On one-parameter Koopman groups

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Abstract

We characterize Koopman one-parameter C_0-groups in the class of all unitary one-parameter C_0-groups on $L_2(X)$ as those that preserve $L_\infty(X)$ and for which the infinitesimal generator is a derivation on the bounded functions in its domain.

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1 Introduction

Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Moreover, let $T: X \to X$ be an (a.e.) invertible, measurable and measure-preserving map, i.e. $\mu(A) = \mu(T^{-1}A)$ for each $A \in \mathcal{B}$. Then T induces on $L_2(X)$ a unitary operator U_T, called a Koopman operator, defined by $U_T f := f \circ T$ for all $f \in L_2(X)$. One can ask for the converse: given a unitary operator U on $L_2(X)$, how to recognize that it is a Koopman operator. The very classical answer says that if U preserves multiplication of bounded functions, i.e. if

$$U(f\,g) = U(f)\,U(g) \tag{1}$$

for all $f, g \in L_\infty(X)$, then U is a Koopman operator by a combination of the multiplication theorem in [Hal] (page 45) and [Kec] Theorem 15.9. Another type of questions one can ask for is, given a unitary operator U on an abstract Hilbert space, how to recognize that it is unitarily equivalent to a Koopman operator, see for example [CR], [Cho], [Rid] and [Den].

The problem which unitary operators can be realized as Koopman operator remains one of important and still unsolved problems in ergodic theory, see e.g. the discussion on this problem in [KL], [KT] and also the survey article [Lem]. Up to unitary equivalence each unitary operator U is determined by the two invariants: the equivalence class $[\sigma]$ of a finite positive Borel measure σ on the circle, called the maximal spectral type σ_U of U, together with the (Borel) multiplicity function $M = M_U: \mathbb{T} \to \{1, 2, \ldots\} \cup \{\infty\}$ which is defined σ-a.e. Once a pair $([\sigma], M)$ is given, it is easy to construct on the abstract level a unitary operator U for which $(\sigma_U, M_U) = ([\sigma], M)$. Nevertheless, it is an open problem whether there exists a (unitary) Koopman operator U such that $(\sigma_U, M_U) = ([\sigma], M)$. (Some restrictions must be imposed on σ, for example σ must be of symmetric type and its topological support must be full if the construction is sought in the class of U_T with T ergodic.) While some progress has been made recently in the spectral theory of single transformation, cf. [Lem], for unitary one-parameter groups still little is known.

A unitary one-parameter C_0-group $(U_t)_{t\in\mathbb{R}}$ is called a Koopman group if for all $t \in \mathbb{R}$ there exists a measurable $T_t: X \to X$ such that $U_t f = f \circ T_t$ for all $f \in L_2(X)$. It is clear that a Koopman group must preserve $L_\infty(X)$, but this latter condition is satisfied also for many unitary one-parameter C_0-groups which are not Koopman groups. By the Stone theorem [Sto], the generator A of a unitary one-parameter C_0-group is skew-adjoint. Therefore each unitary one-parameter C_0-group is determined up to unitary equivalence by (σ_U, M_U), where $\sigma_U = [\sigma]$ for some finite positive Borel measure on $\mathbb{R}$. In order to characterize those pairs $([\sigma], M)$ which can be realized by Koopman groups, it seems to natural to characterize first those generators A for which $(e^{tA})_{t\in\mathbb{R}}$ is equivalent to a Koopman group. Even the problem to characterize in terms of their generator which unitary one-parameter C_0-groups are Koopman groups seems to be, however, far from obvious. Moreover, once such a characterization is done, one can consider the problem whether a perturbation of a Koopman representation remains Koopman. The latter is of independent interest.

In order to formulate the main results of the paper, first recall that if A is an operator in a function space E and $\mathcal{D} \subset D(A)$ is an algebra, then we say that A is a **derivation on** $\mathcal{D}$ if

$$A(f\,g) = (Af)\,g + f\,(Ag)$$

for all $f, g \in \mathcal{D}$. The main result of the paper is the following.

Theorem 1.1. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a unitary one-parameter C_0-group on $L_2(X)$ with generator A. Then the following are equivalent.*

(i) *For all $t \in \mathbb{R}$ there exists an a.e. invertible measurable and measure preserving map $T_t : X \to X$ such that $U_t f = f \circ T_t$ for all $f \in L_2(X)$.*

(ii) *The space $L_\infty(X)$ is invariant under U. Moreover, the space $D(A) \cap L_\infty(X)$ is an algebra and A is a derivation on $D(A) \cap L_\infty(X)$.*

We are also able to prove in Corollary 2.9 a generalisation of the above theorem where the group U is a C_0-group which is not necessarily unitary and we do not require measure preserving in Condition (i). Moreover we have a generalisation where the measure μ is merely σ-finite, see Theorem 2.8 below.

A theorem of the same nature as Theorem 1.1 was given by Gallavotti and Pulvirenti, [GP] Theorem 4.

Theorem 1.2 ([GP]). *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let A be a self-adjoint operator and let $\mathcal{D} \subset D(A) \subset L_\infty(X)$. Suppose that $\mathcal{D}$ is a core for A, $\mathbb{1} \in \mathcal{D}$, $\mathcal{D}$ is an algebra, $\mathcal{D}$ is self-adjoint (that is if $f \in \mathcal{D}$ then $\overline{f} \in \mathcal{D}$), A is a derivation on $\mathcal{D}$ and $\overline{Af} = -A\overline{f}$ for all $f \in \mathcal{D}$. Then* (i) *in Theorem* 1.1 *is valid.*

The theorem of Gallavotti and Pulvirenti does not have an extension where A is merely a C_0-group generator and it is essential in [GP] that the measure μ is finite.

The main application of Theorems 1.1 and 2.8 is a characterization of those C_0-groups on $L_2(X)$ which are pointwise the product of a Koopman operator and a multiplication operator.

Theorem 1.3. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a unitary C_0-group on $L_2(X)$ preserving $L_\infty(X)$. Assume that $\mathbb{1} \in D(A)$ with $A\mathbb{1} \in L_\infty(X)$, where A is the generator of U. Then the following are equivalent.*

(i) *For all $t \in \mathbb{R}$ there exists an a.e. invertible, measurable and measure-preserving map $T_t : X \to X$ and a function $\psi_t : X \to \mathbb{C}$ such that $U_t f = \psi_t \cdot (f \circ T_t)$ for all $f \in L_2(X)$.*

(ii) *For all $t \in \mathbb{R}$ one has $|U_t \mathbb{1}| = 1$ a.e. Moreover, $D(A) \cap L_\infty(X)$ is an algebra and $A - (A\mathbb{1})I$ is a derivation on $D(A) \cap L_\infty(X)$.*

We also have an extension of this theorem for C_0-groups which are not necessarily unitary, see Theorem 3.10. The above result can be viewed as the one-parameter counterpart of the classical Banach–Lamperti theorem, [Lam] Theorem 3.1, classifying that all isometries of $L_p(X)$ for all $p \in [1, \infty) \setminus \{2\}$ are of the form

$$f \mapsto \psi \cdot (f \circ T) \tag{2}$$

for some pointwise map $T : X \to X$ and $\psi : X \to (0, \infty)$. In [GGM] the authors also proved that unitary positivity preserving maps are of the form (2).

In Section 2 we prove Theorem 1.1 and its extension for general C_0-groups. As a tool and byproduct we prove in Theorem 2.5 that if $(X, \mathcal{B}, \mu)$ is a finite measure space and S

is a C_0-group in $L_2(X)$, then S extends consistently to a C_0-group on $L_1(X)$ if and only if the dual group S^* leaves $L_\infty(X)$ invariant. This is a new result in (semi)group theory. In Section 3 we prove Theorem 3.10, characterizing weighted non-singular one-parameter C_0-groups, which has Theorem 1.3 as corollary. It turns out that in Theorem 1.3(i) one has $\psi_t \in L_\infty(X)$ and

$$\psi_{t+s} = \psi_t \cdot (\psi_s \circ T_t) \text{ a.e.}$$

for all $t, s \in \mathbb{R}$. Finally, in Section 4 we determine the form of such ψ, assuming a differentiability condition.

2 Derivations

If $(X, \mathcal{B}, \mu)$ is a measure space and $f, g \in L_2(X)$, then we denote the inner product by $(f, g) = \int_X f \, \overline{g} \, d\mu$. Moreover, if $f \in L_\infty(X)$ and $g \in L_1(X)$ then we denote the duality by $\langle f, g \rangle = \int_X f \, \overline{g} \, d\mu$. If the measure space is clear from the context, then we abbreviate $L_p = L_p(X)$ for all $p \in [1, \infty]$. Further, let $p, q \in [1, \infty]$, let U be a one-parameter (semi)group on $L_p(X)$ and V be a one-parameter (semi)group on $L_q(X)$. We say that U and V are **consistent** if $U_t f = V_t f$ for all $t \in \mathbb{R}$ (or $t \in (0, \infty)$) and $f \in L_p(X) \cap L_q(X)$.

For the proof of Theorem 1.1 we need several lemmas as preparation. The first two seem to be folklore.

Lemma 2.1. *Let $(X, \mathcal{B}, \mu)$ be a measure space, $c > 0$ and $E \colon L_2(X) \to L_2(X)$ be a bounded operator. Suppose that $\|Ef\|_\infty \leq c \, \|f\|_\infty$ for all $f \in L_2(X) \cap L_\infty(X)$. Then there exist unique $\widehat{E} \in \mathcal{L}(L_1(X))$ and $\widetilde{E} \in \mathcal{L}(L_\infty(X))$ such that $\widehat{E} f = E^* f$ for all $f \in L_1(X) \cap L_2(X)$ and $\widetilde{E} f = E f$ for all $f \in L_\infty(X) \cap L_2(X)$. Moreover, $\|\widetilde{E}\|_{\infty \to \infty} \leq c$ and $\widetilde{E} = (\widehat{E})^*$.*

Proof. Let $f \in L_1 \cap L_2$. Then $|(E^* f, g)| = |(f, Eg)| \leq \|f\|_1 \, \|Eg\|_\infty \leq c \, \|f\|_1 \, \|g\|_\infty$ for all $g \in L_2 \cap L_\infty$. So $E^* f \in L_1$ and $\|E^* f\|_1 \leq c \, \|f\|_1$. Since $L_1 \cap L_2$ is dense in L_1 it follows that there exists a unique $\widehat{E} \in \mathcal{L}(L_1)$ such that $\widehat{E} f = E^* f$ for all $f \in L_1 \cap L_2$. Choose $\widetilde{E} = (\widehat{E})^*$. Then the existence follows. The uniqueness on L_∞ is a consequence of the w*-density of $L_2 \cap L_\infty$ in L_∞. □

As a consequence one has the next lemma.

Lemma 2.2. *Let $(X, \mathcal{B}, \mu)$ be a measure space and S a semigroup on $L_2(X)$. Suppose that for all $t \in (0, 1]$ there exists a $c > 0$ such that $\|S_t f\|_\infty \leq c \, \|f\|_\infty$ for all $f \in L_2(X) \cap L_\infty(X)$. Then there exist a unique semigroup $\widetilde{S}$ on $L_\infty(X)$ and a unique semigroup $\widehat{S}$ on $L_1(X)$ such that $\widetilde{S}$ is consistent with S and $\widehat{S}$ is consistent with S^*. Moreover, if there exists a $\tilde{c} \geq 1$ such that $\|S_t f\|_\infty \leq \tilde{c} \, \|f\|_\infty$ for all $t \in (0, 1]$ and $f \in L_2(X) \cap L_\infty(X)$, then $\|\widetilde{S}_t\|_{\infty \to \infty} = \|\widehat{S}_t\|_{1 \to 1} \leq c \, e^{t \log c}$ for all $t \in (0, \infty)$.*

The next lemma is less known.

Lemma 2.3. *Let $(X, \mathcal{B}, \mu)$ be a measure space, $c \geq 1$ and S a C_0-semigroup on $L_2(X)$ with generator A. Suppose that $\|S_t f\|_\infty \leq c \, \|f\|_\infty$ for all $t \in (0, 1]$ and $f \in L_2(X) \cap L_\infty(X)$. Then $D(A) \cap L_\infty(X)$ is dense in $L_2(X)$.*

Proof. Since $L_2 \cap L_\infty$ is dense in L_2, it suffices to show that for all $f \in L_2 \cap L_\infty$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $D(A) \cap L_\infty$ such that $\lim f_n = f$ in L_2. Fix $\varphi \in C_c^\infty(0,\infty)$ with $\int \varphi = 1$. For all $n \in \mathbb{N}$ define $\varphi \in C_c^\infty(0,\infty)$ by $\varphi_n(t) = n\,\varphi(n\,t)$. Let $f \in L_2 \cap L_\infty$ and $n \in \mathbb{N}$. Define $f_n \in L_2$ by

$$f_n = \int_{(0,\infty)} \varphi_n(t)\, S_t f\, dt.$$

Then $f_n \in D(A)$. Moreover, $\lim_{n\to\infty} f_n = f$ in L_2 since S is a continuous semigroup. It remains to show that $f_n \in L_\infty$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $g \in L_1 \cap L_2$. Then

$$|\varphi_n(t)\, (S_t f, g)| \le |\varphi_n(t)|\, \|S_t f\|_\infty\, \|g\|_1 \le |\varphi_n(t)|\, c\, e^{t \log c}\, \|f\|_\infty\, \|g\|_1$$

for all $t \in (0,\infty)$. Hence

$$|(f_n, g)| = \Big| \int_{(0,\infty)} \varphi_n(t)\, (S_t f, g)\, dt \Big| \le M_n\, \|f\|_\infty\, \|g\|_1,$$

where $M_n = \int_{(0,\infty)} |\varphi_n(t)|\, c\, e^{t\log c}\, dt < \infty$. So $f_n \in L_\infty$ as required. □

Remark 2.4. Under the conditions of Lemma 2.3 the space $D(A) \cap L_\infty$ is even a core for A, since the space $D(A) \cap L_\infty$ is invariant under S. See [EN] Proposition II.1.7.

It seems that the next theorem is new. Note that we do not assume a uniform bound of the type (3) in Condition (ii).

Theorem 2.5. *Let $(X, \mathcal{B}, \mu)$ be a finite measure space. Let S be a C_0-group on $L_2(X)$. Then the following are equivalent.*

(i) *The group S extends consistently to a C_0-group on $L_1(X)$.*

(ii) *The space $L_\infty(X)$ is invariant under S^*, that is $S_t^*(L_\infty(X)) \subset L_\infty(X)$ for all $t \in \mathbb{R}$.*

If these conditions are satisfied, then there exist $M \ge 1$ and $\omega \ge 0$ such that

$$\|S_t^* f\|_\infty \le M\, e^{\omega|t|}\, \|f\|_\infty \qquad (3)$$

for all $t \in \mathbb{R}$ and $f \in L_\infty(X)$.

Proof. The implication '(i)⇒(ii)' is trivial. (Cf. the proof of Lemma 2.1.) So it remains to show '(ii)⇒(i)'.

Let $t \in \mathbb{R}$. It follows from the closed graph theorem that there exists a $c > 0$ such that $\|S_t^* f\|_\infty \le c\, \|f\|_\infty$ for all $f \in L_\infty$. Note that we use here that the measure μ is finite. Also note that c depends on t. Hence by Lemma 2.2 there exists a one-parameter group $\widehat{S}$ on L_1 and a one-parameter group $\widetilde{S}$ on L_∞ such that $\widehat{S}$ is consistent with S and $\widetilde{S}$ is consistent with S^*. Moreover, $\widetilde{S}_t = (\widehat{S}_t)^*$ for all $t \in \mathbb{R}$.

We shall show that $\{\widetilde{S}_t : t \in [2,3]\}$ is bounded in $\mathcal{L}(L_\infty)$. By the uniform boundedness principle if suffices to show that $\{\|\widetilde{S}_t f\|_\infty : t \in [2,3]\}$ is bounded for all $f \in L_\infty$. For this we use the arguments as in the first step of the proof of [ABHN] Lemma 3.16.4. Fix $f \in L_\infty$. If $t \in \mathbb{R}$ then

$$\begin{aligned}\|\widetilde{S}_t f\|_\infty &= \sup\{|\langle \widetilde{S}_t f, g\rangle| : g \in L_1 \text{ and } \|g\|_1 \le 1\} \\ &= \sup\{|\langle \widetilde{S}_t f, g\rangle| : g \in L_2 \text{ and } \|g\|_1 \le 1\} \\ &= \sup\{|(f, S_t g)| : g \in L_2 \text{ and } \|g\|_1 \le 1\}.\end{aligned}$$

For each $g \in L_2$ the map $t \mapsto |(f, S_t g)|$ is continuous by the strong continuity of S on L_2. So $t \mapsto \|\widetilde{S}_t f\|_\infty$ is lower semicontinuous and therefore a measurable function on $\mathbb{R}$. This is the key assumption in the first step of the proof of [ABHN] Lemma 3.16.4. In order to have the paper self-contained, we include the proof, with minor modifications. Suppose that $\{\|\widetilde{S}_t f\|_\infty : t \in [2,3]\}$ is not bounded. Then there are $t_0, t_1, t_2, \ldots \in [2,3]$ such that $\lim_{n\to\infty} t_n = t_0$ and $\|\widetilde{S}_{t_n} f\|_\infty \geq n$ for all $n \in \mathbb{N}$. Since $t \mapsto \|\widetilde{S}_t f\|_\infty$ is measurable, there are $M > 0$ and a measurable set $F \subset [0, t_0]$ such that $\mu(F) > 1$ and $\|\widetilde{S}_t f\|_\infty \leq M$ for all $t \in F$. Let $n \in \mathbb{N}$. Then

$$n \leq \|\widetilde{S}_{t_n} f\|_\infty \leq \|\widetilde{S}_{t_n - t}\| \, \|\widetilde{S}_t f\|_\infty \leq M \, \|\widetilde{S}_{t_n - t}\|$$

for all $t \in F$. So $\|\widetilde{S}_s\| \geq M^{-1} n$ for all $s \in E_n$, where

$$E_n = \{t_n - t : t \in F \cap [0, t_n]\}.$$

Note that E_n is measurable and $\mu(E_n) \geq 1$ if $|t_n - t_0| < \mu(F) - 1$. Let $E = \limsup_{n\to\infty} E_n = \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty E_n$. Then E is measurable and $\mu(E) \geq 1$. In particular, $E \neq \emptyset$. Moreover, $\|\widetilde{S}_s\| = \infty$ for all $s \in E$. This is a contradiction.

Thus $\{\widetilde{S}_t : t \in [2,3]\}$ is bounded in $\mathcal{L}(L_\infty)$. Since $\widetilde{S}_t = (\widehat{S}_t)^*$ it follows that $\{\widehat{S}_t : t \in [2,3]\}$ is bounded in $\mathcal{L}(L_1)$. By the group property the set $\{\widehat{S}_t : t \in [-1,1]\}$ is also bounded in $\mathcal{L}(L_1)$. Let $c = \sup\{\|\widehat{S}_t\| : t \in [-1,1]\} < \infty$. Let $g \in L_\infty$. Then $\lim_{t\to 0} \langle g, \widehat{S}_t f\rangle = \lim_{t\to 0} (g, S_t f) = (g, f) = \langle g, f\rangle$ for all $f \in L_2$. Since L_2 is dense in L_1 and $c < \infty$ it follows that $\lim_{t\to 0} \langle g, \widehat{S}_t f\rangle = \langle g, f\rangle$ for all $f \in L_1$. So $\widehat{S}$ is weakly continuous and hence $\widehat{S}$ is a C_0-group. Finally, there are $M \geq 1$ and $\omega \geq 0$ such that $\|\widehat{S}_t\|_{1\to 1} \leq M \, e^{\omega|t|}$ for all $t \in \mathbb{R}$. Then $\|\widetilde{S}_t\|_{\infty\to\infty} = |\widehat{S}_t\|_{1\to 1} \leq M \, e^{\omega|t|}$ for all $t \in \mathbb{R}$. □

We now turn to the proof of Theorem 1.1. The implication (i)⇒(ii) in Theorem 1.1 is a special case of the next proposition.

Proposition 2.6. *Let U be a C_0-group on $L_2(X)$ with generator A, where $(X, \mathcal{B}, \mu)$ is a measure space. Suppose that for every $t \in \mathbb{R}$ there exists a measurable map $T_t \colon X \to X$ such that $U_t f = f \circ T_t$ for all $f \in L_2(X)$. Then U extends consistently to a w*-continuous contraction group on $L_\infty(X)$. Moreover, $D(A) \cap L_\infty(X)$ is an algebra and A is a derivation on $D(A) \cap L_\infty(X)$.*

Proof. Let $t \in \mathbb{R}$. If $f \in L_2 \cap L_\infty$ then $\|U_t f\|_\infty = \|f \circ T_t\|_\infty \leq \|f\|_\infty$. Hence by Lemma 2.2 there exist a unique group $\widetilde{U}$ on L_∞ and a unique group $\widehat{U}$ on L_1 such that $\widetilde{U}$ is consistent with U and $\widehat{U}$ is consistent with U^*. Moreover, $\|\widetilde{U}_t\|_{\infty\to\infty} = \|\widehat{U}_t\|_{1\to 1} = 1$ for all $t \in \mathbb{R}$. Then $\widehat{U}$ is a C_0-group on L_1 by [Voi] Proposition 4. Therefore $\widetilde{U}$ is a w*-continuous group on L_∞.

Let $f, g \in D(A) \cap L_\infty$. Then $U_t(f\,g) = (U_t f)(U_t g)$ for all $t \in \mathbb{R}$. Hence

$$\tfrac{1}{t}\Big(U_t(f\,g) - f\,g\Big) = \tfrac{1}{t}\Big(U_t f - f\Big) U_t g + \tfrac{1}{t}\, f\, \Big(U_t g - g\Big)$$

for all $t > 0$. So $f\,g \in D(A)$ and $A(f\,g) = (Af)\,g + f\,(Ag)$. □

Under more conditions there is a converse of Proposition 2.6. A first step is the next proposition.

Proposition 2.7. *Let U be a C_0-group on $L_2(X)$ with generator A, where $(X, \mathcal{B}, \mu)$ is a measure space. Suppose that $D(A) \cap L_\infty(X)$ is an algebra and A is a derivation on $D(A) \cap L_\infty(X)$. Moreover, suppose that there exists a $c \geq 1$ such that*

$$\|U_t f\|_\infty \leq c\, \|f\|_\infty$$

for all $t \in [-1, 1]$ and $f \in L_2(X) \cap L_\infty(X)$. Then there exists a unique one-parameter group $\widetilde{U}$ on $L_\infty(X)$ which is consistent with U. Moreover,

$$\widetilde{U}_t(f\, g) = (\widetilde{U}_t f)(\widetilde{U}_t g)$$

for all $f, g \in L_\infty$ and $t \in \mathbb{R}$.

Proof. The existence and uniqueness of the one-parameter group $\widetilde{U}$ on L_∞ follows from Lemma 2.2. Then $\|\widetilde{U}_t f\|_\infty \leq c\, e^{\omega|t|}\, \|f\|_\infty$ for all $t \in \mathbb{R}$ and $f \in L_\infty$, where $\omega = \log c$.

Clearly $t \mapsto \langle \widetilde{U}_t f, g\rangle = (U_t f, g)$ is continuous for all $f \in L_\infty \cap L_2$ and $g \in L_1 \cap L_2$. Let $f \in L_\infty \cap L_2$ and $g \in L_1$. Let $t \in \mathbb{R}$ and $\varepsilon > 0$. There exists a $g' \in L_1 \cap L_2$ such that $\|g - g'\|_1 \leq \varepsilon$. Then

$$\begin{aligned} |\langle \widetilde{U}_{t+k} f, g\rangle - \langle \widetilde{U}_t f, g\rangle| &= |\langle \widetilde{U}_{t+k} f, g - g'\rangle + \langle \widetilde{U}_{t+k} f - \widetilde{U}_t f, g'\rangle + \langle \widetilde{U}_t f, g' - g\rangle| \\ &\leq c\, e^{\omega(|t|+1)}\, \varepsilon\, \|f\|_\infty + |\langle \widetilde{U}_{t+k} f - \widetilde{U}_t f, g'\rangle| + c\, e^{\omega|t|}\, \varepsilon\, \|f\|_\infty \end{aligned}$$

for all $k \in [-1, 1]$. Hence

$$\lim_{k \to 0} \langle \widetilde{U}_{t+k} f, g\rangle = \langle \widetilde{U}_t f, g\rangle \tag{4}$$

for all $f \in L_\infty \cap L_2$, $g \in L_1$ and $t \in \mathbb{R}$.

Let $f, g \in D(A) \cap L_\infty$. Define $\alpha\colon \mathbb{R} \to L_2$ by $\alpha(t) = (U_t f)(U_t g)$. Let $h \in L_2$. We shall show that $t \mapsto (\alpha(t), h)$ is differentiable and that

$$\frac{d}{dt}(\alpha(t), h) = \Big((AU_t f)(U_t g) + (U_t f)(AU_t g), h\Big) \tag{5}$$

for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$. If $k \in \mathbb{R} \setminus \{0\}$, then

$$\tfrac{1}{k}\Big((\alpha(t+k), h) - (\alpha(t), h)\Big) = \tfrac{1}{k}\Big((U_{t+k} f - U_t f) U_{t+k} g, h\Big) + \tfrac{1}{k}\Big((U_t f)(U_{t+k} g - U_t g), h\Big). \tag{6}$$

For the first term we shall prove that

$$\lim_{k \to 0} \tfrac{1}{k}\Big((U_{t+k} f - U_t f) U_{t+k} g, h\Big) = \Big((AU_t f)(U_t g), h\Big).$$

Let $k \in \mathbb{R} \setminus \{0\}$. Then

$$\begin{aligned} &\left|\tfrac{1}{k}\Big((U_{t+k} f - U_t f) U_{t+k} g, h\Big) - \Big((AU_t f)(U_t g), h\Big)\right| \\ &\quad = \left|\bigg(\Big(\tfrac{1}{k}(U_{t+k} f - U_t f) - AU_t f\Big) \widetilde{U}_{t+k} g, h\bigg) + \Big((AU_t f)(\widetilde{U}_{t+k} g - \widetilde{U}_t g), h\Big)\right| \\ &\quad \leq \|\tfrac{1}{k}(U_{t+k} f - U_t f) - AU_t f\|_2\, \|\widetilde{U}_{t+k} g\|_\infty\, \|h\|_2 + \left|\langle \widetilde{U}_{t+k} g - \widetilde{U}_t g, h\, \overline{(AU_t f)}\rangle\right|. \end{aligned}$$

Since $\lim_{k\to 0} \|\frac{1}{k}(U_{t+k}f - U_t f) - AU_t f\|_2 = 0$, $\sup_{k\in[-1,1]} \|\widetilde{U}_{t+k}g\|_\infty < \infty$ and $h\,\overline{(AU_t f)} \in L_1$, it follows from (4) that

$$\lim_{k\to 0} \tfrac{1}{k}\Big((U_{t+k}f - U_t f)U_{t+k}g, h\Big) = \Big((AU_t f)(U_t g), h\Big).$$

Similarly one proves for the second term in (6) that

$$\lim_{k\to 0} \tfrac{1}{k}\Big((U_t f)(U_{t+k}g - U_t g), h\Big) = \Big((U_t f)(AU_t g), h\Big).$$

Hence $t \mapsto (\alpha(t), h)$ is differentiable and (5) is valid. Thus α is weakly differentiable, with weak derivative

$$\alpha'(t) = (AU_t f)(U_t g) + (U_t f)(AU_t g).$$

But A is a derivation on $D(A) \cap L_\infty$. Therefore

$$\alpha'(t) = (AU_t f)(U_t g) + (U_t f)(AU_t g) = A\Big((U_t f)(U_t g)\Big) = A(\alpha(t))$$

for all $t \in \mathbb{R}$. Obviously $\alpha(0) = f\,g$. The uniqueness of the Cauchy problem yields $\alpha(t) = e^{tA}(f\,g) = U_t(f\,g)$ for all $t \in \mathbb{R}$.

Fix $t \in \mathbb{R}$. Let $g \in D(A) \cap L_\infty$. Then

$$U_t(f\,g) = (U_t f)(U_t g) = (U_t f)(\widetilde{U}_t g)$$

for all $f \in D(A) \cap L_\infty$. By Lemma 2.3 the space $D(A) \cap L_\infty$ is dense in L_2. Hence it follows by continuity that $U_t(f\,g) = (U_t f)(\widetilde{U}_t g)$ is valid for all $f \in L_2$ and in particular for all $f \in L_2 \cap L_\infty$. So

$$U_t(f\,g) = (\widetilde{U}_t f)(U_t g) \tag{7}$$

for all $f \in L_2 \cap L_\infty$ and $g \in D(A) \cap L_\infty$. Since $D(A) \cap L_\infty$ is dense in L_2, it follows that (7) is valid for all $f, g \in L_2 \cap L_\infty$. So

$$\widetilde{U}_t(f\,g) = (\widetilde{U}_t f)(\widetilde{U}_t g) \tag{8}$$

for all $f, g \in L_2 \cap L_\infty$. Let $f \in L_2 \cap L_\infty$ and $h \in L_1$. Then

$$\langle \widetilde{U}_t(f\,g), h\rangle = \langle f\,g, \widehat{U}_t h\rangle = \langle g, \overline{f}\,\widehat{U}_t h\rangle \tag{9}$$

for all $g \in L_\infty$. Moreover,

$$\langle (\widetilde{U}_t f)(\widetilde{U}_t g), h\rangle = \langle \widetilde{U}_t g, \overline{\widetilde{U}_t f}\, h\rangle = \langle g, \widehat{U}_t(\overline{\widetilde{U}_t f}\, h)\rangle \tag{10}$$

for all $g \in L_\infty$. It follows from (8), (9) and (10) that

$$\langle g, \overline{f}\,\widehat{U}_t h\rangle = \langle g, \widehat{U}_t(\overline{\widetilde{U}_t f}\, h)\rangle \tag{11}$$

for all $g \in L_2 \cap L_\infty$. But $L_2 \cap L_\infty$ is w*-dense in L_∞. So (11) is valid for all $g \in L_\infty$. Using again (9) and (10) one deduces that

$$\langle \widetilde{U}_t(f\,g), h\rangle = \langle (\widetilde{U}_t f)(\widetilde{U}_t g), h\rangle$$

for all $g \in L_\infty$. This is for all $h \in L_1$. So (8) is valid for all $f \in L_2 \cap L_\infty$ and $g \in L_\infty$. Finally, by a similar argument one establishes that (8) is valid for all $f, g \in L_\infty$. □

Theorem 2.8. *Let $(X,\mathcal{B},\mu)$ be a σ-finite measure space such that $(X,\mathcal{B})$ is a standard Borel space. Let U be a C_0-group on $L_2(X)$ with generator A. Then the following are equivalent.*

(i) *For all $t \in \mathbb{R}$ there exists a measurable map $T_t: X \to X$ such that $U_t f = f \circ T_t$ for all $f \in L_2(X)$.*

(ii) *The space $D(A) \cap L_\infty(X)$ is an algebra and A is a derivation on $D(A) \cap L_\infty(X)$. Moreover, there exists a $c > 0$ such that*

$$\|U_t f\|_\infty \le c\, \|f\|_\infty$$

for all $t \in [-1,1]$ and $f \in L_2(X) \cap L_\infty(X)$.

Proof. '(i)⇒(ii)'. This follows from Proposition 2.6.

'(ii)⇒(i)'. By Proposition 2.7 there exists a unique one-parameter group $\widetilde{U}$ on L_∞ which is consistent with U. Moreover,

$$\widetilde{U}_t(f\,g) = (\widetilde{U}_t f)(\widetilde{U}_t g) \tag{12}$$

for all $f,g \in L_\infty$ and $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$. Let $\mathcal{I} = \{B \in \mathcal{B} : \mu(B) = 0\}$. Then $\mathcal{I}$ is a σ-ideal in $\mathcal{B}$. Let $B \in \mathcal{B}$. Then $\widetilde{U}_t \mathbb{1}_B = \widetilde{U}_t(\mathbb{1}_B^2) = (\widetilde{U}_t \mathbb{1}_B)^2$ by (12). Therefore there exists a $B' \in \mathcal{B}$ such that $\widetilde{U}_t \mathbb{1}_B = \mathbb{1}_{B'}$. If also $B'' \in \mathcal{B}$ is such that $\widetilde{U}_t \mathbb{1}_B = \mathbb{1}_{B''}$, then $B' \Delta B'' \in \mathcal{I}$, where Δ denotes the symmetric difference. Define $\Phi(B) = B' \Delta \mathcal{I} \in \mathcal{B}/\mathcal{I}$. Then Φ is a map from $\mathcal{B}$ into $\mathcal{B}/\mathcal{I}$.

Clearly $\Phi(\emptyset) = \emptyset \Delta \mathcal{I}$. Let $B_1, B_2 \in \mathcal{B}$. It follows from (12) that $\Phi(B_1 \cap B_2) = \Phi(B_1) \wedge \Phi(B_2)$. Moreover, if $B_1 \cap B_2 = \emptyset$ and $B_1', B_2' \in \mathcal{B}$ are such that $\widetilde{U}_t \mathbb{1}_{B_1} = \mathbb{1}_{B_1'}$ and $\widetilde{U}_t \mathbb{1}_{B_2} = \mathbb{1}_{B_2'}$, then $\mathbb{1}_{B_1' \cap B_2'} = U_t \mathbb{1}_{B_1 \cap B_2} = 0$, so $\widetilde{U}_t \mathbb{1}_{B_1 \cup B_2} = \widetilde{U}_t(\mathbb{1}_{B_1} + \mathbb{1}_{B_2}) = \mathbb{1}_{B_1'} + \mathbb{1}_{B_2'} = \mathbb{1}_{B_1' \cup B_2'}$. Note that $(\widetilde{U}_t)^{-1} = \widetilde{U}_{-t}$ has the same properties as $\widetilde{U}_t$. Hence there exists a $B \in \mathcal{B}$ such that $\widetilde{U}_{-t}\mathbb{1} = \mathbb{1}_B$. Then $\widetilde{U}_t \mathbb{1}_B = \mathbb{1}$. Consequently

$$\mathbb{1} = \widetilde{U}_t \mathbb{1}_B = \widetilde{U}_t(\mathbb{1}_B\, \mathbb{1}) = (\widetilde{U}_t \mathbb{1}_B)(\widetilde{U}_t \mathbb{1}) = \mathbb{1}\, \widetilde{U}_t \mathbb{1} = \widetilde{U}_t \mathbb{1}.$$

So Φ is a homomorphism. Since $\widetilde{U}_t$ is continuous, it follows that $\widetilde{U}_t$ is a σ-homomorphism of Boolean σ-algebras. By [Kec] Theorem 15.9 there exists a measurable map $T_t: X \to X$ such that $\Phi(B) = T_t^{-1}(B) \Delta \mathcal{I}$ for all $B \in \mathcal{B}$. So $U_t \mathbb{1}_B = \widetilde{U}_t \mathbb{1}_B = \mathbb{1}_B \circ T_t$ for all $B \in \mathcal{B}$ with $\mu(B) < \infty$. Using the continuity of U_t and the image measure under T_t, one deduces that $U_t f = f \circ T_t$, first for all $f \in L_1 \cap L_2$ and then for all $f \in L_2$. □

Corollary 2.9. *Let $(X,\mathcal{B},\mu)$ be a standard Borel probability space. Let U be a C_0-group on $L_2(X)$ with generator A. Then the following are equivalent.*

(i) *For all $t \in \mathbb{R}$ there exists a measurable map $T_t: X \to X$ such that $U_t f = f \circ T_t$ for all $f \in L_2(X)$.*

(ii) *The space $L_\infty(X)$ is invariant under U. Moreover, the space $D(A) \cap L_\infty(X)$ is an algebra and A is a derivation on $D(A) \cap L_\infty(X)$.*

Proof. This is a consequence of Theorems 2.5 and 2.8. □

Note that the map U_t is unitary if and only if the map T_t is measure preserving in Theorem 2.8(i).

Proof of Theorem 1.1. This follows immediately from Corollary 2.9. □

Remark 2.10. Note that in Theorem 1.1 the map $T_t: X \to X$ is measure preserving for all $t \in \mathbb{R}$. Moreover,

$$T_{t_1+T_2} = T_{t_1} \circ T_{t_2} \text{ a.e.}$$

for all $t_1, t_2 \in \mathbb{R}$. Since the one-parameter group U is strongly continuous, it follows from [GTW] page 307 that the group $(T_t)_{t\in\mathbb{R}}$ enjoys the following measurabilty property: there exists a Borel map $F: \mathbb{R} \times X \to X$ such that for all $t \in \mathbb{R}$ one has

$$F(t,x) = T_t x \text{ for a.e. } x \in X.$$

Thus $(T_t)_{t\in\mathbb{R}}$ is a measurable measure preserving flow.

3 Weighted non-singular C_0-groups

Throughout this section let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a one-parameter group on $L_2(X)$ with $U_0 = I$. The group U is called **weighted non-singular** if for each $t \in \mathbb{R}$ there exist a map $T_t: X \to X$ and a function $\psi_t: X \to \mathbb{C}$ such that

$$U_t f = \psi_t \cdot (f \circ T_t) \tag{13}$$

for all $f \in L_2(X)$. By substituting $f = \mathbb{1}$, we obtain that $\psi_t = U_t\mathbb{1}$ for all $t \in \mathbb{R}$, in particular, ψ_t is measurable. Moreover, $\psi_0 = \mathbb{1}$ and the group property of U implies the cocycle identity

$$\psi_{t+t'} = \psi_t \cdot (\psi_{t'} \circ T_t) \tag{14}$$

and the group property

$$T_{t+t'} = T_t \circ T_{t'} \text{ a.e.} \tag{15}$$

for all $t, t' \in \mathbb{R}$. Let $t \in \mathbb{R}$. It follows that $\mathbb{1} = \psi_t \cdot (\psi_{-t} \circ T_t)$, whence $\psi_t \neq 0$ a.e. and

$$\frac{1}{\psi_t} = \psi_{-t} \circ T_t. \tag{16}$$

Therefore

$$f \circ T_t = \frac{1}{\psi_t} \cdot U_t f$$

for all $f \in L_2(X)$, so T_t is measurable. In general, a measurable map $S: X \to X$ is called **non-singular** if $\mu(S^{-1}(A)) = 0$ for all $A \in \mathcal{B}$ with $\mu(A) = 0$. Then note that T_t is a non-singular map of $(X, \mathcal{B}, \mu)$ and that the measure μ and the image measure $T_{t*}\mu$ are equivalent, where

$$(T_{t*}\mu)(A) := \mu(T_t^{-1}A)$$

for all $A \in \mathcal{B}$. Indeed, if $B \in \mathcal{B}$ and $\mu(B) = 0$, then $0 = U_t\mathbb{1}_B = \psi_t \cdot (\mathbb{1}_B \circ T_t) = \psi_t \cdot \mathbb{1}_{T_t^{-1}B}$, hence $(T_{t*}\mu)(B) = \mu(T_t^{-1}B) = 0$.

A weighted non-singular one-parameter group U is called a **weighted Koopman group** if T_t is measure-preserving for all $t \in \mathbb{R}$.

Lemma 3.1. *Let U be a weighted non-singular one-parameter group given by* (13). *Then*

$$\left\| |\psi_t|^2 \cdot \left(\frac{d(T_{t*}\mu)}{d\mu} \circ T_t \right) \right\|_\infty \le \|U_t\|_{2\to 2}^2$$

for all $t \in \mathbb{R}$.

Proof. If $f \in L_2$, then

$$\int |\psi_t|^2 \cdot |f \circ T_t|^2 \, d\mu = \|U_t f\|_2^2 \le c \, \|f\|_2^2,$$

where $c = \|U_t\|_{2\to 2}^2$. Hence

$$\int |\psi_t|^2 \cdot (f \circ T_t) \, d\mu \le c \, \|f\|_1$$

for all $0 \le f \in L_1(X, \mu)$. Equivalently,

$$\int (|\psi_t|^2 \circ T_t^{-1}) \cdot f \cdot \frac{d(T_{t*}\mu)}{d\mu} \, d\mu = \int (|\psi_t|^2 \circ T_t^{-1}) \cdot f \, d(T_{t*}\mu) \le c \, \|f\|_1$$

for all $0 \le f \in L_1(X, \mu)$. Since $(|\psi_t|^2 \circ T_t^{-1}) \cdot \frac{d(T_{t*}\mu)}{d\mu} \ge 0$, one deduces that

$$\left\| (|\psi_t|^2 \circ T_t^{-1}) \cdot \frac{d(T_{t*}\mu)}{d\mu} \right\|_\infty \le c$$

and the result follows by the non-singularity of T_t. □

Lemma 3.2. *Let U be a weighted non-singular one-parameter group given by* (13). *Then the following are equivalent.*

(i) *The representation U preserves $L_\infty(X)$.*

(ii) $\psi_t = U_t \mathbb{1} \in L_\infty(X)$ *for all $t \in \mathbb{R}$.*

(iii) $\frac{d(T_{t*}\mu)}{d\mu} \in L_\infty(X)$ *for all $t \in \mathbb{R}$.*

Proof. '(i)⇒(ii)' is trivial and '(ii)⇒(i)' follows from (13) and the fact that T_t is non-singular for all $t \in \mathbb{R}$.

'(ii)⇒(iii)'. Lemma 3.1 and (16) imply that

$$\left\| \frac{d(T_{t*}\mu)}{d\mu} \circ T_t \right\|_\infty \le \|U_t\|_{2\to 2}^2 \, \|\psi_{-t}\|_\infty^2 < \infty$$

for all $t \in \mathbb{R}$.

'(iii)⇒(ii)'. Since $T_t \circ T_{-t} = I$ a.e., it follows that

$$\Big(\frac{d(T_{t*}\mu)}{d\mu} \circ T_t \Big) \cdot \frac{d(T_{-t*}\mu)}{d\mu} = \mathbb{1}$$

for all $t \in \mathbb{R}$. Then the claim is a consequence of Lemma 3.1. □

Remark 3.3. Let U be a C_0-group which is weighted Koopman and unitary. Let $t \in \mathbb{R}$. Then
$$\int |f|^2 \circ T_t \, d\mu = \int |f|^2 \, d\mu = \|U_t f\|_2^2 = \int |\psi_t|^2 \cdot (|f|^2 \circ T_t) \, d\mu$$
for all $f \in L_2(X)$. Hence $\int (|\psi_t|^2 - 1) \cdot (|f|^2 \circ T_t) \, d\mu = 0$ for all $f \in L_2(X)$ and therefore $|\psi_t| = 1$ a.e.

There are many one-parameter C_0-groups which preserve $L_\infty(X)$, but which are not weighted non-singular.

Example 3.4. Let $B \in \mathcal{B}$ be such that $\mu(B) \neq 0 \neq \mu(X \setminus B)$. Define $A \colon L_2(X) \to L_2(X)$ by $Af = (f, \mathbb{1}_B) \, \mathbb{1}_{X \setminus B}$. Then A is bounded, so it generates a C_0-group U. Since $A^2 = 0$, one deduces that $U_t = I + t \, A$ for all $t \in \mathbb{R}$. Hence obviously U leaves $L_\infty(X)$ invariant. Now choose $t = -\mu(B)^{-1}$. Then
$$U_t \mathbb{1} = \mathbb{1} + t \, (\mathbb{1}, \mathbb{1}_B) \, \mathbb{1}_{X \setminus B} = \mathbb{1} + t \, \mu(B) \, \mathbb{1}_{X \setminus B} = \mathbb{1} - \mathbb{1}_{X \setminus B} = \mathbb{1}_B.$$
Since $\mu(\{x \in X : (U_t \mathbb{1})(x) = 0\}) = \mu(X \setminus B) > 0$, the group U is not weighted non-singular by (16).

We next consider weighted non-singular one-parameter groups which preserve $L_\infty(X)$.

Lemma 3.5. *Let U be a weighted non-singular one-parameter group given by* (13)*. Assume that U preserves $L_\infty(X)$. Then $f \circ T_t \in L_2(X)$ for all $f \in L_2(X)$ and $t \in \mathbb{R}$. Define $V_t \colon L_2(X) \to L_2(X)$ by*
$$V_t f = f \circ T_t.$$
Then one has the following.

(a) $(V_t)_{t \in \mathbb{R}}$ *is a one-parameter group on* $L_2(X)$.

(b) *If U is a C_0-group, then also $(V_t)_{t \in \mathbb{R}}$ is a C_0-group.*

Proof. Note that (13) and (16) imply that
$$V_t f = f \circ T_t = (\psi_{-t} \circ T_t) \, U_t f \in L_2$$
for all $t \in \mathbb{R}$ and $f \in L_2$. Then Statement (a) is a consequence of (15).

'(b)'. By Theorem 2.5 there exist $M \geq 1$ and $\omega \geq 0$ such that $\|U_t f\|_\infty \leq M \, e^{\omega |t|} \, \|f\|_\infty$ for all $t \in \mathbb{R}$ and $f \in L_\infty$.

Fix $f \in L_\infty$. Let $t \in (0,1)$. Then (16) gives
$$\begin{aligned} V_t f - f &= \frac{1}{U_t \mathbb{1}} \Big((U_t f - f) + (\mathbb{1} - U_t \mathbb{1}) f \Big) \\ &= \Big((U_{-t} \mathbb{1}) \circ T_t \Big) \Big((U_t f - f) + (\mathbb{1} - U_t \mathbb{1}) f \Big). \end{aligned} \tag{17}$$
Therefore
$$\begin{aligned} \|V_t f - f\|_2 &\leq \|(U_{-t} \mathbb{1}) \circ T_t\|_\infty \Big(\|U_t f - f\|_2 + \|\mathbb{1} - U_t \mathbb{1}\|_2 \, \|f\|_\infty \Big) \\ &\leq M \, e^\omega \Big(\|U_t f - f\|_2 + \|\mathbb{1} - U_t \mathbb{1}\|_2 \, \|f\|_\infty \Big) \end{aligned} \tag{18}$$
and $\lim_{t \downarrow 0} V_t f = f$. Then the result follows since L_∞ is dense in L_2. □

Proposition 3.6. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a C_0-group on $L_2(X)$ preserving $L_\infty(X)$. Then the following are equivalent.*

(i) *The representation U is weighted non-singular.*

(ii) *For all $t \in \mathbb{R}$ one has $U_t\mathbb{1} \neq 0$ a.e. and $\frac{1}{U_t\mathbb{1}} \in L_\infty(X)$. Moreover, $V = (V_t)_{t\in\mathbb{R}}$ is a C_0-group on $L_2(X)$, where*

$$V_t f = \frac{1}{U_t\mathbb{1}} U_t f \tag{19}$$

for all $t \in \mathbb{R}$. In addition $D(B) \cap L_\infty(X)$ is an algebra and B is a derivation on $D(B) \cap L_\infty(X)$, where B is the generator of V.

Proof. '(i)⇒(ii)'. This follows from (16), Lemma 3.5(b) and Proposition 2.6.

'(ii)⇒(i)'. It follows from (19) that V leaves L_∞ invariant. Then apply Corollary 2.9 to V and the result follows from (19). □

Corollary 3.7. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a unitary C_0-group on $L_2(X)$ preserving $L_\infty(X)$. Then the following are equivalent.*

(i) *The group U is a weighted Koopman group.*

(ii) *For all $t \in \mathbb{R}$ one has $|U_t\mathbb{1}| = 1$ a.e. Moreover, $V = (V_t)_{t\in\mathbb{R}}$ is a unitary C_0-group on $L_2(X)$, where*

$$V_t f = \overline{U_t\mathbb{1}} \cdot U_t f$$

for all $t \in \mathbb{R}$. In addition $D(B) \cap L_\infty(X)$ is an algebra and B is a derivation on $D(B) \cap L_\infty(X)$, where B is the generator of V.

In order to obtain a relationship between the generators of the two C_0-groups in Lemma 3.5(b), we need the following observation.

Lemma 3.8. *Let U be a weighted non-singular one-parameter C_0-group. Let $V = (V_t)_{t\in\mathbb{R}}$ be the group on $L_2(X)$ as in Lemma 3.5. Then*

$$\lim_{t\to 0} \|V_t(U_{-t}\mathbb{1}) \cdot g - g\|_2 = 0$$

for all $g \in L_\infty(X)$.

Proof. It follows from Lemma 3.5(b) that V is a C_0-group. Hence $\sup_{t\in[-1,1]} \|V_t\|_{2\to 2} < \infty$. Let $t \in (-1, 1)$. Then

$$\begin{aligned} \|V_t(U_{-t}\mathbb{1}) \cdot g - g\|_2 &= \|\Big(V_t(U_{-t}\mathbb{1} - \mathbb{1}) + V_t\mathbb{1} - \mathbb{1}\Big) g\|_2 \\ &\leq \Big(\|V_t\|_{2\to 2}\|U_{-t}\mathbb{1} - \mathbb{1}\|_2 + \|V_t\mathbb{1} - \mathbb{1}\|_2\Big) \|g\|_\infty \end{aligned}$$

and the result follows. □

Lemma 3.9. *Let U be a weighted non-singular one-parameter C_0-group. Assume that U preserves $L_\infty(X)$. Let $V = (V_t)_{t\in\mathbb{R}}$ be the C_0-group on $L_2(X)$ as in Lemma 3.5. Denote by A and B the generators of U and V, respectively. Assume that*

$$\mathbb{1} \in D(A).$$

Then $D(A) \cap L_\infty(X) = D(B) \cap L_\infty(X)$ and $Bf = Af - f \cdot A\mathbb{1}$ for each $f \in D(A) \cap L_\infty(X)$.

Proof. Let $f \in D(A) \cap L_\infty$. Since $\mathbb{1} \in D(A)$ it follows from (18) that there exists a $c > 0$ such that $\|V_t f - f\|_2 \le c\,t$ for all $t \in (0,1)$. Therefore $f \in D(B)$ by [EN] Corollary II.5.21. Hence $D(A) \cap L_\infty \subset D(B) \cap L_\infty$. Let $g \in L_\infty$. Then (17) gives

$$\frac{1}{t}(V_t f - f, g) = \Big(\frac{1}{t}(U_t f - f) - f \cdot \frac{1}{t}(U_t \mathbb{1} - \mathbb{1}), \overline{(U_{-t}\mathbb{1}) \circ T_t} \cdot g\Big)$$

for all $t \in (0,1)$. Now take the limit $t \to 0$ and use Lemma 3.8. It follows that

$$(Bf, g) = (Af - f \cdot A\mathbb{1}, g).$$

Therefore $Bf = Af - (A\mathbb{1}) \cdot f$.

Conversely, let $f \in D(B) \cap L_\infty$. Then $U_t f - f = (U_t \mathbb{1})(V_t f - f) + (U_t \mathbb{1} - \mathbb{1}) f$ for all $t \in \mathbb{R}$. The bounds (3) of Theorem 2.5 imply that there exists a $c > 0$ such that

$$\|U_t f - f\|_2 \le \|U_t \mathbb{1}\|_\infty \|V_t f - f\|_2 + \|U_t \mathbb{1} - \mathbb{1}\|_2 \|f\|_\infty \le c\,|t|$$

for all $t \in (0,1)$. Hence $f \in D(A)$ as before. □

We can now prove the main theorem of this section.

Theorem 3.10. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a C_0-group on $L_2(X)$ preserving $L_\infty(X)$. Assume that $\mathbb{1} \in D(A)$ with $A\mathbb{1} \in L_\infty(X)$, where A is the generator of U. Then the following are equivalent.*

(i) *The representation U is weighted non-singular.*

(ii) *The space $D(A) \cap L_\infty(X)$ is an algebra and $A - (A\mathbb{1})I$ is a derivation on $D(A) \cap L_\infty(X)$.*

Proof. '(i)⇒(ii)'. This follows from Proposition 3.6 and Lemma 3.9. Note that this implication does not require the assumption $A\mathbb{1} \in L_\infty$.

'(ii)⇒(i)'. Consider first U^*, which is a C_0-group on L_2 whose generator is A^*. By Theorem 2.5(ii)⇒(i) the one-parameter group U^* extends consistently to a C_0-group $\widehat{U}$ on L_1. Denote by $\widehat{A}$ the generator of this group.

Since $(A\mathbb{1})I$ is a bounded operator the operator $A - (A\mathbb{1})I$ generates a C_0-group V on L_2 by perturbation theory [EN], Theorem III.1.3. Then $A^* - \overline{(A\mathbb{1})}I$ is the generator of V^*. Moreover, again by perturbation theory, $\widehat{A} - \overline{(A\mathbb{1})}I$ is the generator of a C_0-group $\widehat{V}$ on L_1. Let $t \in \mathbb{R}$. The Trotter–Kato formula [EN] Exercise III.5.11(1) gives

$$V_t^* = \lim_{n \to \infty} \Big(e^{-\frac{t}{n}\overline{(A\mathbb{1})}I}\, U^*_{\frac{t}{n}}\Big)^n \quad \text{strongly in } \mathcal{L}(L_2)$$

and

$$\widehat{V}_t = \lim_{n \to \infty} \Big(e^{-\frac{t}{n}\overline{(A\mathbb{1})}I}\, \widehat{U}_{\frac{t}{n}}\Big)^n \quad \text{strongly in } \mathcal{L}(L_1).$$

Let $f \in L_2$. Then $f \in L_1$ and since U^* and $\widehat{U}$ are consistent one deduces that

$$\Big(e^{-\frac{t}{n}\overline{(A\mathbb{1})}I}\, U^*_{\frac{t}{n}}\Big)^n f = \Big(e^{-\frac{t}{n}\overline{(A\mathbb{1})}I}\, \widehat{U}_{\frac{t}{n}}\Big)^n f \text{ a.e.}$$

for all $n \in \mathbb{N}$. Hence $V_t^* f = \widehat{V}_t f$ a.e. and V^* and $\widehat{V}$ are consistent. By Theorem 2.5(i)⇒(ii), applied with $S = V^*$, it follows that V leaves the space L_∞ invariant. By Theorem 2.8 it follows that for all $t \in \mathbb{R}$ there exists a non-singular measurable map $T_t \colon X \to X$ such that $V_t f = f \circ T_t$ for all $f \in L_2$.

Note that

$$\left(V_t \circ e^{s(A\mathbb{1})I}\right) f = V_t(e^{s(A\mathbb{1})} f) = (e^{s(A\mathbb{1})} f) \circ T_t = \left(e^{s((A\mathbb{1})\circ T_s)I} \circ V_t\right) f$$

for all $t, s \in \mathbb{R}$ and $f \in L_2$. Iteration gives

$$\left(V_{\frac{t}{n}} \circ e^{\frac{t}{n}(A\mathbb{1})I}\right)^n = e^{\frac{t}{n}\left((A\mathbb{1})\circ T_{\frac{t}{n}} + \ldots + (A\mathbb{1})\circ T_{\frac{nt}{n}}\right)I} \circ \left(V_{\frac{t}{n}}\right)^n = e^{\frac{t}{n}\left((A\mathbb{1})\circ T_{\frac{t}{n}} + \ldots + (A\mathbb{1})\circ T_{\frac{nt}{n}}\right)I} \circ V_t \quad (20)$$

for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Since $A = (A - (A\mathbb{1})I) + (A\mathbb{1})I$, one can consider the generator of the C_0-group U as a perturbation of the generator of the C_0-group V. Then the Trotter–Kato formula gives

$$U_t = \lim_{n\to\infty} \left(V_{\frac{t}{n}} \circ e^{\frac{t}{n}(A\mathbb{1})I}\right)^n$$

strongly in $\mathcal{L}(L_2)$. Hence (20) gives $U_t = \psi_t \cdot V_t$ for all $t \in \mathbb{R}$, where

$$\psi_t = e^{\int_0^t (A\mathbb{1})\circ T_r \, dr} \in L_\infty.$$

This completes the proof. □

Clearly Theorem 1.3 is a consequence of Theorem 3.10.

The condition $\mathbb{1} \in D(A)$ is not satisfied in general. We give a wide class of examples.

Example 3.11. Let $V = (V_t)_{t\in\mathbb{R}}$ be a unitary C_0-group on $L_2(X)$ given by a measure preserving flow $T = (T_t)_{t\in\mathbb{R}}$ which is ergodic. So $V_t f = f \circ T_t$ for all $t \in \mathbb{R}$ and $f \in L_2(X)$ and the only $f \in L_2(X)$ which are invariant under V_t for all $t \in \mathbb{R}$ are the constants. We will now show that for all $t \in \mathbb{R}$ we can find a measurable $\psi_t \colon X \to \mathbb{R}$, bounded and bounded away from zero, such that $U = (U_t)_{t\in\mathbb{R}}$ is a continuous C_0-group on $L_2(X)$ for which $\mathbb{1} \notin D(A)$, where

$$U_t f = \psi_t \cdot (f \circ T_t)$$

for all $t \in \mathbb{R}$.

Indeed, by Ambrose–Kakutani theorem, see for example [CFS] Theorem 11.2.1, we can represent T as a special flow over an ergodic automorphism S of a standard Borel probability space $(Y, \mathcal{C}, \rho)$, i.e. there exist $F \colon Y \to \mathbb{R}$ and $c > 0$ such that $F > c$, $\int_Y F \, d\rho < \infty$ and

$$X = Y^F := \{(y, s) \in Y \times \mathbb{R} : 0 \le s \le F(y)\}.$$

On Y^F we consider the restriction of the product measurable structure from $Y \times \mathbb{R}$ together with $\rho^F := (\rho \otimes \mathrm{Leb}_\mathbb{R})|_{Y^F}$. The flow T acts as $S^F = (S^F_t)_{t\in\mathbb{R}}$, where under the action of S^F_t (with $t > 0$) a point (y, r) moves up vertically with unit speed until it hits the point $(y, f(y))$ which is identified with $(Sy, 0)$ and this movement is continued until time t. In this way we obtain a unitary C_0-group $V = (V_t)_{t\in\mathbb{R}}$, where $V_t f = f \circ S^F_t$ on $L_2(Y^F, \rho^F)$.

Let $a, b \in \mathbb{R}$ be such that $0 < a < b < c$ and consider the strip $H := Y \times [a, b]$. Then $H \subset Y^F$ and $\rho^F(H) = b - a$. For each $t \in \mathbb{R}$ with $|t| < a \wedge (c - b) \wedge (b - a)$ one has

$$\rho^F(H \triangle S^F_t(H)) = 2|t|. \quad (21)$$

We claim that $g := \mathbb{1}_H \notin D(B)$, where B is the generator of V. Indeed, for all $t \in \mathbb{R}$ with $|t| < a \wedge (c-b) \wedge (b-a)$, it follows from (21) that

$$\|g - g \circ S_t^F\|_2 = \Big(\int_Y |\mathbb{1}_H - \mathbb{1}_H \circ S_t^F|^2 \, d\rho^F \Big)^{1/2} = \big(\rho^F(H \triangle S_t^F(H))\big)^{1/2} = \sqrt{2|t|}.$$

Therefore there is no constant $\kappa > 0$ such that $\|g - g \circ S_t^F\|_2 \le \kappa \, |t|$ for all sufficiently small $|t| > 0$ and hence $g \notin D(B)$.

Let $\theta := g + \mathbb{1}$. Then $\theta \notin D(B)$ and $\theta, \frac{1}{\theta} \in L_\infty(Y^F)$. Set $\psi_t := \frac{\theta}{\theta \circ S_t^F}$ for all $t \in \mathbb{R}$. Then $(\psi_t)_{t \in \mathbb{R}}$ satisfies the cocycle identity (14) and by setting

$$U_t f = \psi_t \cdot (f \circ S_t^F),$$

we obtain a C_0-group $U = (U_t)_{t \in \mathbb{R}}$ on $L_2(Y^F)$. Now

$$\frac{1}{t} \, (U_t \mathbb{1} - \mathbb{1}) = \frac{1}{t} \, (\theta - \theta \circ S_t^F) \cdot \frac{1}{\theta \circ S_t^F} = \frac{1}{t} \, (\theta - V_t \theta) \cdot \frac{1}{\theta \circ S_t^F}$$

and since V is a C_0-group and $\theta \notin D(B)$, we must have $\mathbb{1} \notin D(A)$, where A is the generator of U.

Remark 3.12. By considering the function $\xi = (-1)^{\mathbb{1}_H} = \mathbb{1}_{X \setminus H} - \mathbb{1}_H$, we obtain a measurable function for which $\xi \notin D(B)$ taking values in $\{-1, 1\}$, and if we set $\psi_t := \frac{\xi}{\xi \circ S_t^F}$, then the corresponding group U is weighted Koopman for which $\mathbb{1} \notin D(A)$.

Even if $\mathbb{1} \in D(A)$, then in general $A\mathbb{1} \notin L_\infty(X)$. An example is as follows.

Example 3.13. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the torus with normalized Haar measure. Let

$$E = \{\eta \in L_2(\mathbb{T}) : \int \eta = 0\}.$$

Then E is a closed subspace of $L_2(\mathbb{T})$. We provide E with the norm of $L_2(\mathbb{T})$. For all $\eta \in L_2(\mathbb{T})$ define $\tilde{\eta} \in L_{2,\mathrm{loc}}(\mathbb{R})$ by $\tilde{\eta}(x) = \eta(e^{ix})$.

Fix $\zeta \in E$. For all $t \in \mathbb{R}$ define $\varphi_t \in C(\mathbb{T})$ by

$$\varphi_t(e^{ix}) = \int_x^{x+t} \tilde{\zeta}.$$

Note that φ_t is well defined. Since $\int_\mathbb{T} \zeta = 0$ one deduces that $\|\varphi_t\|_\infty \le 2\pi \|\zeta\|_1$. If $s, t \in \mathbb{R}$ then

$$\tilde{\varphi}_{t+s}(x) = \int_x^{x+t} \tilde{\zeta} + \int_{x+t}^{x+t+s} \tilde{\zeta} = \tilde{\varphi}_t(x) + \tilde{\varphi}_s(x+t)$$

for all $x \in \mathbb{R}$. For all $t \in \mathbb{R}$ define $\psi_t \in C(\mathbb{T})$ by

$$\psi_t = e^{\varphi_t}$$

and define $U_t \colon L_2(\mathbb{T}) \to L_2(\mathbb{T})$ by

$$(U_t f)(z) = \psi_t(z) \, f(e^{it} z).$$

It is easy to verify that $U_t L_\infty(\mathbb{T}) \subset L_\infty(\mathbb{T})$ for all $t \in \mathbb{R}$ and that $U = (U_t)_{t\in\mathbb{R}}$ is a C_0-group. Let A be the generator of U. Clearly $\psi_t = U_t \mathbb{1}$ for all $t \in \mathbb{R}$. Up to now everything also works if $\zeta \in L_1(\mathbb{T})$ with $\int \zeta = 0$, but from now on we use that $\zeta \in L_2(\mathbb{T})$. We shall prove that $\mathbb{1} \in D(A)$ and $A\mathbb{1} = \zeta$.

Let $t \in (0,1)$. Then

$$\begin{aligned} \Big|\frac{1}{t}(U_t\mathbb{1} - \mathbb{1}) - \zeta\Big| &\le \Big|\frac{e^{\varphi_t} - \mathbb{1} - \varphi_t}{t}\Big| + \Big|\frac{1}{t}\,\varphi_t - \zeta\Big| \\ &\le \frac{1}{t}|\varphi_t|^2\, e^{|\varphi_t|} + \Big|\frac{1}{t}\,\varphi_t - \zeta\Big| \\ &\le \frac{1}{t}|\varphi_t|^2\, e^{2\pi\,\|\zeta\|_1} + \Big|\frac{1}{t}\,\varphi_t - \zeta\Big|. \end{aligned} \tag{22}$$

We estimate the terms in (22) separately in $L_2(\mathbb{T})$ in the limit $t \downarrow 0$.

We start with the second term. For all $t \in (0,1)$ define $F_t \colon E \to C(\mathbb{T})$ by

$$(F_t\eta)(e^{ix}) = \frac{1}{t}\int_x^{x+t} \tilde{\eta}.$$

Note that $F_t(\zeta) = \frac{1}{t}\,\varphi_t$. Let $\eta \in E$ and $\tau \in L_2(\mathbb{T})$. Then Fubini and Cauchy–Schwarz give

$$\begin{aligned} |(F_t(\eta), \tau)_{L_2(\mathbb{T})}| &= \frac{1}{t}\Big|\int_0^{2\pi}\int_x^{x+t} \tilde{\eta}(s)\,ds\,\overline{\tilde{\tau}(x)}\,dx\Big| \\ &= \frac{1}{t}\Big|\int_0^{2\pi}\int_0^{t} \tilde{\eta}(x+s)\,ds\,\overline{\tilde{\tau}(x)}\,dx\Big| \\ &= \frac{1}{t}\Big|\int_0^{t}\int_0^{2\pi} \tilde{\eta}(x+s)\,\overline{\tilde{\tau}(x)}\,dx\,ds\Big| \\ &\le \frac{1}{t}\int_0^t 2\pi\,\|\eta\|_{L_2(\mathbb{T})}\,\|\tau\|_{L_2(\mathbb{T})}\,ds \\ &= 2\pi\,\|\eta\|_{L_2(\mathbb{T})}\,\|\tau\|_{L_2(\mathbb{T})}. \end{aligned}$$

So $\|F_t(\eta)\|_{L_2(\mathbb{T})} \le 2\pi\,\|\eta\|_{L_2(\mathbb{T})}$ and the set $\{F_t : t \in (0,1)\}$ is bounded in $\mathcal{L}(E, L_2(\mathbb{T}))$. Clearly $\lim_{t\downarrow 0} F_t(\eta) = \eta$ in $L_2(\mathbb{T})$ for all $\eta \in C(\mathbb{T})$. Since $E \cap C(\mathbb{T})$ is dense in E, it follows that $\lim_{t\downarrow 0} F_t(\eta) = \eta$ in $L_2(\mathbb{T})$ for all $\eta \in E$. In particular for ζ one deduces that

$$\lim_{t\downarrow 0}\Big|\frac{1}{t}\,\varphi_t - \zeta\Big| = 0 \tag{23}$$

in $L_2(\mathbb{T})$. This settles the second term in (22).

Now we consider the first term in (22). We shall show that $\lim_{t\downarrow 0}\frac{1}{t}|\varphi_t|^2 = 0$ in $L_2(\mathbb{T})$. If $t \in (0,1)$, then

$$|\varphi_t(e^{ix})| = \Big|\int_x^{x+t}\tilde{\zeta}\Big| \le \sqrt{2\pi\, t}\,\|\zeta\|_2$$

for all $x \in \mathbb{R}$ by the Cauchy–Schwarz inequality. So $\|\frac{1}{t}|\varphi_t|^2\|_\infty \le 2\pi\,\|\zeta\|_2^2$ for all $t \in (0,1)$. Let $t_1, t_2, \ldots \in (0,1)$ and assume that $\lim_{n\to\infty} t_n = 0$. Then passing to a subsequence if necessary, it follows from (23) that $\lim_{n\to\infty}\frac{1}{t_n}\,\varphi_{t_n}(z) = \zeta(z)$ for a.e. $z \in \mathbb{T}$. Hence

$$\lim_{n\to\infty}\frac{1}{t_n}\,|\varphi_{t_n}(z)|^2 = \lim_{n\to\infty} t_n\,\Big|\frac{1}{t_n}\,\varphi_{t_n}(z)\Big|^2 = 0$$

for a.e. $z \in \mathbb{T}$. Then the bounded convergence theorem of Lebesgue gives $\lim_{n\to\infty} \frac{1}{t_n}|\varphi_{t_n}|^2 = 0$ in $L_2(\mathbb{T})$. Hence $\lim_{t\downarrow 0} \frac{1}{t}|\varphi_t|^2 = 0$ in $L_2(\mathbb{T})$.

Combining the two estimates it follows from (22) that $\mathbb{1} \in D(A)$ and $A\mathbb{1} = \zeta$. Finally, if one chooses $\zeta \in E$ such that $\zeta \notin L_\infty(\mathbb{T})$, then $A\mathbb{1} \notin L_\infty(\mathbb{T})$.

4 Cocycles

In the previous section we started with a group U on $L_2(X)$ and in case U was weighted non-singular as in (13), we defined the representation V given by $V_t f = f \circ T_t$. In that case $U_t = \psi_t \, V_t$. In this section we reverse the order. We start with a representation of the form $V_t f = f \circ T_t$ and wish to construct as general as possible a representation U of the form (13), that is $U_t = \psi_t \, V_t$ for all $t \in \mathbb{R}$.

Throughout this section let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. For all $t \in \mathbb{R}$ let $T_t \colon X \to X$ be a measurable map such that $V = (V_t)_{t\in\mathbb{R}}$ is a C_0-group on $L_2(X)$, where $V_t f := f \circ T_t$ for all $t \in \mathbb{R}$. Let B be the generator of V.

We need a few definitions. A map $\psi \colon \mathbb{R} \to L_\infty(X)$ is said to be a **cocycle (over V)** if

$$\psi_{t+t'} = \psi_t \cdot (\psi_{t'} \circ T_t) \tag{24}$$

for all $t, t' \in \mathbb{R}$, where we write for simplicity $\psi_t = \psi(t)$ for all $t \in \mathbb{R}$. Note that $\psi = 0$ is a cocycle over V. Suppose that ψ is a cocycle. For all $t \in \mathbb{R}$ define $U_t = \psi_t \, V_t \in \mathcal{L}(L_2(X))$. Clearly $\|U_t\|_{2\to 2} \leq \|\psi_t\|_\infty \|V_t\|_{2\to 2}$. If $t, t' \in \mathbb{R}$ then

$$U_{t+t'} f = \Big(\psi_t \cdot (\psi_{t'} \circ T_t) \Big) V_{t+t'} f = U_t(U_{t'} f) = (U_t \circ U_{t'}) f$$

for all $f \in L_2(X)$, so $U = (U_t)_{t\in\mathbb{R}}$ is a one-parameter group on $L_2(X)$, which leaves $L_\infty(X)$ invariant. We call U the **one-parameter group associated with** ψ. Possibly $U_0 = 0$. With a continuity condition this is not the case.

Lemma 4.1. *If* $\lim_{t\to 0} \|\psi_t - \mathbb{1}\|_1 = 0$*, then* $\psi_0 = \mathbb{1}$ *a.e. and* $U_0 = I$*.*

Proof. Let $B \in \mathcal{B}$ and suppose that $\psi_0|_B = 0$ a.e. Then $\psi_t|_B = 0$ a.e. by (24). Since $\lim_{t\to 0} \|\psi_t - \mathbb{1}\|_1 = 0$, one deduces that $\mu(B) = 0$. So $\psi_0 \neq 0$ a.e. In addition, (24) gives $\psi_0 = \psi_{0+0} = \psi_0^2$. Hence $\psi_0 = \mathbb{1}$ a.e. □

The cocycle ψ is called a C_0-**cocycle (over V)** if U is a C_0-group on $L_2(X)$. If $\theta \in L_\infty(X)$ is such that $\theta \neq 0$ a.e., and $\frac{1}{\theta} \in L_\infty(X)$, then it is easy to verify that $t \mapsto \frac{\theta \circ T_t}{\theta}$ is a cocycle. A cocycle ψ is called a **coboundary** if there exists a $\theta \in L_\infty(X)$ such that $\theta \neq 0$ a.e., $\frac{1}{\theta} \in L_\infty(X)$ and

$$\psi_t = \frac{\theta \circ T_t}{\theta}$$

for all $t \in \mathbb{R}$. The function θ is called a **transfer function** of the coboundary. If, in addition, $\theta \in D(B)$ and $B\theta \in L_\infty(X)$, then ψ is called a **coboundary with an L_∞-differentiable transfer function**.

If ψ is a cocycle and $\zeta \in L_2(X)$, then ζ is called the **derivative** of ψ if $\lim_{t\to 0} \frac{1}{t}(\psi_t - \mathbb{1}) = \zeta$ in $L_2(X)$. We say that a cocycle ψ is **differentiable** if there exists an $\zeta \in L_2(X)$ such that ζ is the derivative of ψ.

We start with a characterisation of C_0-cocycles.

Proposition 4.2. *Let $\psi : \mathbb{R} \to L_\infty(X)$ be a cocycle over V. Then the following are equivalent.*

(i) ψ *is a C_0-cocycle.*

(ii) $\lim_{t\to 0} \|\psi_t - \mathbb{1}\|_2 = 0$.

Proof. Let U be the one-parameter group associated with ψ. Since $\psi_t = U_t \mathbb{1}$ for all $t \in \mathbb{R}$, the implication (i)⇒(ii) is trivial. So it remains to prove the converse.

Because $\lim_{t\to 0} \|\psi_t - \mathbb{1}\|_1 = 0$ by (ii), it follows from Lemma 4.1 that $\psi_0 = \mathbb{1}$ a.e. Clearly $\|U_t\|_{2\to 2} \le \|\psi_t\|_\infty \|V_t\|_{2\to 2}$ for all $t \in \mathbb{R}$. If $f \in L_\infty$ then $\|U_t f\|_\infty \le \|\psi_t\|_\infty \|f\|_\infty < \infty$. Hence the operator $\widetilde{U}_t := U_t|_{L_\infty} : L_\infty \to L_\infty$ is bounded. Obviously $(\widetilde{U}_t)_{t\in\mathbb{R}}$ is a one-parameter group on L_∞.

Let $f \in L_\infty$. Then

$$U_t f - f = (\psi_t - \mathbb{1}) V_t f + V_t f - f$$

for all $t \in \mathbb{R}$, so

$$\|U_t f - f\|_2 \le \|\psi_t - \mathbb{1}\|_2 \|V_t f\|_\infty + \|V_t f - f\|_2 = \|f\|_\infty \|\psi_t - \mathbb{1}\|_2 + \|V_t f - f\|_2$$

and therefore

$$\lim_{t\to 0} \|U_t f - f\|_2 = 0 \tag{25}$$

by assumption. Fix $t_0 \in \mathbb{R}$. Then

$$\|U_{t_0+t} f - U_{t_0} f\|_2 \le \|U_{t_0}\|_{2\to 2} \|U_t f - f\|_2$$

for all $t \in \mathbb{R}$. Hence, $\lim_{t\to t_0} U_t f = U_{t_0} f$ in L_2. So $t \mapsto U_t f$ is continuous from $\mathbb{R}$ into L_2. Hence the map $t \mapsto |(U_t f, g)|$ from $\mathbb{R}$ into $\mathbb{R}$ is continuous for all $g \in L_2$.

Let $f \in L_\infty$ and $t \in \mathbb{R}$. Then

$$\begin{aligned}
\|\widetilde{U}_t f\|_\infty &= \sup\{|\langle \widetilde{U}_t f, g\rangle| : g \in L_1 \text{ and } \|g\|_1 \le 1\} \\
&= \sup\{|\langle \widetilde{U}_t f, g\rangle| : g \in L_2 \text{ and } \|g\|_1 \le 1\} \\
&= \sup\{|(U_t f, g)| : g \in L_2 \text{ and } \|g\|_1 \le 1\}.
\end{aligned}$$

Since the map $t \mapsto |(U_t f, g)|$ is continuous for each $g \in L_2$, it follows that the map $t \mapsto \|\widetilde{U}_t f\|_\infty$ is lower semicontinuous, hence it is measurable on $\mathbb{R}$. By the proof of Theorem 2.5, we deduce that the set $\{\widetilde{U}_t : t \in [2,3]\}$ is bounded in $\mathcal{L}(L_\infty)$. Since $(\widetilde{U}_t)_{t\in\mathbb{R}}$ is a one-parameter group on L_∞, also $\{\widetilde{U}_t : t \in [-1,1]\}$ is bounded in $\mathcal{L}(L_\infty)$. Let $c := \sup\{\|\widetilde{U}_t\|_{\infty\to\infty} : t \in [-1,1]\}$. Then $\|\psi_t\|_\infty = \|\widetilde{U}_t \mathbb{1}\|_\infty \le c$ for all $t \in [-1,1]$. Hence $\sup\{\|U_t\| : t \in [-1,1]\} < \infty$. Since L_∞ is dense in L_2, it follows from (25) that U is a C_0-group. □

Corollary 4.3. *Every differentiable cocycle is a C_0-cocycle. Every coboundary is a C_0-cocycle.*

Proposition 4.4. *Let $\zeta \in L_2(X)$. Then there exists at most one function $\psi : \mathbb{R} \to L_\infty(X)$ such that ψ is a cocycle over V and the cocycle ψ is differentiable with derivative ζ.*

Proof. Let $\psi, \widetilde{\psi} \colon \mathbb{R} \to L_\infty$ be cocycles over V which are differentiable with derivative ζ. Then $\psi_0 = \mathbb{1} = \widetilde{\psi}_0$ a.e. Let U be the group associated with ψ. Then U is a C_0-group by Proposition 4.2. Moreover, $\sup\{\|\psi_t\|_\infty : t \in [-1,1]\} < \infty$ by (3) in Theorem 2.5, or the proof of Proposition 4.2. In addition, $\psi_t \neq 0$ a.e. and $\frac{1}{\psi_t} = \psi_{-t} \circ T_t$ for all $t \in \mathbb{R}$ by (16). Define $\eta \colon \mathbb{R} \to L_\infty$ by $\eta(t) = \eta_t := \frac{\widetilde{\psi}_t}{\psi_t}$. Then $\eta_{t+t'} = \eta_t \cdot (\eta_{t'} \circ T_t)$ for all $t, t' \in \mathbb{R}$, so η is a cocycle over V. Moreover,

$$\frac{1}{t}(\eta_t - \mathbb{1}) = \frac{1}{t}\frac{\widetilde{\psi}_t - \psi_t}{\psi_t} = \frac{1}{\psi_t}\left(\frac{\widetilde{\psi}_t - \mathbb{1}}{t} - \frac{\psi_t - \mathbb{1}}{t}\right) = (\psi_{-t} \circ T_t)\left(\frac{\widetilde{\psi}_t - \mathbb{1}}{t} - \frac{\psi_t - \mathbb{1}}{t}\right)$$

for all $t \in \mathbb{R} \setminus \{0\}$. Since $\sup\{\|\psi_{-t}\|_\infty : t \in [-1,1]\} < \infty$, one deduces that the cocycle η is differentiable and $\lim_{t \to 0} \frac{1}{t}(\eta_t - \mathbb{1}) = 0$ in L_2.

Let $t \in \mathbb{R}$ and $h \in \mathbb{R} \setminus \{0\}$. Then

$$\begin{aligned}\frac{1}{h}(\eta_{t+h} - \eta_t) &= \frac{1}{h}(\eta_t \cdot (\eta_h \circ T_t) - \eta_t) \\ &= \eta_t \cdot \left(\frac{\eta_h - \mathbb{1}}{h} \circ T_t\right) = \eta_t \cdot V_t\left(\frac{\eta_h - \mathbb{1}}{h}\right).\end{aligned}$$

It follows that $\lim_{h \to 0} \frac{1}{h}(\eta_{t+h} - \eta_t) = 0$ in L_2. Therefore η is differentiable from $\mathbb{R}$ into L_2 and $\eta'(t) = 0$ for all $t \in \mathbb{R}$. So η is constant and $\eta(t) = \eta(0) = \mathbb{1}$ for all $t \in \mathbb{R}$. Hence $\widetilde{\psi}_t = \psi_t$ for all $t \in \mathbb{R}$, which completes the proof. □

Lemma 4.5. *Let $\zeta \in L_\infty(X)$. Define $\psi \colon \mathbb{R} \to L_\infty(X)$ by $\psi_t := e^{\int_0^t \zeta \circ T_s \, ds}$. Then ψ is a differentiable cocycle with derivative ζ.*

Proof. We first show that ψ is a cocycle over V. Let $t, t' \in \mathbb{R}$. Then

$$\begin{aligned}\int_0^{t+t'} \zeta \circ T_s \, ds &= \int_0^t \zeta \circ T_s \, ds + \int_t^{t+t'} \zeta \circ T_s \, ds \\ &= \int_0^t \zeta \circ T_s \, ds + \int_0^{t'} \zeta \circ T_{s+t} \, ds \\ &= \int_0^t \zeta \circ T_s \, ds + \Big(\int_0^{t'} \zeta \circ T_s \, ds\Big) \circ T_t.\end{aligned}$$

Hence ψ is a cocycle over V.

Next we show that ψ is differentiable. Recall that $|e^z - 1 - z| \leq |z|^2 e^{|z|}$ for all $z \in \mathbb{C}$. Let $t \in [-1,1] \setminus \{0\}$. Then

$$\begin{aligned}\left|\frac{\psi_t - \mathbb{1}}{t} - \zeta\right| &\leq \left|\frac{\psi_t - \mathbb{1} - \int_0^t \zeta \circ T_s \, ds}{t}\right| + \left|\frac{1}{t}\int_0^t \zeta \circ T_s \, ds - \zeta\right| \\ &\leq \frac{1}{|t|}\left(\int_0^t \|\zeta\|_\infty\right)^2 e^{|t| \, \|\zeta\|_\infty} + \left|\frac{1}{t}\int_0^t |\zeta \circ T_s - \zeta| \, ds\right| \\ &\leq |t| \, \|\zeta\|_\infty^2 \, e^{\|\zeta\|_\infty} + \left|\frac{1}{t}\int_0^t |V_s\zeta - \zeta| \, ds\right|.\end{aligned}$$

Therefore

$$\Big\| \frac{1}{t}(\psi_t - \mathbb{1}) - \zeta \Big\|_2 \le |t| \, \|\zeta\|_\infty^2 \, e^{\|\zeta\|_\infty} + \Big| \frac{1}{t} \int_0^t \|V_s\zeta - \zeta\|_2 \, ds \Big|.$$

Since $s \mapsto \|V_s\zeta - \zeta\|_2$ is continuous, one deduces that the cocycle ψ is differentiable with derivative ζ. □

Lemma 4.6. *Let $\zeta \in L_\infty(X)$ and let $\psi\colon \mathbb{R} \to L_\infty(X)$ be a cocycle. Then ψ is differentiable with derivative ζ if and only if $\psi_t = e^{\int_0^t \zeta \circ T_s \, ds}$ for all $t \in \mathbb{R}$.*

Proof. This follows immediately from Proposition 4.4 and Lemma 4.5. □

Next we turn to coboundaries.

Lemma 4.7. *Let ψ be a coboundary with transfer function θ.*

(a) *The coboundary ψ is differentiable if and only if $\theta \in D(B)$. Moreover, if ψ is differentiable, then the derivative is $\frac{B\theta}{\theta}$.*

(b) *If $\theta \in D(B)$ and $B\theta \in L_\infty(X)$, then*

$$\psi_t = e^{\int_0^t \frac{B\theta}{\theta} \circ T_s \, ds}$$

for all $t \in \mathbb{R}$.

Proof. If $t \in \mathbb{R} \setminus \{0\}$, then

$$\frac{\psi_t - \mathbb{1}}{t} = \frac{1}{\theta} \cdot \frac{\theta \circ T_t - \theta}{t} = \frac{1}{\theta} \cdot \frac{V_t\theta - \theta}{t}.$$

Hence ψ is differentiable if and only if $\theta \in D(B)$. Moreover, if ψ is differentiable, then the derivative is $\frac{1}{\theta} B\theta$. This proves Statement (a).

If $\theta \in D(B)$ and $B\theta \in L_\infty$, then $\zeta = \frac{1}{\theta} B\theta \in L_\infty$. Now Statement (b) follows from Lemma 4.6. □

Note that Example 3.11 yields a C_0-cocycle (in fact a coboundary) which is not differentiable. It also gives an example of a coboundary which is not a coboundary with an L_∞-differentiable transfer function.

Lemma 4.8. *Let ψ be a differentiable cocycle with derivative $\zeta \in L_\infty(X)$. Then the following conditions are equivalent.*

(i) *ψ is a coboundary.*

(ii) *ψ is a coboundary with an L_∞-differentiable transfer function.*

(iii) *There exists a $\theta \in D(B) \cap L_\infty(X)$ such that $\theta \neq 0$-a.e., $\frac{1}{\theta} \in L_\infty(X)$, $B\theta \in L_\infty(X)$ and $\zeta = \frac{B\theta}{\theta}$.*

Proof. The implication (ii)⇒(i) is trivial.

'(i)⇒(ii)' and '(i)⇒(iii)'. Let θ be a transfer function of ψ. By definition $\theta \in L_\infty$, $\theta \neq 0$-a.e. and $\frac{1}{\theta} \in L_\infty$. Then Lemma 4.7(a) gives $\theta \in D(B)$ and $\zeta = \frac{B\theta}{\theta}$. So $B\theta = \zeta\,\theta \in L_\infty$.

'(iii)⇒(i)'. Define $\widetilde{\psi}\colon \mathbb{R} \to L_\infty$ by $\widetilde{\psi}_t = \frac{\theta \circ T_t}{\theta}$. Then Lemma 4.7(a) implies that the coboundary $\widetilde{\psi}$ is differentiable with derivative ζ. By the uniqueness of Proposition 4.4 one deduces that $\psi = \widetilde{\psi}$. So ψ is a coboundary. □

We now give an example of a differentiable cocycle which is not a coboundary.

Example 4.9. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the torus with normalized Haar measure. For all $t \in \mathbb{R}$ define $T_t \colon \mathbb{T} \to \mathbb{T}$ by $T_t z = e^{it} z$ and define $V_t \colon L_2(\mathbb{T}) \to L_2(\mathbb{T})$ by $V_t f = f \circ T_t$. Then $V = (V_t)_{t \in \mathbb{R}}$ is a C_0-group. Fix $\zeta \in L_\infty(\mathbb{T})$ with $\int \zeta \notin i\mathbb{Z}$. Define $\psi \colon \mathbb{R} \to L_\infty(\mathbb{T})$ by

$$\psi_t = e^{\int_0^t \zeta \circ T_s \, ds}.$$

Then ψ is a differentiable cocycle by Lemma 4.5. Now suppose that ψ is a coboundary. Let θ be a transfer function. Then $\psi_{2\pi} = \frac{\theta \circ T_{2\pi}}{\theta} = \mathbb{1}$. Hence $\int_0^{2\pi} \zeta \circ T_s \, ds \in 2\pi i \mathbb{Z}$ a.e. But $\int_0^{2\pi} \zeta \circ T_s \, ds = 2\pi \int_{\mathbb{T}} \zeta$ a.e. So $\int_{\mathbb{T}} \zeta \in i\mathbb{Z}$. This is a contradiction.

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