\xi \circ c$ is a *multiplier* (see [Pa], [Le]) for every character $\xi \in \widehat{G}$. It is easy to see that (4-1) has no splitting quotient-extensions if and only if $\xi \circ c$ is *non-trivial*, (i.e. non-2-coboundary) for every character $\xi \in \widehat{G} \setminus \{1_{\widehat{G}}\}$.

Remark 4.9. Let T have pure point spectrum and α be an ergodic G-valued cocycle of T such that $L(T, \alpha) = C(T)$. It is well known that then $\widetilde{C}(T_\alpha) = C(T_\alpha)$ ([Ne], [D1]) and T_α has pure point spectrum [GLS]. Thus we have a short exact sequence of compact Abelian groups

$$1 \to G \to C(T_\alpha) \xrightarrow{\pi} C(T) \to 1. \tag{4-8}$$

Hence G is a trivial $C(T)$-module. Denote by $\widehat{H}^2(C(T), G)$ the set of all 2-cohomology classes of G-valued 2-cocycles of $C(T)$ associated to (4-8) for all possible choices of ergodic α with $L(T, \alpha) = C(T)$. The following follows from the argument used in the proof of Theorem 4.1:

(i) if α, α' are two ergodic T-cocycles which correspond to 2-cocycles c, c' with $[c] = [c'] \in \widehat{H}^2(C(T), G)$ then the difference $\alpha - \alpha'$ is cohomologous to a constant,

(ii) if α, α' are two ergodic T-cocycles which correspond to 2-cocycles c, c' with $[c] \neq [c']$ in $\widehat{H}^2(C(T), G)$ then α and α' are non-cohomologous.

Example 4.10. Let $\mathbb{Z}_2$ stand for the (compact) group of 2-adic integers and $T \in \text{Aut}(\mathbb{Z}_2, \mu_{\mathbb{Z}_2})$ be the ergodic 2-adic translation. Clearly, $C(T) = \mathbb{Z}_2$. Consider two short exact sequences

$$1 \to \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}_2 \to \mathbb{Z}_2 \to 1,$$
$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2 \to \mathbb{Z}_2 \to 1.$$

It follows from Theorem 4.1 that for the first one (resp. the second one) there is a (resp. there is not any) T-cocycle with dense range in $\mathbb{Z}/3\mathbb{Z}$ (resp. $\mathbb{Z}/2\mathbb{Z}$) such that (4-2) commutes.

Example 4.11. Note that $H^2(\mathbb{T}, \mathbb{T}) = 0$, so each ergodic circle-valued cocycle α over an irrational rotation T with $L(T, \alpha) = C(T)$ is cohomologous to a constant, while $\widehat{H}^2(\mathbb{T}, \mathbb{Z}/2\mathbb{Z}) = H^2(\mathbb{T}, \mathbb{Z}/2\mathbb{Z}) \neq 0$, so there is an ergodic $\mathbb{Z}/2\mathbb{Z}$-valued cocycle over T which is not cohomologous to a constant.

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