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MILD MIXING PROPERTY FOR SPECIAL FLOWS UNDER PIECEWISE CONSTANT FUNCTIONS

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Abstract. We give a condition on a piecewise constant roof function and an irrational rotation by α on the circle to give rise to a special flow having the mild mixing property. Such flows will also satisfy Ratner's property. As a consequence we obtain a class of mildly mixing singular flows on the two–torus that arise from quasi-periodic Hamiltonians flows by velocity changes.

1. Introduction.

A finite measure–preserving dynamical system is mildly mixing (see [8]) if its Cartesian product with an arbitrary ergodic conservative (finite or infinite) measure-preserving dynamical system remains ergodic. It is an immediate observation that the (strong) mixing of a dynamical system implies its mild mixing.

The purpose of this paper is to present a certain class of special flows over circle irrational rotations which are mildly mixing but not mixing. The first example of a mildly mixing dynamical system that is not mixing (using Gaussian processes) was constructed by Schmidt in [18]. Another famous example of a mildly mixing but not mixing system is the well–known Chacon transformation (here the mild mixing property follows directly from the minimal self-joining property). Recently, the first two authors proved in [5] that the class of special flows built from piecewise absolutely continuous functions with a non–zero sum of jumps and over a rotation by α with bounded partial quotients is mildly mixing. The absence of mixing follows from a more general result of Kochergin [13] concerning special flows built over any irrational rotation on the circle and under functions of bounded variation.

Opposite to the situation in [5], in this paper we will deal with the case where the sum of jumps of the roof function f is zero. Let $Tx = x+\alpha$ be an irrational rotation on the circle, where α has bounded partial quotients. Let $f : \mathbb{T} \to \mathbb{R}$ be a positive piecewise absolutely continuous function whose derivative is square integrable and

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let $\{\xi_1, \ldots, \xi_p\}$ stand for the set of its discontinuity points. Let $d_i = d(\xi_i)$ stand for the value of the jump at ξ_i, $i = 1, \ldots, p$, i.e.,

$$d(\xi) = \lim_{x \to \xi^-} f(x) - \lim_{x \to \xi^+} f(x).$$

Suppose that the sum of jumps of f, i.e., $S(f) = \sum_{i=1}^{p} d_i$, is equal to zero. Then without loss of generality we can assume that f is piecewise constant. Indeed, this follows from the fact that every absolutely continuous function whose derivative is square integrable is cohomologous (whenever α has bounded partial quotients) to a constant function (the proof of this fact involves the classical small divisor argument).

Let us define now two additional properties that f may satisfy:

(P1): whenever a nontrivial sum $\sum_{i=1}^{p} n_i d_i$ with $n_i \in \mathbb{Z}$, $i = 1, \ldots, p$ equals zero, there exist $1 \leq i < i' \leq p$ such that n_i and $n_{i'}$ are nonzero and $\xi_i - \xi_{i'} \in (\mathbb{Q} + \mathbb{Q}\alpha) \setminus (\mathbb{Z} + \mathbb{Z}\alpha)$;

(P2):

$$\int f(x)\, dx \notin \sum_{i=1}^{p} (\xi_i + \mathbb{Q} + \mathbb{Q}\alpha) d_i.$$

Example. If $f = a + b\chi_{[0,\xi)}$, where $a, b > 0$ and $a/b \notin \mathbb{Q} + \mathbb{Q}\alpha$ and $\xi \in (\mathbb{Q} + \mathbb{Q}\alpha) \setminus (\mathbb{Z} + \mathbb{Z}\alpha)$, then f satisfies (P1) and (P2).

The main result of the paper, the proof of which is postponed until Section 6, is the following condition for the mild mixing.

Theorem 1.1. *If α has bounded partial quotients and $f : \mathbb{T} \to \mathbb{R}^+$ is a piecewise constant function satisfying properties (P1) and (P2), then the special flow T^f is mildly mixing.*

In the case where the sum of jumps is non–zero the proof of the mild mixing property (see [5]) is based upon two tools: a property that emulates the Ratner condition introduced in [17] (we will describe it in detail in Section 5); and the absence of partial rigidity.

In the case of zero sum of jumps only the first property will survive, although Ratner's property will follow from different arguments than those from [5]; the property (P1) will play the main role. In contrast with the method from [5], special flows considered here are partially rigid (special flows under piecewise constant functions are partially rigid for an arbitrary irrational α; see Remark 5) and we cannot apply Lemma 3 in [5].

In Section 3 we deal with a natural class of singular flows on the two–torus that arise from quasi-periodic Hamiltonian flows by velocity changes. We give some conditions, under which they have special representations satisfying (P1) and (P2).

2. Basic definitions and notation.

2.1. **Factors.** Assume that $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$ is an ergodic flow on a standard probability space $(X, \mathcal{B}, \mu)$. By that we mean always a so called *measurable flow*, i.e., we require that the map $\mathbb{R} \ni t \to \langle f \circ S_t, g \rangle \in \mathbb{C}$ is continuous for each $f, g \in L^2(X, \mathcal{B}, \mu)$. Let $(t_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that $t_n \to +\infty$. We say that a flow $\mathcal{S}$ on $(X, \mathcal{B}, \mu)$ is *rigid* along (t_n) if

$$\mu(A \cap S_{-t_n} A) \to \mu(A) \tag{1}$$

for every $A \in \mathcal{B}$. By a *factor* we will mean any $\mathcal{S}$–invariant sub–σ–algebra $\mathcal{A}$ of $\mathcal{B}$. Then a factor $\mathcal{A} \subset \mathcal{B}$ of $\mathcal{S}$ is rigid along (t_n) if the convergence (1) holds for every $A \in \mathcal{A}$.

Recall that a flow is mildly mixing iff it has no non–trivial rigid factor (see [1] p. 84).

A flow $\mathcal{S}$ on $(X, \mathcal{B}, \mu)$ is called *partially rigid* along (t_n) if there exists $0 < u \leq 1$ such that

$$\liminf_{n\to\infty} \mu(A \cap S_{-t_n}A) \geq u\mu(A) \quad \text{for every} \quad A \in \mathcal{B}.$$

Lemma 2.1. *Let $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ be an ergodic measure–preserving flow on a probability standard Borel space $(X, \mathcal{B}, \mu)$. Suppose that $\mathcal{S}$ is not mildly mixing. Then there exists a nontrivial factor $\mathcal{A} \subset \mathcal{B}$ of $\mathcal{S}$ and $(t_n)_{n\in\mathbb{N}}$ with $t_n \to +\infty$ such that*

$$S_{t_n} \to E(\,\cdot\,|\mathcal{A}),$$

weakly in $\mathcal{L}(L^2(X, \mu))$, i.e., for any $g, h \in L^2(X, \mu)$ we have

$$\lim_{n\to\infty} \int_X h \circ S_{t_n} \cdot g \, d\mu = \int_X E(h|\mathcal{A}) \cdot g \, d\mu.$$

This lemma is a continuous time version of a result which can be found in [15] for measure–preserving transformations. For completeness we outline the main arguments from [15]. The proof is based on the theory of joinings of dynamical systems. For the background of this theory we refer the reader to [10]. By a *self–joining* of a flow $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ on $(X, \mathcal{B}, \mu)$ we mean any probability $(S_t \times S_t)_{t\in\mathbb{R}}$–invariant measure on $(X \times X, \mathcal{B} \otimes \mathcal{B})$ whose projections on X are equal to μ. The set of self–joinings of $\mathcal{S}$, denoted by $J(\mathcal{S})$, considered with the weak topology becomes a metrizable compact semitopological semigroup for the operation of composition of self–joinings (see [10] Ch. 6 for definitions). For every $t \in \mathbb{R}$ by $\mu_{S_t} \in J(\mathcal{S})$ we mean the graph joining determined by $\mu_{S_t}(A \times B) = \mu(A \cap S_t^{-1}B)$ for $A, B \in \mathcal{B}$. Suppose that $\mathcal{A}'$ is a non-trivial rigid factor of $\mathcal{S}$. Then we put

$$I(\mathcal{A}') = \{\rho \in J(\mathcal{S}) : \rho|_{\mathcal{A}'\otimes\mathcal{A}'} = (\mu_{Id})_{\mathcal{A}'} \ \text{ and } \ \rho \in \overline{\{\mu_{S_t} : t \in \mathbb{R}\}}\}.$$

Then, as noticed in [15], $I(\mathcal{A}')$ is a semitopological compact semigroup and hence it contains an idempotent, which corresponds to a factor fulfilling the requirements of Lemma 2.1.

Suppose that $\mathcal{A} \subset \mathcal{B}$ is a factor. Let us consider the standard (quotient) probability Borel space $(X/\mathcal{A}, \mathcal{A}, \mu_{\mathcal{A}})$, where $\mu_{\mathcal{A}}$ is the image of μ via the factor map $\pi : (X, \mathcal{B}) \to (X/\mathcal{A}, \mathcal{A})$. Denoting by $\mathcal{P}(X, \mathcal{B})$ the set of probability measures on $(X, \mathcal{B})$, let $\{\mu_{\underline{x}} \in \mathcal{P}(X, \mathcal{B}) : \underline{x} \in X/\mathcal{A}\}$ be the disintegration of μ over the factor σ–algebra $\mathcal{A}$. Then (see [7])

$$\mu_{\underline{x}}(A) = E(\chi_A|\mathcal{A})(\underline{x})$$

and

$$\mu(A) = \int_{X/\mathcal{A}} \mu_{\underline{x}}(A) \, d\mu_{\mathcal{A}}(\underline{x})$$

for every $A \in \mathcal{B}$ and for $\mu_{\mathcal{A}}$–a.e. $\underline{x} \in X/\mathcal{A}$. We say that $\mathcal{A}$ has *finite fibers* if there exists a natural number k such that $\mu_{\underline{x}}$ is the uniform measure on a k-element set for $\mu_{\mathcal{A}}$–a.e. $\underline{x} \in X/\mathcal{A}$.

2.2. **Fraction expansion and discrepancy.** We denote by $\mathbb{T}$ the circle group $\mathbb{R}/\mathbb{Z}$, which we will constantly identify with the interval $[0,1)$ with addition mod 1. Let λ stand for Lebesgue measure on $\mathbb{T}$. For a real number t denote by $\{t\}$ its fractional part and by $\|t\|$ its distance to the nearest integer number. For an irrational $\alpha\in\mathbb{T}$ denote by (q_n) its sequence of denominators (see e.g. [14]); that is, we have

$$\frac{1}{2q_nq_{n+1}}<\left|\alpha-\frac{p_n}{q_n}\right|<\frac{1}{q_nq_{n+1}},$$

where

$$\begin{array}{lll} q_0=1, & q_1=a_1, & q_{n+1}=a_{n+1}q_n+q_{n-1}\\ p_0=0, & p_1=1, & p_{n+1}=a_{n+1}p_n+p_{n-1}\end{array}$$

and $[0;a_1,a_2,\dots]$ stands for the continued fraction expansion of α.

Proposition 1 (Denjoy–Koksma inequality; see [11])**.** *If $f:\mathbb{T}\to\mathbb{R}$ is a function of bounded variation then*

$$\left|\sum_{k=0}^{q_n-1}f(x+k\alpha)-q_n\int_{\mathbb{T}}f\,d\lambda\right|\le\operatorname{Var}f$$

for every $x\in\mathbb{T}$ and $n\in\mathbb{N}$.

We say that α has *bounded partial quotients* if the sequence (a_n) is bounded. If $C=\sup\{a_n:n\in\mathbb{N}\}+1$ then

$$\frac{1}{2Cq_n}<\frac{1}{2q_{n+1}}<\|q_n\alpha\|<\frac{1}{q_{n+1}}<\frac{1}{q_n}$$

for each $n\in\mathbb{N}$.

The following lemma is well–known.

Lemma 2.2. *Let $\alpha\in\mathbb{T}$ be irrational with bounded partial quotients. Then there are constants $C_1,C_2>0$ such that for each $m\in\mathbb{N}$ the lengths of intervals I in the partition of $\mathbb{T}$ arisen from $0,-\alpha,\dots,-(m-1)\alpha$ satisfy*

$$\frac{C_1}{m}\le|I|\le\frac{C_2}{m}.$$

We will also need the following simple fact.

Lemma 2.3. *Let $\alpha\in\mathbb{T}$ be irrational with bounded partial quotients, and let $\beta\in\mathbb{Q}+\mathbb{Q}\alpha$. Then there exists $c_1>0$ such that for each $m\in\mathbb{N}$ the length of each interval in the partition of $\mathbb{T}$ arisen from $0,-\alpha,\dots,-(m-1)\alpha,\beta,\beta-\alpha,\dots,\beta-(m-1)\alpha$ is at least c_1/m.*

Proof. Assume that $\beta=(p_1+p_2\alpha)/q$, where $p_1,p_2\in\mathbb{Z}$ and $q\in\mathbb{N}$. Suppose that $0\le r,s<m$ and $\beta-s\alpha\neq-r\alpha$ mod 1. If $q(r-s)+p_2\neq0$, then from Lemma 2.2 we have

$$\begin{aligned} q\,\|(\beta-s\alpha)+r\alpha\| &\ge \|q\left((\beta-s\alpha)+r\alpha\right)\|=\|(q(r-s)+p_2)\alpha\|\\ &\ge \frac{C_1}{qm+|p_2|}\ge\frac{C_1}{(q+|p_2|)m},\end{aligned}$$

whence $\|(\beta-s\alpha)+r\alpha\|\ge\frac{C_1}{q(q+|p_2|)m}$. If $q(r-s)+p_2=0$, then

$$\|(\beta-s\alpha)+r\alpha\|=\left\|\frac{p_1}{q}\right\|\ge\frac{1}{q}\ge\frac{1}{qm}.$$

□

3. **Singular flows on two–torus.** Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞–quasi–periodic function, i.e.,

$$H(x+m, y+n) = H(x,y) + m\alpha_1 + n\alpha_2$$

for all $(x,y) \in \mathbb{R}^2$ and $m,n \in \mathbb{Z}$, and where $\alpha = \alpha_1/\alpha_2$ is irrational. Points of the two-torus will be denoted as $\overline{x}$, with real local coordinates x and y. Then H determines a (quasi–periodic) Hamiltonian flow $(h_t)_{t\in\mathbb{R}}$ on the torus associated with the following differential equation

$$\frac{d\overline{x}}{dt} = X_H(\overline{x}), \text{ where } X_H = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right).$$

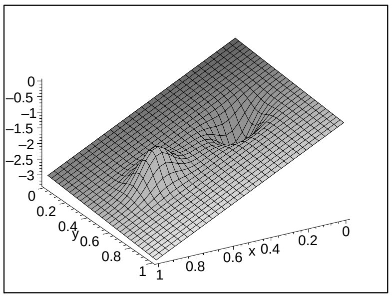

FIGURE 1. The graph of H

Suppose that H has critical points and H is in the general position, i.e., H has no degenerate critical points and has all critical values distinct. An example of the graph of such a Hamiltonian is plotted in Figure 1. In particular, each critical point is either a non–degenerate saddle point or a non–degenerate center. Moreover the critical points repeat periodically (with period 1 in each coordinate) but their critical values are distinct.

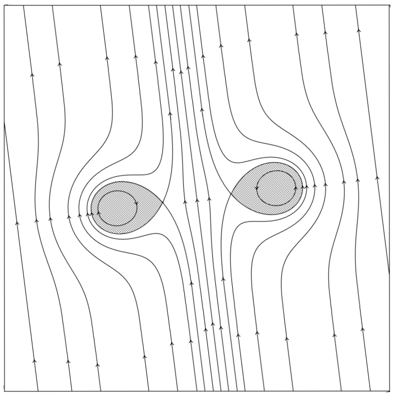

FIGURE 2. The phase portrait of the Hamiltonian flow

Then recall results by Arnold [2]. Any connected component of a level set of H passing through a critical point is either bounded (a point or a lemniscate–like curve) or it has the shape of a folium of Descartes. In the unbounded case, the critical value level set of H separates the plane into two unbounded components and a disk; the closure of the disk is called a *trap*. A trap is homeomorphic to a closed disk and has a critical point on the boundary, called the vertex of the trap (the same terminology applies when we pass to $\mathbb{T}^2$). The phase portrait of the Hamiltonian flow on the torus determined by the quasi–periodic Hamiltonian from Figure 1 is presented in Figure 2. In this case the flow has two traps (shaded areas). Traps with distinct vertices are disjoint. Therefore the phase space of $(h_t)_{t\in\mathbb{R}}$ decomposes into traps filled with fixed points, separatrices and periodic orbits, and an ergodic component EC (white area in Figure 2) of positive Lebesgue measure.

3.1. **From (h_t) to singular flows.** Now we will change velocity in the flow $(h_t)_{t\in\mathbb{R}}$. Let B_i, $i=1,\dots,p$ be all traps and $\overline{x}_i$, $i=1,\dots,p$ their vertices. Suppose $g:\mathbb{T}^2\to\mathbb{R}$ is a non–negative C^∞–function which is positive on the torus except for the points $\{\overline{x}_1,\dots,\overline{x}_p\}$. Let us consider the flow $(\varphi_t)_{t\in\mathbb{R}}$ on $\mathbb{T}^2\setminus\{\overline{x}_1,\dots,\overline{x}_p\}$ associated with the following differential equation

$$\frac{d\overline{x}}{dt}=X(\overline{x}),\ \text{ where } X(\overline{x})=\frac{X_H(\overline{x})}{g(\overline{x})}.$$

Since the orbits of (φ_t) and (h_t) are the same (modulo fixed points of (h_t)), the phase space of $(\varphi_t)_{t\in\mathbb{R}}$ decomposes into traps and the ergodic component EC with positive Lebesgue measure.

Let us denote by $\omega=\omega_X\in\mathcal{F}^1(EC)$ the 1–form of class C^∞ given by $\omega(Y)=\langle X,Y\rangle/\langle X,X\rangle$.

As it was proved in [6], $\int_{EC}d\omega_X$ exists, and if α has partial bounded quotients and $\int_{EC}d\omega_X\neq 0$ then the flow $(\varphi_t)_{t\in\mathbb{R}}$ on EC is weakly mixing and simple in the joining sense.

The aim of this paper is an exploration of the case $\int_{EC}d\omega_X=0$. Let us consider the induced 1–form $i_{X_H}\nu$ on $\mathbb{T}^2$, where $\nu=dx\wedge dy$, i.e.,

$$i_{X_H}\nu(Y)=dx\wedge dy(X_H,Y).$$

Since the flow (h_t) preserves the form ν, $d(i_{X_H}\nu)=0$ (Liouville theorem). Let $\sigma_1,\sigma_2:[0,1]\to\mathbb{T}^2$ stand for the pair of generators of $H_1(\mathbb{T}^2,\mathbb{Z})$ given by

$$\sigma_1(t)=(t,0),\quad \sigma_2(t)=(0,t)\text{ for } t\in[0,1].$$

Then

$$\begin{aligned}\int_{\sigma_1}i_{X_H}\nu &= \int_0^1 dx\wedge dy(X_H(t,0),(1,0))\,dt\\ &= \int_0^1\frac{\partial}{\partial x}H(t,0)\,dt=H(1,0)-H(0,0)=\alpha_1,\end{aligned}$$

and similarly $\int_{\sigma_2}i_{X_H}\nu=\alpha_2$.

Fix $\overline{x}_0\in\mathbb{T}^2$. Then $\xi_i=\int_{\overline{x}_0}^{\overline{x}_i}i_{X_H}\nu$ is well defined up to $\mathbb{Z}\alpha_1+\mathbb{Z}\alpha_2$ for $i=1,\dots,p$. Moreover, for every $i=1,\dots,p$ put

$$d_i=-\int_{\partial B_i}\omega_X.$$

As it was shown in [6], d_i is well defined for $i=1,\dots,p$ and

$$\sum_{i=1}^p d_i=\int_{EC}d\omega_X=0.$$

Theorem 3.1. *Suppose that α_1/α_2 is an irrational number with bounded partial quotients. Assume that $\int_{EC} d\omega_X = 0$. If X satisfies the following properties*

1. *whenever a nontrivial sum $\sum_{i=1}^{p} n_i d_i$ with $n_i \in \mathbb{Z}$, $i = 1, \ldots, p$ is equal to zero, there exist $1 \leq i < i' \leq p$ such that n_i and $n_{i'}$ are nonzero and $\xi_i - \xi_{i'} \in (\mathbb{Q}\alpha_1 + \mathbb{Q}\alpha_2) \setminus (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2)$;*
2. $$\int_{EC} g\, d\nu \notin \sum_{i=1}^{p} (\xi_i + \mathbb{Q}\alpha_1 + \mathbb{Q}\alpha_2) d_i,$$

then the flow (φ_t) is mildly mixing.

Remark 1. Notice that $\xi_i - \xi_{i'} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ does not depend on the choice of $\overline{x}_0$. Since $\sum_{i=1}^{p} d_i = 0$, $\sum_{i=1}^{p}(\xi_i + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2)d_i$ does not depend on the choice of $\overline{x}_0$ either.

3.2. **Special representation of (φ_t).** In this section we will describe a special representation of the flow (φ_t), and we will indicate why Theorem 3.1 follows from Theorem 1.1.

Assume that T is an ergodic automorphism of a standard probability space $(X, \mathcal{B}, \mu)$. If $f : X \to \mathbb{R}$ is a strictly positive integrable function, then by $T^f = (T_t^f)_{t \in \mathbb{R}}$ we will mean the special flow under f (see e.g. [3], Chapter 11) acting on $(X^f, \mathcal{B}^f, \mu^f)$, where $X^f = \{(x, s) \in X \times \mathbb{R} : \ 0 \leq s < f(x)\}$ and $\mathcal{B}^f$ (μ^f) is the restriction of $\mathcal{B} \otimes \mathcal{B}(\mathbb{R})$ ($\mu \otimes \lambda_{\mathbb{R}}$) to X^f ($\lambda_{\mathbb{R}}$ stands for Lebesgue measure on $\mathbb{R}$). Under the action of the flow T^f each point in X^f moves vertically at unit speed, and we identify the point $(x, f(x))$ with $(Tx, 0)$. More precisely, if $(x, s) \in X^f$ then

$$T_t^f(x, s) = (T^n x, s + t - f^{(n)}(x)),$$

where $n \in \mathbb{Z}$ is a unique number such that

$$f^{(n)}(x) \leq s + t < f^{(n+1)}(x)$$

with

$$f^{(m)}(x) = \begin{cases} f(x) + f(Tx) + \ldots + f(T^{m-1}x) & \text{if} \quad m > 0 \\ 0 & \text{if} \quad m = 0 \\ -\big(f(T^m x) + \ldots + f(T^{-1}x)\big) & \text{if} \quad m < 0. \end{cases}$$

We can now describe a special representation of the flow (φ_t) which appeared in Section 3. As it was proved by Arnold in [2], there exists a closed C^∞–curve in EC transversal to the orbits of $(h_t)_{t \in \mathbb{R}}$, and which is homologous to σ_2. Fix a point $\overline{x}_0$ on the transversal. Let $\sigma : [0, \alpha_2] \to \mathbb{T}^2$ be the induced parameterization of the transversal such that $\sigma(0) = \overline{x}_0$ and

$$\int_{\sigma(0)}^{\sigma(s)} i_{X_H}\nu = s \text{ for all } s \in [0, \alpha_2].$$

The first–return map (Poincaré map) is determined everywhere on the curve, except for the finite set $\{\sigma(\beta_i) : i = 1, \ldots, p\}$ of points that are points of the last intersection of the incoming separatrices of vertices of traps with the transversal curve ($\sigma(\beta_i)$ corresponds to $\overline{x}_i$). In the induced parameterization, the first–return map is the rotation by α_1 mod α_2. Moreover, $\beta_i \in \xi_i + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ for $i = 1, \ldots, p$.

For the flow $(\varphi_t)_{t \in \mathbb{R}}$ on EC we will consider the same transversal. Then σ is also the induced parameterization for the vector field X and the form $g(x, y)\, dx \wedge dy$ which is preserved by X. As it was proved in [6], the return time is a C^∞–function of the parameter from $[0, \alpha_2]$ everywhere except for points β_i, $i = 1, \ldots, p$. Moreover, this function $f : [0, \alpha_2] \to \mathbb{R}^+$ is piecewise absolutely continuous with jumps at β_i of size d_i for $i = 1, \ldots, p$, and the derivative of f is square integrable. Next notice that

$$\int_{EC} g(x, y)\, dx\, dy = \int_0^{\alpha_2} f(x)\, dx.$$

Indeed, let us consider the parameterization

$$\theta : \{(s, x) : x \in [0, \alpha_2), 0 \le s < f(x)\} \to EC \ \text{ given by } \ \theta(s, x) = \varphi_s(\sigma(x)).$$

Then

$$\frac{d}{ds}\theta(s, x) = X(\theta(s, x))$$

and

$$\int_{EC} g(x, y)\, dx\, dy = \int_0^{\alpha_2} \int_0^{f(x)} g(\theta(s, x))|\det D\theta(s, x)|\, ds\, dx.$$

Moreover,

$$\begin{aligned} \det D\theta(s, x) &= \det\left[\frac{d}{ds}\theta(s, x)\ \frac{d}{dx}\theta(s, x)\right] = \det\left[X(\theta(s, x))\ \frac{d}{dx}\theta(s, x)\right] \\ &= \frac{1}{g(\theta(s, x))} DH(\theta(s, x))\frac{d}{dx}\theta(s, x) = \frac{1}{g(\theta(s, x))}\frac{d}{dx}(H \circ \theta)(s, x). \end{aligned}$$

Moreover, $H(\theta(s, x)) = H(\varphi_s(\sigma(x))) = H(\sigma(x))$ and

$$x = \int_{\sigma(0)}^{\sigma(x)} i_{X_H}\nu = \int_0^x \nu(X_H, \sigma'(t))\, dt = \int_0^x \frac{d}{dt} H \circ \sigma(t)\, dt = H(\sigma(x)) - H(\sigma(0)).$$

Consequently,

$$\int_{EC} g\, d\nu = \int_0^{\alpha_2} \int_0^{f(x)} ds\, dx = \int_0^{\alpha_2} f(x)\, dx.$$

Finally, rescaling $[0, \alpha_2]$ by the multiplication by $1/\alpha_2$ we conclude that (φ_t) is isomorphic to the special flow built over the circle rotation $Tx = x + \alpha$, where $\alpha = \alpha_1/\alpha_2$ and under a piecewise absolutely continuous function $f : \mathbb{T} \to \mathbb{R}$ whose derivative is square integrable. Moreover, the sum of jumps $S(f)$ equals zero and if $\Xi = \{\xi_1, \ldots, \xi_p\}$ is the set of discontinuity points of f and $d_i = d(\xi_i)$ stand for the value of the jump at ξ_i, $i = 1, \ldots, p$, then f satisfies the properties (P1) and (P2). Without loss of generality we can assume that f is right-continuous and $0 \le \xi_1 < \ldots < \xi_p < 1$.

4. **Properties of special flows under piecewise constant roof functions.** Let $Tx = x + \alpha$ be an irrational rotation and let $f : \mathbb{T} \to \mathbb{R}^+$ be a piecewise absolutely continuous and right–continuous function such that $S(f) = 0$. Let $\Xi = \{\xi_1, \ldots, \xi_p\}$ stand for the set of discontinuity points of f and $d_i = d(\xi_i)$ stand for the value of the jump at ξ_i, $i = 1, \ldots, p$.

Remark 2. Recall that if α has bounded partial quotients and $\zeta : \mathbb{T} \to \mathbb{R}$ is an absolutely continuous function with zero mean such that $\zeta' \in L^2(\mathbb{T}, \lambda)$, then by the classical small divisor argument ζ is a coboundary. It follows that every piecewise absolutely function $f : \mathbb{T} \to \mathbb{R}$ whose derivative is square integrable and such that $S(f) = 0$ is cohomologous to a piecewise constant function whose discontinuities and jumps agree with discontinuities and jumps of f. We will henceforth assume that f is piecewise constant.

Remark 3. Notice that if f satisfies the property (P1) then for every $m \in \mathbb{N}$ the set $\{\xi_i - j\alpha : 1 \le i \le p, 0 \le j < m\}$ is the set of discontinuities of $f^{(m)}$. Indeed, if $f^{(m)}$ has no jump at $\xi_i - j\alpha$ then there exist $1 \le k_1 < \ldots < k_u \le p$ and $0 \le j_l < m$ for $l = 1, \ldots, u$ such that $\xi_{k_l} - j_l\alpha = \xi_i - j\alpha$ for $l = 1, \ldots, u$ and $\sum_{l=1}^{u} d_{k_l} = 0$, which is impossible because $\xi_{k_l} - \xi_{k_{l'}} \in \mathbb{Z} + \mathbb{Z}\alpha$ for all $1 \le l, l' \le u$.

Remark 4. Let a stand for the minimal value of f. Notice that

$$\int f(x)\,dx \in a + \sum_{i=1}^{p} d_i\xi_i + \sum_{i=1}^{p} \mathbb{Z}d_i.$$

Indeed, since $\sum_{i=1}^{p} d_i = 0$, we have

$$\int f(x)\,dx = f(\xi_p) - \sum_{i=1}^{p-1}\left(\sum_{j=1}^{i} d_j\right)(\xi_{i+1} - \xi_i) = a + (f(\xi_p) - a) + \sum_{i=1}^{p} d_i\xi_i.$$

Our claim follows from the fact that differences of all values of f belong to $\sum_{i=1}^{p} \mathbb{Z}d_i$. Using additionally (P2), it follows that

$$a \notin \sum_{i=1}^{p} \mathbb{Q}d_i. \tag{2}$$

We will now show that (P1) together with (P2) imply the weak mixing of T^f. Recall (see e.g. [12]) that all we need to show for the weak mixing of T^f is that for every $r \in \mathbb{R} \setminus \{0\}$ the equation

$$\psi(Tx)/\psi(x) = e^{2\pi i r f(x)}$$

has no measurable solution $\psi : \mathbb{T} \to \mathbb{S}^1$. Let $\sim \subset \Xi \times \Xi$ stand for the equivalence relation given by $x \sim y$ iff $y - x \in \mathbb{Z}\alpha$. Since $\sum_{i=1}^{p} d_i = 0$, the property (P1) implies the existence of at least two nontrivial equivalence classes.

Proposition 2 (see Proposition 1.9 (iii) and Corollary 1.6 in [9])**.** *The equation*

$$\psi(Tx)/\psi(x) = e^{2\pi i f(x)}$$

has a measurable solution $\psi : \mathbb{T} \to \mathbb{S}^1$ *iff*

$$\sum_{\xi' \sim \xi} d(\xi') \in \mathbb{Z} \textit{ for each } \xi \in \Xi \textit{ and } \int f(x)\,dx \in \mathbb{Z} + \mathbb{Z}\alpha.$$

Corollary 1. *If f satisfies (P1) and (P2) then the special flow T^f is weakly mixing.*

Proof. Suppose, contrary to our claim, that T^f is not weakly mixing. Then by Proposition 2 there exists $r \in \mathbb{R} \setminus \{0\}$ such that

$$r\sum_{\xi' \sim \xi} d(\xi') \in \mathbb{Z} \text{ for each } \xi \in \Xi \text{ and } r\int f(x)\,dx \in \mathbb{Z} + \mathbb{Z}\alpha.$$

By (P1), $\sum_{\xi' \sim \xi} d(\xi') \neq 0$ for every $\xi \in \Xi$ and hence $\sum_{\xi' \sim \xi} d(\xi') = n_\xi / r$, where $n_\xi \in \mathbb{Z} \setminus \{0\}$. It follows that $1/r \in \sum_{i=1}^{p} \mathbb{Q}d_i$, and hence

$$\int f(x)\,dx \in \frac{\mathbb{Z} + \mathbb{Z}\alpha}{r} \subset \sum_{i=1}^{p}(\mathbb{Q} + \mathbb{Q}\alpha)d_i. \tag{3}$$

Fix $\eta_1 \in \Xi$ and let η_2 be an element of Ξ such that $\eta_1 \nsim \eta_2$. Since $\sum_{\eta_i' \sim \eta_i} d(\eta') = n_i/r$ for $i = 1, 2$,

$$\sum_{\eta_1' \sim \eta_1} n_2 d(\eta_1') - \sum_{\eta_2' \sim \eta_2} n_1 d(\eta_2') = 0.$$

But $n_i \neq 0$ for $i = 1, 2$, so by (P1) there exist $\eta_1' \sim \eta_1$ and $\eta_2' \sim \eta_2$ such that $\eta_1' - \eta_2' \in \mathbb{Q} + \mathbb{Q}\alpha$, hence $\eta_1 - \eta_2 \in \mathbb{Q} + \mathbb{Q}\alpha$. Since η_1 and η_2 are arbitrary,

$$\sum_{i=1}^{p} d_i \xi_i \in \sum_{i=1}^{p} d_i(\xi_1 + \mathbb{Q} + \mathbb{Q}\alpha) = S(f)\xi_1 + \sum_{i=1}^{p}(\mathbb{Q} + \mathbb{Q}\alpha)d_i = \sum_{i=1}^{p}(\mathbb{Q} + \mathbb{Q}\alpha)d_i,$$

and hence

$$\sum_{i=1}^{p}(\xi_i + \mathbb{Q} + \mathbb{Q}\alpha)d_i = \sum_{i=1}^{p}(\mathbb{Q} + \mathbb{Q}\alpha)d_i.$$

Thus by (3)

$$\int f(x)\, dx \in \sum_{i=1}^{p}(\xi_i + \mathbb{Q} + \mathbb{Q}\alpha)d_i,$$

contrary to (P2). □

Remark 5. Notice that special flows under piecewise constant functions are partially rigid for an arbitrary irrational α. Indeed, suppose that $f : \mathbb{T} \to \mathbb{R}$ is a positive piecewise constant right-continuous function for which $\Xi = \{\xi_1, \ldots, \xi_p\}$ is the set of discontinuity points of f and $d_i = d(\xi_i)$ stand for the value of the jump at ξ_i, $i = 1, \ldots, p$. Then

$$f^{(q_n)}(x) - f^{(q_n)}(0) = -\sum_{i=1}^{p} d_i \sum_{j=0}^{q_n - 1} \chi_{(0,x]}(\xi_i - j\alpha).$$

By the Denjoy-Koksma inequality,

$$\left| \sum_{j=0}^{q_n - 1} \chi_{(0,x]}(\xi_i - j\alpha) - q_n x \right| \leq 2$$

for each $x \in \mathbb{T}$ and $\xi_i \in \Xi$. Since $\sum_{i=1}^{p} d_i = 0$, it follows that

$$f^{(q_n)}(x) - f^{(q_n)}(0) \in D := \sum_{i=1}^{p} d_i \cdot \{-2, -1, 0, 1, 2\}.$$

Put $c_f = \int f(x)\, dx$. By the Denjoy-Koksma inequality, the sequence $(f^{(q_n)} - q_n c_f)_{n \in \mathbb{N}}$ is uniformly bounded, so by passing to a subsequence, if necessary, we can assume that

$$(f^{(q_n)} - q_n c_f)_* \lambda \to P$$

in the space $\mathcal{P}(\mathbb{R})$ of Borel probability measures on $\mathbb{R}$, and $f^{(q_n)}(0) - q_n c_f \to \gamma$. Then P is concentrated on the finite set $\{\gamma\} + D$. Now an application of Proposition 4.1 from [4] yields

$$\lambda^f \left(\left(T^f\right)_{q_n c_f} A \cap B \right) \to \int_{\mathbb{R}} \lambda^f \left(\left(T^f\right)_{-t} A \cap B \right) dP(t)$$

for every $A, B \in \mathcal{B}^f$, which implies the partial rigidity of T^f along the sequence $(q_n c_f + t_0)_{n \in \mathbb{N}}$, where $t_0 \in \{\gamma\} + D$ is a point for which $P(\{t_0\}) > 0$. □

5. **Ratner's property.** Let $Tx = x + \alpha$ be an irrational rotation such that α has bounded partial quotients, and let $f : \mathbb{T} \to \mathbb{R}^+$ be a piecewise constant right–continuous function. In this section, we will show that if f satisfies properties (P1) and (P2), then the special flow T^f satisfies a condition (called Ratner's property) which emulates the Ratner condition from [17].

Definition 5.1 (cf. [5, 17, 20])**.** Let (X, d) be a σ–compact metric space, $\mathcal{B}$ be the σ–algebra of Borel subsets of X, μ be a Borel probability measure on (X, d), and $(S_t)_{t\in\mathbb{R}}$ be a flow on the space $(X, \mathcal{B}, \mu)$. Let $P \subset \mathbb{R} \setminus \{0\}$ be a finite subset and $t_0 \in \mathbb{R} \setminus \{0\}$. The flow $(S_t)_{t\in\mathbb{R}}$ is said to have *the property* $\mathrm{R}(t_0, P)$ if for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exist $\kappa = \kappa(\varepsilon) > 0$, $\delta = \delta(\varepsilon, N) > 0$ and a subset $Z = Z(\varepsilon, N) \in \mathcal{B}$ with $\mu(Z) > 1 - \varepsilon$ such that if $x, x' \in Z$, x' is not in the orbit x and $d(x, x') < \delta$, then there are $M = M(x, x') \geq N$, $L = L(x, x') \geq N$ such that $L/M \geq \kappa$ and there exists $\rho = \rho(x, x') \in P$ such that

$$\frac{\#\{n \in \mathbb{Z} \cap [M, M + L] : d(S_{nt_0}(x), S_{nt_0+\rho}(x')) < \varepsilon\}}{L} > 1 - \varepsilon.$$

Moreover, we say that $(S_t)_{t\in\mathbb{R}}$ has *the property* $\mathrm{R}(P)$ if the set of $s \in \mathbb{R}$ such that the flow $(S_t)_{t\in\mathbb{R}}$ has the $\mathrm{R}(s, P)$–property is uncountable.

As it is known (see [5] or [17] or [20]), this Ratner's property implies that the flow (S_t) is a finite extension of any its nontrivial factor.

Lemma 5.2. *Assume that f satisfies the property (P1). Then there exist a finite set $F \subset \mathbb{R} \setminus \{0\}$, $\kappa > 0$ such that for each $N \geq 1$ we can find $\delta = \delta(N) > 0$ such that for each $x \neq y$, $\|x - y\| < \delta$ there are $L = L(x, y) \geq N$, $M = M(x, y) \geq N$ such that $L/M \geq \kappa$ and for some $\rho \in F$ we have*

$$f^{(n)}(x) - f^{(n)}(y) = \rho \text{ for each integer } n \text{ between } M \text{ and } M + L.$$

Proof. Since α has bounded partial quotients, there exists $0 < c < 1$ such that

$$\|j\alpha\| \geq \frac{c}{|j|} \text{ for } j \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad \frac{q_{n+1}}{q_n} \leq \frac{1}{c} \text{ for all } n \in \mathbb{N}. \tag{4}$$

Let H be a natural number such that if $\xi_i - \xi_{i'} \in (\mathbb{Q} + \mathbb{Q}\alpha) \setminus (\mathbb{Z} + \mathbb{Z}\alpha)$ for some $1 \leq i, i' \leq p$, then there exists a natural number $h \leq H$ such that

$$h(\xi_i - \xi_{i'}) = m_1 + \alpha m_2 \text{ with } m_1, m_2 \in \mathbb{Z} \text{ and } |m_2| \leq H.$$

Put $\kappa := \dfrac{c^{10}}{4pH^2(1 + c^5)}$.

Given $N \geq 1$ we put $\delta = \dfrac{c^7}{2pH^2(1 + c^5)N}$. Now fix $x \neq y$ such that $\|x - y\| < \delta$ and let $s \geq 1$ be a unique number so that

$$\frac{2}{q_{s+1}} \leq \|x - y\| < \frac{2}{q_s}. \tag{5}$$

Then

$$q_{s+1} \geq 2/\delta = \frac{4pH^2(1 + c^5)}{c^7} N. \tag{6}$$

We will now consider $n = q_s, q_s + 1, \ldots, q_{s+4} - 1$ and look at the corresponding values $f^{(n)}(x) - f^{(n)}(y)$. We claim that for each $q_s \leq n < q_{s+4}$ we have

$$f^{(n)}(x) - f^{(n)}(y) \in V := \left\{ \sum_{i=1}^{p} r_i d_i : r_i \in [-R, R] \cap \mathbb{Z} \right\}, \tag{7}$$

where $R = 2/c^5 + 1$. Without loss of generality we can assume that $x < y$. Then the function $f^{(n)}$ is piecewise constant and

$$f^{(n)}(x) - f^{(n)}(y) = \sum_{i=1}^{p} \sum_{j=0}^{n-1} d_i \chi_{(x,y]}(\xi_i - j\alpha).$$

Therefore we only need to show that there is a bounded number (counted with possible multiplicities) of discontinuity points of $f^{(q_{s+4})}$ in the interval $(x, y]$. This, however, is clear because by (4) and (5) we have

$$\|x - y\| \leq \frac{2}{c^4} \frac{1}{q_{s+4}},$$

and for a fixed $1 \leq i \leq p$ and any $0 \leq j \neq j' < q_{s+4}$,

$$\|(\xi_i - j\alpha) - (\xi_i - j'\alpha)\| \geq \frac{c}{q_{s+4}},$$

hence for a fixed $1 \leq i \leq p$ there are at most $R = 2/c^5 + 1$ numbers $0 \leq j < q_{s+4}$ with $\xi_i - j\alpha \in (x, y]$; thus the claim is proved.

Put $F := V \setminus \{0\}$. Notice that it is enough to indicate a sufficiently long integer interval contained in $[q_s, q_{s+4})$ such that for each n in this interval $f^{(n)}(x) \neq f^{(n)}(y)$. Indeed, suppose $J \subset [q_s, q_{s+4})$ a long enough integer interval such that for each $n \in J$ we have $f^{(n)}(x) \neq f^{(n)}(y)$. In view of (7), $f^{(n)}(y) - f^{(n)}(x) \in F$ for $n \in J$. Note that if $f^{(n)}$ and $f^{(n+1)}$ have the same points of discontinuity (counted with possible multiplicities) in the interval $(x, y]$, then $f^{(n+1)}(y) - f^{(n+1)}(x) = f^{(n)}(y) - f^{(n)}(x)$. Since $f^{(q_{s+4})}$ has at most Rp discontinuities in $(x, y]$, we can split J into at most $Rp + 1$ integer intervals so that for every such interval $\widetilde{J} \subset J$ the sequence $(f^{(n)}(y) - f^{(n)}(x))_{n \in \widetilde{J}}$ is constant. Therefore if J' is the longest one then $|J'| \geq |J|/(Rp+1)$, and for some $\rho \in F$ we have $f^{(n)}(y) - f^{(n)}(x) = \rho$ for all $n \in J'$. We will show in the sequel that there exists such an interval J, with $J \subset [q_s, q_{s+4}]$ and $|J| \geq \min(q_{s-1}, \frac{cq_s}{2H^2})$. The bounds M and $M + L$ of an associated interval J' will satisfy $M, L \geq N$ and $L/M \geq \kappa$.

Suppose $0 \leq i \leq q_{s+1}$ is the smallest so that $f^{(q_s+i)}(x) = f^{(q_s+i)}(y)$; if there is no such i we pass directly to the final argument of this proof taking $J = [q_s, q_{s+1}]$. We now look for the smallest $j > 0$ such that for some $1 \leq k \leq p$, $\xi_k - (q_s + i + j)\alpha \in (x, y]$ (that is, we wait until a new discontinuity point comes in $(x, y]$). We have $0 < j < q_{s+1}$. Indeed, by basic properties of the discrepancy $D_{q_{s+1}}(\alpha)$, for each $z \in \mathbb{T}$ in each interval of length $2/q_{s+1}$ there must be a point of the form $z - w\alpha$ for some $0 \leq w < q_{s+1}$, and we apply this to the interval $(x, y]$ and $z = \xi_k - (q_s + i)\alpha$ (for each $k = 1, \ldots, p$).

Hence, in view of the definition of j,

$$f^{(q_s+i+j+1)}(x) \neq f^{(q_s+i+j+1)}(y).$$

We now consider $f^{(q_s+i+j+v)}(y) - f^{(q_s+i+j+v)}(x)$ for $v = 1, \ldots, q_{s+4} - (q_s + i + j)$ until (if it exists) v_0 for which $f^{(q_s+i+j+v_0)}(x) = f^{(q_s+i+j+v_0)}(y)$. We have

$$f^{(q_s+i+j+v_0)}(x) = f^{(q_s+i+j)}(x) + f^{(v_0)}(T^{q_s+i+j}x)$$

(with the same formula for y), so by the definition of j we obtain

$$f^{(v_0)}(T^{q_s+i+j}x) = f^{(v_0)}(T^{q_s+i+j}y).$$

Moreover, by the definition of j, we have $\xi_k \in (T^{q_s+i+j}x, T^{q_s+i+j}y]$ (we consider intervals on the circle) for some $1 \leq k \leq p$. This implies that f, hence $f^{(v_0)}$, has some discontinuity points between $x+(q_s+i+j)\alpha$ and $y+(q_s+i+j)\alpha$. Therefore there exist $n_i \in \mathbb{N}\setminus\{0\}$ for $i=1,\dots,u$, with $1\le u\le p$ and some $1\le k_1<\ldots<k_u\le p$ such that

$$0=f^{(v_0)}(T^{q_s+i+j}x)-f^{(v_0)}(T^{q_s+i+j}y)=\sum_{i=1}^{u} n_i d_{k_i},$$

which by the assumption (P1) means that there are $l\neq l'$ such that

$$\xi_{k_l}-\xi_{k_{l'}}\in(\mathbb{Q}+\mathbb{Q}\alpha)\setminus(\mathbb{Z}+\mathbb{Z}\alpha).$$

Moreover, $\|T^{q_s+i+j}x-T^{q_s+i+j}y\|<\frac{2}{q_s}$ and for some $0\le w,w'<v_0$ we have

$$\xi_{k_l}-w\alpha,\xi_{k_{l'}}-w'\alpha\in(T^{q_s+i+j}x,T^{q_s+i+j}y],$$

and in particular

$$\|(\xi_{k_l}-w\alpha)-(\xi_{k_{l'}}-w'\alpha)\|<\frac{2}{q_s}.$$

However,

$$(\mathbb{Q}+\mathbb{Q}\alpha)\setminus(\mathbb{Z}+\mathbb{Z}\alpha)\ni\xi_{k_l}-\xi_{k_{l'}}=(m_1+m_2\alpha)/h,$$

where $m_1,m_2\in\mathbb{Z}$ with $|m_2|\le H$ and $h\in\mathbb{N}$ with $h\le H$. Therefore

$$\begin{aligned}\frac{\|\left(h(w-w')-m_2\right)\alpha\|}{h}&\le\left\|\frac{m_1-(h(w-w')-m_2)\alpha}{h}\right\|\\&=\|(\xi_{k_l}-w\alpha)-(\xi_{k_{l'}}-w'\alpha)\|<\frac{2}{q_s}.\end{aligned}\tag{8}$$

Moreover, $h(w-w')-m_2\neq0$. Indeed, if $h(w-w')-m_2=0$ then $\|m_1/h\|<2/q_s<1/H\le1/h$, therefore $m_1=0$. But that gives

$$\xi_{k_l}-\xi_{k_{l'}}=\frac{m_2\alpha}{h}=(w-w')\alpha\in\mathbb{Z}\alpha,$$

which has been excluded. In view of (4), $\|\left(h(w-w')-m_2\right)\alpha\|\ge\frac{c}{h|w-w'|+|m_2|}$. Hence (by (8))

$$\frac{c}{Hv_0}\le\frac{c}{H(|w-w'|+1)}\le\frac{c}{h|w-w'|+|m_2|}\le\frac{2H}{q_s}$$

and finally $v_0\ge\frac{cq_s}{2H^2}$. It is now enough to take $J=[q_s+i+j,q_s+i+j+v_0]$ to conclude.

If $f^{(q_s+i+j+v)}(y)\neq f^{(q_s+i+j+v)}(x)$ for $v=1,\dots,q_{s+4}-(q_s+i+j)$, then we can take $J=[q_s+i+j,q_{s+4}]$.

Now let $J'\subset J$ be an integer interval such that $|J'|\ge|J|/(Rp+1)$ and for some $\rho\in F$ we have $f^{(n)}(y)-f^{(n)}(x)=\rho$ for all $n\in J'$. Let M,L be natural numbers such that $J'=[M,M+L]$. Then in view of (6) we obtain

$$\begin{aligned}L&\ge|J|/(Rp+1)\ge\frac{c^5}{2p(c^5+1)}\min\left(q_{s+1}-q_s,\frac{cq_s}{2H^2},q_{s+4}-2q_{s+1}-q_s\right)\\&\ge\frac{c^5}{2p(c^5+1)}\min\left(q_{s-1},\frac{cq_s}{2H^2},q_{s+2}\right)\ge\frac{c^7}{4pH^2(c^5+1)}q_{s+1}\ge N,\\M&\ge q_s\ge cq_{s+1}\ge N,\end{aligned}$$

and

$$\frac{L}{M}\ge\frac{c^7}{4pH^2(c^5+1)}\frac{q_{s+1}}{q_{s+4}}\ge\frac{c^{10}}{4pH^2(c^5+1)}=\kappa.$$

□

By Corollary 1, Lemma 5.2 and Lemma 6 in [5] we have the following.

Theorem 5.3. *Assume that α has bounded partial quotients and $f:\mathbb{T}\to\mathbb{R}^+$ is a piecewise constant function satisfying properties (P1) and (P2). Then the special flow T^f has the $\mathrm{R}(\gamma,P)$–property for every $\gamma\neq 0$.*

Now from Theorem 4 in [5] and the subsequent remark we have the following.

Corollary 2. *Assume that α has bounded partial quotients and $f:\mathbb{T}\to\mathbb{R}^+$ is a piecewise constant function satisfying properties (P1) and (P2). Then T^f is a finite extension of each of its non–trivial factors.*

6. **Mild mixing.** Assume that T is an irrational rotation by α which has bounded partial quotients and $f:\mathbb{T}\to\mathbb{R}^+$ is a piecewise constant function satisfying properties (P1) and (P2). In this section, using Ratner's property, we will prove that T^f is mildly mixing.

Lemma 6.1. *Let $Tx=x+\alpha$ be an irrational rotation and let $f:\mathbb{T}\to\mathbb{R}$ be a positive integrable function with respect to the measure λ. Suppose that $\mathcal{A}\subset\mathcal{B}^f$ is a factor of T^f with finite fibers, say k–point in a.e. fiber. Assume that $\mathcal{A}$ is rigid and, moreover, that $T^f_{t_n}\to E(\,\cdot\,|\mathcal{A})$, for a fixed real sequence (t_n) going to $+\infty$. Then for each $\varepsilon>0$ we can find a family $\{I_1,\ldots,I_l\}$ of pairwise disjoint intervals on $\mathbb{T}$ of the same length $\leq\varepsilon$ for which $\sum_{j=1}^{l}|I_j|>1-\varepsilon$, and $n_0\in\mathbb{N}$ such that for every $n\geq n_0$ and each $1\leq j\leq l$ there exists a measurable set $I_j^{(n)}\subset I_j$ such that $\lambda(I_j^{(n)})\geq\frac{1}{2}\frac{1}{k}|I_j|$ and if $x\in I_j^{(n)}$, then there exists $m\in\mathbb{N}$ such that $t_n\in f^{(m)}(x)+(-\varepsilon,\varepsilon)$ and $\|m\alpha\|<\varepsilon$.*

Proof. We divide the set $\mathbb{T}\times(0,\varepsilon)$ into consecutive vertical strips (squares) $S_1,\ldots,S_l$ of width ε and the last strip which may have smaller width. The family $I_1,\ldots,I_l$ is defined as the base intervals of those squares. Then $\sum_{j=1}^{l}|I_j|>1-\varepsilon$. By the assumption we know that

$$\lambda^f(T^f_{-t_n}(S_j)\cap S_j)\to\int E(S_j|\mathcal{A})\chi_{S_j}\,d\lambda^f,$$

for $j=1,\ldots,l$. Moreover, $E(S_j|\mathcal{A})(x,r)=\left(\lambda^f\right)_{\pi_{\mathcal{A}}(x,r)}(S_j)$, where

$$\lambda^f=\int\left(\lambda^f\right)_{\underline{x}}\,d\left(\lambda^f\right)_{\mathcal{A}}(\underline{x})$$

denotes the disintegration of λ^f over the factor $(\mathbb{T}^f/\mathcal{A},\left(\lambda^f\right)_{\mathcal{A}})$ (and each $\left(\lambda^f\right)_{\underline{x}}$ is the uniform measure on a k-element set) and $\pi_{\mathcal{A}}:\mathbb{T}^f\to\mathbb{T}^f/\mathcal{A}$ is the natural projection. For each $S\in\mathcal{B}^f$, for a.a. $(x,r)\in S$ we have $\left(\lambda^f\right)_{\pi_{\mathcal{A}}(x,r)}(S)\geq 1/k$. Applied to $S=S_j$, this gives

$$\int E(S_j|\mathcal{A})\chi_{S_j}\,d\lambda^f\geq\frac{1}{k}\lambda^f(S_j).$$

Therefore there exists $n_0\in\mathbb{N}$ such that for $n\geq n_0$ we have

$$\lambda^f(T^f_{-t_n}(S_j)\cap S_j)\geq\frac{1}{2k}\lambda^f(S_j)$$

for all $j=1,\ldots,l$. Suppose that $(x,t)\in T^f_{-t_n}(S_j)\cap S_j$ and $m\in\mathbb{N}$ is a unique number such that $f^{(m)}(x)\leq t+t_n<f^{(m+1)}(x)$. Then

$$T^f_{t_n}(x,t)=(T^mx,t+t_n-f^{(m)}(x))\in S_j\subset X\times(0,\varepsilon),$$

and hence $t_n \in f^{(m)}(x) + (-\varepsilon, \varepsilon)$. Since $(x,t), (T^m x, t + t_n - f^{(m)}(x)) \in S_j$ and the width of S_j is less than ε, we obtain $\|m\alpha\| = \|T^m x - x\| < \varepsilon$. Taking $I_j^{(n)} = \{x \in I_j : \exists_{0 \le t < \varepsilon}\, (x,t) \in T^f_{-t_n}(S_j) \cap S_j\}$ completes the proof. □

Assume that α has bounded partial quotients and $f : \mathbb{T} \to \mathbb{R}^+$ is a piecewise constant function satisfying properties (P1) and (P2).

Lemma 6.2. *Under the assumptions on $\mathcal{A}$ of Lemma 6.1 there exist a finite set $0 \notin F \subset \mathbb{R}$ and $0 < \xi < 1$ such that for each (sufficiently small) $\varepsilon > 0$ and $n \ge n_0$ ($n_0 = n_0(\varepsilon)$), there exists a family $(I_j)_{1 \le j \le l}$ of pairwise disjoint intervals of length $\le \varepsilon$ such that $\sum_{j=1}^{l} |I_j| > 1 - \varepsilon$ and a family $\left(\widetilde{I}_j^{(n)}\right)_{1 \le j \le l}$ of measurable sets such that $\widetilde{I}_j^{(n)} \subset I_j$ and $\lambda(\widetilde{I}_j^{(n)}) > \xi |I_j|$ for $j = 1, \dots, l$, and for every $x \in \widetilde{I}_j^{(n)}$ there exists $s \in \mathbb{N}$ for which*

$$f^{(s)}(x) \in t_n + F + (-\varepsilon, \varepsilon) \text{ and } \|s\alpha\| < \varepsilon.$$

Let us insist on the fact that the novelty of Lemma 6.2 after Lemma 6.1 is the fact that 0 does not belong to F. Roughly speaking, the proof of Lemma 6.2 goes along the following lines: a little ε and a large n are fixed. Since f is piecewise constant, by Lemma 6.1 and its proof, there is a maximal family of intervals $K_1, K_2, \dots, K_l$ such that, for each j, there exists s such that for all $x \in K_j$, $|f^{(s)}(x) - t_n| < \varepsilon$. The measure of $\cup_{j=1}^{L} K_j$ is not too small. Also, the distances between these K_j are not too small. We associate to each K_j a right adjacent interval K_j'. These new intervals will do the job.

Proof. Let a and b stand for the minimal and maximal value of f respectively. Let F be the set of sums

$$d_{k_1} + d_{k_2} + \dots + d_{k_u}$$

such that $1 \le k_1 < \dots < k_u \le p$ satisfy $\xi_{k_i} - \xi_{k_{i'}} \notin (\mathbb{Q} + \mathbb{Q}\alpha) \setminus (\mathbb{Z} + \mathbb{Z}\alpha)$ for all $1 \le i, i' \le u$. By (P1), $0 \notin F$. Let

$$F_1 = \{n_0 a + n_1 d_1 + \dots + n_p d_p : 0 < |n_0| \le 2pb/a, |n_i| \le 2pb/a + 1, i = 1, \dots, p\}.$$

In view of (2), $0 \notin F_1$. Recall that (see Remark 3) for every $m \in \mathbb{N}$ the set $\{\xi_i - j\alpha : 1 \le i \le p, 0 \le j < m\}$ is the set of discontinuities of $f^{(m)}$.

From Lemmas 2.2 and 2.3 there exist $c_1, c_2 > 0$ such that for any natural m the lengths of subintervals given by the partition of $\mathbb{T}$ into the points of discontinuity of $f^{(m)}$ are at most c_2/m, and if $\xi_i - \xi_j \in \mathbb{Q} + \mathbb{Q}\alpha$ and $\xi_i - r\alpha \ne \xi_j - s\alpha$ then

$$\|(\xi_i - r\alpha) - (\xi_j - s\alpha)\| \ge \frac{c_1}{m} \text{ for all } 0 \le r, s < m. \tag{9}$$

Fix $0 < \varepsilon < \min(a/2, 1/2)$ such that

$$(-2\varepsilon, 2\varepsilon) \cap (F \cup F_1) = \emptyset. \tag{10}$$

By Lemma 6.1 we get the family of intervals $\{I_1, \dots, I_l\}$ and the natural number $n_0 = n_0(\varepsilon)$. We consider now $n \ge n_0$ such that

$$t_n > \max(16klc_2 b, 3\varepsilon). \tag{11}$$

We now estimate for how many j's the graph of $f^{(j)}$ has a non-empty intersection with the rectangle $\mathbb{T} \times [t_n - \varepsilon, t_n + \varepsilon]$. If m is the maximal, and m_1 is the minimal such as j, then $f^{(m)} \ge am$ and $f^{(m_1)} \le m_1 b$, so

$$m_1 \ge m \cdot \frac{a}{2b} \quad \text{and} \quad t_n/2 \le m_1 b. \tag{12}$$

Furthermore, the length of any interval on which $f^{(s)}$ is constant for some $m_1 \leq s \leq m$ is at most c_2/m_1. Moreover, for every $m_1 \leq s \leq m$,

$$\begin{aligned} &\text{if } 0 < x' - x < c_1/m, \;\; x, x' \text{ are continuity points of } f^{(s)} \\ &\text{and } f^{(s)} \text{ is not constant on } [x, x'], \text{ then } f^{(s)}(x') - f^{(s)}(x) \in F. \end{aligned} \tag{13}$$

Indeed,

$$f^{(s)}(x) - f^{(s)}(x') = \sum_{i=1}^{p} \sum_{j=0}^{s-1} d_i \chi_{(x,x')}(\xi_i - j\alpha),$$

and by (9) if $\xi_i - j\alpha, \xi_{i'} - j'\alpha \in (x, x')$ then either $\xi_i - \xi_{i'} \notin \mathbb{Q} + \mathbb{Q}\alpha$ or $\xi_i - j\alpha = \xi_{i'} - j'\alpha$. It follows that for every $1 \leq i \leq p$ there exists at most one $0 \leq j < s$ such that $\xi_i - j\alpha \in (x, x')$, and hence

$$f^{(s)}(x) - f^{(s)}(x') = d_{k_1} + d_{k_2} + \ldots + d_{k_u},$$

where $1 \leq k_1 < \ldots < k_u \leq p$ and $\xi_{k_i} - \xi_{k_{i'}} \notin (\mathbb{Q} + \mathbb{Q}\alpha) \setminus (\mathbb{Z} + \mathbb{Z}\alpha)$ for all $1 \leq i, i' \leq u$.

Let us consider the set

$$\widehat{I}_j^{(n)} := \bigcup_{m_1 \leq s \leq m} \mathrm{Int}\{x \in I_j : f^{(s)}(x) \in (t_n - \varepsilon, t_n + \varepsilon), \|s\alpha\| < \varepsilon\}.$$

Then $\widehat{I}_j^{(n)}$ is the union of disjoint consecutive intervals $J_1, \ldots, J_{N+1}$, i.e., J_i lies on the left hand side of J_{i+1} for all $1 \leq i \leq N$. Since $I_j^{(n)} \subset \bigcup_{i=1}^{N+1} \overline{J_i}$, we have $\lambda(\widehat{I}_j^{(n)}) > \frac{1}{2k}|I_j|$.

Suppose that $x \in \widehat{I}_j^{(n)}$ and $f^{(s)}(x), f^{(s')}(x) \in (t_n - \varepsilon, t_n + \varepsilon)$ for some $m_1 \leq s \leq s' \leq m$. Then

$$(s' - s)a \leq f^{(s'-s)}(T^s x) = f^{(s')}(x) - f^{(s)}(x) \in [0, 2\varepsilon) \subset [0, a),$$

hence $s = s'$.

Suppose that $x \in J_i$ and s is a natural number such that $f^{(s)}(x) \in t_n + (-\varepsilon, \varepsilon)$. Then $f^{(s)}$ is constant on J_i. Indeed, otherwise there exist continuity points $x, x' \in J_i$ of $f^{(s)}$ such that $0 < x' - x < c_1/m$ and $f^{(s)}$ is not constant on $[x, x']$; then by (13),

$$f^{(s)}(x') - f^{(s)}(x) \in F \cap (-2\varepsilon, 2\varepsilon),$$

which is impossible. It follows also that for every $i = 1, \ldots, N+1$ there exists $m_1 \leq s_i \leq m$ such that $\|s_i\alpha\| < \varepsilon$ and $f^{(s_i)}(x) \in t_n + (-\varepsilon, \varepsilon)$ for all $x \in J_i$. It is also clear that the ends of the interval J_i are discontinuities of $f^{(s_i)}$ for $i = 1, \ldots, N+1$, with the possible exceptions of the left hand end of J_1 and the right hand end of J_{N+1}. Therefore $|J_i| \leq c_2/m_1$ for $i = 1, \ldots, N+1$. Moreover,

$$\frac{1}{4kl} \leq \frac{1}{2k}|I_j| \leq \lambda(I_j^{(n)}) \leq \lambda(\bigcup_{i=1}^{N+1} J_i) \leq (N+1)\frac{c_2}{m_1}. \tag{14}$$

Hence, by (12) and (11) we obtain

$$N + 1 \geq \frac{m_1}{4klc_2} \geq \frac{t_n}{8klc_2 b} \geq 2$$

and hence $N \geq 1$.

Notice that for every $1 \leq i \leq N$ the distance between J_i and J_{i+1} is at least c_1/m. Otherwise, there exist continuity points $x \in J_i$, $x' \in J_{i+1}$ for $f^{(s_{i+1})}$ such that $0 < x' - x < c_1/m$. Since $f^{(s_{i+1})}$ is not constant on $[x, x']$, by (13), we have $f^{(s_{i+1})}(x') - f^{(s_{i+1})}(x) \in F$. Then

$$\begin{aligned} f^{(s_{i+1}-s_i)}(T^{s_i}x) &= f^{(s_{i+1})}(x) - f^{(s_i)}(x) \in f^{(s_{i+1})}(x') - f^{(s_i)}(x) - F \\ &\subset -F + (-2\varepsilon, 2\varepsilon) \subset (-2pb, 2pb), \end{aligned}$$

since $F \subset (-pb, pb)$. It follows that $|s_{i+1} - s_i| \leq 2pb/a$ and hence

$$f^{(s_{i+1}-s_i)}(T^{s_i}x) = (s_{i+1} - s_i)a + n_1 d_1 + \ldots + n_p d_p,$$

where $|n_j| \leq 2pb/a$ for $j = 1, \ldots, p$. If $s_{i+1} \neq s_i$ then

$$(-2\varepsilon, 2\varepsilon) \ni f^{(s_{i+1})}(x') - f^{(s_i)}(x) \in f^{(s_{i+1}-s_i)}(T^{s_i}x) + F \subset F_1,$$

which is a contradiction. Therefore $s_{i+1} = s_i$ and then

$$f^{(s_i)}(x') - f^{(s_i)}(x) \in F \cap (-2\varepsilon, 2\varepsilon),$$

which is also impossible.

For every $i = 1, \ldots, N$ let J_i' stand for the open right adjacent to J_i interval of length c_1/m. If $x' \in J_i'$ is a continuity point for $f^{(s_i)}$ then we can find $x \in J_i$ such that $0 < x' - x < c_1/m$. So we have

$$f^{(s_i)}(x') \in f^{(s_i)}(x) + F \subset F + t_n + (-\varepsilon, \varepsilon), \tag{15}$$

by (13), because $f^{(s_i)}$ is not constant on $[x, x']$. Clearly, (15) holds also for possible discontinuity points of $f^{(s_i)}$. Moreover,

$$\lambda(\bigcup_{i=1}^{N} J_i') \geq \frac{N}{N+1} \frac{c_1/m}{c_2/m_1} \lambda(\bigcup_{i=1}^{N+1} J_i) \geq \frac{c_1 a}{8kc_2 b}|I_j|,$$

by (14) and (12). Set

$$\widetilde{I}_j^{(n)} := \bigcup_{i=1}^{N} J_i'.$$

Then $\lambda(\widetilde{I}_j^{(n)}) > \xi|I_j|$, where $\xi = \frac{c_1 a}{8kc_2 b}$. □

Lemma 6.3. *Under the assumptions on $\mathcal{A}$ of Lemma 6.1 there exist a finite set F, with $0 \notin F \subset \mathbb{R}$ and a real ξ, with $0 < \xi \leq 1$ such that for each $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and a finite family $\{P_{i,j,\varepsilon}\}_{i,j}$ of pairwise disjoint rectangles in $\mathbb{T}^f$ with diameter less than ε and whose union is of λ^f-measure at least $1 - \varepsilon$ such that for each $P_{i,j,\varepsilon}$ and $n \geq n_0$, there exists $\widetilde{P}_{i,j,\varepsilon}^{(n)} \subset P_{i,j,\varepsilon}$ with $\lambda^f(\widetilde{P}_{i,j,\varepsilon}^{(n)}) \geq \xi\lambda^f(P_{i,j,\varepsilon})$ such that if $(x,t) \in \widetilde{P}_{i,j,\varepsilon}^{(n)}$ then*

$$d(T^f_{t_n+\rho}(x,t), (x,t)) < \varepsilon \text{ for some } \rho \in F.$$

Proof. We fix $\varepsilon > 0$ and consider the family $\{I_i\}_{i=1}^{l}$ as in Lemma 6.1 (taking $\varepsilon/2$ instead of ε). Let us consider the family of all rectangles of the form $P_{i,j,\varepsilon} = I_i \times (j\varepsilon/2, (j+1)\varepsilon/2)$ which are subsets of

$$\begin{aligned} \mathbb{T}^f_\varepsilon &:= \{(x,t) \in \mathbb{T}^f : \varepsilon < t < f(x) - \varepsilon\} \\ &\quad \setminus \left(\bigcup_{u=1}^{p} [\xi_u - \varepsilon, \xi_u + \varepsilon] \times [\liminf_{x \to \xi_u} f(x) - \varepsilon, \limsup_{x \to \xi_u} f(x)] \right). \end{aligned}$$

Taking ε small enough, we can make $\lambda^f(\cup_{i,j} P_{i,j,\varepsilon})$ arbitrarily close to 1. By Lemma 6.2, there exist $0 < \xi < 1$ (independent of ε) and n_0 such that for each $n \geq n_0$ there exists a family of measurable sets $\{\widetilde{I}_1^{(n)}, \ldots, \widetilde{I}_l^{(n)}\}$ such that $\widetilde{I}_i^{(n)} \subset I_i$

and $\lambda(\widetilde{I}_i^{(n)}) > \xi|I_i|$ for $i = 1, \ldots, l$ and for every $x \in \widetilde{I}_i^{(n)}$ there exist $s \in \mathbb{N}$ and $\rho \in F$ for which

$$f^{(s)}(x) \in t_n + \rho + (-\varepsilon/2, \varepsilon/2) \text{ and } \|s\alpha\| < \varepsilon/2. \tag{16}$$

Suppose that $(x,t) \in P_{i,j,\varepsilon}$ and $x \in \widetilde{I}_i^{(n)}$. Let s be a natural number for which (16) holds. Since $(x,t) \in \mathbb{T}^f_\varepsilon$ and $\|s\alpha\| < \varepsilon$, we have $(T^s x, t) \in \mathbb{T}^f$ and $\varepsilon < t < f(T^s x) - \varepsilon$. Let $\varepsilon_n = t_n + \rho - f^{(s)}(x)$. Then $|\varepsilon_n| < \varepsilon/2$ and $0 < t + \varepsilon_n < f(T^s x)$. Therefore $(T^s x, t + \varepsilon_n) \in \mathbb{T}^f$ and

$$(T^s x, t + \varepsilon_n) = T^f_{\varepsilon_n + f^{(s)}(x)}(x,t) = T^f_{t_n+\rho}(x,t).$$

Consequently,

$$d(T^f_{t_n+\rho}(x,t),(x,t)) = \|T^s x - x\| + |\varepsilon_n| = \|s\alpha\| + |\varepsilon_n| < \varepsilon.$$

Now taking $\widetilde{P}^{(n)}_{i,j,\varepsilon} = \{(x,t) \in P_{i,j,\varepsilon} : x \in \widetilde{I}_i^{(n)}\}$ completes the proof. □

Lemma 6.4. *Under the assumptions on $\mathcal{A}$ of Lemma 6.1 there exist a finite set F, with $0 \notin F \subset \mathbb{R}$ and a real ξ, with $0 < \xi \leq 1$ such that for every $A \in \mathcal{B}^f$ we have*

$$\sum_{\rho \in F} \lambda^f(T^f_{-t_n-\rho}A \cap A) \geq \frac{\xi}{2}\lambda^f(A)$$

for each n large enough.

Proof. First suppose that $A \subset \mathbb{T}^f$ is a rectangle. For every $\varepsilon > 0$ let

$$A_\varepsilon = \bigcup_{i,j}\{P_{i,j,\varepsilon} \subset A : d(P_{i,j,\varepsilon}, \partial A) > \varepsilon\}.$$

Fix $\varepsilon > 0$ such that $\lambda^f(A_\varepsilon) \geq \frac{1}{2}\lambda^f(A)$. By Lemma 6.3, for all n large enough there exists $\widetilde{P}^{(n)}_{i,j,\varepsilon} \subset P_{i,j,\varepsilon}$ with $\lambda^f(\widetilde{P}^{(n)}_{i,j,\varepsilon}) \geq \xi\lambda^f(P_{i,j,\varepsilon})$ such that if $(x,t) \in \widetilde{P}^{(n)}_{i,j,\varepsilon}$ then

$$d(T^f_{t_n+\rho}(x,t),(x,t)) < \varepsilon \text{ for some } \rho \in F.$$

Let

$$A_{\varepsilon,n} = \bigcup_{i,j}\{\widetilde{P}^{(n)}_{i,j,\varepsilon} \subset A_\varepsilon : d(\widetilde{P}^{(n)}_{i,j,\varepsilon}, \partial A) > \varepsilon\}.$$

Then $\lambda^f(A_{\varepsilon,n}) > \xi\lambda^f(A_\varepsilon)$. If for each $\rho \in F$ we put

$$A^\rho_{\varepsilon,n} = \{(x,t) \in A_{\varepsilon,n} : d(T^f_{t_n+\rho}(x,t),(x,t)) < \varepsilon\},$$

then $A^\rho_{\varepsilon,n} \subset T^f_{-(t_n+\rho)}A \cap A$ and $A_{\varepsilon,n} = \bigcup_{\rho\in F} A^\rho_{\varepsilon,n}$, which yields

$$\frac{\xi}{2}\lambda^f(A) \leq \lambda^f(A_{\varepsilon,n}) \leq \sum_{\rho\in F}\lambda^f(A^\rho_{\varepsilon,n}) \leq \sum_{\rho\in F}\lambda^f(T^f_{-t_n-\rho}A \cap A).$$

From this inequality, it is immediate that

$$\frac{\xi}{2}\lambda^f(A) \leq \sum_{\rho\in F}\lambda^f(T^f_{-t_n-\rho}A \cap A)$$

is still true for any finite union of pairwise disjoint rectangles. By standard approximation of a Borel set in $\mathbb{T}\times\mathbb{R}$ by such unions, the inequality extends to any element of $\mathcal{B}^f$. □

Proof of Theorem 1.1. Suppose, contrary to our claim, that T^f is not mildly mixing. By Lemma 2.1, there exists a nontrivial rigid factor $\mathcal{A} \subset \mathcal{B}^f$ of T^f and $(t_n)_{n\in\mathbb{N}}$ with $t_n \to +\infty$ such that $T^f_{t_n} \to E(\,\cdot\,|\mathcal{A})$ weakly. From Corollary 2 it follows that $\mathcal{A} \subset \mathcal{B}^f$ is a factor of T^f with finite fibers. Using Lemma 6.4, for every $A \in \mathcal{A}$ we have

$$\sum_{\rho\in F} \lambda^f(T^f_{-t_n-\rho}A \cap A) \geq \frac{\xi}{2}\lambda^f(A)$$

for each n large enough. Since $\lambda^f(T^f_{t_n}A \triangle A) \to 0$, we conclude that

$$\sum_{\rho\in F} \lambda^f(T^f_{-\rho}A \cap A) \geq \frac{\xi}{2}\lambda^f(A)$$

for every $A \in \mathcal{A}$. Let us consider the flow T^f on $\mathbb{T}^f/\mathcal{A}$. First notice that $\mathbb{T}^f/\mathcal{A}$ cannot be finite. Otherwise, since the extension $\mathbb{T}^f \to \mathbb{T}^f/\mathcal{A}$ is finite, $\mathbb{T}^f$ is finite, which is a contradiction. As T^f on $\mathbb{T}^f/\mathcal{A}$ is ergodic and $\mathbb{T}^f/\mathcal{A}$ is not finite, a Rokhlin lemma for flows insures the existence of $A \in \mathcal{A}$ of positive measure such that $T^f_{-\rho}A \cap A = \emptyset$ for all $\rho \in F$. □

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