# APPROXIMATE ORTHOGONALITY OF POWERS FOR ERGODIC AFFINE UNIPOTENT DIFFEOMORPHISMS ON NILMANIFOLDS

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ABSTRACT. Let $G$ be a connected, simply connected nilpotent Lie group and $\Gamma < G$ a lattice. We prove that each ergodic diffeomorphism $\phi(x\Gamma) = uA(x)\Gamma$ on the nilmanifold $G/\Gamma$, where $u \in G$ and $A : G \to G$ is a unipotent automorphism satisfying $A(\Gamma) = \Gamma$, enjoys the property of asymptotically orthogonal powers (AOP). Two consequences follow:

(i) Sarnak's conjecture on Möbius orthogonality holds in every uniquely ergodic model of an ergodic affine unipotent diffeomorphism;

(ii) For ergodic affine unipotent diffeomorphisms themselves, the Möbius orthogonality holds on so called typical short interval:

$$\frac{1}{M} \sum_{M \leq m < 2M} \left| \frac{1}{H} \sum_{m \leq n < m+H} f(\phi^n(x\Gamma)) \boldsymbol{\mu}(n) \right| \to 0$$

as $H \to \infty$ and $H/M \to 0$ for each $x\Gamma \in G/\Gamma$ and each $f \in C(G/\Gamma)$.

In particular, the results in (i) and (ii) hold for ergodic nil-translations. Moreover, we prove that each nilsequence is orthogonal to the Möbius function $\boldsymbol{\mu}$ on a typical short interval.

We also study the problem of lifting of the AOP property to induced actions and derive some applications on uniform distribution.

## CONTENTS

## 1. INTRODUCTION

Following [Sar11], we say that a homeomorphism $T$ of a compact metric space $X$ is *Möbius orthogonal* if for each $f \in C(X)$ and $x \in X$, we have

$$\frac{1}{N} \sum_{n \le N} f(T^n x) \boldsymbol{\mu}(n) \to 0,$$

as $N \to \infty$. Here $\boldsymbol{\mu} : \mathbb{N} \to \mathbb{C}$ stands for the Möbius function defined by: $\boldsymbol{\mu}(1) = 1$, $\boldsymbol{\mu}(n) = (-1)^k$, if $n = p_1 p_2 \cdots p_k$ for distinct prime numbers $p_1, \ldots, p_k$, and $\boldsymbol{\mu}(n) = 0$ otherwise. In 2010, P. Sarnak formulated the following conjecture, referred to as *Sarnak's conjecture on Möbius orthogonality*:

> Every zero entropy homeomorphism of a compact metric space
> is Möbius orthogonal.

Since then, the Möbius orthogonality has been proved in many particular cases of zero entropy homeomorphisms: [EAKL16], [EALdlR14], [Bou13], [BSZ13], [DDM15], [DK15], [FKLM15], [FM], [GT12], [Kar15], [Mï6], [Pec15].

Sarnak's conjecture is mainly motivated by some open problems in number theory, for example, the famous Chowla conjecture on auto-correlations of the Möbius function implies Sarnak's conjecture (see [HKLD14], [Sar11], [Tao12] for more details).

As stated above, Sarnak's conjecture deals with topological dynamical systems. However, we may consider it by looking at all invariant measures for the homeomorphim $T$. (By the variational principle, all such measures have measure-theoretic entropy zero.) Then we will deal with an ergodic theory problem considered in the class of measure-theoretic dynamical systems with zero measure-theoretic entropy.

Reversing the problem, we may start with an ergodic, zero entropy automorphism $R$ of a probability standard Borel space $(Z, \mathcal{D}, \nu)$ and suppose that $T$ is a *uniquely ergodic model of $R$*, that is, $T$ is a homeomorphism of a compact metric space $X$ with a unique $T$-invariant probability Borel measure $\mu$ such that the measure-theoretic systems $(Z, \mathcal{D}, \nu, R)$ and $(X, \mathcal{B}(X), \mu, T)$ are measure-theoretically isomorphic. We recall that, by the Jewett-Krieger theorem, each ergodic system $R$ has many uniquely ergodic models, some of which

may even be topologically mixing (see also [Leh87]). In such a setting it is natural to ask if there are some ergodic properties of $R$ which are sufficient to guarantee that **every** uniquely ergodic model $T$ of $R$ will be Möbius orthogonal.

For example, irrational rotations are well-known to be Möbius orthogonal [Dav37] but the fact that the Möbius orthogonality holds in all uniquely ergodic topological models of irrational rotations has been proved only recently in [HLD15]. In [HLD15], such a result has been achieved by introducing a new invariant of measure-theoretic isomorphism, namely, the AOP property (which, in particular, holds for all irrational rotations). The Möbius orthogonality holds true in every uniquely ergodic model of a measure-theoretic automorphism $R$ satisfying the AOP property.

Let us recall briefly the definition of the AOP property (see Section 2.1, Definition 2.1 and Remark 2.1 for more details). An ergodic automorphism $R$ of a probability standard Borel space $(Z, \mathscr{D}, \nu)$ is said to have *asymptotically orthogonal powers* (AOP), or to satisfy the *AOP property*, if the Hausdorff distance of the set $J(R^p, R^q)$ of joinings between $R^p$ and $R^q$ from the singleton $\{\nu \otimes \nu\}$ goes to zero when $p$ and $q$ are different prime numbers tending to infinity.

The AOP property for the automorphism $R$ implies that all non-zero powers of it are ergodic, i.e. $R$ is *totally ergodic.*

Denoting by $\mathscr{P}$ the set of prime numbers, a consequence of the AOP property is that in each uniquely ergodic model $(X, T)$ of $(Z, \mathscr{D}, \nu, R)$, we have

$$(1.1) \qquad \limsup_{\substack{p,q \to \infty \\ p,q \in \mathscr{P} \\ p \neq q}} \limsup_{K \to \infty} \left| \frac{1}{b_{K+1}} \sum_{k \leq K} \sum_{b_k \leq n < b_{k+1}} f(T^{pn} x_k) \overline{f(T^{qn} x_k)} \right| = 0$$

for each increasing sequence of integers $(b_k)$ with $b_{k+1} - b_k \to \infty$, for each sequence of points $(x_k) \subset X$ and a zero mean function $f \in C(X)$. By the Kátai-Bourgain-Sarnak-Ziegler criterion[1] ([BSZ13], [Kát86]), we obtain

$$(1.2) \qquad \frac{1}{b_{K+1}} \sum_{k \leq K} \left( \sum_{b_k \leq n < b_{k+1}} f(T^n x_k) \boldsymbol{u}(n) \right) \to 0, \text{ when } K \to \infty,$$

for every multiplicative function $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}$ bounded by 1 and all $(b_k)$, $(x_k)$ and $f$ as above. If we set $x_k = x$ for all $k \geq 1$ and $\boldsymbol{u} = \boldsymbol{\mu}$, then (1.2) means that the Möbius orthogonality holds for $T$. However, in many concrete situations, from (1.2), we can deduce the stronger conclusion

$$\lim_{K \to \infty} \frac{1}{b_{K+1}} \sum_{k \leq K} \left| \sum_{b_k \leq n < b_{k+1}} f(T^n x) \boldsymbol{u}(n) \right| = 0,$$

or, equivalently (see [HLD15]),

$$\frac{1}{M} \sum_{M \leq m < 2M} \left| \frac{1}{H} \sum_{m \leq n < m+H} f(T^n x) \boldsymbol{u}(n) \right|, \text{ when } H \to \infty, H/M \to 0,$$

for each zero mean function $f \in C(X)$, $x \in X$ and $\boldsymbol{u}$ as above.

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[1]This criterion (in the form used in [HLD15]) says that if $(a_n) \subset \mathbb{C}$ is bounded and $\limsup_{p \neq q \to \infty, p,q \in \mathscr{P}} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \leq N} a_{pn} \overline{a}_{qn} \right| = 0$ then $\frac{1}{N} \sum_{n \leq N} a_n \boldsymbol{u}(n) \to 0$ for each multiplicative $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}, |\boldsymbol{u}| \leq 1$.

We use this criterion by considering $(a_n)$ given by $(f(x_1), \ldots, f(T^{b_1-1} x_1), f(T^{b_1} x_2), \ldots)$.

The AOP property has been proved in [HLD15] for so-called quasi-discrete spectrum automorphisms [Abr62]. This implies the Möbius orthogonality of all uniquely ergodic models of quasi-discrete spectrum automorphisms. Some uniquely ergodic models of such automorphisms are given by affine transformations on compact abelian (metric) groups. For **these** particular models the Möbius orthogonality has been proved earlier [GT12], [LS15]. Recall that the affine transformations are examples of distal homeomorphisms.[2] Another natural class of distal (uniquely ergodic) homeomorphisms is given by nil-translations and, more generally, affine unipotent diffeomorphisms of nilmanifolds. Recall that if $G$ is a connected, simply connected, nilpotent Lie group, $\Gamma < G$ is a lattice and $u \in G$, then each rotation $l_u(x\Gamma) := ux\Gamma$ acting on $G/\Gamma$ is called a *nil-translation by $u$*. More generally, we may consider *affine unipotent* diffeomorphisms on $G/\Gamma$, that is maps of $G/\Gamma$ of the form $\phi(x\Gamma) := uA(x)\Gamma$, where $A$ is a unipotent automorphism of $G$ such that $A(\Gamma) = \Gamma$ and $u \in G$. For such maps, the Möbius orthogonality has been proved in [GT12], where even some quantitative version of it has been proved. Liu and Sarnak in [LS15] suggest that perhaps instead of proving Sarnak's conjecture, it will be easier first to prove the Möbius orthogonality in the distal case (see [KPL15], [LS15] and [Wan15] for some results in this direction). Instead, we can ask if, assuming total ergodicity, we have the AOP property.

In the present paper, we confirm that the AOP approach may bring some fruits by proving the following.

**Theorem A.** *Let $G$ be a connected, simply connected, nilpotent Lie group and $\Gamma < G$ a lattice. Then, every ergodic affine unipotent diffeomorphism $\phi : G/\Gamma \to G/\Gamma$ has the AOP property.*

By the previous discussion, the following result is immediate:

**Corollary B.** *The Möbius orthogonality holds in every uniquely ergodic model of any affine unipotent diffeomorphism of a compact connected nilmanifold.*

Moreover, the algebraic structure of the underlying space $G/\Gamma$ will let us prove the following.

**Corollary C.** *Let $G$ be a connected, simply connected, nilpotent Lie group and $\Gamma < G$ a lattice. Let $\phi : G/\Gamma \to G/\Gamma$ be an ergodic affine unipotent diffeomorphism. Then, for every $x \in G$, for every zero mean function $f \in C(G/\Gamma)$ and for every multiplicative function $\mathbf{u} : \mathbb{N} \to \mathbb{C}$ bounded by $1$, we have*

$$(1.3) \qquad \frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le h < m+H} f(\phi^h(x\Gamma)) \, \mathbf{u}(h) \right| \to 0$$

*as $H \to \infty$ and $H/M \to 0$. For $\mathbf{u} = \boldsymbol{\mu}$ the result holds for arbitrary $f \in C(G/\Gamma)$.*

The property expressed by (1.3) will be referred to as the *Möbius orthogonality on typical short interval*. One more consequence of Theorem A is the following sample of a result when this property takes place.

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[2]A homeomorphism $T$ is distal if the orbit $(T^n x, T^n y)$, $n \in \mathbb{N}$, is bounded away from the diagonal in $X \times X$ for each $x \ne y$.

**Proposition D.** *Assume that $P \in \mathbb{R}[x]$ is a non-zero degree polynomial with irrational leading coefficient. Then for all $\gamma \in \mathbb{R} \setminus \{0\}$ and $\varrho \in \mathbb{R}$, we have*

$$(1.4) \qquad \frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le h < m+H} e^{2\pi i P([\gamma h + \varrho])} \boldsymbol{\mu}(h) \right| \longrightarrow 0$$

*as $H \to \infty$ and $H/M \to 0$.*

Recall that a sequence $(a_n) \subset \mathbb{C}$ is called a *nilsequence* if it is a uniform limit of sequences of the form $(f(l_u^n(x\Gamma)))$, where $G/\Gamma$ is a compact nilmanifold.[3] Green and Tao [GT12] proved that all nilsequences are orthogonal to $\boldsymbol{\mu}$ (their result is quantitative). We will prove the Möbius orthogonality on typical short interval:

**Theorem E.** *For each nilsequence $(a_n) \subset \mathbb{C}$, we have*

$$\frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le h < m+H} a_h \boldsymbol{\mu}(h) \right| \to 0$$

*when $H \to \infty$, $H/M \to 0$.*

As Leibman in [Lei10] proved that all polynomial multi-correlation sequences are nilsequences in the Weyl pseudo-metric (see Section 4.7), we obtain the following result.

**Corollary F.** *For every automorphism $T$ of a probability standard Borel space $(X, \mathcal{B}, \mu)$, the polynomial multi-correlations functions are orthogonal to the Möbius function on a typical short interval, that is, for every $g_i \in L^\infty(X, \mu)$, $p_i \in \mathbb{Z}[x]$, $i = 1, \ldots, k$ $(k \ge 1)$, we have*

$$\frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le h < m+H} \boldsymbol{\mu}(h) \int_X g_1 \circ T^{p_1(h)} \cdot \ldots \cdot g_k \circ T^{p_k(h)} \, d\mu \right| \longrightarrow 0$$

*when $H \to \infty$ and $H/M \to 0$.*

In some classical cases (e.g. when $G$ is $H_3(\mathbb{R})$), it is well-known that nil-translations are time automorphisms of the suspension flows over affine automorphisms of the torus. Since affine automorphisms on tori enjoy the AOP property [HLD15], a natural question arises whether the AOP property of an automorphism implies the AOP property of the suspension flow. This, in fact, motivates a more general question. The AOP property can be studied for actions of (general) abelian groups, see Section 5.2. Passing from automorphisms (i.e. $\mathbb{Z}$-actions) to their suspensions ($\mathbb{R}$-actions) is a particular case of inducing [Zim78]. Our aim will be to prove the following result about the "relative" AOP property for induced actions.

**Proposition G.** *Let $H$ be a closed cocompact subgroup of a locally compact second countable abelian group $G$. Assume that $H$ has no non-trivial compact subgroups. Assume, moreover, that an $H$-action $\mathcal{S}$ on a probability standard*

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[3]$G/\Gamma$ need not be connected and $l_u$ need not be ergodic.

*Borel space* $(Y, \mathscr{C}, \nu)$ *has the AOP property. Then, for each* $E, F \in L^2(Y \times G/H) \ominus$ $L^2(G/H)$, *we have*

$$\lim_{p \neq q, p, q \in \mathscr{P}, p, q \to \infty} \sup_{\kappa \in J^e(\widetilde{\mathscr{S}}^{(p)}, \widetilde{\mathscr{S}}^{(q)})} \left| \int_{Y \times G/H \times Y \times G/H} E \otimes F \, d\kappa \right| = 0$$

*for the induced G-action* $\widetilde{\mathscr{S}}$, *i.e. the induced G-action has the "relative" AOP property.*

An application of this result for $H = k\mathbb{Z}$ and $G = \mathbb{Z}$ yields the following:

**Corollary H.** *Assume that* $T$ *is a uniquely ergodic homeomorphism of* $X$, *with the unique invariant measure* $\mu$. *Assume moreover, that* $(X, \mu, T)$ *has the AOP property. Then, for each multiplicative function* $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}$, $|\boldsymbol{u}| \leq 1$, *for each* $k \geq 1$ *and* $0 \leq j < k$, *we have*

$$(1.5) \qquad \frac{1}{N} \sum_{n \leq N} f(T^n x) \boldsymbol{u}(kn + j) \to 0$$

*for each* $f \in C(X)$ *of zero mean and each* $x \in X$. *In particular,*

$$(1.6) \qquad \frac{1}{N} \sum_{n \leq N} f(T^n x) \boldsymbol{\mu}(kn + j) \to 0.$$

**Corollary I.** *Let* $(a_n)$ *be a sequence* $\left( f(\phi^n(x\Gamma)) \right)$, $\left( e^{2\pi i P([\gamma n + \varrho])} \right)$, *an arbitrary nil-sequence or* $\left( \int_X g_1 \circ T^{P_1(n)} \cdot \ldots \cdot g_r \circ T^{P_r(n)} \, d\mu \right)$ *as in Corollary* C, *Proposition* D, *Theorem* E *and Corollary* F, *respectively. Then, for each* $k, j \in \mathbb{N}$, *we have*

$$(1.7) \qquad \frac{1}{M} \sum_{M \leq m < 2M} \left| \sum_{m \leq n < m + H} a_n \boldsymbol{\mu}(kn + j) \right| \to 0$$

*when* $H \to \infty$, $H/M \to 0$.

Finally, we will also give new examples of AOP flows with partly continuous singular spectra.

The second part of the paper has a form of an appendix in which we provide some more information on Lie groups, but also we elucidate the approach to prove the AOP property for nil-translations through Lie group apparatus as special properties of measure-theoretical distal automorphisms.

## 2. ON THE AOP PROPERTY AND THE MÖBIUS ORTHOGONALITY

2.1. **Joinings. The AOP property.** Assume that $T$ and $S$ are ergodic automorphisms of probability standard Borel spaces $(X, \mathscr{B}, \mu)$ and $(Y, \mathscr{C}, \nu)$, respectively. A $T \times S$-invariant probability measure $\rho$ on $(X \times Y, \mathscr{B} \otimes \mathscr{C})$ is called a *joining* of $T$ and $S$ if the marginals of $\rho$ on $X$ and $Y$ are equal to $\mu$ and $\nu$, respectively. The product measure $\mu \otimes \nu$ is a joining of $T$ and $S$, called the *product joining*. In particular, the set $J(T, S)$ of joinings of $T$ and $S$ is not empty. If $\rho \in J(T, S)$ is ergodic for $T \times S$, then $\rho$ is called an *ergodic joining* and we write $\rho \in J^e(T, S)$. The set $J(T, S)$, endowed with the vague topology, is a closed simplex for the natural affine structure on the space of probability Borel measures on $(X \times Y, \mathscr{B} \otimes \mathscr{C})$. Then $J^e(T, S)$ is the set of extremal points of $J(T, S)$.

The automorphisms $T$ and $S$ are called *disjoint* if their only joining is the product joining $\mu \otimes \nu$, (see [Fur67]); in this case we write $T \perp S$.

All above notions have obvious generalizations to measure preserving actions of groups and semi-groups.

The following definition can be introduced for ergodic actions of more general groups (see Section 5.2); later on, we shall consider the AOP property for flows.

Let $L_0^2(X, \mathscr{B}, \mu)$ stand for the subspace of zero mean function in $L^2(X, \mathscr{B}, \mu)$ and recall that $\mathscr{P}$ stands for the set of prime natural numbers.

**Definition 2.1** ([HLD15]). A totally ergodic automorphism $T$ of a probability standard Borel space $(X, \mathscr{B}, \mu)$ has *asymptotically orthogonal powers* (AOP) if for each $f, g \in L_0^2(X, \mathscr{B}, \mu)$, we have

$$(2.1) \qquad \limsup_{\substack{p,q \to \infty \\ p,q \in \mathscr{P} \\ p \neq q}} \sup_{\rho \in J^e(T^p, T^q)} \left| \int_{X \times X} f \otimes g \, d\rho \right| = 0.$$

In this case, we also say that $T$ has the AOP property.

Clearly, if the prime powers of $T$ are pairwise disjoint, then $T$ enjoys the AOP property. There are, however, other natural examples, see [HLD15], [KPL15]; in fact, it may even happen that all non-zero powers of automorphism having AOP are isomorphic.

**Remark 2.1.** By definition, $\rho_n \to \rho$ for the vague topology if and only if we have $\int_{X \times Y} f \otimes g \, d\rho_n \to \int_{X \times Y} f \otimes g \, d\rho$ for all $f, g \in L_0^2(X, \mathscr{B}, \mu)$. Then, since $L_0^2(X, \mathscr{B}, \mu)$ is separable, the vague topology on $J(T, S)$ is metrizable. Then the AOP property for an automorphism $T$ states that, when $p$ and $q$ are distinct large primes, all ergodic joinings (and hence all joinings) of $T^p$ and $T^q$ are uniformly close to product measure.

Moreover, in order to show the AOP property for $T$, we only need to check the property (2.1) for $f, g$ belonging to a linearly dense subset in $L_0^2(X, \mathscr{B}, \mu)$.

Assume that automorphisms $T$ and $S$ have a *common factor*, i.e. there exists an ergodic automorphism $R$ defined on a probability space $(Z, \mathscr{D}, \kappa)$ with equivariant factor maps $\pi_{X,Z} : (X, \mathscr{B}, \mu) \to (Z, \mathscr{D}, \kappa)$ and $\pi_{Y,Z} : (Y, \mathscr{C}, \nu) \to (Z, \mathscr{D}, \kappa)$. Let

$$\mu = \int_Z \mu_z \, d\kappa(z), \ \nu = \int_Z \nu_z \, d\kappa(z)$$

be the disintegrations of the measures $\mu$ and $\nu$. Then every joining $\rho \in J(R, R)$ has a natural extension to a joining $\widehat{\rho} \in J(T, S)$ determined by[4]

$$(2.2) \qquad \int_{X \times Y} f \otimes g \, d\widehat{\rho} = \int_{Z \times Z} \left( \int_X f \, d\mu_{z_1} \int_Y g \, d\nu_{z_2} \right) d\rho(z_1, z_2)$$

for all $f, g \in L^\infty(X, \mathscr{B}, \mu)$. The joining $\widehat{\rho}$ is called the *relatively independent extension* of $\rho$. The joining $\widehat{\rho}$ need not be ergodic, even if $\rho$ is; however, the image via $\pi_{X,Z} \times \pi_{Y,Z}$ of almost every ergodic component of $\widehat{\rho}$ is equal to $\rho$.

## 2.2. A criterion for AOP and Möbius orthogonality on typical short intervals.
Theorem A and Corollaries B and C will be proved by showing that affine unipotent diffeomorphisms of compact nilmanifolds satisfy the hypotheses of the following theorem.

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[4]Recall that, up to an abuse of notation, $\int_X f \, d\mu_{z_1} = \mathbb{E}(f | \pi_{X,Z}^{-1}(\mathscr{D}))(z_1)$ and similarly $\int_Y g \, d\nu_{z_2} = \mathbb{E}(g | \pi_{Y,Z}^{-1}(\mathscr{D}))(z_2)$.

**Theorem 2.2.** *Let $T : X \to X$ be a homeomorphism of a compact metric space. Assume that $T$ is totally and uniquely ergodic for the $T$-invariant probability Borel measure $\mu$. Let $\mathscr{C} \subset C(X) \cap L_0^2(X, \mu)$ be a set whose linear span is dense in $L_0^2(X, \mu)$.*

(i) *Assume that, for all $f_1, f_2 \in \mathscr{C}$ and for all but a finite number of pairs of distinct prime numbers $(r, s)$, we have $\rho(f_1 \otimes \bar{f}_2) = 0$ for all ergodic joinings $\rho$ of $T^r$ and $T^s$.*

   *Then $T$ satisfies the AOP property and the Möbius orthogonality holds in every uniquely ergodic model of $(X, \mu, T)$.*

(ii) *Assume further that for all $f \in \mathscr{C}$ and all $\omega \in \mathbb{C}$ with $|\omega| = 1$ there exists a homeomorphism $S : X \to X$ such that $f(T^n(Sx)) = \omega f(T^n x)$ for every $x \in X$ and $n \in \mathbb{Z}$.*

   *Then, for every $x \in X$, for every every zero mean $f \in C(X)$ and every multiplicative $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}$ bounded by 1, we have*

$$(2.3) \qquad \frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le n < m+H} f(T^n x) \boldsymbol{u}(n) \right| \longrightarrow 0$$

   *when $H \to \infty$ and $H/M \to 0$. If $\boldsymbol{u} = \boldsymbol{\mu}$ the result holds for arbitrary $f \in C(X)$.*

In view of Remark 2.1, since the set $\mathscr{C}$ is linearly dense in $L_0^2(X, \mu)$, Part (i) of the above theorem was proved in [HLD15] (Theorem 2). So we only need to prove Part (ii).

**Remark 2.3.** If $f$ is constant or more general $f(T^n x) = \exp(ina)$ for some $a \in \mathbb{R}$, then (2.3) holds true, when $\boldsymbol{u}$ is the Möbius function, by Theorem 1.7 in [MRT15] (see also the discussion preceding this theorem).

*Proof of Theorem 2.2.* Suppose that the hypotheses of (i) and (ii) are satisfied. Let $(b_k)_{k \ge 1}$ be an increasing sequence of natural numbers such that $b_{k+1} - b_k \to +\infty$.

Fix $x \in X$, $f \in \mathscr{C} \subset C(X) \cap L_0^2(X, \mu)$ and a multiplicative function $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}$ bounded by 1. For every $k \ge 1$, let $\omega_k \in \mathbb{C}$ be the number of modulus 1 such that

$$\left| \sum_{b_k \le n < b_{k+1}} f(T^n x) \boldsymbol{u}(n) \right| = \omega_k \sum_{b_k \le n < b_{k+1}} f(T^n x) \boldsymbol{u}(n).$$

By hypothesis, there exists a homeomorphism $S_k : X \to X$ such that $f(T^n S_k x) = \omega_k f(T^n x)$ for every $n \in \mathbb{Z}$. Let $x_k := S_k x$. Then

$$(2.4) \qquad \left| \sum_{b_k \le n < b_{k+1}} f(T^n x) \boldsymbol{u}(n) \right| = \sum_{b_k \le n < b_{k+1}} f(T^n x_k) \boldsymbol{u}(n).$$

By Theorem 3 in [HLD15], the AOP property implies that for all zero mean $f \in C(X)$, for all sequences $(x_k)_{k \ge 1}$ in $X$ and for all multiplicative functions $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}$ bounded by 1, we have

$$\frac{1}{b_{K+1}} \sum_{k \le K} \left( \sum_{b_k \le n < b_{k+1}} f(T^n x_k) \boldsymbol{u}(n) \right) \to 0 \text{ when } K \to \infty.$$

In view of (2.4), it follows that

$$(2.5) \qquad \frac{1}{b_{K+1}} \sum_{k \le K} \left| \sum_{b_k \le n < b_{k+1}} f(T^n x) \boldsymbol{u}(n) \right| \to 0 \text{ when } K \to \infty.$$

As $f \in \mathscr{C}$ was arbitrary, it follows that (2.5) holds also for every $f \in \mathrm{span}\,\mathscr{C}$.

Let $f$ be an arbitrary continuous function on $X$ with zero mean. Since the space $\mathrm{span}\,\mathscr{C}$ is dense in $L_0^2(X, \mu)$, for every $\varepsilon > 0$ there exists $f_\varepsilon \in \mathrm{span}\,\mathscr{C}$ such that $\int_X |f - f_\varepsilon|\,d\mu < \varepsilon$. Then

$$\frac{1}{b_{K+1}} \sum_{k \le K} \Big| \sum_{b_k \le n < b_{k+1}} f(T^n x)\boldsymbol{u}(n) \Big| \le \frac{1}{b_{K+1}} \sum_{k \le K} \Big| \sum_{b_k \le n < b_{k+1}} f_\varepsilon(T^n x)\boldsymbol{u}(n) \Big|$$
$$+ \frac{1}{b_{K+1}} \sum_{n \le b_{K+1}} |f(T^n x) - f_\varepsilon(T^n x)|.$$

Since $T$ is uniquely ergodic and $|f - f_\varepsilon|$ is continuous, we have

$$\frac{1}{b_{K+1}} \sum_{n \le b_{K+1}} |f(T^n x) - f_\varepsilon(T^n x)| \to \int_X |f - f_\varepsilon|\,d\mu < \varepsilon \text{ as } K \to \infty.$$

It follows that

$$\limsup_{K \to \infty} \frac{1}{b_{K+1}} \sum_{k \le K} \Big| \sum_{b_k \le n < b_{k+1}} f(T^n x)\boldsymbol{u}(n) \Big| \le \varepsilon$$

for each $\varepsilon > 0$ which proves (2.5) for all continuous functions $f$ with zero mean. According to [HLD15] (see the proof of Theorem 5) this implies (1.3) for every continuous function $f$ with zero mean. $\qquad \square$

**Remark 2.4.** Let $T$ be a uniquely ergodic homeomorphism of a compact metric space $X$ and let $\mu$ be the unique $T$-invariant probability measure. Let $\boldsymbol{u}$ be a multiplicative function bounded by 1. Suppose that the conclusion of Theorem 2.2 (ii) holds true: for each continuous functions $f : X \to \mathbb{C}$ with zero mean and $x \in X$, we have

$$(2.6) \qquad A(f, M, H) := \frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le n < m+H} f(T^n x)\boldsymbol{u}(n) \right| \longrightarrow 0$$

as $H \to \infty$ and $H/M \to 0$. Then the approximation argument used in the proof of Theorem 2.2 (ii) yields the validity of (2.6) for a more general class of functions.

Let $D(X)$ be the space of bounded measurable functions $f : X \to \mathbb{C}$ such that the closure of the set of discontinuity point of $f$ has zero measure. First note that every $f \in D(X)$ satisfies the following equidistribution condition

$$(2.7) \qquad \frac{1}{N} \sum_{n \le N} f(T^n x) \to \int_X f\,d\mu \text{ for all } x \in X.$$

Indeed, since the closure $F$ of the set of discontinuities of $f$ has zero measure, for every $\varepsilon > 0$ there exists an open set $F \subset U$ such that $\mu(\overline{U}) < \varepsilon$. By Tietze's extension theorem and the continuity of $f$ restricted to $X \setminus U$, there exists a continuous function $f_\varepsilon : X \to \mathbb{C}$ such that $f(x) = f_\varepsilon(x)$ for $x \in X \setminus U$ and $\|f_\varepsilon\|_{\sup} \le \|f\|_{\sup}$. Then

$$\left| \frac{1}{N} \sum_{n \le N} f(T^n x) - \int_X f\,d\mu \right| \le \left| \frac{1}{N} \sum_{n \le N} f_\varepsilon(T^n x) - \int_X f_\varepsilon\,d\mu \right|$$
$$+ 2\|f\|_{\sup} \Big( \frac{1}{N} \sum_{n \le N} \chi_U(T^n x) + \mu(U) \Big).$$

By unique ergodicity, $\frac{1}{N} \sum_{n \leq N} \delta_{T^n x} \to \mu$ weakly. Therefore,

$$\frac{1}{N} \sum_{n \leq N} f_\varepsilon(T^n x) \to \int_X f_\varepsilon \, d\mu$$

and, by the regularity of $\mu$,

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \chi_U(T^n x) \leq \limsup_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \delta_{T^n x}(\overline{U}) \leq \mu(\overline{U}) < \varepsilon.$$

It follows that

$$\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n \leq N} f(T^n x) - \int_X f \, d\mu \right| \leq 4\varepsilon \|f\|_{\sup}$$

for each $\varepsilon > 0$ which proves (2.7).

Finally, for each $f \in D(X)$ with zero mean and $\varepsilon > 0$, we can find a continuous function $f_\varepsilon$ with zero mean such that $\int_X |f - f_\varepsilon| \, d\mu < \varepsilon$. Since $|f - f_\varepsilon| \in D(X)$, by (2.7), we have

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n \leq N} |f(T^n x) - f_\varepsilon(T^n x)| < \varepsilon.$$

The quantity above bounds the asymptotical difference between $A(f, M, H)$ and $A(f_\varepsilon, M, H)$. Therefore (2.6) holds for each $f \in D(X)$ with zero mean and for arbitrary $x \in X$.

## 3. AOP PROPERTY FOR NIL-TRANSLATIONS

In this section we shall prove that ergodic nil-translations on compact connected nilmanifolds satisfy the hypotheses (i) and (ii) of the criterion provided by Theorem 2.2, thereby proving Theorem A and Corollaries B and C for $\phi$ a nil-translation on a compact connected nilmanifold.

3.1. **Background on nilpotent Lie groups.** Let $G$ be a connected simply connected $k$-step nilpotent Lie group with Lie algebra $\mathfrak{g}$, and let $\Gamma$ be a lattice in $G$. The quotient $M = G/\Gamma$ is then a compact nilmanifold on which $G$ acts on the left by translations. Denote by $\lambda = \lambda_M$ the $G$-invariant probability measure on $M$ (locally given by a Haar measure of $G$). Let

$$\mathfrak{g} = \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots \supset \mathfrak{g}^{(k)} \supset \mathfrak{g}^{(k+1)} = \{0\}, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}], \ i = 1, \ldots, k,$$

be the descending central series of $\mathfrak{g}$ (with $\mathfrak{g}^{(k)} \neq \{0\}$) and let

$$G = G^{(1)} \supset G^{(2)} \supset \cdots \supset G^{(k)} \supset G^{(k+1)} = \{e_G\}, \ G^{(i+1)} = [G, G^{(i)}], \ i = 1, \ldots, k,$$

be the corresponding series for $G$. In this setting, there exists a strong Malcev basis through the filtration $(\mathfrak{g}^{(i)})_{i=1}^k$ strongly based at the lattice $\Gamma$, that is a basis $X_1, \ldots, X_{\dim \mathfrak{g}}$ of $\mathfrak{g}$ such that for an increasing sequence of integers $0 = \ell_0 < \ell_1 < \cdots < \ell_k = \dim \mathfrak{g}$ we have:

(1) The elements $X_{\ell_{i-1}+1}, X_{\ell_{i-1}+2}, \ldots, X_{\dim \mathfrak{g}}$ form a basis of $\mathfrak{g}^{(i)}$;
(2) For each $i \in \{1, \ldots, \dim \mathfrak{g}\}$ the elements $X_i, X_{i+1}, \ldots, X_{\dim \mathfrak{g}}$ span an ideal of $\mathfrak{g}$;
(3) The lattice $\Gamma$ is given by

$$\Gamma = \{\exp(n_1 X_1) \cdots \exp(n_{\dim \mathfrak{g}} X_{\dim \mathfrak{g}}) \mid n_i \in \mathbb{Z}, i = 1, \ldots, \dim \mathfrak{g}\}.$$

It will be convenient to set $d_i = \ell_i - \ell_{i-1}$ for $i = 1, \ldots, k$, so that $d_1$ denotes the dimension of the abelianized group $G/[G, G]$ and $d_k$ is the dimension of the $k$ derived group $G^{(k)}$, which, we recall, is included in the center $Z(G)$ of $G$.

Since, for all $i = 1, \ldots, k-1$, the group $G^{(i+1)}$ is a closed normal subgroup of $G$, we have natural epimorphisms $\pi^{(i)} \colon G \to G/G^{(i+1)}$. The group $G^{(i+1)} \cap \Gamma$ is a lattice in $G^{(i+1)}$ (Theorem 2.3 in [Rag72, Corollary 1]); equivalently, $G^{(i+1)}\Gamma$ is a closed subgroup of $G$. Moreover, $\pi^{(i)}(\Gamma)$ is a lattice in $G/G^{(i+1)}$ and $M^{(i)} := G/G^{(i+1)}\Gamma \approx (G/G^{(i+1)})/\pi^{(i)}(\Gamma)$ is a nilmanifold. It follows that the short exact sequences

$$0 \hookrightarrow G^{(i+1)} \hookrightarrow G \twoheadrightarrow G/G^{(i+1)} \twoheadrightarrow 0, \qquad i = 1, \ldots, k-1,$$

induce $G$-equivariant fiber bundles of nilmanifolds

$$(3.1) \qquad \pi^{(i)} \colon M = G/\Gamma \twoheadrightarrow M^{(i)} = G/G^{(i+1)}\Gamma, \qquad i = 1, \ldots, k-1,$$

whose fibers are the orbits of $G^{(i+1)}$ on $G/\Gamma$, hence homeomorphic to the nilmanifolds $G^{(i+1)}/(G^{(i+1)} \cap \Gamma)$.

Two cases are of most interest for us. When $i = 1$ the base nilmanifold $M^{(1)} = G/[G, G]\Gamma$ is a torus of dimension $d_1$. As $G/[G, G]$ is an abelian group, it is identified by the exponential map with (the additive group) $\mathfrak{g}/\mathfrak{g}^{(2)} \approx \mathbb{R}^{d_1}$. Then the vectors

$$\bar{X}_i = X_i \mod \mathfrak{g}^{(2)}, \qquad i = 1, \ldots, d_1,$$

form a set of generators of the lattice $\pi^{(2)}(\Gamma) \approx \Gamma/(G^{(2)} \cap \Gamma)$. We shall use an additive notation for the abelian group $G/[G, G]$; thus we identify $M^{(1)}$ with $\mathbb{R}^{d_1}/\mathbb{Z}^{d_1}$ by means of the basis $\bar{X}_i$, $(i = 1, \ldots, d_1)$.

At the opposite extreme, when $i = k-1$, the group $G^{(k)}$ is abelian, thus isomorphic to $\mathbb{R}^{d_k}$. Let us consider the action of $G^{(k)}$ by left translations on $M = G/\Gamma$. As the stability group of any point in $M$ is the subgroup $G^{(k)} \cap \Gamma$, the action of $G^{(k)}$ on $M$ induces a free action of the torus group $\mathbb{T}^{(k)} = G^{(k)}/(G^{(k)} \cap \Gamma)$ on $M$. Ssnce the group $\mathbb{T}^{(k)}$ acts transitively on the fibers of the fibration $\pi^{(k-1)} \colon M \to M^{(k-1)}$, the map $\pi^{(k-1)}$ is a principal $\mathbb{T}^{(k)}$-bundle.

It follows from the above that the Hilbert space $L^2(M, \lambda)$ decomposes into a sum of $G$-invariant orthogonal Hilbert subspaces

$$L^2(M, \lambda) = \bigoplus_{\chi \in \widehat{\mathbb{T}^{(k)}}} H_\chi,$$

where we have set, for each character $\chi$ of the torus $\mathbb{T}^{(k)}$,

$$(3.2) \qquad H_\chi := \{f \in L^2(M) \mid f(zx) = \chi(z)f(x), \, \forall x \in M, \, \forall z \in G^{(k)}\}.$$

Let $\lambda^{(k)}$ be the probability Haar measure on $\mathbb{T}^{(k)}$. Denote by $C_\chi$ the subspace of continuous functions in $H_\chi$. The linear operator $\mathscr{F}_\chi \colon L^2(M, \lambda) \to H_\chi$ given by

$$\mathscr{F}_\chi(f)(x) = \int_{\mathbb{T}^{(k)}} \chi(z) f(z^{-1}x) \, d\lambda^{(k)}(z)$$

is the orthogonal projector on $H_\chi$. It follows that

**Lemma 3.1.** *The space $C_\chi$ is dense in $H_\chi$.*

3.2. **Dynamics and joinings for nil-translations.** For $u \in G$, let $l_u \colon M \to M$ be the left translation by $u$. Then $l_u$ is a measure preserving automorphism of $(M, \lambda)$, which for simplicity we call a *nil-translation.*

By the normality of $G^{(i+1)}$ in $G$, the projection $\pi^{(i)} \colon M \to M^{(i)}$ intertwines the map $l_u \colon M \to M$ with the nil-translation $l_{\pi^{(i)}(u)} \colon M^{(i)} \to M^{(i)}$.

The celebrated theorem by Auslander, Green and Hahn (see [AGH63]), states that the nil-translation $l_u$ is ergodic (minimal and uniquely ergodic) if and only if the translation $l_{\pi^{(1)}(u)}$ by $\pi^{(1)}(u)$ on the torus $M^{(1)}$ is ergodic (minimal and uniquely ergodic). The latter condition is equivalent to saying that the rotation vector $\alpha = (\alpha_1, \ldots, \alpha_{d_1}) \in \mathbb{R}^{d_1} / \mathbb{Z}^{d_1}$ defined by

$$\pi^{(1)}(u) = \exp(\alpha_1 \bar{X}_1 + \cdots + \alpha_{d_1} \bar{X}_{d_1})$$

is irrational with respect to the lattice $\mathbb{Z}^{d_1}$; equivalently, that the real numbers $1, \alpha'_1, \ldots, \alpha'_{d_1}$ are linearly independent over $\mathbb{Q}$, for any lift $(\alpha'_1, \ldots, \alpha'_{d_1})$ of $\alpha$ to $\mathbb{R}^{d_1}$. Using an additive notation we denote the translation $l_{\pi^{(1)}(u)}$ of the torus $M^{(1)}$ by $R_\alpha$, as it is uniquely determined by the rotation vector $\alpha$. With a slight abuse of language, and coherently with the choice of using an additive notation for the torus group $M^{(1)}$, we shall then say that the projection $\pi^{(1)}$ intertwines the translation $l_u$ with the rotation $R_\alpha$.

Let $l_u \colon M \to M$ be the nil-translation (not necessary ergodic with respect to $\lambda$) by $u$ and let $\mu$ be an $l_u$-invariant ergodic probability measure on $M$. Let us consider the stabilizer of the measure $\mu$

$$\Lambda(\mu) = \{g \in G : (l_g)_* \mu = \mu\}.$$

Then $\Lambda(\mu) < G$ is a closed subgroup of $G$. The following celebrated theorem of Ratner [Rat90, Theorem 1] (first proved independently, in the context of nil-flows, by Starkov in [Sta84] and Lesigne in [Les89]) describes all such ergodic measures.

**Theorem 3.2** ([Sta84], [Les89],[Rat90])**.** *If $\mu$ is a $l_u$-invariant ergodic measure on $M$ then there exists $x \in M$ such that the orbit $\Lambda(\mu)x \subset M$ is closed and it is the topological support of $\mu$.*

*Let $U \in \mathfrak{g}$ and let $\mu$ be a probability measure on $M$ which is invariant and ergodic for the nilflow $(l_{u^t})_{t \in \mathbb{R}}$ ($u^t = \exp(tU)$). Then the stabilizer $\Lambda(\mu)$ is a connected group and the topological support of $\mu$ is an orbit $\Lambda(\mu)x \subset M$.*

From now on, we fix an ergodic translation $l_u$ of $M$ (with respect to $\lambda$) projecting to the corresponding ergodic rotation $R_\alpha$ of the torus $M^{(1)}$.

We now consider the group $G \times G$ with the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$, the lattice $\Gamma \times \Gamma$ and the corresponding nilmanifold $M \times M = (G \times G)/(\Gamma \times \Gamma)$ on which the group $G \times G$ acts by left translations. We have natural projections $p_1$ and $p_2$ of $M \times M$ onto $M$, by selecting the first or the second coordinate of a point $(x_1, x_2) \in M \times M$, respectively.

Then we have $(G \times G)^{(i)} = G^{(i)} \times G^{(i)}$, $(\mathfrak{g} \oplus \mathfrak{g})^{(i)} = \mathfrak{g}^{(i)} \oplus \mathfrak{g}^{(i)}$ for all $i = 1, \ldots, k$. Clearly $M^{(i)} \times M^{(i)} = (G \times G)/(G \times G)^{(i)}(\Gamma \times \Gamma)$ for all $i = 1, \ldots, k-1$.

Let $r, s \in \mathbb{N}$ be relatively prime numbers. Let $\rho$ be an ergodic joining of the measure theoretical ergodic systems $(M, l_u^r, \lambda)$ and $(M, l_u^s, \lambda)$. We recall that, by definition, $\rho$ is a measure on $M \times M$, with marginals $(p_i)_* \rho = \lambda$, $(i = 1, 2)$, which

is invariant and ergodic for the product transformation $l_u^r \times l_u^s$. The transformation $l_u^r \times l_u^s$ on $M \times M$ is the left translation on $M \times M$ by $(u^r, u^s)$:

$$l_u^r \times l_u^s (x_1, x_2) = (u^r x_1, u^s x_2), \quad \forall (x_1, x_2) \in M \times M.$$

Then the projection map $\pi^{(1)} \times \pi^{(1)}$ of $M \times M$ onto the $2d_1$-dimensional torus $M^{(1)} \times M^{(1)}$ intertwines the map $l_u^r \times l_u^s$ with the rotation $R_\alpha^r \times R_\alpha^s$ of $M^{(1)} \times M^{(1)}$. Moreover, the image measure $\rho^{(1)} := (\pi^{(1)} \times \pi^{(1)})_* \rho$ is $R_\alpha^r \times R_\alpha^s$-invariant and ergodic.

**Lemma 3.3.** *Let $R_\alpha$ a minimal rotation of the torus $d$-dimensional $\mathbb{T}^d$ and rotation vector $\alpha$. Let $r, s$ relatively prime integers and let $\eta$ be an $R_\alpha^r \times R_\alpha^s$ -invariant and ergodic probability measure on $\mathbb{T}^d \times \mathbb{T}^d$. Then the stabilizer $\Lambda(\eta)$ of $\eta$ is the group*

$$\mathbb{T}_{r,s} = \{(t_1, t_2) \in \mathbb{T}^d \times \mathbb{T}^d \mid s t_1 = r t_2\},$$

*with Lie algebra $\{(r v, s v) \mid v \in \mathbb{R}^d\}$.*

*Proof.* As the map $R_\alpha^r \times R_\alpha^s$ is the rotation $R_{(r\alpha, s\alpha)}$ with rotation vector $(r\alpha, s\alpha)$ of the $2d$ dimensional torus $\mathbb{T}^d \times \mathbb{T}^d$, it preserves the orbits of the closed subgroup $\mathbb{T}_{r,s}$ that is, the cosets

$$\mathbb{T}_{r,s,c} = \{(t_1, t_2 + c) \in \mathbb{T}^d \times \mathbb{T}^d \mid s t_1 = r t_2\}, \quad c \in \mathbb{T}^d.$$

We claim that the action of $R_{(r\alpha, s\alpha)}$ on each coset $\mathbb{T}_{r,s,c}$ is minimal and uniquely ergodic. Indeed, let $a, b \in \mathbb{Z}$ be such that $ar + bs = 1$. The map $I_c : \mathbb{T}_{r,s,c} \to \mathbb{T}^d$, given by $I_c(t_1, t_2 + c) = a t_1 + b t_2$, is well defined. Its inverse is given by the formula $t \in \mathbb{T}^d \mapsto (r t, s t + c) \in \mathbb{T}_{r,s,c}$. It follows that the map $I_c$ is a homeomorphism intertwining the action of $R_{(r\alpha, s\alpha)}$ on $\mathbb{T}_{r,s,c}$ with the action of $R_\alpha$ on $\mathbb{T}^d$; it also intertwines also the action of $\mathbb{T}_{r,s}$ on $\mathbb{T}_{r,s,c}$ with the action of $\mathbb{T}^d$ on itself. Since $R_\alpha$ is minimal and uniquely ergodic, so is the action of $R_{(r\alpha, s\alpha)}$ on $\mathbb{T}_{r,s,c}$. Since $\eta$ is an ergodic measure for $R_{(r\alpha, s\alpha)}$, there exists a $c \in \mathbb{T}$ such that $\mathrm{supp}(\eta) = \mathbb{T}_{r,s,c} = \mathbb{T}_{r,s}(0, c)$. It follows that $\Lambda(\eta) = \mathbb{T}_{r,s}$, which completes the proof. $\qquad\square$

Applying the above lemma to our setting we have:

**Corollary 3.4.** *Let $\eta$ be an $R_\alpha^r \times R_\alpha^s$ -invariant and ergodic probability measure on $M^{(1)} \times M^{(1)}$. Then the Lie algebra of the stabilizer $\Lambda(\eta)$ is t*

$$\mathfrak{h}_{r,s} := \{(r \bar{X}, s \bar{X}) \mid \bar{X} \in \mathfrak{g}/\mathfrak{g}^{(2)}\} \subset (\mathfrak{g} \oplus \mathfrak{g})/(\mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(2)}).$$

Denote by $H < G \times G$ the stabilizer of the joining $\rho$ and let $\mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{g}$ be its Lie algebra.

**Lemma 3.5.** *There exist elements $X_1', X_1'', \ldots, X_{d_1}', X_{d_1}''$ in $\mathfrak{g}$ satisfying*

$$X_i' \equiv X_i'' \equiv X_i \mod \mathfrak{g}^{(2)}, \text{ for all } i = 1, \ldots, d_1$$

*and such that*

$$\overline{Y}_i = (r X_i', s X_i'') \in \mathfrak{h}.$$

*Proof.* Let us consider the image measure $\rho^{(1)} := (\pi^{(1)} \times \pi^{(1)})_* \rho$ which is $R_\alpha^r \times R_\alpha^s$-invariant and ergodic. By Theorem 3.2, the topological support of the measure $\rho$ is a closed coset $H\bar{x}$ for some $\bar{x} \in M \times M$. Then the topological support of $\rho^{(1)}$ is $(\pi^{(1)} \times \pi^{(1)})(H)\bar{t}$ for some $\bar{t} \in M^{(1)} \times M^{(1)}$.

Since $\rho^{(1)}$ is $R_\alpha^r \times R_\alpha^s$-invariant and ergodic, by Corollary 3.4, the topological support of $\rho^{(1)}$ is $\Lambda(\rho^{(1)})\bar{t}$ and the Lie algebra of $\Lambda(\rho^{(1)})$ is $\mathfrak{h}_{r,s}$. As the orbits $(\pi^{(1)} \times \pi^{(1)})(H)\bar{t}$ and $\Lambda(\rho^{(1)})\bar{t}$ are closed and equal, it follows that the Lie algebras of $(\pi^{(1)} \times \pi^{(1)})(H)$ and $\Lambda(\rho^{(1)})$ are the same and equal to $\mathfrak{h}_{r,s}$. Hence the Lie algebra $\mathfrak{h}$ of the Lie group $H$ projects under the tangent map $d(\pi^{(1)} \times \pi^{(1)})$ onto the algebra $\mathfrak{h}_{r,s} = \{(r\bar{X}, s\bar{X}) \mid \bar{X} \in \mathfrak{g}/\mathfrak{g}^{(2)}\}$. It follows that for every $i = 1, \ldots, d_1$ there exists $(X_i', X_i'') \in \mathfrak{h}$ such that

$$d(\pi^{(1)} \times \pi^{(1)})(X_i', X_i'') = (r\, d\pi^{(1)}(X_i), s\, d\pi^{(1)}(X_i)),$$

which yields the required elements $X_1', X_1'', \ldots, X_{d_1}', X_{d_1}'' \in \mathfrak{g}$. $\qquad\square$

**Lemma 3.6.** *The Lie algebra* $\mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{g}$ *satisfies*

$$\{(r^k Z, s^k Z) \mid Z \in \mathfrak{g}^{(k)}\} \subset \mathfrak{h} \cap (\mathfrak{g}^{(k)} \oplus \mathfrak{g}^{(k)}).$$

*It follows that the group $H$ contains the subgroup*

$$L := \{\exp(r^k Z, s^k Z) \mid Z \in \mathfrak{g}^{(k)}\} < G^{(k)} \times G^{(k)}.$$

*Proof.* Let $X_i', X_i'', \overline{Y}_i$ for $i = 1, \ldots, d_1$ be given by Lemma 3.5. Consider the $k$-fold Lie products of elements of the sets $\overline{S} = \{\overline{Y}_i \mid i = 1, \ldots, d_1\}$ and $S = \{X_i \mid i = 1, \ldots, d_1\}$; that is, for every $(i_1, \ldots, i_k) \in \{1, \ldots, d_1\}^k$, we set

$$\overline{S}_{i_1, i_2, \ldots, i_k} := [\overline{Y}_{i_1}, [\overline{Y}_{i_2}, \ldots, [\overline{Y}_{i_{k-1}}, \overline{Y}_{i_k}] \ldots]] \in \mathfrak{g}^{(k)} \oplus \mathfrak{g}^{(k)}$$

and

$$S_{i_1, i_2, \ldots, i_k} := [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]] \in \mathfrak{g}^{(k)}.$$

Then, by definition and Lemma D.3, we have

$$\begin{aligned}
\overline{S}_{i_1, i_2, \ldots, i_k} &= [(rX_{i_1}', sX_{i_1}''), [(rX_{i_2}', sX_{i_2}''), \ldots, [(rX_{i_{k-1}}', sX_{i_{k-1}}''), (rX_{i_k}', sX_{i_k}'')] \ldots]] \\
&= \big([rX_{i_1}', [rX_{i_2}', \ldots, [rX_{i_{k-1}}', rX_{i_k}'] \ldots]], [sX_{i_1}'', [sX_{i_2}'', \ldots, [sX_{i_{k-1}}'', sX_{i_k}''] \ldots]]\big) \\
&= \big(r^k [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]], s^k [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]]\big) \\
&= \big(r^k S_{i_1, i_2, \ldots, i_k}, s^k S_{i_1, i_2, \ldots, i_k}\big).
\end{aligned}$$

Since $X_1 + \mathfrak{g}^{(2)}, \ldots, X_{d_1} + \mathfrak{g}^{(2)}$ is a basis of $\mathfrak{g}/\mathfrak{g}^{(2)}$, by Lemma D.1, the set $\{X_1, \ldots, X_{d_1}\}$ generates the Lie algebra $\mathfrak{g}$. Therefore, in view of Lemma D.2, the family of $k$-fold products $S_{i_1, i_2, \ldots, i_k}$ spans $\mathfrak{g}^{(k)}$.

By Lemma 3.5, $\overline{Y}_i \in \mathfrak{h}$ for every $i = 1, \ldots, d_1$. It follows that

$$\big(r^k S_{i_1, i_2, \ldots, i_k}, s^k S_{i_1, i_2, \ldots, i_k}\big) = \overline{S}_{i_1, i_2, \ldots, i_k} \in \mathfrak{h} \cap (\mathfrak{g}^{(k)} \oplus \mathfrak{g}^{(k)})$$

for every $(i_1, \ldots, i_k) \in \{1, \ldots, d_1\}^k$. Consequently, for all $Z \in \mathfrak{g}^{(k)}$ we have $(r^k Z, s^k Z) \in \mathfrak{h} \cap (\mathfrak{g}^{(k)} \oplus \mathfrak{g}^{(k)})$, which completes the proof.

$\qquad\square$

**Proposition 3.7.** *Let* $\chi_1$, $\chi_2$ *be characters of the torus* $\mathbb{T}^{(k)}$ *so that at least one is non-trivial. Then, if* $\chi_1^{r^k} \neq \chi_2^{s^k}$, *for any* $f_i \in H_{\chi_i}$, $i = 1, 2$, *and any ergodic joining* $\rho$ *of* $(M, l_u^r, \lambda)$ *and* $(M, l_u^s, \lambda)$, *we have*

$$\rho(f_1 \otimes \bar{f}_2) = 0.$$

*Proof.* Recall there exists a closed subgroup $H < G \times G$ such that the measure $\rho$ is $H$-invariant and $H$ contains the subgroup

$$L = \{\exp(r^k Z, s^k Z) \mid Z \in \mathfrak{g}^{(k)}\} < G^{(k)} \times G^{(k)}.$$

Since $f_1 \in H_{\chi_1}$ and $f_2 \in H_{\chi_2}$, for every $(\exp(r^k Z), \exp(s^k Z)) \in L$ and all $(x_1, x_2) \in M \times M$ we have

$$f_1 \otimes \bar{f}_2((\exp(r^k Z), \exp(s^k Z))(x_1, x_2)) = f_1(\exp(r^k Z)x_1) \cdot \bar{f}_2(\exp(s^k Z)x_2)$$

$$= \chi_1(\exp(r^k Z))\overline{\chi_2(\exp(s^k Z))} \cdot f_1 \otimes \bar{f}_2(x_1, x_2)$$

$$= (\chi_1^{r^k} \cdot \chi_2^{-s^k})(\exp(Z)) \cdot f_1 \otimes \bar{f}_2(x_1, x_2).$$

As the probability measure $\rho$ is invariant under the action of the group $L < H$, it follows that

$$\rho(f_1 \otimes \bar{f}_2) = (\chi_1^{r^k} \cdot \chi_2^{-s^k})(z)\rho(f_1 \otimes \bar{f}_2)$$

for every $z \in G^{(k)}/(G^{(k)} \cap \Gamma) = \mathbb{T}^{(k)}$. By assumption, the character $\chi_1^{r^k} \cdot \chi_2^{-s^k}$ is non-trivial, so we can find $z \in \mathbb{T}^{(k)}$ with $(\chi_1^{r^k} \cdot \chi_2^{-s^k})(z) \neq 1$. This yields $\rho(f_1 \otimes \bar{f}_2) = 0$, which completes the proof. □

The following theorem shows that Theorem 2.2 applies to nil-translations. Consequently, Theorem A and Corollaries B and C are true for nil-translations.

**Theorem 3.8.** *Let $l_u$ be an ergodic nil-translation of a compact connected nil-manifold $M = G/\Gamma$. Then there exists a set $\mathscr{C} \subset C(M) \cap L_0^2(M, \lambda)$ whose span is dense in $L_0^2(M, \lambda)$ such that:*

    (i) *for any pair $f_1, f_2 \in \mathscr{C}$, for all but at most one pair of relatively prime natural numbers $(r, s)$ we have $\rho(f_1 \otimes \bar{f}_2) = 0$ for all ergodic joinings $\rho$ of $l_u^r$ and $l_u^s$;*

    (ii) *for every $f \in \mathscr{C}$ and every $\omega \in \mathbb{C}$ with $|\omega| = 1$ there exists an element $g \in G$ such that $f(l_u^n(l_g x)) = \omega f(l_u^n x)$ for every $x \in M$ and every $n \in \mathbb{Z}$.*

*Proof.* The proof is by induction on the class of nilpotency $k$ of $G$.

If $k = 1$ then $M$ is the torus $\mathbb{T}^{(1)}$ and $H_\chi = \mathbb{C}\chi$ for every character $\chi$ of $\mathbb{T}^{(1)}$. Let $\mathscr{C}$ be the set of nontrivial characters. Then $\mathscr{C}$ is linearly dense in $L_0^2(M, \lambda)$. Moreover, for every pair of nontrivial characters $\chi_1, \chi_2$ there is at most one pair of relatively prime natural numbers $r, s$ such that $\chi_1^r = \chi_2^s$. In view of Proposition 3.7, the hypotheses (i) of Theorem 2.2 are verified. Moreover, if $\chi \in \mathscr{C}$ then for every $z \in \mathbb{T}^{(1)}$ we have $\chi(l_u^n(l_z x)) = \chi(l_z(l_u^n x)) = \chi(z)\chi(l_u^n x)$ which verifying the hypotheses (ii) of Theorem 2.2.

Suppose that for every ergodic nil-translation on any compact connected nilmanifold of class $k - 1$ the required set of continuous functions does exist. Let us consider an ergodic nil-translation $l_u$ on a compact connected nilmanifold $M = G/\Gamma$ of class $k$. Then $M^{(k-1)}$ is a compact connected nilmanifold of class $k - 1$ and $\pi^{(k-1)} \colon M \to M^{(k-1)}$ intertwines $l_u$ with the ergodic rotation $l_{\pi^{(k-1)}u}$ on $M^{(k-1)}$. Denote by $\mathscr{C}^{(k-1)}$ the subset of continuous functions on $M^{(k-1)}$ derived from the induction hypothesis. Next we use

$$L^2(M, \lambda) = \bigoplus_{\chi \in \widehat{\mathbb{T}^{(k)}}} H_\chi.$$

If $\chi = 1$ is the trivial character then $H_1$ consists of $G^{(k)}$-periodic functions and $H_1$ can be naturally identified with $L^2(M^{(k-1)})$. Since the identification is $G$-equivariant, for functions from $H_1$ the dynamics given by the rotation $l_u$ coincides with the dynamic of $l_{\pi^{(k-1)}u}$ on $M^{(k-1)}$. Let

$$\mathscr{C} := \mathscr{C}^{(k-1)} \cup \bigcup_{\chi \in \widehat{\mathbb{T}^{(k)}} \setminus \{1\}} C_\chi.$$

By Lemma 3.1 and by the induction hypothesis, the set $\mathscr{C}$ is linearly dense in the space $L^2_0(M)$.

First note that for all functions from the subset $\mathscr{C}^{(k-1)} \subset \mathscr{C}$ both properties (i) and (ii) follows from the induction hypothesis. To complete (i) we need to take $f_1 \in H_{\chi_1}$ and $f_2 \in H_{\chi_2}$ with at least one non-trivial character $\chi_1$, $\chi_2$ of $\mathbb{T}^{(k)}$. As there is at most one pair of relatively prime natural numbers $r, s$ such that $\chi_1^{r^k} = \chi_2^{s^k}$, Proposition 3.7 implies (i).

Let $f \in H_\chi$ for a non-trivial character $\chi$. Since $G^{(k)}$ is a subgroup of the center $Z(G)$, the action $z \in \mathbb{T}^{(k)} \mapsto l_z$ commutes with the nil-translation $l_u$. It follows that

$$f(l_u^n(l_z x)) = f(l_z(l_u^n x)) = \chi(z) f(l_u^n x).$$

The non-triviality of $\chi$ completes the proof of (ii).                  □

## 4. AOP FOR AFFINE DIFFEOMORPHISMS OF NILMANIFOLDS

In this section we shall prove Theorem A and Corollaries B and C for zero entropy ergodic affine diffeomorphisms on compact connected nilmanifolds. Recall that each such affine diffeomorphism is unipotent, see [Par69] and [BD00]. In fact, we shall show that the hypotheses of Theorem 2.2 apply to such maps.

### 4.1. **On affine diffeomorphisms of nilmanifolds.**
For any Lie group $G$, an *affine diffeomorphism of $G$* is a mapping of the form $g \mapsto uA(g)$, where $u \in G$ and $A$ is an automorphism of $G$.

Let $l_u$ be the left translation on $G$ by an element $u \in G$. Then the above affine map is the composition $l_u \circ A$. We shall however use the shortened notation $uA$, whenever convenient.

Let $M = G/\Gamma$ be a compact nilmanifold, with $\Gamma$ a lattice in $G$ and $G$ a connected, simply connected (nilpotent) Lie group. An affine diffeomorphism $uA$ of $G$ induces a quotient diffeomorphism of $M$ if and only if $A(\Gamma) = \Gamma$; an *affine diffeomorphism of $M$* is a map of $M$ that arises as such a quotient. For simplicity, whenever the condition $A(\Gamma) = \Gamma$ is satisfied, the symbol $\phi = uA$ (or $\phi = l_u \circ A$) will denote both an affine diffeomorphism of $G$ and the induced quotient affine diffeomorphism of $M$.

We recall that the group $\mathrm{Aut}(G)$ of automorphisms of $G$ is identified, via the exponential map, with the group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of the Lie algebra $\mathfrak{g}$ of $G$; thus, for $A \in \mathrm{Aut}(G) \approx \mathrm{Aut}(\mathfrak{g})$, we have $\exp(A(X)) = A(\exp X)$, for all $X \in \mathfrak{g}$.

An affine diffeomorphism $uA$ of $M$ (or of $G$) is *unipotent*, if $A : \mathfrak{g} \to \mathfrak{g}$ is a unipotent automorphism. (Recall that an ergodic affine diffeomorphism of $M$ has zero entropy if and only if it is unipotent.) In this case we can write $A = \exp B$, with $B : \mathfrak{g} \to \mathfrak{g}$ a nilpotent derivation of $\mathfrak{g}$; by definition of derivation we have

$$(4.1) \qquad\qquad B([X, Y]) = [BX, Y] + [X, BY] \quad \text{for} \ \ X, Y \in \mathfrak{g}.$$

4.1.1. *Some rationality issues.* Let $\Gamma$ be a lattice in $G$. Then $\Gamma$ determines a rational structure on the Lie algebra $\mathfrak{g}$. In fact, by Theorem 5.1.8 in [CG90], the vector space $\mathfrak{g}_\Gamma := \mathbb{Q}\text{-span}(\log\Gamma)$ is a Lie algebra over $\mathbb{Q}$ such that $\mathfrak{g} = \mathfrak{g}_\Gamma \otimes_\mathbb{Q} \mathbb{R}$. Indeed, any strong Malcev basis strongly based on the lattice $\Gamma$ is a $\mathbb{Q}$-basis of $\mathfrak{g}_\Gamma$.

Recall that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is *rational (with respect to $\Gamma$)* if $(\mathfrak{h} \cap \mathfrak{g}_\Gamma) \otimes_\mathbb{Q} \mathbb{R} = \mathfrak{h}$, i.e. if $\mathfrak{h}$ has a basis contained in $\mathfrak{g}_\Gamma$. For example, each $\mathfrak{g}^{(i)}$ is a rational ideal with respect to any lattice of $G$.

By definition, a connected closed subgroup $H = \exp(\mathfrak{h})$ is *a rational subgroup of $G$* if the subalgebra $\mathfrak{h}$ is rational.

In view of Theorem 5.1.11 in [CG90], a connected closed subgroup $H$ is a rational subgroup of $G$ if and only if the intersection subgroup $H \cap \Gamma$ is a lattice in $H$. If furthermore $H$ is normal in $G$, then $G/\Gamma H$ is a compact nilmanifold; thus we obtain a smooth $G$-equivariant factor map $\pi_H : G/\Gamma \to G/\Gamma H$.

If $uA$ is an affine unipotent diffeomorphism of $M = G/\Gamma$, since $A(\Gamma) = \Gamma$, we have $A(\mathfrak{g}_\Gamma) = \mathfrak{g}_\Gamma$. As the logarithm of a unipotent automorphims is a rational map, if $A = \exp B$ we have $B(\mathfrak{g}_\Gamma) \subset \mathfrak{g}_\Gamma$. Hence $B$ is a rational endomorphism of $\mathfrak{g}_\Gamma$.

## 4.2. **Suspensions.** 

We consider the usual construction turning an affine diffeomorphism into a translation [Dan77]. Let $\phi = uA$ be a unipotent affine diffeomorphism of the nilmanifold $M = G/\Gamma$ such that $A \neq I$. Let $B$ be the (nilpotent) derivation of $\mathfrak{g}$ such that $A = \exp B$ and let $\mathscr{A} = \{A^t\}_{t \in \mathbb{R}}$, with $A^t = \exp tB$, be the one-parameter subgroup of automorphisms of $G$ generated by $B$. (We have a natural identification $A^t \in \mathscr{A} \mapsto t \in \mathbb{R}$.)

The semi-direct product $\widetilde{G} = G \rtimes \mathscr{A}$ is a simply connected, connected subgroup of the affine diffeomophisms of $G$ with the inherited product rule defined by

$$(4.2) \qquad (g_1 A^{t_1}) \cdot (g_2 A^{t_2}) = g_1 A^{t_1}(g_2) A^{t_1 + t_2}.$$

We regard $G$ as a normal subgroup of $\widetilde{G}$ via the inclusion map $l : g \in G \mapsto l_g \in \widetilde{G}$. Thus we have a split exact sequence

$$(4.3) \qquad 0 \to G \overset{l}{\hookrightarrow} \widetilde{G} \overset{\check{p}}{\to} \mathbb{R} \to 0$$

where $\check{p}(gA^t) = t$, for all $gA^t \in \widetilde{G}$.

The set $\widetilde{\Gamma} = \{\gamma A^n \mid \gamma \in \Gamma, n \in \mathbb{Z}\}$ is a closed lattice subgroup of $\widetilde{G}$; thus $\widetilde{M} = \widetilde{G}/\widetilde{\Gamma}$ is a compact nilmanifold. By formulas (4.3) and (4.2), since $\check{p}(\widetilde{\Gamma}) = \mathbb{Z}$, the map $\check{p}$ induces a fibration (in fact a $G$-bundle)

$$(4.4) \qquad p : gA^t\widetilde{\Gamma} \in \widetilde{G}/\widetilde{\Gamma} \mapsto t \in \mathbb{R}/\mathbb{Z}$$

with compact fibers $p^{-1}\{t\} = \{gA^t\widetilde{\Gamma} \mid g \in G\}$ parametrized by $t \in \mathbb{R}/\mathbb{Z}$. Since $gA^t\widetilde{\Gamma} = g'A^t\widetilde{\Gamma}$, for $t \in \mathbb{R}/\mathbb{Z}$ and $g, g' \in G$, if and only if $gA^t(\Gamma) = g'A^t(\Gamma)$ we identify $p^{-1}\{t\} \approx G/A^t(\Gamma)$. In particular, $p^{-1}\{0\} \approx G/\Gamma$, the identification being given by the embedding $i : G/\Gamma \hookrightarrow \widetilde{G}/\widetilde{\Gamma}$ defined by $i(g\Gamma) = g\widetilde{\Gamma}$. Thus we shall consider $G/\Gamma$ as a subset of $\widehat{G}/\widetilde{\Gamma}$.

Let $\widetilde{\mathfrak{g}}$ be the Lie algebra of the group $\widetilde{G}$. Since this group is generated by the normal subgroup $G$ and by $\mathscr{A} = (\exp tB)_{t \in \mathbb{R}}$, the Lie algebra $\widetilde{\mathfrak{g}}$ is the semi-direct product $\mathfrak{g} \rtimes \mathbb{R}B$; in addition to the commutation rules of elements of Lie algebra

$\mathfrak{g}$, we have the rule[5]

$$(4.5) \qquad [B, X] = B(X), \quad \text{for all } X \in \mathfrak{g}.$$

Let $\widetilde{u} = uA$. Since the derivation $B$ is nilpotent, the group $\widetilde{G}$ is nilpotent. Thus the exponential map of $\widetilde{G}$ is a homeomorphism and we can write $\widetilde{u} = \exp(B + v)$ for some $v \in \mathfrak{g}$. Indeed, since $\mathfrak{g}$ is an ideal in $\widetilde{\mathfrak{g}}$, by the Baker-Campbell-Hausdorff formula, setting $u = \exp v_1$, we have $uA = \exp v_1 \cdot \exp B = \exp(B + v)$ for some $v \in \mathfrak{g}$. On the compact nilmanifold $\widetilde{M} = \widetilde{G}/\widetilde{\Gamma}$ we consider the nil-translation $l_{\widetilde{u}}$. Then

$$(4.6) \qquad l_{\widetilde{u}}(gA^s\widetilde{\Gamma}) = \widetilde{u} \cdot (gA^s)\widetilde{\Gamma} = uA(g)A^{s+1}\widetilde{\Gamma} = uA(g)A^s\widetilde{\Gamma},$$

where in the last line we used the observation that $A^{-1} \in \widetilde{\Gamma}$. It follows that the nil-translation $l_{\widetilde{u}}$ preserves the fibres $p^{-1}\{t\}$ of the fibration (4.4). By the formula (4.6), the homeomorphism $i \colon G/\Gamma \to p^{-1}\{0\} \subset \widetilde{G}/\widetilde{\Gamma}$ intertwines the affine diffeomorphism $uA \colon G/\Gamma \to G/\Gamma$ with the homeomorphism $(l_{\widetilde{u}})_{|p^{-1}\{0\}} \colon p^{-1}\{0\} \to p^{-1}\{0\}$. We shall therefore identify these two maps.

Let $\{\widetilde{u}^t\}_{t \in \mathbb{R}} < \widetilde{G}$ be the one-parameter group defined by $\widetilde{u}^t = \exp t(B + v)$. Again, the Baker-Campbell-Hausdorff formula $\exp t(B + v) \exp(-tB) = u_t \in G$ for every $t \in \mathbb{R}$. Therefore, $\widetilde{u}^t = u_t A^t$ for every $t \in \mathbb{R}$.

Denote by $(\widetilde{\phi}_t)_{t \in \mathbb{R}}$ the nilflow on $\widetilde{G}/\widetilde{\Gamma}$ defined by the one-parameter group $\{\widetilde{u}^t\}_{t \in \mathbb{R}}$. By definition, $\widetilde{\phi}_1 = l_{\widetilde{u}}$ on $\widetilde{G}/\widetilde{\Gamma}$ and $p \circ \widetilde{\phi}_t = p + t \mod \mathbb{Z}$. From the invariance of the set $p^{-1}\{0\}$ under the map $l_{\widetilde{u}}$ and since the first return time of any point in $p^{-1}\{0\}$ for the flow $(\widetilde{\phi}_t)_{t \in \mathbb{R}}$ is equal to 1, we conclude that $p^{-1}\{0\}$ is a Poincaré section of the flow $(\widetilde{\phi}_t)_{t \in \mathbb{R}}$ (with constant return time 1). Moreover, $p^{-1}\{t\} = \widetilde{\phi}_t p^{-1}\{0\}$ for every $t \in \mathbb{R}/\mathbb{Z}$. As the map $l_{\widetilde{u}}$ restricted to $p^{-1}\{0\}$ is identified with $\phi \colon M \to M$, we conclude, by standard arguments, that any $\phi$ invariant measure $\mu$ on $G/\Gamma$ defines a unique invariant measure $\widetilde{\mu}$ for the flow $(\widetilde{\phi}^t)$ on $\widetilde{G}/\widetilde{\Gamma}$ given by

$$\widetilde{\mu}(f) = \int_0^1 dt \int_{G/\Gamma} d\mu(x) \, f(\widetilde{\phi}^t(i(x))) = \int_0^1 dt \int_{p^{-1}\{t\}} d\widetilde{\mu}_t(y) \, f(y).$$

where $\widetilde{\mu}_t$ is a probability measure on $p^{-1}\{t\}$ given by $\widetilde{\mu}_t = \widetilde{\phi}^t_* i_* \mu$. The above formula shows that the family of measures $(\widetilde{\mu}_t)_{t \in [0,1)}$ form the conditional measures of the measure $\widetilde{\mu}$ with respect to the projection $p$.

**Definition 4.1.** The measure preserving nilflow $(\widetilde{G}/\widetilde{\Gamma}, (\widetilde{\phi}^t), \widetilde{\mu})$ is called the *suspension* of the measure preserving affine unipotent diffeomorphism $(G/\Gamma, \phi, \mu)$.

A simple application of these ideas is the following lemma.

**Lemma 4.1.** *Let $\phi$ be a unipotent affine diffeomorphims of the nilmanifold $G/\Gamma$ preserving a measure $\mu$. Let $(\widetilde{G}/\widetilde{\Gamma}, (\widetilde{\phi}^t), \widetilde{\mu})$ be the measure preserving nilflow suspension of $(G/\Gamma, \phi, \mu)$ and $Y$ the one-parameter group of $\widetilde{G}$ generating the flow. Let $\Lambda(\widetilde{\mu}) < \widetilde{G}$ be the stabilizer of $\widetilde{\mu}$. Then $\Lambda(\widetilde{\mu}) = H \rtimes \{\exp tY\}$ where $Y = B + v$ and $H$ is a closed subgroup of $G$ satisfying $\Lambda(\mu)_0 < H < \Lambda(\mu)$, where $\Lambda(\mu) < G$ is the stabilizer of $\mu$ and $\Lambda(\mu)_0$ denotes the connected component of the identity of $\Lambda(\mu)$.*

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[5]In fact, from (4.2), $(\exp tB) \cdot \exp X \cdot (\exp -tB) = (\exp tB)(X)$ for every $t \in \mathbb{R}$ and every $X \in \mathfrak{g}$.

*Proof.* As usual, we identify the fiber $p^{-1}(\{0\})$ of the fiber bundle $p \colon \widetilde{G}/\widetilde{\Gamma} \to \mathbb{R}/\mathbb{Z}$ with $G/\Gamma$ and the measure $\mu$ with a measure $\widetilde{\mu}_0$ supported on $p^{-1}(\{0\}) \approx G/\Gamma$.

The one-parameter group $\{\exp tY\}$ generating the flow $\widetilde{\phi}^t$ is clearly contained in $\Lambda(\widetilde{\mu})$; hence $H := \Lambda(\widetilde{\mu}) \cap G$ is a normal closed subgroup of $\Lambda(\widetilde{\mu})$ and $\Lambda(\widetilde{\mu}) = H \rtimes \{\exp tY\}$.

For each $h \in H$, the left translation $l_h \colon \widetilde{G}/\widetilde{\Gamma} \to \widetilde{G}/\widetilde{\Gamma}$ is a smooth diffeomorphism fibering over the circle $\mathbb{R}/\mathbb{Z}$, i.e. $p \circ l_h = p$ and $l_h(p^{-1}(\{t\})) = p^{-1}(\{t\})$ for $t \in \mathbb{R}/\mathbb{Z}$. Thus, for all $h \in H$, from $(l_h)_* \widetilde{\mu} = \widetilde{\mu}$ we obtain that $(l_h)_* \widetilde{\mu}_t = \widetilde{\mu}_t$ for almost all $t \in \mathbb{R}/\mathbb{Z}$. Since the mapping $(h,t) \in H \times \mathbb{R}/\mathbb{Z} \to (l_h)_* \widetilde{\mu}_t = (l_h)_* \phi_*^t \widetilde{\mu}_0$ is continuous, we have $(l_h)_* \widetilde{\mu}_t = \widetilde{\mu}_t$ for all $h \in H$ and *all* $t \in \mathbb{R}/\mathbb{Z}$. In particular,

$$(l_h)_* \mu = (l_h)_* \widetilde{\mu}_0 = \widetilde{\mu}_0 = \mu.$$

This shows that $H$ is contained in the stabilizer $\Lambda(\mu)$ of $\mu$.

The measure $\widetilde{\mu}_0$ on $\widetilde{G}/\widetilde{\Gamma}$ is preserved by the time one map $\widetilde{\phi}^1$, since this map restricted to $p^{-1}\{0\}$ coincides with the affine map $\phi$. Hence, for all $h \in \Lambda(\mu)$ and all $n \in \mathbb{Z}$, $(\widetilde{\phi}^n \circ l_h \circ \widetilde{\phi}^{-n})_* \widetilde{\mu}_0 = \widetilde{\mu}_0$, that is $\exp(nY)\Lambda(\mu)\exp(-nY) = \Lambda(\mu)$. Since the adjoint action of $\widetilde{G}$ on $\mathfrak{g}$ is algebraic, we obtain the identity

$$\exp(tY)\Lambda(\mu)_0 \exp(-tY) = \Lambda(\mu)_0,$$

for all $t \in \mathbb{R}$. It follows that if $h \in \Lambda(\mu)_0$ then $\widetilde{\mu}_0$ is also $\widetilde{\phi}^{-t} \circ l_h \circ \widetilde{\phi}^t$-invariant for every $t \in \mathbb{R}$. Therefore, $(l_h)_* \widetilde{\mu}_t = \widetilde{\mu}_t$ for every $t \in \mathbb{R}$ and $(l_h)_* \widetilde{\mu} = \widetilde{\mu}$. Consequently, $\Lambda(\mu)_0 < H$. $\qquad\square$

4.2.1. *A bit of categorical thinking.* Nilmanifolds are the objets of a category **NilMan** which we could formalise in the following way: the objects of this category are pairs $(G, \Gamma)$ with $G$ a connected, simply connected nilpotent Lie group and $\Gamma$ a lattice in $G$; a morphism $f$ from $(G, \Gamma)$ to $(G', \Gamma')$ is a smooth homomorphism $f \colon G \to G'$ such that $f(\Gamma) \subset \Gamma'$. Thus a morphism $f$ determines a smooth map $\bar{f} \colon G/\Gamma \to G'/\Gamma'$.

This category can enriched by adding new structures to an object $(G, \Gamma)$; for example, we may add an element $X \in \mathfrak{g}$, (or, equivalently, the one-parameter group $\{\exp tX\}$, or the flow determined by $\{\exp tX\}$ on $G/\Gamma$). Such category will be called the *category of nilflows*.

Morphisms for these enriched categories are morphisms **NilMan** respecting the additional structures. As another example, we may consider the *category of measure preserving nilflows*, whose objects are quadruples consisting of a nilmanifold $(G, \Gamma)$, an element $X \in \mathfrak{g}$ and a probability measure $\mu$ on $G/\Gamma$ invariant by the flow determined by $\{\exp tX\}$ on $G/\Gamma$. A morphism $f$ from the measure preserving nilflow $(G, \Gamma, X, \mu)$ to $(G', \Gamma', X', \mu')$ is a morphism of nilmanifolds $f \colon (G, \Gamma) \to (G', \Gamma')$ such that $f_* X = X'$ and $\bar{f}_* \mu = \mu'$, where, as before, $\bar{f}$ denotes the quotient map $\bar{f} \colon G/\Gamma \to G'/\Gamma'$.

The reader will be easily define in an analogous way the category **MpUniAff** of *measure preserving unipotent affine diffeomorphisms of nilmanifolds*.

The interest of these categories lies in the fact that we have already seen some functorial constructions.

The *abelianization functor **Ab*** from the category of measure preserving nilflows to itself associates to each measure preserving nilflow $F = (G, \Gamma, X, \mu)$ the toral flow $\textbf{\textit{Ab}}(F) = (G/G^{(2)}, \Gamma/\Gamma^{(2)}, \bar{X}, \bar{\mu})$ where $\bar{X} = X + \mathfrak{g}^{(2)}$ and $\bar{\mu} = \pi_*^{(2)} \mu$ is image of $\mu$ by the projection $\pi^{(2)} \colon G/\Gamma \to G/G^{(2)}\Gamma$ (see (3.1)). We define, for any

homomorphism $f: G \to G_1$, the abelianized homomorphism $\textbf{\textit{Ab}}(f)$ as the in-duced homomorphism $G/G^{(2)} \to G_1/G_1^{(2)}$. It is routine to check that $\textbf{\textit{Ab}}$ is a well defined functor from the category of measure preserving nilflows to itself. The Auslander- Green-Hahn criterion states that the object $\textbf{\textit{Ab}}(F)$ is ergodic if and only if $F$ is ergodic.

The suspension construction we discussed above is in fact an isomorphism of categories. Let us define the category **FiberedMpNilfl** of *measure preserving nilflows fibering over the circle flow*. The objects of this category are quadruples $(\widetilde{G}, \widetilde{\Gamma}, p, Y, \widetilde{\mu})$ where $p$ is a morphism from nilmanifolds $(\widetilde{G}, \widetilde{\Gamma})$ to the nilmanifold $(\mathbb{R}, \mathbb{Z})$, $Y$ is an element in $\widetilde{\mathfrak{g}}$ such that $p_* Y = d/dt$, and $\widetilde{\mu}$ is a $Y$-invariant measure on $\widetilde{G}/\widetilde{\Gamma}$. (For short, we write such an object as $(p, Y, \widetilde{\mu})$ since $(\widetilde{G}, \widetilde{\Gamma})$ is implied by $p$). Morphisms for this category are defined in the obvious way (this is in fact a slice category).

The suspension construction associates to each measure preserving unipo-tent affine diffeomorphisms of nilmanifolds $(G, \Gamma, uA, \mu)$ a nilmanifold $(\widetilde{G}, \widetilde{\Gamma})$ and a morphism $\breve{p}: (\widetilde{G}, \widetilde{\Gamma}) \to (\mathbb{R}, \mathbb{Z})$ inducing a fiber bundle $p: \widetilde{G}/\widetilde{\Gamma} \to \mathbb{R}/\mathbb{Z}$; it further defines a vector $Y = B + v = \log(uA) \in \widetilde{\mathfrak{g}}$, satisfying $\pi_* Y = d/dt$ and a $Y$-invariant measure $\widetilde{\mu}$ on $\widetilde{G}/\widetilde{\Gamma}$. Thus to each object $(G, \Gamma, uA, \mu)$ in the category **MpUniAff** of measure preserving unipotent affine diffeomorphisms of nilman-ifolds we have associated the object $(\pi, Y, \widetilde{\mu}) = \textbf{\textit{Susp}}(G, \Gamma, uA, \mu)$ in the category **FiberedMpNilfl** of measure preserving nilflows fibering over the circle flow.

Suppose $f: (G, \Gamma, uA, \mu) \to (G', \Gamma', u'A', \mu')$ is a morphism of **MpUniAff**. By definition $f$ is a homomorphism $f: G \to G'$ such that $f(\Gamma) \subset \Gamma'$ and such that the induced map $\bar{f}: G/\Gamma \to G'/\Gamma'$ is a morphism of the measure pre-serving dynamical systems $(G/\Gamma, uA, \mu)$ and $(G'/\Gamma', u'A', \mu')$. Let $(\widetilde{G}, \widetilde{\Gamma}, \breve{p}, Y, \widetilde{\mu})$ and $(\widetilde{G}', \widetilde{\Gamma}', \breve{p}', Y', \widetilde{\mu}')$ be the suspensions of the systems $(G, \Gamma, uA, \mu)$ and $(G', \Gamma', u'A', \mu')$. By definition $\widetilde{G} = G \rtimes \{A^t\}$, $Y = \log uA$ etc. Define $\widetilde{f} = \textbf{\textit{Susp}}(f)$ by setting

$$\widetilde{f}: \widetilde{G} \to \widetilde{G}', \quad \widetilde{f}(gA^t) = f(g)(A')^t \qquad \text{for all } g \in G.$$

Since $A^t g A^{-t} = A^t(g)$ and $f \circ A = A' \circ f$, we have

$$\widetilde{f}(A^t g A^{-t}) = \widetilde{f}(A^t(g)) = f(A^t(g)) = (A')^t f(g)$$
$$= (A')^t f(g)(A')^{-t} = \widetilde{f}(A^t) \widetilde{f}(g) \widetilde{f}(A^{-t});$$

thus $\widetilde{f}$ is a homomorphism. We leave to the reader the care of checking that $\widetilde{f}$ is a morphism of **FiberedMpNilfl**, that is that $\widetilde{f}(\widetilde{\Gamma}) \subset \widetilde{\Gamma}'$; that $p' \circ \widetilde{f} = p$; that $\widetilde{f}_* Y = Y'$; and that $\overline{\widetilde{f}}_* \widetilde{\mu} = \widetilde{\mu}'$.

We have showed that $\textbf{\textit{Susp}}$ is a functor. The functor $\textbf{\textit{Susp}}$ is in fact an isomor-phism of categories. In fact given $(\breve{p}, Y, \widetilde{\mu})$, where $\breve{p}: (\widetilde{G}, \widetilde{\Gamma}) \to (\mathbb{R}, \mathbb{Z})$ induces a fiber bundle $p: \widetilde{G}/\widetilde{\Gamma} \to \mathbb{R}/\mathbb{Z}$, we recover $G$ as $\ker \breve{p}$ and $\Gamma$ as $\widetilde{\Gamma} \cap G$; the affine map $uA$ is obtained as the first return map of the flow generated by $Y$ to the fiber $p^{-1}(\{0\}) = G/\Gamma$; finally the measure measure $\mu$ is obtained as the conditional measure of $\mu$ on the fiber $p^{-1}(\{0\}) = G/\Gamma$. If $F: (\breve{p}, Y, \widetilde{\mu}) \to (\breve{p}', Y', \widetilde{\mu}')$ is a mor-phism of **FiberedMpNilfl** from $\breve{p}' \circ F = p$ we obtain that $F(\ker p) \subset \ker p'$; thus $F$ restricted to $G := \ker p$ is a homomorphism of $G$ into $G' := \ker p'$. We let to the reader the verification that $F_{|G}$ is a morphism of **MpUniAff** from $\textbf{\textit{Susp}}^{-1}(\breve{p}, Y, \widetilde{\mu})$ to $\textbf{\textit{Susp}}^{-1}(\breve{p}', Y', \widetilde{\mu}')$,

Just as for the abelianization functor, the functor **Susp** has the property that the object $(G, \Gamma, uA, \mu)$ is ergodic if and only if the image **Susp**$(G, \Gamma, uA, \mu)$ is ergodic.

For nil-translations the proof of the AOP property was based on the functorial properties of the abelianization. For affine unipotent maps it is based on the interplay of the two functors **Ab** and **Susp**; more precisely on study of the functor **Susp**$^{-1} \circ$ **Ab** $\circ$ **Susp**.

4.3. **Invariant measures for unipotent affine diffeomorphisms.** The construction above gave a correspondence, which preserves ergodicity, between affine unipotent diffeomorphisms of nilmanifolds and nilflows. Criteria of ergodicity for these dynamical systems were given by Parry [Par69] and by Hahn [Hah63, Hah64], for affine diffeomorphims and, as previouly mentioned, by Auslander, Green and Hahn for the nilflows. We shall exploit this correspondence and generalize it to non-ergodic measures. To simplify matters and to clarify a main point in the proof let us start examining the simplest case.

Assume $G = \mathbb{R}^d$ and let $\phi = uA$ be an affine unipotent diffeomorphism of the a torus $G/\Gamma$. The suspension of $(G/\Gamma, \phi)$ yields a nilmanifold $\widetilde{G}/\widetilde{\Gamma}$ and a flow $\widetilde{\phi}^t$. If $A = \exp B$, the Lie algebra of $\widetilde{G}$ is $\widetilde{\mathfrak{g}} = \mathbb{R}^n \oplus \mathbb{R}B$ with the only commutation relation $[B, X] = B(X)$ for any $X \in \mathbb{R}^n$ (here we identified $\mathfrak{g}$ with $\mathbb{R}^n$). In particular the class of nilpotency of $\widetilde{G}$ is equal to the class of nilpotency of the endomorphism $B$.

The first derived subalgebra $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ is therefore the subspace of $\mathfrak{g}$ image of $B$, denoted $\mathfrak{v}$, and the group $[\widetilde{G}, \widetilde{G}]$ is the subgroup $V$ of $G$ generated by $\mathfrak{v}$. The subspace $\mathfrak{v}$ is rational because the automorphism of the torus $A$ maps the lattice $\Gamma$ to itself (hence it is an element of $\mathrm{SL}_n(\mathbb{Z})$ is a suitable integral basis for $\Gamma$) and because the log of a unipotent automorphism is a rational map. Hence the orbits of the subgroup $V$ are closed and the space of orbits $G/V\Gamma$ is a lower dimensional quotient torus of the torus $G/\Gamma$.

The affine map $\phi = uA$ passes to the quotient torus $G/V\Gamma$, (since $\mathfrak{v} = B(\mathbb{R}^n)$ and $A(\mathfrak{v}) = (\exp B)(\mathfrak{v}) \subset \mathfrak{v})$, yielding a quotient affine map $\bar{\phi}$ of the torus $G/V\Gamma$ onto itself. However, since $B$ is nilpotent, the automorphism induced by $A$ on the quotient torus $G/V\Gamma$ is the identity automorphism. It follows that the map $\bar{\phi}$ is a pure translation by the element $\bar{u}$, projection of $u$ in $G/V$.

Coming back to the suspended nilflow $(\widetilde{G}/\widetilde{\Gamma}, \widetilde{\phi}^t)$, we know by the Auslander-Green-Hahn criterion that this flow is ergodic if and only if the abelianized flow on the torus $\widetilde{G}/[\widetilde{G}, \widetilde{G}]\widetilde{\Gamma}$ is. Since $\widetilde{G}/[\widetilde{G}, \widetilde{G}]\widetilde{\Gamma} = \widetilde{G}/V\widetilde{\Gamma}$, the abelianized flow is the suspension of the translation $l_{\bar{u}}$ on the torus $G/V\Gamma$. (The latter assertion can be easily verified going through the steps of the construction of the suspension, but it is, in fact, a consequence of the functoriality of the construction.)

The reader will have no difficulty in generalizing the above discussion to the case where $G/\Gamma$ is a general connected nilmanifold. The conclusion is the following lemma.

**Lemma 4.2.** *Let $\phi$ be a unipotent affine diffeomorphism of the nilmanifold $G/\Gamma$ preserving a probability measure $\mu$. Let $(\widetilde{G}/\widetilde{\Gamma}, (\widetilde{\phi}^t), \widetilde{\mu})$ be the measure preserving nilflow suspension of $(G/\Gamma, \phi, \mu)$.*

*Let $F$ be a rational subgroup of $G$ normal in $\widetilde{G}$ (hence normal in $G$). Then*

(1) *The affine map $\phi$ project to an affine map $\phi_F$ on $G/F\Gamma$, preserving the measure $\mu_F$ image of $\mu$ by the quotient map $G/\Gamma \to G/F\Gamma$.*

(2) *The suspension of the unipotent affine diffeomorphim $(G/F\Gamma, \phi_F, \mu_F)$ is the flow $(\widetilde{G}/F\widetilde{\Gamma}, (\widetilde{\phi_F}^t), \widetilde{\mu}_F)$, where $(\widetilde{\phi_F}^t)$, $\widetilde{\mu}_F$ are the images of $(\widetilde{\phi}^t)$, $\widetilde{\mu}$ by the quotient map $\widetilde{G}/\widetilde{\Gamma} \to \widetilde{G}/F\widetilde{\Gamma}$.*

We remark that the above lemma applies to the case $F = [\widetilde{G}, \widetilde{G}^{(i)}]$. In fact we have:

**Lemma 4.3.** *The commutator $[\widetilde{G}, \widetilde{G}]$ is a rational subgroup of $G$ normal in $\widetilde{G}$. In fact $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = [\mathfrak{g}, \mathfrak{g}] + B\mathfrak{g}$ is a rational sub-algebra of $G$. Furthemore the descending central series $[\widetilde{G}, \widetilde{G}^{(i)}]$ and $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}^{(i)}]$ of $\widetilde{G}$ and of $\widetilde{\mathfrak{g}}$ form a descending series of rational normal subgroups of $G$ and of rational normal ideals of $\mathfrak{g}$.*

*Proof.* The equality $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = [\mathfrak{g}, \mathfrak{g}] + B\mathfrak{g}$ and the inclusion $[\widetilde{G}, \widetilde{G}] \lhd G$ are immediate consequences of the definition of suspended flow $(\widetilde{G}/\widetilde{\Gamma}, \widetilde{\phi}^t)$ and of the commutation relations (4.5). Since $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ is a rational sub-algebra of $\widetilde{\mathfrak{g}}$ the orbits of $[\widetilde{G}, \widetilde{G}]$ are closed in $\widetilde{G}/\widetilde{\Gamma}$. Thus the intersections of the $[\widetilde{G}, \widetilde{G}]$ orbits with $G/\Gamma \subset \widetilde{G}/\widetilde{\Gamma}$ are also closed. It follows that $[\widetilde{G}, \widetilde{G}]$ is a rational subgroup of $G$.

Alternatively we may argue as in paragraph 4.1.1 that both $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ and $B(\mathfrak{g})$ are rational sub-algebras of $\widetilde{\mathfrak{g}}$ and use the fact that the sum of rational sub-algebras is rational.

*Mutatis mutandis,* the same arguments apply to $[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}^{(i)}]$ and to $[\widetilde{G}, \widetilde{G}^{(i)}]$.    □

**Corollary 4.4.** *Let $\Sigma = (G, \Gamma, \phi, \mu)$ be a measure preserving unipotent affine diffeomorphism with $\phi = uA$. Let $(\widetilde{G}/\widetilde{\Gamma}, (\widetilde{\phi}^t), \widetilde{\mu})$ be the measure preserving nilflow corresponding to the suspension $\widetilde{\Sigma} = \boldsymbol{Susp}(\Sigma)$.*

(1) *The affine map $\phi$ projects to a translation $\phi_T$ on the torus $G/[\widetilde{G}, \widetilde{G}]\Gamma$ by the element $\bar{u} = u[\widetilde{G}, \widetilde{G}]$. The translation $\phi_T$ preserves the measure $\mu_T$ image of $\mu$ by the quotient map $G/\Gamma \to G/[\widetilde{G}, \widetilde{G}]\Gamma$.*

(2) *The suspension of the toral translation $\Sigma_T := (G/[\widetilde{G}, \widetilde{G}]\Gamma, \phi_T, \mu_T)$ is the linear flow $(\widetilde{G}/[\widetilde{G}, \widetilde{G}]\widetilde{\Gamma}, (\widetilde{\phi_T}^t), \widetilde{\mu}_T)$, where $(\widetilde{\phi_T}^t)$, $\widetilde{\mu}_T$ are the images of $(\widetilde{\phi}^t)$, $\widetilde{\mu}$ by the quotient map $\widetilde{G}/\widetilde{\Gamma} \to \widetilde{G}/[\widetilde{G}, \widetilde{G}]\widetilde{\Gamma}$.*

*Proof.* The nilmanifold $\widetilde{G}/[\widetilde{G}, \widetilde{G}]\widetilde{\Gamma}$ is clearly a torus. The nilmanifold $G/[\widetilde{G}, \widetilde{G}]\Gamma$ is a torus because it is a quotient of the torus $G/[\widetilde{G}, \widetilde{G}]\Gamma$. Thus the nilflow $(\widetilde{\phi_T}^t)$ is a linear flow. The affine unipotent map $\phi_T$ is indeed a translation on $G/[G, G]\Gamma$: in fact, since $B(\mathfrak{g}) \subset [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ the derivation $B$ is trivial on $\mathfrak{g}/[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$, which implies that the automorphims $A$ projects to the identity automorphism of $G/[\widetilde{G}, \widetilde{G}]$.

The other statements of the corollary are proved in Lemmata 4.2 and 4.3.    □

**Remark 4.5.** Observe that the nilflow $\widetilde{\Sigma}_T = (\widetilde{G}/[\widetilde{G}, \widetilde{G}]\widetilde{\Gamma}, (\widetilde{\phi_T}^t), \widetilde{\mu}_T)$ is just the abelianization $\boldsymbol{Ab}(\widetilde{\Sigma})$ of $\widetilde{\Sigma} = \boldsymbol{Susp}(\Sigma)$ and that

$$\Sigma_T = \boldsymbol{Susp}^{-1}(\widetilde{\Sigma}_T) = \boldsymbol{Susp}^{-1}(\boldsymbol{Ab}(\boldsymbol{Susp}(\Sigma))).$$

Thus the main content of the above corollary is the toral rotation $\Sigma_T$ is obtained from $\Sigma$ via the quotient morphism $G/\Gamma \to G/[\widetilde{G}, \widetilde{G}]\Gamma$.

4.4. **Joinings of unipotent affine diffeomorphisms of nilmanifolds.** Let $\phi = uA$, with $u \in G$ and $A \in \text{Aut}(G)$, be an ergodic affine unipotent diffeomorphism

of the connected nilmanifold $G/\Gamma$ such that $A \neq I$. Denote by $\lambda$ the uniquely $\phi$-invariant probability measure on $G/\Gamma$, i.e. the Haar measure on $G/\Gamma$.

Let $r, s \in \mathbb{N}$ be relatively prime positive integers. Let us consider the product diffeomorphism $\phi^r \times \phi^s$ on $G^2/\Gamma^2 = (G \times G)/(\Gamma \times \Gamma)$. The map $\phi^r \times \phi^s$ is an affine unipotent diffeomorphism of $G/\Gamma \times G/\Gamma$. Indeed, $\phi^r \times \phi^s = u_{r,s} A_{r,s}$ with $u_{r,s} = (u_r, u_s) \in G^2$ and $A_{r,s} = A^r \times A^s \in \operatorname{Aut}(G^2)$. Note that $B_{r,s} := \log A_{r,s} = (rB, sB)$, with $B := \log A$. Thus

Let $\rho$ be an ergodic joining of the measure theoretical systems $\Sigma_r = (G/\Gamma, \phi^r, \lambda)$ and $\Sigma_s = (G/\Gamma, \phi^s, \lambda)$. By definition $\rho$ is a measure on $G/\Gamma \times G/\Gamma$ with marginals $\lambda$ on each factor, invariant and ergodic for the transformation $\phi^r \times \phi^s$. Thus we have an measure preserving unipotent affine diffeomorphism $\Sigma_{r,s} = (G^2, \Gamma^2, \phi^r \times \phi^s, \rho)$ and morphisms $q_i \colon \Sigma_{r,s} \to \Sigma_i$, $(i = r, s)$.

Let $\widetilde{\Sigma}_{r,s} = (\widetilde{G}_{r,s}, \widetilde{\Gamma}_{r,s}, p_{r,s}, Y_{r,s}, \widetilde{\rho}_{r,s}) = \boldsymbol{Susp}(\Sigma_{r,s})$. Here $\widetilde{G}_{r,s} = G^2 \rtimes \{\exp t B_{r,s}\}$ and $Y_{r,s} = \log(\phi^r \times \phi^s) = (rY, sY)$, with $Y = \log \phi = \log u A = B + v$, for some $v \in \mathfrak{g}$. In the sequel we shall write $\widetilde{\Sigma}_{r,s} = (\widetilde{G}_{r,s}/\widetilde{\Gamma}_{r,s}, Y_{r,s}, \widetilde{\rho}_{r,s})$, for short. Let $\widetilde{\mathfrak{g}}_{r,s}$ denote the Lie algebra of $\widetilde{G}_{r,s}$. Then $\widetilde{\mathfrak{g}}_{r,s} = \mathfrak{g}^2 \oplus \mathbb{R} B_{r,s}$.

For $i = r, s$, let $\widetilde{\Sigma}_i = (\widetilde{G}_i, \widetilde{\Gamma}_i, p_i, Y_i, \lambda_i)$ be the suspensions of $\Sigma_i$ and denote by $\mathfrak{g}_i$ the Lie algebra of $G_i$. Observe that $\lambda_r$ and $\lambda_s$ are the Haar measures on the corresponding nilmanifolds and that $Y_r = rY$ and $Y_s = sY$.

Note that, since $\widetilde{\mathfrak{g}}_{r,s}$ is the semi-direct product of $\mathfrak{g}^2 = \mathfrak{g} \oplus \mathfrak{g}$ and the span of the derivation $B_{r,s} = (rB, sB)$ of $\mathfrak{g}^2$ we have

$$\widetilde{\mathfrak{g}}_{r,s} = (\mathfrak{g} \oplus \mathfrak{g}) \rtimes \mathbb{R}(rB, sB) = (\mathfrak{g} \rtimes \mathbb{R}rB) \oplus (\mathfrak{g} \rtimes \mathbb{R}sB) = \widetilde{\mathfrak{g}}_r \oplus \widetilde{\mathfrak{g}}_s = \widetilde{\mathfrak{g}} \oplus \widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}^2,$$
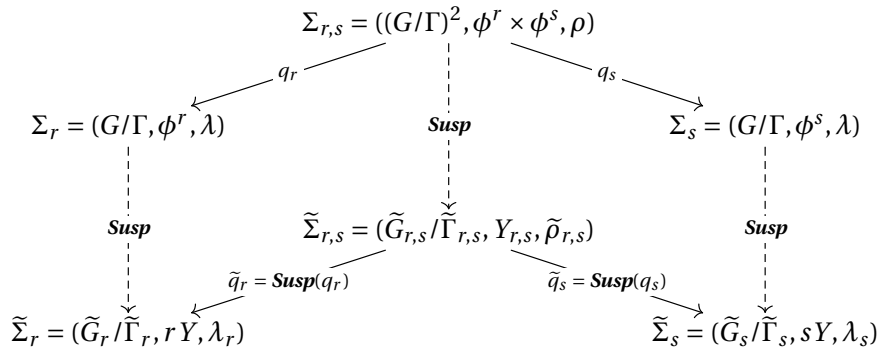
where $\widetilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathbb{R}B$. Hence $\widetilde{G}_{r,s} = \widetilde{G}_r \times \widetilde{G}_s = \widetilde{G} \times \widetilde{G}$, with $\widetilde{G} = G \rtimes \{\exp tB\}$. Let $k$ be the class of nilpotency of the group $\widetilde{G}$. Then, for all $i = 2, \dots, k$, we have:

$$\widetilde{\mathfrak{g}}_{r,s}^{(i)} = \widetilde{\mathfrak{g}}_r^{(i)} \oplus \widetilde{\mathfrak{g}}_s^{(i)} = \widetilde{\mathfrak{g}}^{(i)} \oplus \widetilde{\mathfrak{g}}^{(i)} \subset \mathfrak{g} \oplus \mathfrak{g}$$

and

$$\widetilde{G}_{r,s}^{(i)} = \widetilde{G}_r^{(i)} \times \widetilde{G}_s^{(i)} = \widetilde{G}^{(i)} \times \widetilde{G}^{(i)} < G \times G.$$

We have[6]:

$$\Sigma_{r,s} = ((G/\Gamma)^2, \phi^r \times \phi^s, \rho)$$

$$\Sigma_r = (G/\Gamma, \phi^r, \lambda) \quad\quad \boldsymbol{Susp} \quad\quad \Sigma_s = (G/\Gamma, \phi^s, \lambda)$$

$$\widetilde{\Sigma}_{r,s} = (\widetilde{G}_{r,s}/\widetilde{\Gamma}_{r,s}, Y_{r,s}, \widetilde{\rho}_{r,s})$$

$$\widetilde{q}_r = \boldsymbol{Susp}(q_r) \quad\quad \widetilde{q}_s = \boldsymbol{Susp}(q_s)$$

$$\widetilde{\Sigma}_r = (\widetilde{G}_r/\widetilde{\Gamma}_r, rY, \lambda_r) \quad\quad\quad \widetilde{\Sigma}_s = (\widetilde{G}_s/\widetilde{\Gamma}_s, sY, \lambda_s)$$

By Lemma 4.1 we have:

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[6]Remark that $\widetilde{\Sigma}_{r,s}$ is not a joining of $\widetilde{\Sigma}_r$ and $\widetilde{\Sigma}_s$, since $\widetilde{G}_{r,s}/\widetilde{\Gamma}_{r,s} \neq \widetilde{G}_r/\widetilde{\Gamma}_r \times \widetilde{G}_s/\widetilde{\Gamma}_s$. In fact $\dim \widetilde{G}_{r,s} = 2 \dim G + 1$ and $\dim \widetilde{G}_r = \dim \widetilde{G}_s = \dim G + 1$; thus $\dim \widetilde{G}_{r,s} \neq \dim \widetilde{G}_r + \dim \widetilde{G}_s$.

**Lemma 4.6.** *Denote by* $\widetilde{\mathfrak{h}}_{r,s} \subset \widetilde{\mathfrak{g}}_{r,s}$ *and by* $\mathfrak{h}_{r,s}$ *the Lie algebras of the stabilisers* $\Lambda(\widetilde{\rho}_{r,s}) < \widetilde{G}_{r,s}$ *and* $\Lambda(\rho_{r,s}) < G^2$ *of the measures* $\widetilde{\rho}_{r,s}$ *and* $\rho$*. Then*

$$(4.7) \qquad\qquad\qquad \widetilde{\mathfrak{h}}_{r,s} = \mathfrak{h}_{r,s} \rtimes \mathbb{R}Y_{r,s}.$$

By Corollary 4.4, applying the functor $\boldsymbol{Ab'} = \boldsymbol{Susp}^{-1} \circ \boldsymbol{Ab} \circ \boldsymbol{Susp}$ to the triangle $\Sigma_{r,s}$, $\Sigma_r$, $\Sigma_s$ we obtain a diagram

$$\boldsymbol{Ab'}(\Sigma_{r,s}) = (T^2, l_{\bar{u}}^r \times l_{\bar{u}}^s, \rho')$$

$$\boldsymbol{Ab'}(q_r) \swarrow \qquad\qquad \boldsymbol{Ab'}(q_s) \searrow$$

$$\boldsymbol{Ab'}(\Sigma_r) = (T, l_{\bar{u}}^r, \lambda_{\mathbb{T}}) \qquad\qquad\qquad \boldsymbol{Ab'}(\Sigma_s) = (T, l_{\bar{u}}^s, \lambda_{\mathbb{T}})$$

where $T$ is the torus $G/[\widetilde{G},\widetilde{G}]\Gamma$, $\lambda_T$ is the Lebesgue/Haar measure on $G/[\widetilde{G},\widetilde{G}]\Gamma$, $\boldsymbol{Ab'}(q_i)$ are the projections of $T^2 = T \times T$ on the corresponding factors, $l_{\bar{u}}$ is the left translation by $\bar{u} = u[\widetilde{G},\widetilde{G}]$ and $\rho'$ is the projection of the ergodic joining $\rho$ via the map $(G/\Gamma)^2 \to (G/[\widetilde{G},\widetilde{G}]\Gamma)^2$. In particular the measure $\rho'$ is a joining of the ergodic and minimal toral rotations $\boldsymbol{Ab'}(\Sigma_r)$ and $\boldsymbol{Ab'}(\Sigma_s)$.

Since the measure $\rho$ is ergodic, the measure $\rho'$, image of $\rho$ by the morphism $\Sigma_{r,s} \to \boldsymbol{Ab'}(\Sigma_{r,s})$ is an ergodic measure for the for the rotation $l_{\bar{u}}^r \times l_{\bar{u}}^s$ of $\mathbb{T}^2$. By Lemma 3.3, the stabilizer $\Lambda(\rho')$ of the measure $\rho'$ is the connected group

$$\Lambda(\rho') = \left\{ \exp(rX, sX) \mid X \in \text{Lie}(T) = \mathfrak{g}/[\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}] \right\}.$$

Now we focus on the quadrangle of morphisms given by Corollary 4.4.

$$\Sigma_{r,s} = ((G/\Gamma)^2, \phi^r \times \phi^s, \rho) \xrightarrow{\qquad\qquad} \boldsymbol{Ab'}(\Sigma_{r,s}) = (T^2, l_{\bar{u}}^r \times l_{\bar{u}}^s, \rho')$$

$$\Big\downarrow \boldsymbol{Susp} \qquad\qquad\qquad\qquad \Big\uparrow \boldsymbol{Susp}^{-1}$$

$$\widetilde{\Sigma}_{r,s} = (\widetilde{G}_{r,s}/\widetilde{\Gamma}_{r,s}, Y_{r,s}, \widetilde{\rho}_{r,s}) \longrightarrow \boldsymbol{Ab}(\widetilde{\Sigma}_{r,s}) = (\widetilde{G}_{r,s}/[\widetilde{G}_{r,s},\widetilde{G}_{r,s}]\widetilde{\Gamma}_{r,s}, Y'_{r,s}, \widetilde{\rho}'_{r,s})$$

**Lemma 4.7.** *The stabilizer* $\Lambda(\widetilde{\rho}'_{r,s})$ *of the measure* $\widetilde{\rho}'_{r,s}$*, image of* $\widetilde{\rho}_{r,s}$ *by the morphism* $\widetilde{\Sigma}_{r,s} \to \boldsymbol{Ab}(\widetilde{\Sigma}_{r,s})$*, is the connected group*

$$\Lambda(\widetilde{\rho}'_{r,s}) = \left\{ \exp(rX, sX) \mid X \in \widetilde{\mathfrak{g}}/[\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}] \right\}.$$

*Proof.* Since the toral flow $\boldsymbol{Ab}(\Sigma_{r,s})$ is a suspension of the toral rotation $\boldsymbol{Ab'}(\Sigma_{r,s})$, it follows by elementary reasons of by Lemma 4.1 that we have

$$\Lambda(\widetilde{\rho}'_{r,s}) = \Lambda(\rho') \times \{\exp tY'_{r,s}\} = \left\{ \exp(rX, sX) \mid X \in \mathfrak{g}/[\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}] \right\} \times \{\exp tY'_{r,s}\},$$

where $Y'_{r,s} = Y_{r,s} \mod [\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}]^2$. Thus, $Y'_{r,s} = (rY', sY')$, with $Y' = Y \mod [\widetilde{\mathfrak{g}},\widetilde{\mathfrak{g}}]$. Since the span of $\mathfrak{g}$ and of $Y$ is the Lie algebra $\widetilde{\mathfrak{g}}$, the statement follows. $\square$

By Lesigne-Starkov-Ratner Theorem 3.2, since the measure $\widetilde{\rho}_{r,s}$ is ergodic, the stabilizer $\Lambda(\widetilde{\rho}_{r,s}) < \widetilde{G}_{r,s}$ of $\widetilde{\rho}_{r,s}$ is a connected closed subgroup of $\widetilde{G}_{r,s}$ such that the topological support of $\widetilde{\rho}_{r,s}$ is a $\Lambda(\widetilde{\rho}_{r,s})$-orbit in $\widetilde{G}_{r,s}/\widetilde{\Gamma}_{r,s}$. Clearly the same properties hold for the stabilizer $\Lambda(\widetilde{\rho}'_{r,s})$ of the measure $\widetilde{\rho}'_{r,s}$. Since the action of $\widetilde{G}_{r,s}/[\widetilde{G}_{r,s},\widetilde{G}_{r,s}]$ on the torus $\widetilde{G}_{r,s}/[\widetilde{G}_{r,s},\widetilde{G}_{r,s}]\widetilde{\Gamma}_{r,s}$ is locally free, we obtain:

**Lemma 4.8.** *The projection of the stabilizer* $\Lambda(\widetilde{\rho}_{r,s}) < \widetilde{G}_{r,s}$ *onto* $\widetilde{G}_{r,s}/[\widetilde{G}_{r,s},\widetilde{G}_{r,s}]$ *is the stabilizer* $\Lambda(\widetilde{\rho}'_{r,s})$ *of the measure* $\widetilde{\rho}'_{r,s}$*.*

**Lemma 4.9.** *The Lie algebra $\widetilde{\mathfrak{h}}_{r,s} \subset \widetilde{\mathfrak{g}} \oplus \widetilde{\mathfrak{g}}$ satisfies*

$$\{(r^k Z, s^k Z) \mid Z \in \widetilde{\mathfrak{g}}^{(k)}\} \subset \widetilde{\mathfrak{h}}_{r,s} \cap (\widetilde{\mathfrak{g}}^{(k)} \oplus \widetilde{\mathfrak{g}}^{(k)}).$$

*It follows that the stabilizer $\Lambda(\rho) < G \times G$ of the joining $\rho$ contains the subgroup*

$$L := \{\exp(r^k Z, s^k Z) \mid Z \in \widetilde{\mathfrak{g}}^{(k)}\} < \widetilde{G}^{(k)} \times \widetilde{G}^{(k)} = \widetilde{G}_{r,s}^{(k)}.$$

*Proof.* The first statement follows from the previous two lemmata and from Lemma D.3 as in the proof of Lemma 3.6. The second is an application of Lemma 4.6.  □

Recall that $\widetilde{G}^{(k)} < G$ is a closed normal subgroup in the center of $G$ such that its Lie algebra $\widetilde{\mathfrak{g}}^{(k)}$ is a rational ideal in $\mathfrak{g}$ include in the kernel of $B : \mathfrak{g} \to \mathfrak{g}$. It follows that $M^{(k-1)} := G/\Gamma\widetilde{G}^{(k)}$ is a nilmanifold and $\mathbb{T}^{(k)} := \widetilde{G}^{(k)}/(\widetilde{G}^{(k)} \cap \Gamma)$ is a torus. The group $\mathbb{T}^{(k)}$ acts on $G/\Gamma$ by left translation. Thus, we have an orthogonal decomposition

$$L^2(M, \lambda) = \bigoplus_{\chi \in \widehat{\mathbb{T}^{(k)}}} H_\chi$$

where, for each character $\chi$ of the torus $\mathbb{T}^{(k)}$ we set

$$H_\chi = \left\{f \in L^2(M, \lambda) \mid f(zx) = \chi(z)f(x), \forall z \in \mathbb{T}^{(k)}, \forall x \in M\right\}.$$

**Proposition 4.10.** *Let $\chi_1$, $\chi_2$ be characters of the torus $\mathbb{T}^{(k)}$ so that at least one is non-trivial. Then, if $\chi_1^{r^k} \neq \chi_2^{s^k}$, for any $f_i \in H_{\chi_i}$, $i = 1, 2$, and any ergodic joining $\rho$ of $(M, \phi^r, \lambda)$ and $(M, \phi^s, \lambda)$, we have*

$$(4.8) \qquad\qquad\qquad \rho(f_1 \otimes \bar{f}_2) = 0.$$

*Moreover, if $f \in H_\chi$ for nontrivial $\chi$ then for every $\omega \in \mathbb{C}$ with $|\omega| = 1$ there exists $g \in G$ such that*

$$f(\phi^n(l_g x)) = \omega f(\phi^n x) \text{ for all } x \in M, \ n \in \mathbb{Z}.$$

*Proof.* Suppose that $\chi_1^{r^k} \neq \chi_2^{s^k}$ and $f_1 \in H_{\chi_1}$ and $f_2 \in H_{\chi_2}$. Let $\rho$ be any ergodic joining of $(M, \phi^r, \lambda)$ and $(M, \phi^s, \lambda)$. By Lemma 4.9, every element of the group $L < \widetilde{G}^{(k)} \times \widetilde{G}^{(k)}$ stabilizes the measure $\rho$. It follows that the rotation by $(z^{r^k}, z^{s^k}) \in \mathbb{T}^{(k)} \times \mathbb{T}^{(k)}$ on $M \times M$ preserves $\rho$ for every $z \in \mathbb{T}^{(k)}$. Therefore

$$\rho(f_1 \otimes \bar{f}_2) = (\chi_1^{r^k} \chi_2^{-r^k})(z)\rho(f_1 \otimes \bar{f}_2),$$

which gives (4.8).

Since $\widetilde{\mathfrak{g}}^{(k)} \subset \ker B$, for every $h \in \widetilde{G}^{(k)}$ we have $A(h) = h$, so

$$\phi(l_h g\Gamma) = uA(hg)\Gamma = l_h\phi(g\Gamma) \text{ for every } g\Gamma \in M.$$

It follows that, if $f \in H_\chi$ for nontrivial $\chi$ then for every $\omega \in \mathbb{C}$ with $|\omega| = 1$ there exists $g \in \widetilde{G}^{(k)}$ such that

$$f(\phi^n(l_g x)) = f(l_g\phi^n(x)) = \chi(g)f(\phi^n x) = \omega f(\phi^n x) \text{ for all } x \in M, \ n \in \mathbb{Z},$$

and the proof is complete.  □

Finally, proceeding by induction on $k$ the clas of nilpotency of the group $\widetilde{G}$, we obtain the following result.

**Theorem 4.11.** *Let $\phi$ be an ergodic affine unipotent diffeomorphism of a compact connected nilmanifold $M = G/\Gamma$. Then there exists a set $\mathscr{C} \subset C(M) \cap L_0^2(M, \lambda)$ whose span is dense in $L_0^2(M, \lambda)$ such that:*

(i) *for any pair $f_1, f_2 \in \mathscr{C}$, for all but at most one pair of relatively prime natural numbers $(r, s)$ we have $\rho(f_1 \otimes \bar{f_2}) = 0$ for all ergodic joinings $\rho$ of $\phi^r$ and $\phi^s$;*

(ii) *for every $f \in \mathscr{C}$ and any $\omega \in \mathbb{C}$ with $|\omega| = 1$ there exists $g \in G$ such that we have $f(\phi^n(l_g x)) = \omega f(\phi^n x)$ for every $x \in M$ and $n \in \mathbb{Z}$.*

*Proof.* First note that if $\chi$ is the trivial character, then every element of $H_\chi$ is invariant under the action of $\widetilde{G}^{(k)}$, hence it can be treated as an element of $L^2(M^{(k-1)}, \lambda)$. Furthermore, we can define the affine unipotent diffeomorphism $\phi^{(k-1)} : M^{(k-1)} \to M^{(k-1)}$ by

$$\phi^{(k-1)}(g\Gamma\widetilde{G}^{(k)}) = uA(g)\Gamma\widetilde{G}^{(k)}.$$

As $A(\Gamma) = \Gamma$ and $A(\widetilde{G}^{(k)}) = \widetilde{G}^{(k)}$, this map is well defined and is a factor of $\phi$ via the natural $G$-equivariant projection $\pi^{(k)} : G/\Gamma \to G/\Gamma\widetilde{G}^{(k)}$. It allows us to proceed by induction.

The inductive step is given by Proposition 4.10. Note that Proposition 4.10 requires the non-triviality of the automorphism $A$. Therefore induction must start from an affine diffeomorphism with trivial automorphism, so from a nil-translation. Thus the basis case of induction is derived from Theorem 3.8. Finally, the construction of the set $\mathscr{C}$ runs as in the proof of Theorem 3.8. □

*Proof of Theorem A and Corollaries B and C.* Theorem A as well as the Corollaries B and C are direct consequences of Theorem 4.11 and Theorem 2.2. □

4.5. **Nil-translations on non-connected nilmanifolds.** Suppose that $G$ is a nilpotent Lie group and $\Gamma < G$ its lattice. Then $M = G/\Gamma$ is a compact nilmanifold which we assume to be non-connected. Let $l_u$ for $u \in G$ be a uniquely ergodic translation on $M$. Denote by $G_0 < G$ the identity component of $G$. The group $G_0$ is closed (and open) normal, and we may assume that it is simply connected. Moreover, $M_0 = (G_0\Gamma)/\Gamma \approx G_0/(G_0 \cap \Gamma)$ is a connected component of $M$. Since $l_u$ is minimal and $M$ is compact, there is a minimal integer $m > 1$ such that $M$ is the disjoint union of clopen sets $l_u^k M_0$ for $0 \le k < m$ and $l_u^m M_0 = M_0$. Moreover, $l_u^m : M_0 \to M_0$ is a uniquely ergodic diffeomorphism which is a unipotent affine diffeomorphism of the compact connected nilmanifold $M_0 = G_0/(G_0 \cap \Gamma)$. Indeed, as $l_u^m((G_0\Gamma)/\Gamma) = (G_0\Gamma)/\Gamma$, we have $u^m \in G_0\Gamma$, and hence there are $u_0 \in G_0$ and $\gamma \in \Gamma$ such that $u^m = u_0\gamma$. It follows that $l_u^m = l_{u_0} \circ \mathrm{Ad}_\gamma$ on $M_0$, where $l_{u_0}$ is a nil-translation on the nilmanifold $G_0/(G_0 \cap \Gamma)$ and $\mathrm{Ad}_\gamma : G_0 \to G_0$ is a unipotent automorphism of $G_0$ with $\mathrm{Ad}_\gamma(G_0 \cap \Gamma) = G_0 \cap \Gamma$. Therefore $l_u^m : M_0 \to M_0$ is an ergodic unipotent affine diffeomorphism which will be denoted by $\phi : M_0 \to M_0$

Let us consider the factor map $p : M \to \mathbb{Z}/m\mathbb{Z}$ defined by

$$p(l_u^k x) = k \mod m\mathbb{Z}, \quad \text{if } k \in \mathbb{Z} \text{ and } x \in M_0.$$

Then the map $p$ intertwines the nil-translation $l_u$ on $M$ with the rotation on $\mathbb{Z}/m\mathbb{Z}$ by 1. Denote by $H^+$ the space of $L^2(M, \lambda)$ of functions constant on fibers of $p$ and by $H^-$ its orthocomplement in $L^2(M, \lambda)$. Then $H^-$ coincides with the

space of $L^2$-functions which have zero mean on each fiber $M_k := l_u^k M_0$ for $0 \leq k < m$.

**Proposition 4.12.** *Let $l_u$ be an ergodic nil-translation of a compact nilmanifold $M = G/\Gamma$ with $m$ connected components. Then there exists a set $\mathscr{C} \subset C(M) \cap H^-$ whose span is dense in $H^-$ such that:*

- (i) *for any $f_1, f_2 \in \mathscr{C}$, for all but a finite number of pairs of distinct prime numbers $(r, s)$, we have $\rho(f_1 \otimes \bar{f}_2) = 0$ for all ergodic joinings $\rho$ of $l_u^r$ and $l_u^s$;*
- (ii) *for every $f \in \mathscr{C}$ and any $\omega \in \mathbb{C}$ with $|\omega| = 1$ there exists a homeomorphism $S : M \to M$ such that $f(l_u^n(Sx)) = \omega f(l_u^n x)$ for every $x \in M$ and $n \in \mathbb{Z}$.*

*Proof.* Let us consider the ergodic unipotent affine diffeomorphism $\phi := l_{u_0} \mathrm{Ad}_\gamma$ on $M_0$. By Theorem 4.11 applied to $\phi$, there exists a set $\mathscr{C}_0 \subset C(M_0) \cap L_0^2(M_0)$ whose span is dense in $L_0^2(M_0, \lambda)$ and which satisfies the conditions (i) and (ii) of that theorem. We consider $\mathscr{C}_0$ as a set of continuous functions on $M$ by extending its elements as 0 on $M \setminus M_0$. For every $k \in \mathbb{Z}/m\mathbb{Z}$ the map $l_u^k : M \to M$ establishes a homeomorphism between $M_0$ and $M_k$. Set $\mathscr{C}_k = \{f \circ l_u^{-k} \mid f \in \mathscr{C}_0\}$. Elements of $\mathscr{C}_k$ are continuous functions on $M$ supported on $M_k$. Then $\mathscr{C} := \bigcup_{k \in \mathbb{Z}/m\mathbb{Z}} \mathscr{C}_k$ is a linearly dense set in $H^-$.

Fix $f \in \mathscr{C}_k$ for some $0 \leq k < m$ and $\omega \in \mathbb{C}$ with $|\omega| = 1$. Then $f$ is supported on $M_k$ and there exists $h \in \mathscr{C}_0$ such that the restriction of $f$ to $M_k$ is equal to $h \circ l_u^{-k}$. By (ii) in Theorem 4.11, there exists $g \in G_0$ such that $h(\phi^n(l_g x)) = \omega h(\phi^n x)$ for every $x \in M_0$ and $n \in \mathbb{Z}$. Let $S : M \to M$ be the homeomorphism defined by $Sx = l_u^{k'} \circ l_g \circ l_u^{-k'} x$ if $x \in M_{k'}$. Then

$$(4.9) \qquad f(l_u^n(Sx)) = \omega f(l_u^n x) \text{ for all } n \in \mathbb{Z} \text{ and } x \in M.$$

Indeed, let $x \in M_{k'}$: if $n \neq k - k' \mod m$ then both $l_u^n x$ and $l_u^n(Sx)$ do not belong to $M_k$, and hence both sides of (4.9) vanish. If $n = k - k' + n'm$ for some $n' \in \mathbb{Z}$ then $x = l_u^{k'} y$ for some $y \in M_0$ and

$$f(l_u^n(Sx)) = h(l_u^{n+k'-k} l_g y) = h(\phi^{n'} l_g y) = \omega h(\phi^{n'} y) = \omega h(l_u^{n+k'-k} y) = \omega f(l_u^n x).$$

Note that there exits only finitely many pairs of distinct prime numbers $(r, s)$ such that $m, r, s$ are not pairwise coprime. Therefore to prove we can assume that $(r, s)$ is a pair such that $m, r, s$ are pairwise coprime. Then both the nil-translations $l_u^r$ and $l_u^s$ are also uniquely ergodic.

Let $\rho$ be an ergodic joining of $l_u^r$ and $l_u^s$. Choose any pair $f_1, f_2$ of functions in $\mathscr{C}$. Suppose that $f_1 \in \mathscr{C}_{k_1}$ and $f_2 \in \mathscr{C}_{k_2}$ for some $0 \leq k_1, k_2 < m$. Then, for $i = 1, 2$, $f_i$ is supported on $M_{k_i}$ and $f_i = h_i \circ l_u^{-k_i}$ for some $h_i \in \mathscr{C}_0$. Denote by $\rho_{k_1, k_2}$ the restriction of $\rho$ to the set $M_k \times M_{k'}$. Then the measure $(l_u^{-k_1} \times l_u^{-k_2})_* \rho_{k_1, k_2}$ is supported on $M_0 \times M_0$ and coincides with the restriction of $(l_u^{-k_1} \times l_u^{-k_2})_* \rho$ to $M_0 \times M_0$. Then we have

$$\rho(f_1 \otimes \bar{f}_2) = (l_u^{-k_1} \times l_u^{-k_2})_* \rho_{k_1, k_2}(h_1 \otimes \bar{h}_2).$$

If the measure $\rho_{k_1, k_2}$ vanishes then $\rho(f_1 \otimes \bar{f}_2) = 0$. Thus we may assume that $\rho_{k_1, k_2}$ is not zero. As $l_u^r$ and $l_u^s$ are uniquely ergodic and $l_u^{-k_1} \times l_u^{-k_2}$ commutes with $l_u^r \times l_u^s$, the measure $(l_u^{-k_1} \times l_u^{-k_2})_* \rho$ is also an ergodic joining of $l_u^r$ and $l_u^s$.

Since $r, s, m$ are pairwise coprime and $(l_u^{-k_1} \times l_u^{-k_2})_* \rho$ does not vanish on $M_0 \times M_0$, the measure $(l_u^{-k_1} \times l_u^{-k_2})_* \rho$ is supported on $\bigcup_{k=0}^{m-1} (l_u^r \times l_u^s)^k (M_0 \times M_0)$, where the elements of the union are pairwise disjoint and $(l_u^r \times l_u^s)^m (M_0 \times M_0) = M_0 \times M_0$. It follows that the restriction $(l_u^{-k_1} \times l_u^{-k_2})_* \rho_{k_1,k_2}$ is an $l_u^{rm} \times l_u^{sm}$-invariant ergodic measure on $M_0 \times M_0$. By (i) in Theorem 4.11, it follows that

$$\rho(f_1 \otimes \bar{f}_2) = (l_u^{-k_1} \times l_u^{-k_2})_* \rho_{k_1,k_2}(h_1 \otimes \bar{h}_2) = 0$$

for all but one pair $(r, s)$, which completes the proof. $\qquad\square$

**Theorem 4.13.** *Let $l_u$ be a nil-translation of the compact nilmanifold $M = G/\Gamma$. Then*

$$(4.10) \qquad \frac{1}{M} \sum_{M \leq m < 2M} \left| \frac{1}{H} \sum_{m \leq n < m+H} f(l_u^n x) \boldsymbol{\mu}(n) \right| \to 0$$

*when $H \to \infty$, $H/M \to 0$ for each $x \in M$ and each $f \in C(M)$. If additionally $l_u$ is ergodic then (4.10) hold for all $f \in D(M)$.*

*Proof.* Assume that the nil-translation is ergodic. Denote by $\mathscr{B}_{rat} \subset \mathscr{B}$ the maximal factor of $l_u$ with the rational discrete spectrum. Then $H^+ = L^2(M, \mathscr{B}_{rat}, \lambda)$. In view of (i) in Proposition 4.12, the nil-translation satisfies the AOP property (i.e. (2.1)) for all functions $f, g \in H^- = L^2(M, \mathscr{B}_{rat}, \lambda)^\perp$. Since the space $H^-$ does not depend on the choice of the topological model of the nil-translation, similar arguments to those used in the proof of Theorem 3 in [HLD15] show that for each zero mean $f \in C(M) \cap H^-$, for any sequence $(x_k)_{k \geq 1}$ in $M$ and any $(b_k)_{k \geq 1}$ with $b_{k+1} - b_k \to +\infty$ we have

$$\frac{1}{b_{K+1}} \sum_{k \leq K} \left( \sum_{b_k \leq n < b_{k+1}} f(l_u^n x_k) \boldsymbol{\mu}(n) \right) \to 0 \text{ when } K \to \infty.$$

Together with (ii) in Proposition 4.12 this gives (4.10) for every $f \in C(M) \cap H^-$. If $f \in H^+$ then the sequence $(f(l_u^n x))$ is periodic and the property (4.10) follows from Theorem 1.7 in [MRT15] applied to the Möbius function (cf. Remark 2.3). If follows that (4.10) holds for every $f \in C(M)$, and finally, by Remark 2.4, also for every $f \in D(M)$.

We need now to consider the case where the nil-translation $l_u$ is not ergodic. Then, for any $x \in M$ denote by $M_x$ its orbit closure. By [Rat91], the restriction of $l_u$ to $M_x$ is topologically isomorphic to an ergodic nil-translation on a compact (not necessary connected) nilmanifold. Therefore, our claim is reduced to the ergodic case. $\qquad\square$

### 4.6. **Polynomial type sequences.**

**Theorem 4.14.** *Let $\phi$ be an ergodic affine unipotent diffeomorphism of a compact connected nilmanifold $M = G/\Gamma$. Then for all $\gamma \in \mathbb{R} \setminus \{0\}$ and $\varrho \in \mathbb{R}$, we have*

$$(4.11) \qquad \frac{1}{M} \sum_{M \leq m < 2M} \left| \frac{1}{H} \sum_{m \leq n < m+H} f(\phi^{[\gamma n + \varrho]} x) \boldsymbol{\mu}(n) \right| \to 0$$

*when $H \to \infty$, $H/M \to 0$ for each $x \in M$ and each $f \in C(M)$.*

*Proof.* Let $\widetilde{\Sigma} = (\widetilde{G}, \widetilde{\Gamma}, p, (\widetilde{\phi}_t)_{t \in \mathbb{R}}, \widetilde{\lambda})$ be the suspension of the ergodic affine unipotent diffeomorphism $\Sigma = (G, \Gamma, \phi, \lambda)$, with $\lambda$ and $\widetilde{\lambda}$ the Haar measures on the corresponding nilmanifolds. The nilflow $\widetilde{\Sigma}$ is uniquely ergodic. As usual we

identify $G/\Gamma$ with $p^{-1}\{0\}$. For each continuous function $f$ on $G/\Gamma$ let $\widetilde{f}$ be the unique function on $\widetilde{G}/\widetilde{\Gamma}$ defined by the condition

$$\widetilde{f}(\widetilde{\phi}_t x) = f(x), \quad \forall x \in G/\Gamma \approx p^{-1}\{0\}, \ \forall t \in [0,1),$$

or, equivalently, by the condition

(4.12) $$\widetilde{f}(\widetilde{\phi}_t x) = f(\phi^{\lfloor t \rfloor} x), \quad \forall x \in G/\Gamma \approx p^{-1}\{0\}, \ \forall t \in \mathbb{R}.$$

Since the set of discontinuities of $\widetilde{f}$ is contained in $p^{-1}\{0\}$, the function $\widetilde{f}$ belong so the class $D(\widetilde{M})$ defined in Remark 2.4.

The map $\widetilde{\phi}_\gamma$ is a nil-translation $l_{\widetilde{u}}$ on $\widetilde{G}/\widetilde{\Gamma}$ by an element $\widetilde{u} \in \widetilde{G}$; thus we shall consider two cases.

**Ergodic case:** Suppose that the nil-translation $\widetilde{\phi}_\gamma = l_{\widetilde{u}}$ on $\widetilde{G}/\widetilde{\Gamma}$ is ergodic. Then $\widetilde{\phi}_{\gamma n+\varrho} = \widetilde{\phi}_\varrho \circ l_{\widetilde{u}}^n$ and, by Theorem 4.13 and formula (4.12), we have, for all $x \in G/\Gamma$,

$$\frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le n < m+H} f(\phi^{[\gamma n+\varrho]} x) \boldsymbol{\mu}(n) \right|$$

$$= \frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le n < m+H} \widetilde{f} \circ \widetilde{\phi}_\varrho (l_{\widetilde{u}}^n(x)) \boldsymbol{\mu}(n) \right| \to 0.$$

**Non-ergodic case:** Assume that the nil-translation $\widetilde{\phi}_\gamma = l_{\widetilde{u}}$ on $\widetilde{G}/\widetilde{\Gamma}$ is not ergodic. Then, by [Rat91], for every $x = g\Gamma \in G/\Gamma$ there exists a closed subgroup $H < \widetilde{G}$ so that $\widetilde{u} \in H$, the $l_{\widetilde{u}}$-orbit closure of $\widetilde{x} := g\widetilde{\Gamma}$ coincides with the orbit $H\widetilde{x}$ and the restriction of $l_{\widetilde{u}}$ to the sub-nilmanifold $\widetilde{W} := H\widetilde{x} \approx H/(H \cap g\widetilde{\Gamma}g^{-1})$ is uniquely ergodic. The sub-nilmanifold $\widetilde{W}$ projects via $p$ to a sub-nilmanifold of $\mathbb{R}/\mathbb{Z}$, that is to a closed subgroup $W$ of $\mathbb{R}/\mathbb{Z}$. There are two possibilities: either $W$ is finite or $W = \mathbb{R}/\mathbb{Z}$. In the first case the nilmanifold $\widetilde{W}$ has a finite number of components and the restriction $\widetilde{f} \circ \widetilde{\phi}_{\varrho|\widetilde{W}}$ of $\widetilde{f} \circ \widetilde{\phi}_\varrho$ to $\widetilde{W}$ is continuous. In the second case, the discontinuities of $\widetilde{f} \circ \widetilde{\phi}_{\varrho|\widetilde{W}}$ belong to the lower dimensional sub-nilmanifold $\widetilde{W} \cap p^{-1}\{-\varrho\}$.

In summary, in both cases we have $\widetilde{f} \circ \widetilde{\phi}_{\varrho|\widetilde{W}} \in D(\widetilde{W})$ and it can be treated as a function on the compact nilmanifold $\widetilde{W} = H/(H \cap g\widetilde{\Gamma}g^{-1})$. Then, by Theorem 4.13, we have

$$\frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le n < m+H} \widetilde{f} \circ \widetilde{\phi}_\varrho (l_{\widetilde{u}}^n(x)(\widetilde{x})) \boldsymbol{\mu}(n) \right| \to 0,$$

which completes the proof in the non-ergodic case. $\qquad\square$

*Proof of Proposition D.* Let $P(x) = a_d x^d + \ldots + a_1 x + a_0 \in \mathbb{R}[x]$ have the leading coefficient $a_d$ irrational. Let $\alpha = a_d \cdot d!$ and let $\phi \colon \mathbb{T}^d \to \mathbb{T}^d$ be given by

$$\phi(x_1, x_2, \ldots, x_d) = (x_1 + \alpha, x_1 + x_2, \ldots, x_{d-1} + x_d).$$

Then $\phi$ is an ergodic affine unipotent diffeomorphism of $\mathbb{T}^d$. Following [Fur67] (see also [EW11]), we can now find $x_1, \ldots, x_d \in \mathbb{T}$ so that

$$P(n) = \binom{n}{d} \alpha + \binom{n}{d-1} x_1 + \ldots + n x_{d-1} + x_d \bmod 1 \text{ for each } n \in \mathbb{Z}.$$

Moreover, notice that (mod 1) we have

$$\binom{n}{d}\alpha + \binom{n}{d-1}x_1 + \ldots + nx_{d-1} + x_d = f(\phi^n(x_1,\ldots,x_d)),$$

where $f\colon \mathbb{T}^d \to \mathbb{T}$ is the projection on the last coordinate. Finally, the assertion of the theorem follows directly from Theorem 4.14 applied to the function $\exp(\iota f)$.

□

4.7. **Nilsequences and polynomial multiple correlations.** Following [BHK05] and [Lei10], recall that a bounded sequence $(c_n) \in \mathbb{C}^{\mathbb{N}}$ is called a *basic nilsequence* if there exist a compact but not necessarily connected nilmanifold $G/\Gamma$, a continuous function $f \in C(G/\Gamma)$, an element $u \in G$ and a point $x \in G/\Gamma$ such that

$$c_n = f(l_u^n x), \qquad \forall n \in \mathbb{N},$$

where as usual $l_u$ denotes the nil-translation $G/\Gamma \ni x \mapsto ux \in G/\Gamma$. A sequence $(d_n) \in \mathbb{C}^{\mathbb{N}}$ is called a *nilsequence*, if it is a uniform limit of basic nilsequences, i.e. for every $\varepsilon > 0$, there exists a basic nil-sequence $(c_n)$ such that

$$|c_n - d_n| < \varepsilon \text{ for all } n \in \mathbb{N}.$$

*Proof of Theorem E.* Theorem E follows directly from Theorem 4.13.          □

**Theorem** (Leibman, [Lei10]). *Let $T$ be an automorphism of a probability standard Borel space $(X,\mathcal{B},\mu)$. Given, for all $i = 1,\ldots,k$, functions $g_i \in L^{\infty}(X,\mu)$ and polynomials $p_i \in \mathbb{Z}[x]$, there exists a nilsequence $(d_n)$ such that*

$$(4.13) \qquad \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \left| d_n - \int_X g_1 \circ T^{p_1(n)} \cdot \ldots \cdot g_k \circ T^{p_k(n)} \, d\mu \right| = 0.$$

*Proof of Corollary F.* By Theorem E, we have

$$\frac{1}{M} \sum_{M \le m < 2M} \left| \frac{1}{H} \sum_{m \le n < m+H} \boldsymbol{\mu}(n) d_n \right| \to 0$$

for a nilsequence $(d_n)$ satisfying the assertion (4.13) of Leibman's theorem. The statement follows now by an immediate application of the triangle inequality.

□

## 5. Lifting AOP to induced action

5.1. **Induced actions.** In this section, we follow [Mac52], [Zim78]. Assume that $G$ is a locally compact second countable (lcsc) group. Assume that $H \subset G$ is a closed subgroup of $G$. The quotient space $G/H$ is lcsc for the quotient topology. Let $\pi : G \to G/H$ be the canonical, continuous quotient map. Let $\tau = (\tau_g)_{g \in G}$ denote the $G$-action on $G/H$ by left translations: $\tau_g(xH) = gxH$. The action $\tau$ is continuous, transitive, hence ergodic for any $G$-quasi-invariant Borel probability measure $m$ on $G/H$ (such measures always exist). We say that $G/H$ is a finite volume space if $G/H$ supports a $G$-invariant probability measure $m_{G/H}$,

Let $\mathscr{S} = (S_h)_{h \in H}$ be an almost free Borel left action of $H$ on a probability standard Borel space $(Y,\mathscr{C},\nu)$. The group $H$ acts on the right on the product $Y \times G$, by

$$(y,g)h = (S_{h^{-1}} y, gh)$$

Let $(Y \times G)/H$ be the orbit space endowed with the quotient measurable structure. Since $G/H$ is Hausdorff, since $(Y, \mathscr{C}, \nu)$ is a standard Borel space, and since the projection $(Y \times G)/H \to G/H$ measurable, the Borel space $(Y \times G)/H$ is countably separated (i.e. the $H$-action on $Y \times G$ is smooth, in the sense of [Zim78, Def. 2.1.9]).

The $G$-action $((\tau_{\mathscr{S}})_g)_{g \in G}$ on $(Y \times G)/H$ is just given by left translation on the second factor:

$$(5.1) \qquad (\tau_{\mathscr{S}})_g : (Y \times G)/H \to (Y \times G)/H, \quad (\tau_{\mathscr{S}})_g (y, g_1) H = (y, g g_1) H$$

is called the *G-action induced from $\mathscr{S}$*.

If $m$ is any $G$-quasi-invariant Borel probability measure on $G/H$, then we may define a Borel probability measure $\nu \otimes_{\mathscr{S}} m$ on $(Y \times G)/H$, by setting, for any positive measurable function $F$ on $(Y \times G)/H$,

$$\nu \otimes_{\mathscr{S}} m(F) = \int_{G/H} \left( \int_Y \widetilde{F}(y, g) \, \mathrm{d}\nu(y) \right) \mathrm{d}m(gH)$$

where $\widetilde{F}$ is the $H$-invariant lift of $F$ to $Y \times G$. The measure $\nu \otimes_{\mathscr{S}} m$ is quasi-invariant for the $G$-action induced from $\mathscr{S}$ and it is ergodic if the action $\mathscr{S}$ is ergodic on $(Y, \mathscr{C}, \nu)$. Furthermore, if $G/H$ admits a $G$-invariant probability measure $m_{G/H}$, then $\nu \otimes_{\mathscr{S}} m_{G/H}$ is a $G$-invariant Borel probability measure on $(Y \times G)/H$.

Let

$$(5.2) \qquad s : G/H \to G, \text{ a Borel map, } \pi \circ s = \mathrm{Id}_{G/H}$$

be a (measurable) selector for $\pi$. The map $\theta$ defined on $G \times G/H$ by setting

$$(5.3) \qquad \theta(g, xH) = s(gxH)^{-1} g \, s(xH).$$

takes its values in $H$, since

$$s(gxH)H = gxH, \quad \text{and} \quad g \, s(xH) = gxH.$$

**Lemma 5.1.** *The map $\theta : G \times G/H \to H$ is a (left) cocycle for the G-action $\tau$.*

*Proof.* We have

$$\begin{aligned} \theta(g_1 g_2, xH) &= s(g_1 g_2 xH)^{-1} g_1 g_2 s(xH) \\ &= s(g_1 g_2 xH)^{-1} g_1 s(g_2 xH)^{-1} s(g_2 xH) g_2 s(xH) \\ &= \theta(g_1, g_2 xH) \theta(g_2, xH). \qquad \square \end{aligned}$$

The skew product $G$-action $\tau_{\theta, \mathscr{S}} = ((\tau_{\theta, \mathscr{S}})_g)_{g \in G}$, defined by

$$(5.4) \quad (\tau_{\theta, \mathscr{S}})_g : Y \times G/H \to Y \times G/H, \qquad (\tau_{\theta, \mathscr{S}})_g (y, xH) = (S_{\theta(g, xH)} y, gxH)$$

is isomorphic to the $G$-action $\tau_{\mathscr{S}}$ on $(Y \times G)/H$ induced from $\mathscr{S}$, via the Borel isomorphism $\Phi : Y \times G/H \to (Y \times G)/H$ defined by

$$\Phi(y, gH) = (S_{g^{-1} s(gH)} y, g) H = (y, s(gH)) H.$$

**Remark 5.2.** In the vocabulary of [LL01], [LP03], the induced $G$-action is a Rokhlin cocycle extension of the $G$-action $\tau$ through the $H$-valued cocycle $\theta$ and an ergodic $H$-action $\mathscr{S}$. Both descriptions (5.1) and (5.4) for the $G$-action induced from $\mathscr{S}$ have their advantages and disadvantages: the first is natural and intrinsic; the second, albeit depending of a arbitrary selector $\theta$, yields an easier description of the measure $\nu \otimes_{\mathscr{S}} m$ which is just $\nu \otimes m$ on the space $Y \times$

$G/H$; the second definition also make apparent that $(Y \times G)/H$ endowed with the measure $\nu \otimes m$ is a standard Borel probability space.

5.2. **The AOP property.** Assume now that $G$ is an abelian group without torsion elements, that $H < G$ is closed and co-compact and let $m_{G/H}$ be the $G$-invariant probability measure on $G/H$. Assume also that we have an ergodic $H$-action $\mathscr{S} = (S_h)_{h \in H}$ on $(Y, \mathscr{C}, \nu)$ and let $p \in \mathbb{Z}$. Then $p$ determines another $H$-action $\mathscr{S}^{(p)} = (S_h^{(p)})_{h \in H}$, where

$$(5.5) \qquad\qquad S_h^{(p)}(y) = (S_h)^p(y) = S_{h^p}(y);$$

indeed the map $h \mapsto h^p$ is a group homomorphism.

Following [HLD15], we now define:

**Definition 5.1.** An ergodic $H$ action $\mathscr{S}$ is said to have the AOP property if for each $f, g \in L_0^2(Y, \mathscr{C}, \nu)$

$$(5.6) \qquad \lim_{p \neq q, p, q \in \mathscr{P}, p, q \to \infty} \sup_{\kappa \in J^e(\mathscr{S}^{(p)}, \mathscr{S}^{(q)})} \left| \int_{Y \times Y} f \otimes g \, d\kappa \right| = 0.$$

If (5.6) is satisfied for $f, g$, then we say that $f, g$ satisfy the AOP property.

**Remark 5.3.** (i) In order to talk about the AOP property, we need to know that the $H$-actions are "totally" ergodic, that is that $\mathscr{S}^{(p)}$ is ergodic for each prime $p$. To see its meaning let us pass to the character group $\widehat{H}$ and consider the endomorphism $E_p : \chi \mapsto \chi^p$. We need to assume that the kernel of $E_p$ is a set of measure zero for the maximal spectral type of $\mathscr{S}$ on $L_0^2(Y, \mathscr{C}, \nu)$. If $H$ equals $\mathbb{Z}$ then it simply means that $T$ has no roots of unity as eigenvalues. If $\widehat{H}$ is torsion free, as for example it holds for $H = \mathbb{R}$, then each ergodic action is totally ergodic.

(ii) If (5.6) holds then it also holds if $f \in L^2(Y, \mathscr{C}, \nu)$ and $g \in L_0^2(Y, \mathscr{C}, \nu)$ (or vice versa); indeed, we write $f = f_0 + c$, with $c = \int_Y f \, d\nu$ then $\int_{Y \times Y} c \otimes g \, d\kappa = \int_Y g \, d\nu = 0$.

(iii) If (5.6) holds, then, in fact, we can replace the set of ergodic joinings $J^e(\mathscr{S}^{(p)}, \mathscr{S}^{(q)})$ by $J(\mathscr{S}^{(p)}, \mathscr{S}^{(q)})$; indeed,

$$\sup_{\kappa \in J^e(\mathscr{S}^{(p)}, \mathscr{S}^{(q)})} \left| \int_{Y \times Y} f \otimes g \, d\kappa \right| = \sup_{\kappa \in J(\mathscr{S}^{(p)}, \mathscr{S}^{(q)})} \left| \int_{Y \times Y} f \otimes g \, d\kappa \right|$$

for each $f, g \in L^2(Y, \mathscr{C}, \nu)$.

(iv) If $L$ is a closed subgroup of $H$ and the $L$-subaction of $\mathscr{S}$ has the AOP property, then the original action has the AOP property. Indeed, any ergodic invariant measure for $S_{h^p} \times S_{h^q}$, $h \in H$ is an invariant measure for the subaction $S_{\ell^p} \times S_{\ell^q}$, $\ell \in L$. Then use (iii).

We will constantly assume that

$$(5.7) \qquad\qquad H \text{ has no non-trivial compact subgroups.}^{[7]}$$

Since $H$ is cocompact, there is a relatively compact fundamental domain, that is, a subset $A \subset G$ such that

$$(5.8) \qquad \overline{A} \text{ is compact, } hA \cap h'A = \emptyset \text{ whenever } h \neq h' \text{ and } \bigcup_{h \in H} hA = G.$$

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[7]Equivalently, we assume that the dual group $\widehat{H}$ is connected. It may happen that $G$ has non-trivial compact subgroups.

We can always assume that (cf. (5.2))

$$(5.9) \qquad\qquad s: H \to A \text{ and } s(H) = e.$$

5.3. **Ergodic components for the $G$-action $\tau^{(p)} \times \tau^{(q)}$ and regularity of cocycles.** Recall $\tau_g^{(p)}(xH) = g^p x H$. Assume that $(p,q) = 1$ and let $a, b \in \mathbb{Z}$ be so that

$$ap + bq = 1.$$

The following result is a more general version of Lemma 2.2.1 in [KPL15] and Lemma 3.3. Since the proof runs along similar lines, we omit it.

**Lemma 5.4.** *The ergodic components of $\tau^{(p)} \times \tau^{(q)} = (\tau_g^{(p)} \times \tau_g^{(q)})_{g \in G}$ are of the form*

$$\mathcal{H}_c := \{(x_1 H, x_2 c H) : x_1^q H = x_2^p H\}, \, c \in G.^{[8]}$$

*Moreover, the action of $\tau^{(p)} \times \tau^{(q)}$ on $\mathcal{H}_c$ is (topologically)[9] isomorphic to $\tau$.*

We are now interested in the (infinite) skew product $G$-action $(\tau_\theta)^{(p)} \times (\tau_\theta)^{(q)}$,

$$\begin{aligned}
\left((\tau_\theta)^{(p)} \times (\tau_\theta)^{(q)}\right)_g &(x_1 H, h_1, x_2 H, h_2) = \\
&\left(g^p x_1 H, \theta(g^p, x_1 H) h_1, g^q x_2 H, \theta(g^q, x_2 H) h_2\right)
\end{aligned}$$

restricted to $\mathcal{H}_c \times H \times H$ (up to a natural interchange of coordinates).

**Lemma 5.5.** *For each $c \in G$, the map $\widetilde{R}(x_1 H, h_1, x_2 c H, h_2) := (x_1^a x_2^b H, h_1, h_2)$ establishes an isomorphism of $(\tau_\theta)^{(p)} \times (\tau_\theta)^{(q)}|_{\mathcal{H}_c}$ with $\tau_{\theta^{(p,q)}}$, where the cocycle*

$$\theta^{(p,q)} : G \times G/H \to H \times H,$$

*is given by the formula*

$$(5.10) \qquad \theta^{(p,q)}(g, yH) := \left(\theta(g^p, y^p H), \theta(g^q, y^q c H)\right).$$

*Proof.* First notice that indeed $\theta^{(p,q)}$ is a cocycle, that is, we have

$$\begin{aligned}
\theta^{(p,q)}(g_1 g_2, yH) &= \left(\theta((g_1 g_2)^p, y^p H), (\theta((g_1 g_2)^q, y^q c H)\right) \\
&= \left(\theta(g_1^p, y^p H), \theta(g_1^q y^q c H)\right) \cdot \left(\theta(g_2^p, g_1^p y^p H), \theta(g_2^q, g_1^q y^q c H)\right) \\
&= \theta^{(p,q)}(g_1, yH) \cdot \theta^{(p,q)}(g_2, \tau_{g_1}(yH)).
\end{aligned}$$

Since

$$\theta^{(p,q)}(g, x_1^a x_2^b H) = \left(\theta(g^p, x_1^{pa} x_2^{pb} H), \theta(g^q, x_1^{qa} x_2^{qb} c H)\right)$$

and

$$x_1^{pa} x_2^{pb} H = x_1^{pa} x_1^{qb} H = x_1 H$$

with a similar observation concerning the second coordinate, the equivariance easily follows. $\qquad\square$

**Lemma 5.6.** *The cocycle $\theta^{(p,q)}$ is regular.*

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[8] Note that since $G$ is abelian, $x^m H = (xH)^m$ in $G/H$

[9] The action of $\tau$ on $G/H$ is uniquely ergodic and so is the action of $\tau^{(p)} \times \tau^{(q)}$ on $\mathcal{H}_c$.

*Proof.* Let $J$ be an algebraic automorphism of $H \times H$ given by the matrix $\begin{bmatrix} q & -p \\ a & b \end{bmatrix}$[10]
and consider the cocycle $J \circ \theta^{(p,q)} =: (\Psi_1, \Psi_2)$, where (in view of (5.3))

$$\begin{aligned} \Psi_1(g, yH) &= \left( \theta(g^p, y^p H) \right)^q \left( \theta(g^q, y^q cH) \right)^{-p} \\ &= \left( g^p s(g^p y^p H)^{-1} s(y^p H) \right)^q \left( g^q s(g^q y^q cH)^{-1} s(y^q cH) \right)^{-p} \\ &= \left( s(g^p y^p H)^{-1} s(y^p H) \right)^q \left( s(g^q y^q cH)^{-1} s(y^q cH) \right)^{-p}. \end{aligned}$$

In view of (5.9) the values of the cocycle $\Psi_1$ belong to the set $(A^{-1}A)^{-p+q}$ which is relatively compact. It follows by (5.7) and Theorem 5.2 in [MS80] that the cocycle $\Psi_1$ is a coboundary. Now,

$$\begin{aligned} \Psi_2(g, yH) &= \left( \theta(g^p, y^p H) \right)^a \left( \theta(g^q, y^q cH) \right)^b \\ &= g^{pa} \left( s(g^p y^p H)^{-1} s(y^p H) \right)^a g^{qb} \left( s(g^q y^q cH)^{-1} s(y^q cH) \right)^b \\ &= g \left( s(g^p y^p H)^{-1} s(y^p H) \right)^a \left( s(g^q y^q cH)^{-1} s(y^q cH) \right)^b. \end{aligned}$$

Note that for each $h \in H$, we have

$$\Psi_2(h, yH) = h.$$

It follows immediately that $\tau_{\Psi_2}$ is transitive,[11] whence $\Psi_2$ is an ergodic cocycle.

It follows that our cocycle $J \circ \theta^{(p,q)}$ is cohomologous to a cocycle taking values in the group $\{e\} \times H$ and the cocycle is ergodic (the corresponding skew product is transitive). If we write

$$J \circ \theta^{(p,q)}(g, yH) = \left( \eta(g yH)^{-1} \eta(yH), \Psi_2(g, yH) \right)$$

then

$$\theta^{(p,q)}(g, yH) = \left( \Psi_2(x, yH)^p \left( \eta(g yH)^{-1} \eta(yH) \right)^b, \Psi_2(x, yH)^q \left( \eta(g yH)^{-1} \eta(yH) \right)^{-a} \right).$$

It follows that $\theta^{(p,q)}$ is cohomologous to the cocycle

$$(g, yH) \mapsto \left( \left( \Psi_2(g, yH) \right)^p, \left( \Psi_2(x, yH) \right)^q \right)$$

taking values in the subgroup

$$\mathscr{F}^{(p,q)} := \left\{ (h_1, h_2) \in H \times H : h_1^q = h_2^p \right\} = \left\{ (h^p, h^q) : h \in H \right\}.$$

The latter cocycle is ergodic,[12] which completes the proof. $\qquad\square$

5.4. **Ergodic joinings of the $G$-action $(\tau_{\theta,\mathscr{S}})^{(p)}$ with $(\tau_{\theta,\mathscr{S}})^{(q)}$.** We are interested in a description of $\tilde{\lambda} \in J^e((\tau_{\theta,\mathscr{S}})^{(p)}, (\tau_{\theta,\mathscr{S}})^{(q)})$. Recall that

$$\begin{aligned} \left( (\tau_{\theta,\mathscr{S}})^{(p)} \times (\tau_{\theta,\mathscr{S}})^{(q)} \right)_g &\left( (x_1 H, y_1), (x_2 H, y_2) \right) \\ &= \left( g^p x_1 H, S_{\theta(g^p, x_1 H)}(y_1), g^q x_2 H, S_{\theta(g^q, x_2 H)}(y_2) \right). \end{aligned}$$

But $\tilde{\lambda}|_{G/H \times G/H} =: \lambda$ is an ergodic joining of $\tau^{(p)}$ with $\tau^{(q)}$, and by unique ergodicity, it is an ergodic component of the $G$-action $\tau^{(p)} \times \tau^{(q)}$. Hence we can

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[10]$J(h_1, h_2) = (h_1^q h_2^{-p}, h_1^a h_2^b)$. The inverse of it is given by $\begin{bmatrix} b & p \\ -a & q \end{bmatrix}$.

[11]Indeed, we have

$$\left( \tau_{\Psi_2} \right)_{hg}(H, e) = \left( \tau_{\Psi_2} \right)_h \left( \tau_{\Psi_2} \right)_g (H, e) = \left( \tau_{\Psi_2} \right)_h (gH, \Psi_2(g, H)) = (gH, h\Psi_2(g, H)).$$

[12]The map $(xH, (h_1, h_2)) \mapsto (xH, h_1^a h_2^b)$ settles an isomorphism of $\tau_{(\Psi_2^p, \Psi_2^q)}$ with $\tau_{\Psi_2}$.

pass to $\mathcal{H}_c$ replacing $(G/H \times G/H, \lambda)$ by $(G/H, m_{G/H})$, with $c \in G/H$ in this notation implicit. It follows that we now consider the $G$-action $\tau_{\theta^{(p,q)}, \mathscr{S} \otimes \mathscr{S}}$ with an ergodic measure $\widetilde{\lambda}$ (whose projection on the first and the second, and the first and the third coordinates are equal to $m_{G/H} \otimes \nu$). Here, $\mathscr{S} \otimes \mathscr{S}$ denotes the product $H \times H$-action $(S_h \times S_{h'})_{(h,h')} \in H \times H$. Moreover,

$$\left(\tau_{\theta^{(p,q)}, \mathscr{S} \otimes \mathscr{S}}\right)_g (xH, y_1, y_2) = \left(\tau_g(xH), S_{\theta(g^p, x^p H)}(y_1), S_{\theta(g^p, x^p cH)}(y_2)\right).$$

But, by Lemma 5.6, $\theta^{(p,q)}$ is cohomologous to the cocycle $(\Psi_2^p, \Psi_2^q)$ taking values in $\mathscr{F}^{(p,q)} \subset H \times H$, and the latter cocycle is ergodic. As a matter of fact:

$$\theta^{(p,q)} \cdot \left(\eta(g, \cdot)^b, \eta(g, \cdot)^{-a}\right) = (\eta^b, \eta^{-a}) \cdot \left(\Psi_2^p, \Psi_2^q\right).$$

This yields an isomorphism (an equivariant map)

$$Id_{(S_{\eta^b}, S_{\eta^{-a}})}(xH, y_1, y_2) = \left(xH, S_{\eta(xH)^b}(y_1), S_{\eta(xH)^{-a}}(y_2)\right)$$

between the $G$-actions $\tau_{\theta^{(p,q)}, \mathscr{S} \otimes \mathscr{S}}$ and $\tau_{(\Psi_2^p, \Psi_2^q), \mathscr{S}^{(p)} \times \mathscr{S}^{(q)}}$.

Now, by Theorem 3 in [LMN03], since $(\Psi_2^p, \Psi_2^q)$ is ergodic, we obtain the following.

**Lemma 5.7.** *We have*

(5.11) $$\left(\left(Id_{(S_{\eta^b}, S_{\eta^{-a}})}\right)^{-1}\right)_* (\widetilde{\lambda}) = m_{G/H} \otimes \kappa,$$

*where* $\kappa \in J^e(\mathscr{S}^{(p)}, \mathscr{S}^{(q)})$.

5.5. **Proof of Proposition G.** Before we start proving Proposition G, let us make the following observation. Assume that $\mathscr{R} = (R_g)_{g \in G}$ is a $G$-action on $(Z, \mu)$ and let $(W, \xi)$ be a probability standard Borel space. Assume that we have two Rokhlin cocycles:

$$\Theta, \Psi : G \times Z \to \mathrm{Aut}(W, \xi)$$

and a measurable map $\Sigma : Z \to \mathrm{Aut}(W, \xi)$. Assume that $Id_\Sigma$ establishes an isomorphism of $(\mathscr{R}_\Psi, Z \times W, \mu \otimes \xi)$ with $(\mathscr{R}_\Theta, Z \times W, \rho)$, where $\rho|_Z = \mu$. Assume that $f \in L^1(Z, \mu)$, $g \in L^1(Y, \xi)$, then

$$\int_{Z \times W} f(z) g(w) \, d\rho(z, w) = \int_{Z \times W} (f \otimes g) \circ Id_\Sigma(z, w) \, d\mu(z) d\xi(w)$$

$$= \int_Z f(z) \left(\int_W g(\Sigma_z(w)) \, d\xi(w)\right) d\mu(z)$$

$$= \int_Z f(z) \left(\int_W g(w) \, d\left((\Sigma_z)_*(\xi)\right)(w)\right) d\mu(z).$$

It follows that

(5.12) $$\left|\int_{Z \times W} f(z) g(z) \, d\rho(z, w)\right| \leq \|f\|_{L^1(\mu)} \sup_{z \in Z} \left|\int_W g \, d\left((\Sigma_z)_*(\xi)\right)\right|.$$

*Proof of Proposition G.* Consider now the situation in which $\mathscr{R} = \tau$, $Z = G/H$, $\mu = m_{G/H}$, $W = Y \times Y$, $\xi = \kappa$: here $\mathscr{R}_\Theta = \tau_{\theta^{(p,q)}, \mathscr{S} \otimes \mathscr{S}}$ acts on $G/H \times Y \times Y$ and preserves $\rho = \widetilde{\lambda}$ (the parameter $c$ of the ergodic component is implicit), and the other $G$-action $\mathscr{R}_\Psi = \tau_{(\Psi_2^p, \Psi_2^q), \mathscr{S}^{(p)} \times \mathscr{S}^{(q)}}$ preserves $m_{G/H} \otimes \kappa$. In view of (5.11), we consider the isomorphism $Id_{(S_{\eta^b}, S_{\eta^{-a}})}$ between $\tau_{\theta^{(p,q)}, \mathscr{S} \otimes \mathscr{S}}$ and $\tau_{(\Psi_2^p, \Psi_2^q), \mathscr{S}^{(p)} \times \mathscr{S}^{(q)}}$. Hence, in our notation,

$$\Sigma_{xH} = S_{\eta(xH)^b} \times S_{\eta(xH)^{-a}}.$$

It follows that the fiber automorphisms are of the form $S_{h^b} \times S_{h^{-a}}$. Therefore, each fiber automorphism commute with the $G$-action $\mathscr{S}^{(p)} \times \mathscr{S}^{(q)}$. It easily follows that

$$(\Sigma_{xH})_* \kappa \in J^e(\mathscr{S}^{(p)}, \mathscr{S}^{(q)})$$

and from (5.12), we obtain that for each $f \in L^2(G/H, m_{G/H})$ and $F_1, F_2 \in L^2(Y, \nu)$, we have

$$(5.13) \qquad \left| \int_{G/H \times Y \times Y} f \otimes F_1 \otimes F_2 \, d\widetilde{\lambda} \right| \leq \|f\|_1 \sup_{\kappa' \in J^e(\mathscr{S}^{(p)}, \mathscr{S}^{(q)})} \left| \int_{Y \times Y} F_1 \otimes F_2 \, d\kappa' \right|.$$

To deduce the assertion, it is now enough to notice that, although on the LHS of (5.13), the constant $c$ is implicit, it has no influence since if we consider (as we should) $f_1 \otimes F_1 \otimes f_2 \otimes F_2$ as a member of $L^2(G/H \times Y \times G/H \times Y, \widetilde{\lambda})$ (with $\widetilde{\lambda}$ an arbitrary ergodic joining of $(\tau_{\theta,\mathscr{S}})^{(p)}$ and $(\tau_{\theta,\mathscr{S}})^{(q)}$) then by the Schwarz inequality, the $L^1(\widetilde{\lambda})$-norm of $f_1 \otimes f_2$ will be bounded by $\|f_1\|_{L^2(m_{G/H})} \|f_2\|_{L^2(m_{G/H})}$, hence does not depend on $\widetilde{\lambda}$. The result follows. $\qquad\square$

5.6. **Examples and applications.** Assume that we have an ergodic $G$-action $\mathscr{S} = (S_g)_{g \in G}$ on $(Y, \mathscr{C}, \nu)$. If $H \subset G$ is cocompact then, by [Zim78], the induced $G$-action from the subaction $(S_h)_{h \in H}$ is isomorphic to $\mathscr{S} \times \tau$ (by an isomorphism being the identity on the $\tau$-coordinates).

It follows now from Proposition G that:[13]

**Corollary 5.8.** *If $\mathscr{S} = (S_g)_{g \in G}$ is as above and the subaction $(S_h)_{h \in H}$ has the AOP property, then the $G$-action $\mathscr{S} \times \tau = (S_g \times \tau_g)_{g \in G}$ on $(Y \times G/H, \nu \otimes m_{G/H})$ has the relative (with respect to $\tau$) AOP property.*

**Example 5.1.** If we consider $\mathbb{Z} \subset \mathbb{R}$, then a natural choice of the selector $s : \mathbb{R}/Z \to \mathbb{R}$ satisfying (5.2), (5.8) and (5.9) is to set $s(r) := \{r\}$ being the fractional part of $r \in \mathbb{R}$. Then

$$\theta(t, \{r\}) = -\{t + \{r\}\} + t + \{r\} = [t + \{r\}] \in \mathbb{Z}.$$

Assume now that $S$ is an ergodic automorphism of $(Y, \mathscr{C}, \nu)$. It follows that the induced $\mathbb{R}$-action $\widetilde{S}$ is given by

$$\widetilde{S}_t(y, \{r\}) = (S^{\lfloor t + \{r\} \rfloor} y, \{t + r\}),$$

hence, it is the standard *suspension* construction over $S$.

**Example 5.2.** Consider $n\mathbb{Z} \subset \mathbb{Z}$. Then $s : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$ is given by $s(k + n\mathbb{Z}) = k \bmod n$. Then, by writing $m + k = tn + r$ with $0 \leq r < n$, we have

$$\theta(m, k \bmod n) = -(m + (k \bmod n)) + m + (k \bmod n) = -r + m + k = tn \in n\mathbb{Z}.$$

If we are now given an ergodic $n\mathbb{Z}$-action[14] we can induce. Instead of doing this directly, first write the obvious automorphism $n\ell = \ell$ between $n\mathbb{Z}$ and $\mathbb{Z}$ and then rewrite the cocycle $\theta$, which now becomes

$$\theta(m, x) = \left\lfloor \frac{x + m}{n} \right\rfloor.$$

Since this is a $\mathbb{Z}$-cocycle, it is entirely determined by the function $\theta = \theta(1, \cdot)$ given by: $\theta(x) = 0$ if $x = 0, \ldots, n - 2$ and $\theta(n - 1) = 1$. If now $S$ is an ergodic automorphism of $(Y, \mathscr{C}, \nu)$ then its ($n$-discrete) induced is given by $\widetilde{S}$ acting

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[13]This should be compared with Remark 5.3 (iv).

[14]This is of course a $\mathbb{Z}$-action.

on $Y \times \{0, 1, \ldots, n-1\}$ by the formula: $\widetilde{S}(y, j) = (y, j + 1)$ if $j = 0 \ldots, n - 2$, and $\widetilde{S}(y, n-1) = (Sy, 0)$. Hence, we obtain the classical $n$-discrete suspension.

*Proof of Corollary* H. The $k$-discrete suspension $\widetilde{T}$ of $T$ is a uniquely ergodic homeomorphism of the space $X \times \{0, 1, \ldots, k-1\}$. Define $F(z, i) = 0$ if $i \neq j$ and $z \in X$, and $F(z, j) = f(z)$. Then $F \perp L^2(\{0, 1, \ldots, k-1\})$. Now, by the relative AOP property, we have

$$\frac{1}{N} \sum_{n \leq N} F(\widetilde{T}^n(x, 0)) \boldsymbol{u}(n) \to 0.$$

It is however not hard to see that by the definition of $F$,

$$\frac{1}{N} \sum_{n \leq N} F(\widetilde{T}^n(x, 0)) \boldsymbol{u}(n) = \frac{1}{N} \sum_{n \leq N, n = j \text{ mod } k} f(T^n x) \boldsymbol{u}(n)$$

and the result follows. $\qquad\square$

**Lemma 5.9.** *Let $T$ be a uniquely ergodic homeomorphism of $X$, with the unique invariant measure $\mu$. Assume that $(X, \mu, T)$ has the AOP property. Assume additionally that there exists a set $\mathscr{C} \subset C(X) \cap L_0^2(X, \mu)$ whose linear span is dense in $L_0^2(X, \mu)$ such that for all $f \in \mathscr{C}$ and all $\omega \in \mathbb{C}$ with $|\omega| = 1$ there exists a homeomorphism $S: X \to X$ such that $f(T^n(Sx)) = \omega f(T^n x)$ for every $x \in X$ and $n \in \mathbb{Z}$.*

*Let $\widetilde{T}: \widetilde{X} \to \widetilde{X}$ be a discrete suspension of $T$. Then for every $F \in C(\widetilde{X})$ and every $\widetilde{x} \in \widetilde{X}$ we have*

$$(5.14) \qquad \frac{1}{M} \sum_{M \leq m < 2M} \left| \frac{1}{H} \sum_{m \leq n < m + H} F(\widetilde{T}^n \widetilde{x}) \boldsymbol{\mu}(n) \right| \longrightarrow 0$$

*when $H \to \infty$ and $H/M \to 0$.*

*Proof.* Suppose that $\widetilde{T}$ the $k$-discrete suspension of $T$. Let us consider the subspace $H := L^2(\widetilde{X}) \ominus L^2(\{0, \ldots, k-1\})$. By Proposition G, the subspace $H$ satisfies the AOP property.

For every $f \in \mathscr{C}$ and $0 \leq j < k$ denote by $f_j : \widetilde{X} \to \mathbb{C}$ the continues function which vanishes on all level sets except $X \times \{j\}$, where is given by $f_j(x, j) = f(x)$. Functions of such form establish a linearly dense subset of $H$.

Suppose that $S: X \to X$ is a homeomorphism such that $f(T^n(Sx)) = \omega f(T^n x)$ for every $x \in X$ and $n \in \mathbb{Z}$. Then for the homeomorphism $\widetilde{S}: \widetilde{X} \to \widetilde{X}$ given by $\widetilde{S}(x, l) = (Sx, l)$ we have $f_j(\widetilde{T}^n(\widetilde{S}\widetilde{x})) = \omega f_j(\widetilde{T}^n \widetilde{x})$ for every $\widetilde{x} \in \widetilde{X}$ and $n \in \mathbb{Z}$. Now the arguments used in the proof of Theorem 2.2 applied to the family of functions $\{f_j\}$ show that (5.14) holds for every continuous function $F \in H$. If $F \in L^2(\{0, \ldots, k-1\})$ then the sequence $(F(\widetilde{T}^n \widetilde{x}))$ is periodic and (5.14) follows from Remark 2.3. It follows that (5.14) holds for every $F \in C(\widetilde{X})$. $\qquad\square$

**Lemma 5.10.** *Let $\widetilde{T}: \widetilde{X} \to \widetilde{X}$ be a homomorphism coming from Lemma 5.9. Then for every $k \geq 1$, $0 \leq j \leq k$, $F \in C(\widetilde{X})$ and $\widetilde{x} \in \widetilde{X}$ we have*

$$(5.15) \qquad \frac{1}{M} \sum_{M \leq m < 2M} \left| \frac{1}{H} \sum_{m \leq n < m + H} F(\widetilde{T}^n \widetilde{x}) \boldsymbol{\mu}(kn + j) \right| \longrightarrow 0$$

*when $H \to \infty$ and $H/M \to 0$.*

*Proof.* Following the proof of Corollary H we consider the $k$-discrete suspension $\widetilde{\widetilde{T}}$ of $\widetilde{T}$. By Proposition 2.4 in [Zim78], $\widetilde{\widetilde{T}}$ is also a discrete suspension of $T$.

For every $F \in C(\widetilde{X})$ define $\widetilde{F}(\widetilde{z}, j) = F(\widetilde{z})$ and $\widetilde{F}(\widetilde{z}, l) = 0$ for $l \neq j$. Since $\widetilde{F} \in C(\widetilde{\widetilde{X}})$, by Lemma 5.9 applied to $\widetilde{\widetilde{T}}$ as a discrete suspension of $T$, we have

$$\frac{1}{kM} \sum_{kM \leq m < 2kM} \left| \frac{1}{kH} \sum_{m \leq n < m+kH} \widetilde{F}(\widetilde{\widetilde{T}}^n(\widetilde{x}, 0)) \boldsymbol{\mu}(n) \right| \longrightarrow 0.$$

Moreover,

$$\frac{1}{M} \sum_{M \leq m < 2M} \left| \frac{1}{H} \sum_{m \leq n < m+H} F(\widetilde{T}^n \widetilde{x}) \boldsymbol{\mu}(kn+j) \right|$$

$$= k^2 \frac{1}{kM} \sum_{kM \leq km < 2kM} \left| \frac{1}{kH} \sum_{km \leq n < km+kH} \widetilde{F}(\widetilde{\widetilde{T}}^n(\widetilde{x}, 0)) \boldsymbol{\mu}(n) \right|$$

$$\leq k^2 \frac{1}{kM} \sum_{kM \leq m < 2kM} \left| \frac{1}{kH} \sum_{m \leq n < m+kH} \widetilde{F}(\widetilde{\widetilde{T}}^n(\widetilde{x}, 0)) \boldsymbol{\mu}(n) \right| \longrightarrow 0,$$

which completes the proof. $\qquad \square$

*Proof of Corollary I.* First observe the it suffices to focus only on basic nilsequences, since all considered sequences are or are appropriately approximated by basic nil-sequences. Moreover, every nilsequence is of the form $(F(\widetilde{T}^n \widetilde{x}))$, where $T$ is a homeomorphism satisfying the assumption of Lemma 5.9 and $F \in C(\widetilde{X})$. Indeed, $T$ is an ergodic affine unipotent diffeomorphism on a compact connected nilmanifold. Therefore, by Lemma 5.9, the sequence $(F(\widetilde{T}^n \widetilde{x}))$ meets (1.7). $\qquad \square$

All examples of AOP actions that appeared so far in the paper have either purely discrete or, they have mixed spectrum: discrete and Lebesgue. We will now give examples of AOP flows that have mixed spectrum: discrete and continuous singular, making use of recent results from [KL16].

Below, we use Remark 5.2 and (5.6).

**Lemma 5.11.** *Let $T$ be an ergodic automorphism of probability a standard Borel space $(X, \mathscr{B}, \mu)$ and $\varphi : X \to K$ a cocycle with values in an abelian lcsc group. Then the formula*

$$\widetilde{\varphi}(t, (x, s)) := \varphi^{(\lfloor t+s \rfloor)}(x)$$

*defines a cocycle for the $\mathbb{R}$-action of the suspension $\widetilde{T}$ of $T$.*

*Proof.* First notice that

$$\widetilde{T}_t(x, s) = (T^{\lfloor t+s \rfloor} x, \{t+s\}),$$

whence $\widetilde{T}_{t_1+t_2}(x, s) = \widetilde{T}_{t_2}(\widetilde{T}_{t_1}(x, s))$ implies

(5.16) $$\lfloor t_1 + t_2 + s \rfloor = \lfloor t_1 + s \rfloor + \lfloor t_2 + \{t_1 + s\} \rfloor.$$

Now, we have

$$\widetilde{\varphi}(t_1 + t_2, (x, s)) = \varphi^{(\lfloor t_1+t_2+s \rfloor)}(x) \overset{(5.16)}{=} \varphi^{(\lfloor t_1+s \rfloor + \lfloor t_2+\{t_1+s\} \rfloor)}(x)$$

$$= \varphi^{(\lfloor t_1+s \rfloor)}(x) + \varphi^{(\lfloor t_2+\{t_1+s\} \rfloor)}(T^{\lfloor t_1+s \rfloor} x)$$

$$= \widetilde{\varphi}(t_1, (x, s)) + \widetilde{\varphi}(t_2, \widetilde{T}_{t_1}(x, s)).$$

$\qquad \square$

Assume now that $K = \mathbb{R}$, so $\varphi : X \to \mathbb{R}$. We will now consider a special class of extensions of $T$. Namely, let $\mathscr{S} = (S_t)$ be an ergodic flow on a probability standard Borel space $(Y, \mathscr{C}, \nu)$. Then consider the (probability) space $(X \times Y, \mathscr{B} \otimes \mathscr{C}, \mu \otimes \nu)$ on which we consider the (measure-preserving) automorphism:

$$T_{\varphi, \mathscr{S}}(x, y) := (Tx, S_{\varphi(x)}(y)).$$

Note that if $\mathscr{S}$ is a continuous flow acting on $Y$ compact and if $\varphi$ is continuous then $T_{\varphi, \mathscr{S}}$ is a homeomorphism of $X \times Y$. Our special interest in this class of actions come from the following result.

**Theorem 5.12** ([KL16])**.** *For each $\varphi : \mathbb{T} \to \mathbb{R}$ of class $C^2$, $\varphi$ different from a trigopolynomial, there is $\alpha$ irrational that if $Tx = x + \alpha$ and $\mathscr{S}$ is an arbitrary uniquely ergodic flow then $T_{\varphi, \mathscr{S}}$ has the AOP property. If $\varphi(x) = x - \frac{1}{2}$ then for each $\alpha$ with bounded partial quotients and each uniquely ergodic $\mathscr{S}$ which has no non-trivial rational eigenvalues, $T_{\varphi, \mathscr{S}}$ has the AOP property.*

It follows now from Theorem G that the suspension of the $T_{\varphi, \mathscr{S}}$ with the AOP property will also enjoy the same property (for flows and relatively). We will now try to say a little bit more on the structure of these suspensions (including a relationship with nilflows, see Proposition 5.15).

**Lemma 5.13.** *We have (up to natural isomorphism) $\widetilde{T_{\varphi, \mathscr{S}}} = \widetilde{T}_{\widetilde{\varphi}, \mathscr{S}}$.*

*Proof.* We have

$$\left(\widetilde{T_{\varphi, \mathscr{S}}}\right)_t ((x, y), s) = \left(\left(T_{\varphi, \mathscr{S}}\right)^{\lfloor t+s \rfloor}(x, y), \{t + s\}\right)$$
$$= \left(T^{\lfloor t+s \rfloor}x, S_{\varphi^{(\lfloor t+s \rfloor)}(x)}(y), \{t + s\}\right)$$
$$= \left(T^{\lfloor t+s \rfloor}x, \{t + s\}, S_{\varphi^{(\lfloor t+s \rfloor)}(x)}(y)\right) = \left(\widetilde{T}_t(x, s), S_{\widetilde{\varphi}(t, (x, s))}(y)\right)$$
$$= \left(\widetilde{T}_{\widetilde{\varphi}, \mathscr{S}}\right)_t ((x, s), y).$$

$\square$

By its definition, each suspension flow has the linear flow $\mathscr{L}: L_t x = x + t$ on $\mathbb{T}$, as its factor, [15] so if we think about good ergodic properties of the suspension, it should be considered relatively to this factor $\mathscr{L}$. For example, if $T$ is weakly mixing then $\widetilde{T}$ is relatively weakly mixing over $\mathscr{L}$. Indeed,

$$\left(\widetilde{T} \times_{\mathscr{L}} \widetilde{T}\right)_t (x_1, x_2, s) = \left(T^{\lfloor t+s \rfloor}x_1, T^{\lfloor t+s \rfloor}x_2, \{s + t\}\right)$$
$$= \left((T \times T)^{\lfloor t+s \rfloor}(x_1, x_2), \{s + t\}\right),$$

so the relative product is just the suspension over $T \times T$. Similar calculation can be done if we want to compute some relative properties of $\widetilde{T_{\varphi, \mathscr{S}}}$ over $\widetilde{T}$. Indeed, for the relative product we have:

$$(\widetilde{T_{\varphi, \mathscr{S}}}) \times_{\widetilde{T}} (\widetilde{T_{\varphi, \mathscr{S}}})((x, y_1), (x, y_2), s)$$
$$= \left((T_{\varphi, \mathscr{S}})^{\lfloor t+s \rfloor}(x, y_1), (T_{\varphi, \mathscr{S}})^{\lfloor t+s \rfloor}(x, y_2), \{s + t\}\right)$$
$$= \left(\left(T_{\varphi, \mathscr{S}} \times_T T_{\varphi, \mathscr{S}}\right)^{\lfloor t+s \rfloor}(x, y_1, y_2), \{s + t\}\right),$$

so again the relative product of the suspension is the suspension of the relative product. Now, the examples of $T_{\varphi, \mathscr{S}}$ in Theorem 5.12 are relatively weakly mixing over $T$ whenever $\mathscr{S}$ is weakly mixing. It follows that, we have the following.

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[15]The factor map is given by $(x, s) \mapsto s$.

**Corollary 5.14.** *Assume that $T$ and $\varphi$ are as in Theorem 5.12. Assume that $\mathscr{S}$ is weakly mixing. Then $\widetilde{T_{\varphi,\mathscr{S}}}$ have the relative AOP property and they are relatively weakly mixing extensions of $\widetilde{T}$.* [16]

The situation changes if on the fibers, instead of $\mathscr{S}$ weakly mixing, we put a distal flow. For example, if we apply the above to $\varphi(x) = x - \frac{1}{2}$ and $S_t = x + t$, we obtain a nilflow (the Heisenberg case), but because of restrictions on $\mathscr{S}$ (which in this case has rational eigenvalues), we cannot apply [KL16] directly. Consider $S_t x = x + \beta t$ with $\beta$ irrational. Here we obtain that $T_{\varphi,\mathscr{S}}$ has the AOP property. The natural question arises whether in case of $\beta$ irrational, we also obtain a nilflow. The answer is negative as the following result shows.

**Proposition 5.15.** *If $\varphi(x) = x - \frac{1}{2}$ and $S_t^{(\beta)}(x) := x + t\beta$ with $\beta \notin \mathbb{Q}$ then $\widetilde{T_{\varphi,\mathscr{S}}}$ has singular spectrum.*

*Proof.* We have $\widetilde{T}_{\widetilde{\varphi},\mathscr{S}^{(\beta)}} = \widetilde{T}_{\beta\widetilde{\varphi},\mathscr{S}^{(1)}}$. Then

$$\widetilde{T}_{\beta\widetilde{\varphi},\mathscr{S}^{(1)}} = \widetilde{T_{\beta\varphi,\mathscr{S}^{(1)}}}.$$

We have to now argue that the maximal spectral type of $T_{\beta\varphi,\mathscr{S}^{(1)}}$ is singular. In view of [LP12], to compute the maximal spectral type of $T_{\beta\varphi,\mathscr{S}^{(1)}}$, we need to calculate the maximal spectral types of weighted unitary operators given by the multiples $r\beta\varphi$, $r \in \mathbb{R}$, and then integrate this against the maximal spectral type of $\mathscr{S}^{(1)}$. The latter is simply a purely atomic measure whose atoms are in $\mathbb{Z}$. This means that we are interested only in the weighted operators given by $m\beta\varphi$ and they are all with singular spectrum by [ILM99]. □

<div align="center">APPENDIX</div>

<div align="center">APPENDIX A.  COCYCLES AND GROUP EXTENSIONS</div>

Assume that $T$ is an ergodic automorphism of a probability standard Borel space $(X, \mathscr{B}, \mu)$. Let $K$ be a compact (metric) abelian group. Each Borel function $\varphi : X \to K$ is called a *cocycle*. Actually, $\varphi$ determines $\varphi^{(\cdot)}(\cdot) : \mathbb{Z} \times X \to K$:

(A.1)     $$\varphi^{(n)}(x) = \begin{cases} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{n-1}x) & \text{if} \quad n \geq 1 \\ 0 & \text{if} \quad n = 0 \\ -(\varphi(T^n x) + \ldots + \varphi(T^{-1}x)) & \text{if} \quad n < 0, \end{cases}$$

satisfying the cocycle identity $\varphi^{(m+n)}(x) = \varphi^{(m)}(x) + \varphi^{(n)}(T^m x)$ for all $m, n \in \mathbb{Z}$ and $\mu$-a.e. $x \in X$. Having $T$ and $\varphi$, we can define the corresponding *group extension* $T_\varphi$ of $T$ by setting:

$$T_\varphi : X \times K \to X \times K, \ T_\varphi(x, k) = (Tx, \varphi(x) + k).$$

Clearly $T_\varphi$ is an automorphism of $(X \times K, \mathscr{B} \otimes \mathscr{B}(K), \mu \otimes \lambda_K)$, where $\lambda_K$ stands for Haar measure on $K$. Then $T_\varphi$ is ergodic if and only if the only measurable solutions $\xi : X \to \mathbb{S}^1$ of the equations

$$\chi \circ \varphi = \xi \circ T/\xi, \ \chi \in \widehat{K}$$

(i.e. $\chi \circ \varphi$ is a *coboundary*) exist when $\xi$ is a constant function and $\chi = \mathbb{1}$ [Anz51].

Let $\sigma_k : X \times K \to X \times K$ be defined by $\sigma_k(x, k') = (x, k' + k)$. Then $\sigma_k$ $(k \in K)$ is an automorphism of $(X \times K, \mathscr{B} \otimes \mathscr{B}(K), \mu \otimes \lambda_K)$ and $\sigma_k$ is an element of the

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[16]Note that this is completely different case than nilflows which are distal extensions of of linear flows.

centralizer $C(T_\varphi)$. Then, by a slight abuse of notation, we have $K \subset C(T_\varphi)$ and $K$ is a compact (abelian) subgroup in the weak topology. The reciprocal is also true.

**Proposition A.1** (e.g. [dJR87]). *Assume that $\overline{T}$ is an ergodic automorphism of a probability standard Borel space $(\overline{X}, \overline{\mathscr{B}}, \overline{\mu})$. Assume that $K \subset C(\overline{T})$ is a compact abelian subgroup (we assume that $K$ acts freely on $X$) of the centralizer. Let*

$$\overline{\mathscr{A}} := \{\overline{A} \in \overline{\mathscr{B}} : k\overline{A} = \overline{A} \text{ for each } k \in K\}.$$

*Then $\overline{\mathscr{A}}$ is an $\overline{T}$-invariant $\sigma$-algebra and*

$$K = \{\overline{R} \in C(\overline{T}) : \overline{R}\overline{A} = \overline{A} \text{ for each } \overline{A} \in \overline{\mathscr{A}}\}.$$

*If $T = \overline{T}|_{\overline{\mathscr{A}}}$ is the factor automorphism acting on the factor space*

$$(X, \mathscr{B}, \mu) := (\overline{X}/\overline{\mathscr{A}}, \overline{\mathscr{A}}, \overline{\mu}|_{\overline{\mathscr{A}}})$$

*then there is a cocycle $\varphi : X \to K$ such that $\overline{T}$ is isomorphic to $T_\varphi$ (with an isomorphism being the identity on $\overline{\mathscr{A}}$).*

**Remark A.2.** We recall how to define $\varphi$. Let $\pi : \overline{X} \to X$ be the factor map. Then the fibers $\pi^{-1}(x)$ are copies of $K$. Let $\xi : X \to \overline{X}$ be a measurable selector (for $\pi$). Then for each $\overline{x} \in \pi^{-1}(x)$ there is a unique $k_{\overline{x}} \in K$ such that $k_{\overline{x}}\overline{x} = \xi(x)$. Note that then

$$k_{\overline{x}}(\overline{T}\overline{x}) = \overline{T}(k_{\overline{x}}\overline{x}) = \overline{T}\xi(x),$$

whence if we define $\varphi(x) \in K$ as the only element of $K$ such that $\varphi(x)(\overline{T}\xi(x)) = \xi(Tx)$ then the map $\overline{x} \mapsto (\pi(\overline{x}), k_{\overline{x}})$ establishes an isomorphism between $\overline{T}$ and $T_\varphi$.

## APPENDIX B. INVARIANT MEASURES FOR GROUP EXTENSIONS

If $T_\varphi$ is ergodic, then $\mu \otimes \lambda_K$ is the only $T_\varphi$-invariant probability measure $\kappa$ whose projection $(\pi_X)_*(\kappa)$ is $\mu$ [Fur67]. This result has a refinement as follows (see, e.g. [KN74], [LM90]). If $\varphi : X \to K$ is not ergodic then there is a (unique) closed subgroup $K' \subset K$, a cocycle $\varphi' : X \to K'$ and a Borel map $j : X \to K$ such that

(B.1)     $T_{\varphi'}$ is ergodic as an automorphism of $\left(X \times K', \mathscr{B} \otimes \mathscr{B}(K'), \mu \otimes \lambda_{K'}\right)$

and

(B.2)     $$\varphi(x) = \varphi'(x) + j(Tx) - j(x)$$

for a measurable $j : X \to K$ (i.e. $\varphi$ is *cohomologous* to a cocycle $\varphi'$ which is taking values in a smaller closed subgroup $K'$ and $T_{\varphi'}$ is ergodic). The group $K'$ is called the *group of essential values* of $\varphi$ and is denoted by $E(\varphi)$. Moreover,

(B.3)     $$E(\varphi') = E(\varphi) = \Lambda(\varphi)^\perp,$$

where

$$\Lambda(\varphi) := \left\{\chi \in \widehat{K} : \chi \circ \varphi \text{ is a coboundary}\right\}.$$

In particular,

(B.4)     $$T_\varphi \text{ is ergodic if and only if } E(\varphi) = K.$$

We intend to describe the set $\mathcal{M}(X \times K, T_\varphi; \mu)$ of all $T_\varphi$-invariant probability measures $\kappa$ on $X \times K$ that $(\pi_X)_*(\kappa) = \mu$. It is again a simplex (with its natural affine structure). The set of extremal points is equal to $\mathcal{M}^e(X \times K, T_\varphi; \mu)$ of ergodic members of $\mathcal{M}(X \times K, T_\varphi; \mu)$. We have the following.

**Proposition B.1** ([LM90])**.** *If $\kappa \in \mathcal{M}^e(X \times K, T_\varphi; \mu)$ then $(\sigma_k)_*(\kappa) \in \mathcal{M}^e(X \times K, T_\varphi; \mu)$ for each $k \in K$. Moreover, if $\kappa, \kappa' \in \mathcal{M}^e(X \times K, T_\varphi; \mu)$ then $(\sigma_{k_0})_*(\kappa) = \kappa'$ for some $k_0 \in K$.*

It follows from Proposition B.1 that to describe all ergodic members of $\mathcal{M}(X \times K, T_\varphi; \mu)$, we need to describe just one. Let $\kappa \in \mathcal{M}^e(X \times K, T_\varphi; \mu)$. Set

$$\operatorname{stab}(\kappa) := \{k \in K : (\sigma_k)_*(\kappa) = \kappa\}.$$

**Lemma B.2** ([LM90])**.** *For each $\kappa \in \mathcal{M}^e(X \times K, T_\varphi; \mu)$, we have $\operatorname{stab}(\kappa) = E(\varphi)$.*

By (B.1),

(B.5)  $$\mathcal{M}\big(X \times E(\varphi), T_{\varphi'}; \mu\big) = \mathcal{M}^e\big(X \times E(\varphi), T_{\varphi'}; \mu\big) = \{\mu \otimes \lambda_{E(\varphi)}\}.$$

Moreover, by (B.2), the map $\Theta : X \times E(\varphi) \to X \times K$ given by

$$\Theta(x, k') = (x, k' + j(x))$$

is equivariant, i.e. $\Theta \circ T_{\varphi'} = T_\varphi \circ \Theta$. It follows that $\kappa := \Theta_*(\mu \otimes \lambda_{E(\varphi)}) \in \mathcal{M}^e(X \times K, T_\varphi; \mu)$. Finally, notice that if $j : X \to K$ satisfies (B.2) then, for each $k \in K$, also $j_k(x) := j(x) + k$ satisfies (B.2) (moreover, each Borel $\overline{j} : X \to K$ satisfying (B.2) is of the form $j_k$). Taking into account this and Proposition B.1, we obtain the following.

**Proposition B.3** ([LM90])**.** *If $\kappa \in \mathcal{M}^e(X \times K, T_\varphi; \mu)$ then there exists a Borel $j : X \to K$ satisfying (B.2) for which*

$$\kappa = \int_X \big(\sigma_{j(x)}\big)_* (\delta_x \otimes \lambda_{E(\varphi)}) \, d\mu(x).$$

If we take $f \in L^2(X, \mathcal{B}, \mu)$ and a character $\chi \in \widehat{K}$ then, once $\chi$ is non-trivial, the tensor $f \otimes \chi$ has the zero mean for $\mu \otimes \lambda_K$. We will now show that the zero mean phenomenon holds for many characters and ergodic members of $\mathcal{M}(X \times K, T_\varphi; \mu)$.

**Corollary B.4.** *Assume that $f \in L^2(X, \mathcal{B}, \mu)$. If $\chi \notin \Lambda(\varphi)$ then $\int_{X \times K} f \otimes \chi \, d\kappa = 0$ for each $\kappa \in \mathcal{M}^e(X \times K, T_\varphi; \mu)$.*

*Proof.* By (B.3), $\chi \notin \Lambda(\varphi)$ if and only if $\chi \notin E(\varphi)^\perp$ and the latter is equivalent to saying that $\chi|_{E(\varphi)} \neq \mathbb{1}$. By Proposition B.3,

(B.6)  $$\kappa = \int_X \big(\sigma_{j(x)}\big)_* (\delta_x \otimes \lambda_{E(\varphi)}) \, d\mu(x).$$

But if $\chi|_{E(\varphi)} \neq \mathbb{1}$ then $\int_K \chi \, d\lambda_{E(\varphi)} = 0$ and the same integral vanishes if we integrate against an arbitrary shift of $\lambda_{E(\varphi)}$. The result now follows from (B.6).  $\square$

Assume now that $S$ is an ergodic automorphism of a probability standard Borel space $(Y, \mathcal{C}, \nu)$ and let $\psi : Y \to L$ be a cocycle, where $L$ is a compact (metric) abelian group. Assume that $T_\varphi$ and $S_\psi$ are ergodic and let $\widetilde{\rho} \in J^e(T_\varphi, S_\psi)$. Denote $\rho = \widetilde{\rho}|_{X \times Y}$. Then $\rho \in J^e(T, S)$. We also have

$$\widetilde{\rho} \in \mathcal{M}^e\big((X \times Y) \times (K \times L), (T \times S)_{\varphi \times \psi}; \rho\big),$$

where $\varphi \times \psi : (X \times Y, \rho) \to K \times L$:

$$(\varphi \times \psi)(x, y) = (\varphi(x), \psi(y)) \text{ for } \rho - \text{a.e. } (x, y) \in X \times Y.$$

In other words,

$$(B.7) \qquad J^e(T_\varphi, S_\psi) \subset \mathcal{M}^e\big((X \times Y) \times (K \times L), (T \times S)_{\varphi \times \psi}; \rho\big).$$

We can now take up a more general problem and study the form of

$$\overline{\rho} \in \mathcal{M}^e\big(((X \times Y) \times (K \times L), (T \times S)_{\varphi \times \psi}; \rho\big)$$

(with $\rho = \overline{\rho}|_{X \times Y}$). This problem is equivalent to investigating the invariant measures for the group extensions $(T \times S, \rho)_{\varphi \times \psi}$ with $\rho \in J^e(T, S)$. By denoting $E_\rho(\varphi \times \psi)$ the corresponding group of essential values, the problem is reduced to study the form of such groups.

**Lemma B.5.** *If $\rho \in J^e(T, S)$ and the cocycles $\varphi$ and $\psi$ considered over $(T \times S, \rho)$ are ergodic then $\pi_K(E_\rho(\varphi \times \psi)) = K$ and $\pi_L(E_\rho(\varphi \times \psi)) = L$.*

*Proof.* It is not hard to see that $E(\pi_K \circ (\varphi \times \psi)) = \pi_K(E_\rho(\varphi \times \psi))$ and since $\pi_K \circ (\varphi \times \psi) = \varphi$ is ergodic over $(T \times S, \rho)$ by assumption, the result follows from (B.4). $\square$

**Remark B.6.** Although the problem of describing all elements of the set $\big\{E_\rho(\varphi \times \psi) : \rho \in J^e(T, S)\big\}$ looks as a problem more general than the problem of description of the members of $J^e(T_\varphi, S_\psi)$, in fact, quite often they are the same. We will see the equality of the two sets in (B.7) when we study nil-cocycles in forthcoming sections.

## APPENDIX C. A CRITERION TO LIFT AOP TO A GROUP EXTENSION

Assume again that $T$ is a totally ergodic automorphism of a probability standard Borel space $(X, \mathcal{B}, \mu)$. Let $K$ be a compact (metric) abelian group and let $\varphi : X \to K$ be a cocycle. We assume also that $T_\varphi$ is also totally ergodic. [17] We want to study the AOP property for $T_\varphi$ (assuming that $T$ enjoys it). To study this property, we will have to consider the set $J^e\big((T_\varphi)^r, (T_\varphi)^s\big)$ for $r, s$ coprime. Note that $(T_\varphi)^r = (T^r)_{\varphi^{(r)}}$, so we are in the framework of the previous subsection in which $T$, $\varphi$, $S$, $\psi$ are $T^r$, $\varphi^{(r)}$, $T^s$ and $\varphi^{(s)}$, respectively.

Assume that $K = \mathbb{T}^d$ and identify characters of $K$ with $\mathbb{Z}^d$: given $\overline{m} \in \mathbb{Z}^d$, we set $\chi_{\overline{m}}(\overline{x}) = e^{2\pi i \langle \overline{m}, \overline{x} \rangle}$. Given $k \geq 1$ and $r, s$ coprime, let

$$A_{k,r,s} := \big\{(\overline{m}, \overline{n}) \in \mathbb{Z}^d \times \mathbb{Z}^d : s^k \overline{m} = r^k \overline{n}\big\} \subset \widehat{K} \times \widehat{K}.$$

**Proposition C.1.** *Let $T$ be a totally ergodic automorphism of a probability standard Borel space $(X, \mathcal{B}, \mu)$. Let $\varphi : X \to \mathbb{T}^d$ be a cocycle so that $T_\varphi$ is totally ergodic. Assume that for some $k \geq 1$ and for all $r, s$ coprime, we have*

$$(C.1) \qquad \Lambda_{(\varphi^{(r)} \times \varphi^{(s)}, \rho)} \subset A_{k,r,s} \text{ for each } \rho \in J^e(T^r, T^s).$$

*If $T$ enjoys the AOP property, so does $T_\varphi$.*

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[17]This means that $T_\varphi$ has no root of unity in its spectrum. This is equivalent to saying that for no $n \geq 2$, no character $\chi \in \widehat{K}$, we can solve the functional equation $\chi \circ \varphi = e^{2\pi i/n} \xi \circ T/\xi$ for a measurable $\xi : X \to \mathbb{S}^1$ [Anz51].

*Proof.* Fix $f, g \in L^2(X, \mathcal{B}, \mu)$ and $\overline{m}_0, \overline{n}_0 \in \mathbb{Z}^d$ such that at least one is no-zero. It is enough to show that for all $r, s$ coprime and sufficiently large, we have

$$\text{(C.2)} \qquad \int f \otimes \chi_{\overline{m}_0} \otimes g \otimes \chi_{\overline{n}_0} \, d\widetilde{\rho} = 0 \text{ for each } \widetilde{\rho} \in J^e \left( (T_\varphi)^r, (T_\varphi)^s \right),$$

cf. Remark 2.1. By (B.7), it is sufficient that (C.2) holds for each $\widetilde{\rho} \in \mathcal{M}^e\big((X \times X) \times (\mathbb{T}^d \times \mathbb{T}^d), (T^r \times T^s)_{\varphi^{(r)} \times \varphi^{(s)}}; \rho\big)$. By Corollary B.4, we only need to show that for all $r, s$ coprime and sufficiently large, $(\overline{m}_0, \overline{n}_0) \notin \Lambda_{\varphi^{(r)} \times \varphi^{(s)}}$ for all $\rho \in J^e(T^r, T^s)$. Hence, by assumption, it is sufficient to show that $(\overline{m}_0, \overline{n}_0) \notin A_{k,r,s}$ for all $r, s$ coprime and sufficiently large. This is however clear, if $(\overline{m}_0, \overline{n}_0) \in A_{k,r,s}$ then $s^k \overline{m}_0 = r^k \overline{n}_0$, whence the coordinates of $\overline{m}_0$ must be multiples of $r^k$ and the coordinates of $\overline{n}_0$ must be multiples of $s^k$ and this can hold only for finitely many $r, s$. The result follows. $\qquad \square$

Since $(r, s) = 1$, we have $A_{k,r,s} = \{(r^k \overline{j}, s^k \overline{j}) : \overline{j} \in \mathbb{Z}^d\}$. Then, it follows that

$$A_{k,r,s}^{\perp} = \left\{ (\overline{x}, \overline{y}) \in \mathbb{T}^d \times \mathbb{T}^d : r^k \overline{x} + s^k \overline{y} = \overline{0} \right\}.$$

Using Proposition C.1, Lemma B.2 and (B.3), we obtain the following.

**Corollary C.2.** *Let $T$ be a totally ergodic automorphism of a probability standard Borel space $(X, \mathcal{B}, \mu)$. Let $\varphi : X \to \mathbb{T}^d$ be a cocycle so that $T_\varphi$ is totally ergodic. Assume that for some $k \geq 1$ and for all $r, s$ coprime, we have*

$$\text{(C.3)} \qquad\qquad\qquad \text{stab}(\widetilde{\rho}) \supset A_{k,r,s}^{\perp}$$

*for some $\widetilde{\rho} \in \mathcal{M}^e\big((X \times X) \times (\mathbb{T}^d \times \mathbb{T}^d), (T^r \times T^s)_{\varphi^{(r)} \times \varphi^{(s)}}; \rho\big)$ and for every $\rho \in J^e(T^r, T^s)$. If $T$ enjoys the AOP property, so does $T_\varphi$.*

## APPENDIX D. SOME LEMMATA ON NILPOTENT LIE ALGEBRAS

In the sequel $\mathfrak{g}$ is a $k$-step nilpotent Lie algebra and

$$\mathfrak{g} = \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots \supset \mathfrak{g}^{(k)}$$

its descending central series. By Lemma 1.1.1 in [CG90],

$$[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)} \quad \text{for all } i, j \geq 1.$$

**Lemma D.1.** *A set $S = \{X_1, \ldots, X_j\} \subset \mathfrak{g}$ is a minimal set of generators for a nilpotent Lie algebra $\mathfrak{g}$ if and only if $S[\mathfrak{g}, \mathfrak{g}]$ is a basis of the vector space $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.*

*Proof.* The proof is by induction on $k$. The statement is true for $\mathfrak{g}$ abelian ($k = 1$). Suppose the statement true for all nilpotent Lie algebras of class of nilpotency $\ell \leq k$ and assume $\mathfrak{g}$ nilpotent of class of nilpotency $k + 1$.

Let $S \subset \mathfrak{g}$ be a set such that $S[\mathfrak{g}, \mathfrak{g}]$ is a basis of the vector space $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{g} = \langle S \rangle + [\mathfrak{g}, \mathfrak{g}]$, where $\langle S \rangle$ denotes the linear span of $S$. It follows that $\mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}] = [\langle S \rangle, \mathfrak{g}^{(k)}]$ as $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(k+2)} = 0$. Furthermore, the set

$$(S\mathfrak{g}^{(k+1)})\big[(\mathfrak{g}/\mathfrak{g}^{(k+1)}), (\mathfrak{g}/\mathfrak{g}^{(k+1)})\big] \approx S[\mathfrak{g}, \mathfrak{g}]$$

is a basis of the vector space $(\mathfrak{g}/\mathfrak{g}^{(k+1)})/\big[(\mathfrak{g}/\mathfrak{g}^{(k+1)}), (\mathfrak{g}/\mathfrak{g}^{(k+1)})\big] \approx \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{g}/\mathfrak{g}^{(k+1)}$ is nilpotent of class of nilpotency $k$, the induction hypothesis applies to the Lie algebra $\mathfrak{g}/\mathfrak{g}^{(k+1)}$ and its subset $S\mathfrak{g}^{(k+1)}$, so $S\mathfrak{g}^{(k+1)}$ is a minimal set of generators of $\mathfrak{g}/\mathfrak{g}^{(k+1)}$. Thus the Lie subalgebra $\mathfrak{g}_S$ generated by $S$ projects onto $\mathfrak{g}/\mathfrak{g}^{(k+1)}$ under the quotient mapping $\mathfrak{g} \mapsto \mathfrak{g}/\mathfrak{g}^{(k+1)}$. Thus it suffices to show that $S$ generates a set spanning $\mathfrak{g}^{(k+1)}$. Let $T \subset \mathfrak{g}_S$ be a finite set projecting to a

spanning set of $\mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$. Then $\mathfrak{g}^{(k)} = \langle T \rangle + \mathfrak{g}^{(k+1)}$. By definition $[\langle S \rangle, \langle T \rangle] \subset \mathfrak{g}_S$. It follows that

$$\mathfrak{g}^{(k+1)} = [\langle S \rangle, \mathfrak{g}^{(k)}] = [\langle S \rangle, \langle T \rangle + \mathfrak{g}^{(k+1)}] \subset [\langle S \rangle, \langle T \rangle] + \mathfrak{g}^{(k+2)} = [\langle S \rangle, \langle T \rangle] \subset \mathfrak{g}_S.$$

Finally observe that $S$ is minimal, since any proper subset of $S$ does not project to a basis of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, i.e. does not generate $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. A fortiori it does not generate $\mathfrak{g}$.

Assume $S$ is a minimal set of generators of $\mathfrak{g}$. Then $S$ projects to a generating set of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, that is a finite spanning set for the vector space $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Let $S_1 \subset S$ be a subset projecting to a basis of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. By the above $S_1$ generates $\mathfrak{g}$. By the minimality of $S$ we have $S = S_1$, concluding the proof. $\square$

Let $S = \{X_1, \ldots, X_j\}$ be a minimal generating set for the $k$-step nilpotent Lie algebra $\mathfrak{g}$. For every $(i_1, \ldots, i_k) \in \{1, \ldots, j\}^k$ define the Lie $k$-fold product

$$S_{i_1, i_2, \ldots, i_k} := [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]]$$

and let $V_k(S) \subset \mathfrak{g}^{(k)}$ be the linear span of the set of $k$-fold products.

**Lemma D.2.** *Let $\mathfrak{g}$ be a $k$-step nilpotent Lie algebra and $S$ a minimal generating set for $\mathfrak{g}$. Then $\mathfrak{g}^{(k)} = V_k(S)$.*

*Proof.* The proof is by induction on $k$. When $\mathfrak{g}$ is abelian the statement is obvious. Suppose that the Lemma is true for all nilpotent Lie algebras of class $\ell < k$ and let $\mathfrak{g}$ be of class $k$ and $S$ be a minimal generating set for $\mathfrak{g}$. By the previous Lemma, the projection $\bar{S} = \{\bar{X}_1, \ldots, \bar{X}_j\}$ of $S$ into $\mathfrak{g}/\mathfrak{g}^{(k)}$ is a set of generators for $\mathfrak{g}/\mathfrak{g}^{(k)}$. Then, by the induction hypothesis, the set of $(k-1)$-fold products of the elements $\bar{S}$ span $\mathfrak{g}^{(k-1)}/\mathfrak{g}^{(k)}$. It follows that $\mathfrak{g}^{(k-1)} = \mathfrak{g}^{(k)} + V_{k-1}(S)$. Since $V_{k-1}(S) \subset \mathfrak{g}^{(k-1)}$, this gives

$$\mathfrak{g}^{(k)} = [\mathfrak{g}, \mathfrak{g}^{(k-1)}] = [\langle S \rangle + \mathfrak{g}^{(2)}, \mathfrak{g}^{(k)} + V_{k-1}(S)] = [\langle S \rangle, V_{k-1}(S)]$$

The proof is concluded by the observations that $[\langle S \rangle, V_{k-1}(S)] \subset V_k(S)$ and that, as $\mathfrak{g}$ is of class $k$, the opposite inclusion is true by definition of $\mathfrak{g}^{(k)}$. $\square$

**Lemma D.3.** *Let $S = \{X_1, \ldots, X_j\}$ and $S' = \{X_1', \ldots, X_j'\}$ be subsets of a nilpotent $k$-step Lie algebra $\mathfrak{g}$ such that $X_i = X_i' \mod [\mathfrak{g}, \mathfrak{g}]$ for all $i = 1, \ldots, j$. Then for every $(i_1, \ldots, i_k) \in \{1, \ldots, j\}^k$ we have*

$$S_{i_1, i_2, \ldots, i_k} = S'_{i_1, i_2, \ldots, i_k}$$

*Proof.* Again the proof is by induction on the class of nilpotency. The statement being trivially true in the abelian case, we assume that the Lemma is true for all nilpotent Lie algebras of class $\ell < k$. Let $\mathfrak{g}$ be of class $k$. Let $X_i' = X_i + Y_i$ with $Y_i \in [\mathfrak{g}, \mathfrak{g}]$, for all $i = 1, \ldots, j$. By the induction hypothesis, applied to the algebra $\mathfrak{g}/\mathfrak{g}^{(k)}$ for all $(i_2, \ldots, i_k) \in \{1, \ldots, j\}^{k-1}$, the elements $S_{i_2, \ldots, i_k}$ and $S'_{i_2, \ldots, i_k}$ coincide modulo $\mathfrak{g}^{(k)}$, i.e. differ by an element $\alpha_{i_2, \ldots, i_k} \in \mathfrak{g}^{(k)}$. Thus we have

$$\begin{aligned} S'_{i_1, i_2, \ldots, i_k} &= [X_{i_1}', S'_{i_2, \ldots, i_k}] = [X_{i_1} + Y_{i_1}, S_{i_2, \ldots, i_k} + \alpha_{i_2, \ldots, i_k}] \\ &= [X_{i_1}, S_{i_2, \ldots, i_k}] = S_{i_1, i_2, \ldots, i_k} \end{aligned}$$

concluding the proof. $\square$

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