

ODOMETER ACTIONS OF THE HEISENBERG GROUP

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ABSTRACT. Let $H_3(\mathbb{R})$ denote the 3-dimensional real Heisenberg group. Given a family of lattices $\Gamma_1 \supset \Gamma_2 \supset \dots$ in it, let T stand for the associated uniquely ergodic $H_3(\mathbb{R})$ -odometer, i.e. the inverse limit of the $H_3(\mathbb{R})$ -actions by rotations on the homogeneous spaces $H_3(\mathbb{R})/\Gamma_j$, $j \in \mathbb{N}$. The decomposition of the underlying Koopman unitary representation of $H_3(\mathbb{R})$ into a countable direct sum of irreducible components is explicitly described. The ergodic 2-fold self-joinings of T are found. It is shown that in general, the $H_3(\mathbb{R})$ -odometers are neither isospectral nor spectrally determined.

0. INTRODUCTION

Let $T = (T_g)_{g \in G}$ be an ergodic measure preserving action of a locally compact second countable group G on a standard probability space (X, \mathfrak{B}, μ) . Denote by $U_T = (U_T(g))_{g \in G}$ the associated Koopman unitary representation of G in $L^2(X, \mu)$:

$$U_T(g)f := f \circ T_g^{-1}, \quad f \in L^2(X, \mu).$$

Suppose first that G is Abelian. If U_T is a direct countable sum of 1-dimensional unitary sub-representations (generated by the U_T -eigenfunctions) then T is said to *have a pure point spectrum*. In 1932, J. von Neumann [Ne] developed a theory of such actions in the case $G = \mathbb{R}$. We highlight three main aspects of this theory:

- (A1) *isospectrality*: two ergodic flows with pure point spectrum are isomorphic if and only if the associated Koopman unitary representations are unitarily equivalent,
- (A2) *classification by simple algebraic invariants*: the ergodic flows with pure point spectrum considered up to isomorphism are in one-to-one correspondence with the countable subgroups in $\widehat{\mathbb{R}}$ which is the dual of \mathbb{R} ,
- (A3) *structure*: if an ergodic flow has pure point spectrum then it is isomorphic to a flow by rotations on a compact metric Abelian group endowed with the Haar measure.

We also draw attention to a special subclass—apparently the simplest one—of flows with pure point *rational* spectrum. They are precisely those flows which admit a structure of inverse limit of transitive flows.

The results analogous to (A1)–(A3) hold for the general Abelian G with similar proofs. The non-Abelian case—considered in this paper—is more intricate. G. Mackey in [Ma3] extended the concept of pure point spectrum to actions of

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non-Abelian groups in the following way: T has a pure point spectrum if U_T is a direct sum of countably many finite dimensional unitary representations of G . He established a structure for these actions: an ergodic action T has pure point spectrum if and only if it is isomorphic to a G -action by rotations on a homogeneous space of a compact group. However, in general, the G -actions with pure point spectrum are not isospectral even in the case of finite G . G. Mackey refers to [To] for a counterexample. (See also a discussion in [Le–We, Section 6].) Hence no classification for them is obtained.

In the present paper we consider the case where G is the 3-dimensional real Heisenberg group $H_3(\mathbb{R})$ which is apparently the ‘simplest’ non-Abelian nilpotent connected Lie group. Moreover, we single out a special class of actions of $H_3(\mathbb{R})$ which we call *odometers*. They are inverse limits of transitive $H_3(\mathbb{R})$ -actions on homogeneous spaces by lattices in $H_3(\mathbb{R})$. In this connection, we note that for discrete finitely generated groups G , the G -odometers were considered by M. Cortez and S. Petit in [CoPe] in the context of topological dynamics. We define G -odometers for arbitrary locally compact second countable groups G and study them as measure preserving dynamical systems. Thus, by construction, the $H_3(\mathbb{R})$ -odometers are counterparts of the \mathbb{Z} -actions with pure point *rational* spectrum. However, by taking into account definitions, the Heisenberg odometers are neither actions with pure point spectrum nor weakly mixing (the Cartesian square of a Heisenberg odometer is not ergodic).

Our purpose here is to investigate whether von Neumann’s theory of flows with pure point spectrum extends (or partially extends) to the Heisenberg odometers.

We compute explicitly the spectrum of a Heisenberg odometer T , i.e. we describe the decomposition of U_T into the direct countable sum of its irreducible components and we calculate their multiplicities in terms of the underlying sequence of lattices in $H_3(\mathbb{R})$ (Theorem 5.2). These components consist of two families: 1-dimensional unitary representations which occur in U_T with multiplicity 1 and infinitely dimensional unitary representations. In the non-degenerate case (i.e. where the underlying \mathbb{R}^2 -odometer is not transitive) every infinitely dimensional irreducible unitary representation that occurs in U_T has there infinite multiplicity. To prove this result we use Kirillov’s orbit method [Ki] and Howe-Richardson spectral multiplicity formula [Ho], [Ri]. The 1-dimensional family is parameterized by a so-called *off-rational*¹ subgroup in \mathbb{R}^2 and the infinite dimensional family is parameterized by an off-rational subgroup in \mathbb{R} . Besides, there is a certain relation between these two subgroups. We prove that conversely, given two off-rational subgroups in \mathbb{R}^2 and \mathbb{R} connected with this relation, there is a Heisenberg odometer whose Koopman representation is determined by these subgroups (Proposition 5.5). We show by example that the $H_3(\mathbb{R})$ -odometers are not isospectral (Example 5.11).

Furthermore, we explicitly describe all ergodic 2-fold self-joinings of the Heisenberg odometers in Theorem 6.5. We provide an example of an ergodic 2-fold self-joining of a transitive Heisenberg odometer which is neither transitive nor “odometric” at all (Example 6.4). We note that this is in marked contrast to the properties of locally compact transitive systems with pure point spectrum: each ergodic 2-fold self-joining of such a system is transitive itself (and hence has a pure point spectrum).

Since our definition of the Heisenberg odometers is “structural”, it seems natural

¹See Definition 5.3.

to investigate whether there is an equivalent spectral definition (as in the Abelian case). Thus we come to the following problem.

- (A4) *Spectral determinacy*: if an $H_3(\mathbb{R})$ -action is spectrally equivalent to a Heisenberg odometer, is it isomorphic to a Heisenberg odometer?

Answering (A3), we prove the spectral determinacy of the subclass of transitive Heisenberg odometers and, more generally, the subclass of Heisenberg odometers with transitive underlying \mathbb{R}^2 -odometers (Theorem 7.1). However, the entire class of Heisenberg odometers is not spectrally determined. Nevertheless, we show that if the maximal spectral type of an $H_3(\mathbb{R})$ -action R coincides with the maximal spectral type of an $H_3(\mathbb{R})$ -odometer then R has the structure of a compact skew product extension of an $H_3(\mathbb{R})$ -action with pure point spectrum. We obtain a criterion for when R is conjugate to a $H_3(\mathbb{R})$ -odometer in terms of the corresponding “extending” cocycle (Theorem 7.7). Using this criterion plus the orbit theory of amenable group actions we construct an example of a probability preserving $H_3(\mathbb{R})$ -system which generates the same (up to the unitary equivalence) Koopman unitary representation as an $H_3(\mathbb{R})$ -odometer but which is not isomorphic to any Heisenberg odometer (Example 7.11).

We also consider $H_3(\mathbb{Z})$ -odometers and compare their properties with the properties of $H_3(\mathbb{R})$ -odometers. In this connection it is interesting to note that $H_3(\mathbb{Z})$ is not of type I and hence neither the unitary dual $\widehat{H_3(\mathbb{Z})}$ is a standard Borel space nor the spectral theorem (i.e. a unique decomposition of an arbitrary unitary representation of $H_3(\mathbb{Z})$ into a direct integral of irreducible representations) holds. However for the class of $H_3(\mathbb{Z})$ -odometers the situation is different. Every such odometer has a pure point spectrum in the sense of [Ma3] and the decomposition of the corresponding Koopman unitary representations of $H_3(\mathbb{Z})$ into irreducible components is well defined. Such decompositions for a subclass of *normal* free $H_3(\mathbb{Z})$ -odometers were explicitly computed in [Li-Ug]. We note that the normal $H_3(\mathbb{Z})$ -odometers are isospectral.

The outline of the paper is as follows. In Section 1 we discuss basic properties of $H_3(\mathbb{R})$, compute the group of its automorphisms and classify the lattices in $H_3(\mathbb{R})$. The odometer actions of locally compact groups are introduced in Section 2. Some general structural properties and the freeness of odometers are studied there. In Section 3 a convenient criterion for the freeness of Heisenberg odometers is found. In Section 4, for a transitive $H_3(\mathbb{R})$ -odometer, the decomposition of the Koopman unitary representation U_T into irreducible unitary representations of $H_3(\mathbb{R})$ is described. A similar decomposition for a general $H_3(\mathbb{R})$ -odometer is obtained in Section 5. The corresponding measure of maximal spectral type sits on the union of two countable subgroups (in \mathbb{R}^2 and \mathbb{R}). All admissible pairs of such subgroups are described in Section 5. The fact that the $H_3(\mathbb{R})$ -odometers are not isospectral is also proved in Section 5. Section 6 is devoted to a description of the ergodic 2-fold self-joinings of $H_3(\mathbb{R})$ -odometers. In Section 7 we investigate spectral determinacy of the $H_3(\mathbb{R})$ -odometers. In Section 8 we consider the $H_3(\mathbb{Z})$ -odometers. Concluding remarks and open problems are discussed in the final Section 9.

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1. PRELIMINARIES ON $H_3(\mathbb{R})$

We recall that $H_3(\mathbb{R})$ consists of 3×3 upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b, c are arbitrary reals. The Heisenberg group endowed with the natural topology is a connected, simply-connected nilpotent Lie group. We now let

$$a(t) := \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad c(t) := \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\{a(t) \mid t \in \mathbb{R}\}$, $\{b(t) \mid t \in \mathbb{R}\}$ and $\{c(t) \mid t \in \mathbb{R}\}$ are three closed one-parameter subgroups in $H_3(\mathbb{R})$. The last is the center of $H_3(\mathbb{R})$. Every element g of $H_3(\mathbb{R})$ can be written uniquely as the product $g = c(t_3)b(t_2)a(t_1)$ for some $t_1, t_2, t_3 \in \mathbb{R}$. We also note that $[a(t_1), b(t_2)] := a(t_1)b(t_2)a(t_1)^{-1}b(t_2)^{-1} = c(t_1t_2)$ and the commutator of $H_3(\mathbb{R})$ equals the center of $H_3(\mathbb{R})$.

The subgroups $H_{2,a} := \{a(t_1)c(t_3) \mid t_1, t_3 \in \mathbb{R}\}$ and $H_{2,b} := \{b(t_2)c(t_3) \mid t_2, t_3 \in \mathbb{R}\}$ are both Abelian, normal and closed in $H_3(\mathbb{R})$. The corresponding group extensions

$$\begin{aligned} 0 &\rightarrow H_{2,a} \rightarrow H_3(\mathbb{R}) \rightarrow H_3(\mathbb{R})/H_{2,a} \rightarrow 0 \quad \text{and} \\ 0 &\rightarrow H_{2,b} \rightarrow H_3(\mathbb{R}) \rightarrow H_3(\mathbb{R})/H_{2,b} \rightarrow 0 \end{aligned}$$

both split. This implies that $H_3(\mathbb{R})$ is isomorphic to the semidirect product $\mathbb{R}^2 \rtimes_B \mathbb{R}$, where the homomorphism $B : \mathbb{R} \rightarrow \text{GL}_2(\mathbb{R})$ is given by $B(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}$.

The subgroups $H_{2,a}$ and $H_{2,b}$ are automorphic in $H_3(\mathbb{R})$, i.e. there is an isomorphism θ of $H_3(\mathbb{R})$ with $\theta(H_{2,a}) = H_{2,b}$. We define θ by setting $\theta(a(t)) := b(t)$, $\phi(b(t)) := a(t)$ and $\theta(c(t)) := c(-t)$ for all $t \in \mathbb{R}$. To put it another way,

$$\theta \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b & ab - c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}.$$

We call θ the *flip* in $H_3(\mathbb{R})$. We note that $\theta^2 = \text{id}$.

The set of unitarily equivalent classes of irreducible (weakly continuous) representations of $H_3(\mathbb{R})$ is called the *unitary dual* of $H_3(\mathbb{R})$. It is denoted by $\widehat{H_3(\mathbb{R})}$. The irreducible unitary representations of $H_3(\mathbb{R})$ are well known. They consist (up to unitary equivalence) of a family of 1-dimensional representations $\pi_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{R}$, and a family of infinite dimensional representations π_γ , $\gamma \in \mathbb{R} \setminus \{0\}$, as follows [Ki]:

$$\begin{aligned} \pi_{\alpha,\beta}(c(t_3)b(t_2)a(t_1)) &:= e^{2\pi i(\alpha t_1 + \beta t_2)} \quad \text{and} \\ (\pi_\gamma(c(t_3)b(t_2)a(t_1))f)(x) &:= e^{2\pi i\gamma(t_3 + t_2x)} f(x + t_1), \quad f \in L^2(\mathbb{R}, \lambda_{\mathbb{R}}). \end{aligned}$$

Thus we can identify $\widehat{H_3(\mathbb{R})}$ with the disjoint union $\mathbb{R}^2 \sqcup \mathbb{R}^*$. We recall that there is a natural Borel structure on the unitary dual of each locally compact second

countable group [Ma2]. In the case of the Heisenberg group this Borel σ -algebra coincides with the standard σ -algebra of Borel subsets in $\mathbb{R}^2 \sqcup \mathbb{R}^*$.

Given an arbitrary unitary representation $U = (U(g))_{g \in H_3(\mathbb{R})}$ of $H_3(\mathbb{R})$ in a separable Hilbert space \mathcal{H} , there are a measure σ_U on $\widehat{H_3(\mathbb{R})}$ (i.e. two measures $\sigma_U^{1,2}$ on \mathbb{R}^2 and σ_U^3 on \mathbb{R}^*) and a map $l_U : \widehat{H_3(\mathbb{R})} \rightarrow \mathbb{N} \cup \{\infty\}$ (i.e. two maps $l_U^{1,2} : \mathbb{R}^2 \ni (x, y) \mapsto l_U^{1,2}(x, y) \in \mathbb{N} \cup \{\infty\}$ and $l_U^3 : \mathbb{R}^* \ni z \mapsto l_U^3(z) \in \mathbb{N} \cup \{\infty\}$) such that the following decompositions hold (up to the unitary equivalence):

$$\begin{aligned} \mathcal{H} &= \int_{\mathbb{R}^2}^{\oplus} l_U^{1,2}(\alpha, \beta) \bigoplus_{j=1} \mathbb{C} d\sigma_U^{1,2}(\alpha, \beta) \oplus \int_{\mathbb{R}^*}^{\oplus} l_U^3(\gamma) \bigoplus_{j=1} L^2(\mathbb{R}, \lambda_{\mathbb{R}}) d\sigma_U^3(\gamma) \quad \text{and} \\ U(g) &= \int_{\mathbb{R}^2}^{\oplus} l_U^{1,2}(\alpha, \beta) \bigoplus_{j=1} \pi_{\alpha, \beta}(g) d\sigma_U^{1,2}(\alpha, \beta) \oplus \int_{\mathbb{R}^*}^{\oplus} l_U^3(\gamma) \bigoplus_{j=1} \pi_{\gamma}(g) d\sigma_U^3(\gamma). \end{aligned}$$

The equivalence class of σ_U is called the *maximal spectral type* of U . The map l_U is called the *multiplicity function* of U . The essential range of l_U is called the *set of spectral multiplicities* of U . The maximal spectral type and the multiplicity function of U (σ_U -mod 0) are both determined uniquely by the unitary equivalence class of U .

Below, in this section we:

- (a) describe explicitly the group of all continuous automorphisms of the Heisenberg group and
- (b) classify the lattices in $H_3(\mathbb{R})$ up to automorphism of $H_3(\mathbb{R})$.

Although these results are not new, we think it is easier to reprove them here than to find them in the literature and “adjust” to our notation.

From now on we denote by p the natural projection

$$H_3(\mathbb{R}) \ni g \mapsto p(g) := (t_1, t_2) \in \mathbb{R}^2$$

whenever $g = c(t_3)b(t_2)a(t_1)$.

Let $\tilde{\theta}$ be an automorphism of $H_3(\mathbb{R})$. Of course, $\tilde{\theta}$ preserves the center of $H_3(\mathbb{R})$. Hence there exists $\tau \in \mathbb{R}^*$ such that

$$(1-1) \quad \tilde{\theta}(c(t)) = c(\tau t), \quad t \in \mathbb{R}.$$

Since the center of $H_3(\mathbb{R})$ is the kernel of p , $\tilde{\theta}$ passes through p , i.e. there is a matrix $A = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} \in GL_2(\mathbb{R})$ such that $p \circ \tilde{\theta} = A \circ p$. Therefore we can write $\tilde{\theta}(a(t))$ as

$$\tilde{\theta}(a(t)) = c(f(t))b(\xi_2 t)a(\xi_1 t), \quad t \in \mathbb{R},$$

for some continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$. Since $\tilde{\theta}(a(t_1 + t_2)) = \tilde{\theta}(a(t_1))\tilde{\theta}(a(t_2))$, it follows that

$$f(t_1 + t_2) = f(t_1) + f(t_2) + \xi_1 \xi_2 t_1 t_2$$

for all $t_1, t_2 \in \mathbb{R}$. Therefore $f(t) = \frac{\xi_1 \xi_2}{2} t^2 + \xi t$ for some real $\xi \in \mathbb{R}$. Thus,

$$(1-2) \quad \tilde{\theta}(a(t)) = c\left(\frac{\xi_1 \xi_2}{2} t^2 + \xi t\right) b(\xi_2 t) a(\xi_1 t), \quad t \in \mathbb{R}.$$

In a similar way,

$$(1-3) \quad \tilde{\theta}(b(t)) = c\left(\frac{\eta_1\eta_2}{2}t^2 + \eta t\right)b(\eta_2 t)a(\eta_1 t), \quad t \in \mathbb{R},$$

for some $\eta \in \mathbb{R}$. Since $\tilde{\theta}$ is a homomorphism of $H_3(\mathbb{R})$, we have

$$\tilde{\theta}(a(t_1))\tilde{\theta}(b(t_2)) = \tilde{\theta}(a(t_1)b(t_2)) = \tilde{\theta}(b(t_2)a(t_1)c(t_1t_2)) = \tilde{\theta}(b(t_2))\tilde{\theta}(a(t_1))\tilde{\theta}(c(t_1t_2))$$

for all $t_1, t_2 \in \mathbb{R}$. Applying (1-2) and (1-3) and then (1-1), we obtain

$$c(\xi_1\eta_2t_1t_2) = c(\eta_1\xi_2t_1t_2 + \tau t_1t_2),$$

which yields $\tau = \det A$. Thus, the three parameters A, ξ, η determine $\tilde{\theta}$ completely. Conversely, given $A \in GL_2(\mathbb{R})$ and $\xi, \eta \in \mathbb{R}$, the formulas (1-1)–(1-3) with $\tau = \det A$ determine completely an automorphism of $H_3(\mathbb{R})$. We will denote it by $\theta_{A,\xi,\eta}$.

Thus, we have proved the following proposition.

Proposition 1.1. $\text{Aut}(H_3(\mathbb{R})) = \{\theta_{A,\xi,\eta} \mid A \in GL_2(\mathbb{R}), \xi, \eta \in \mathbb{R}\}$.

We also note that $N := \{\theta_{I,\xi,\eta} \mid \xi, \eta \in \mathbb{R}\}$ is a normal subgroup in $\text{Aut}(H_3(\mathbb{R}))$. It is isomorphic naturally to \mathbb{R}^2 . In fact, N is the subgroup of inner automorphisms of $H_3(\mathbb{R})$. We obtain a short exact sequence

$$(1-4) \quad \{1\} \leftarrow GL_2(\mathbb{R}) \leftarrow \text{Aut}(H_3(\mathbb{R})) \leftarrow N \leftarrow \{1\}.$$

Moreover, a direct calculation shows that the homomorphism

$$GL_2(\mathbb{R}) \ni A \mapsto \theta_{A,0,0} \in \text{Aut}(H_3(\mathbb{R}))$$

is a cross-section of the natural projection in (1-4).

Corollary 1.2. $\text{Aut}(H_3(\mathbb{R}))$ is isomorphic to the semidirect product $\mathbb{R}^2 \rtimes GL_2(\mathbb{R})$, where the corresponding action of $GL_2(\mathbb{R})$ on \mathbb{R}^2 is given by $Av := \det A \cdot (A^*)^{-1}v$.

We now describe the structure of lattices in the Heisenberg group and classify them up to group automorphism. Recall that a lattice is a discrete subgroup of finite covolume. In the case of a simply connected nilpotent Lie group, every lattice is cocompact [Ra]. Fix a lattice Γ in $H_3(\mathbb{R})$. There is a real $\xi_\Gamma > 0^2$ such that

$$\Gamma \cap \{c(t) \mid t \in \mathbb{R}\} = \{c(m\xi_\Gamma) \mid m \in \mathbb{Z}\}.$$

The central extension

$$\{0\} \leftarrow \mathbb{R}^2 \xleftarrow{P} H_3(\mathbb{R}) \xleftarrow{c} \mathbb{R} \leftarrow \{0\}$$

²If $\xi_\Gamma = 0$ then the intersection of Γ with the center of $H_3(\mathbb{R})$ is trivial. This yields that Γ is Abelian and hence there is a line (i.e. a 1-dimensional subspace) \mathcal{L} in \mathbb{R}^2 such that Γ is contained in a normal closed subgroup $H := \{c(t)b(t_2)a(t_1) \mid (t_1, t_2) \in \mathcal{L}, t \in \mathbb{R}\}$ of $H_3(\mathbb{R})$. Therefore the quotient group $H_3(\mathbb{R})/H$ being a quotient space of $H_3(\mathbb{R})/\Gamma$ is of finite Haar measure, i.e. compact. However, it is straightforward to verify that $H_3(\mathbb{R})/H$ is isomorphic to \mathbb{R} , a contradiction.

induces a short exact sequence

$$\{0\} \longleftarrow p(\Gamma) \xleftarrow{p} \Gamma \xleftarrow{c} \xi_\Gamma \mathbb{Z} \longleftarrow \{0\}.$$

We note that $p(\Gamma)$ is a lattice in \mathbb{R}^2 .³ Therefore there is a matrix $A \in GL_2(\mathbb{R})$ such that $p(\theta_{A,0,0}(\Gamma)) = Ap(\Gamma) = \mathbb{Z}^2$. The commutator subgroup $[\Gamma, \Gamma]$ is of a finite index $k_\Gamma > 0$ in $p^{-1}(0) \cap \Gamma$.

It is easy to see that $k_{\theta(\Gamma)} = k_\Gamma$ for each $\theta \in \text{Aut}(H_3(\mathbb{R}))$. Let ζ and η be the smallest non-negative reals such that $c(\zeta)b(0)a(1) \in \theta_{A,0,0}(\Gamma)$ and $c(\eta)b(1)a(0) \in \theta_{A,0,0}(\Gamma)$. Then $\theta_{I,-\zeta,-\eta}(c(\zeta)a(1)) = a(1)$ and $\theta_{I,-\zeta,-\eta}(c(\eta)a(1)) = b(1)$. We let $\tilde{\Gamma} := \theta_{I,-\zeta,-\eta}(\theta_{A,0,0}(\Gamma))$. Then $a(1), b(1) \in \tilde{\Gamma}$. Hence $[a(1), b(1)] = c(1) \in \tilde{\Gamma}$. On the other hand, $[a(1), b(1)] = c(k_{\tilde{\Gamma}}\xi_{\tilde{\Gamma}}) = c(k_\Gamma\xi_\Gamma)$. It follows that $\xi_{\tilde{\Gamma}} = 1/k_\Gamma$. The elements $a(1), b(1)$ and $c(k_\Gamma^{-1})$ generate the lattice $\tilde{\Gamma}$. Thus we have proved the following proposition.

Proposition 1.3. *Given a lattice Γ in $H_3(\mathbb{R})$, there is an automorphism θ of $H_3(\mathbb{R})$ such that*

$$\theta(\Gamma) = \left\{ \left(\begin{array}{ccc} 1 & l & \frac{n}{k_\Gamma} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{array} \right) \mid l, m, n \in \mathbb{Z} \right\}.$$

Hence two lattices Γ_1 and Γ_2 in $H_3(\mathbb{R})$ are automorphic if and only if $k_{\Gamma_1} = k_{\Gamma_2}$. Two lattices Γ_1 and Γ_2 in $H_3(\mathbb{R})$ are conjugate if and only if $k_{\Gamma_1} = k_{\Gamma_2}$ and $p(\Gamma_1) = p(\Gamma_2)$.

Corollary 1.4. *If an element $g \in H_3(\mathbb{R})$ commutes with each element of a lattice Γ in $H_3(\mathbb{R})$ then g belongs to the center of $H_3(\mathbb{R})$.*

The following lemma will be used in the proof of Theorem 5.7 which is one of the main results of this paper.

Lemma 1.5. *Let Γ and Λ be two lattices in $H_3(\mathbb{R})$ such that $\Lambda \subset \Gamma$. Let $p(\Gamma) = A(\mathbb{Z}^2)$ for a matrix $A \in GL_2(\mathbb{R})$. Then for each positive real ξ such that $\xi/\xi_\Gamma \in \mathbb{N}$, $\xi_\Lambda/\xi \in \mathbb{N}$ and $|\det A|/\xi \in \mathbb{N}$, there is a lattice Θ such that $\Gamma \supset \Theta \supset \Lambda$, $p(\Theta) = p(\Gamma)$ and $\xi_\Theta = \xi$.*

Proof. In view of Proposition 1.3, we may assume without loss of generality that $\Gamma = \{c(j_3/k)b(j_2)a(j_1) \mid j_1, j_2, j_3 \in \mathbb{Z}\}$ for some $k \in \mathbb{N}$. It follows from the condition of the lemma that there are positive integers k_1, k_2, k_3 such that $k = k_1k_2$, $\xi = 1/k_2$ and $\xi_\Lambda = k_3/k_2$. Let Θ denote the group generated by Λ and the lattice $\{c(j_3/k_2)b(j_2)a(j_1) \mid j_1, j_2, j_3 \in \mathbb{Z}\}$. Then Θ is a lattice in $H_3(\mathbb{R})$. Of course, $\Gamma \supset \Theta \supset \Lambda$ and $p(\Theta) = p(\Gamma)$. It is easy to see that

$$\xi_\Theta \mathbb{Z} = \xi_\Lambda \mathbb{Z} + \frac{1}{k_2} \mathbb{Z} = \frac{1}{k_2} \mathbb{Z},$$

i.e. $\xi_\Theta = \xi$. \square

³Indeed, the flow $F : \mathbb{R} \times H_3(\mathbb{R})/\Gamma \ni (t, g\Gamma) \mapsto c(t)g\Gamma \in H_3(\mathbb{R})/\Gamma$ is periodic with period ξ_Γ . Hence the Borel space O of F -orbits is standard. On the other hand, O is isomorphic as a Borel space to $H_3(\mathbb{R})/p^{-1}(p(\Gamma))$. The later space is, in turn, isomorphic to the Borel space $\mathbb{R}^2/p(\Gamma)$. Hence $\mathbb{R}^2/p(\Gamma)$ endowed with the natural quotient Borel structure is standard. Hence $p(\Gamma)$ is closed in \mathbb{R}^2 by [Ma2].

Remark 1.6. It is straightforward to verify that for each lattice Γ in $H_3(\mathbb{R})$, we have $k_\Gamma = \frac{|\det A|}{\xi_\Gamma}$, where the matrix A is defined by $p(\Gamma) = A(\mathbb{Z}^2)$. Indeed, since $[\Gamma, \Gamma] = \{c(k_\Gamma \xi_\Gamma m) \mid m \in \mathbb{Z}\}$, it follows that $[\gamma_1, \gamma_2] = c(\pm k_\Gamma \xi_\Gamma)$ for each pair of elements $\gamma_1, \gamma_2 \in \Gamma$ whose images $p(\gamma_1)$ and $p(\gamma_2)$ in \mathbb{R}^2 generate the entire lattice $p(\Gamma)$.

Remark 1.7. There is a natural action of $\text{Aut}(H_3(\mathbb{R}))$ on $\widehat{H_3(\mathbb{R})}$. It is not difficult to compute that $\pi_{\alpha, \beta} \circ \theta_{A, \xi, \eta} = \pi_{\xi_1 \alpha + \xi_2 \beta, \eta_1 \alpha + \eta_2 \beta}$ and $\pi_\gamma \circ \theta_{A, \xi, \eta} \sim \pi_{\gamma \cdot \det A}$.

2. ODOMETER ACTIONS OF LOCALLY COMPACT GROUPS

In this section we introduce the concept of odometer action for locally compact second countable groups. We study some general structural properties of these actions and discuss when they are free.

Let G and K be locally compact second countable groups. Let $T = (T_g)_{g \in G}$ be an action of G on a standard Borel space Z . A Borel map $c : G \times Z \rightarrow K$ is called a cocycle of T if

$$c(g_1 g_2, z) = c(g_1, T_{g_2} z) c(g_2, z)$$

for all $g_1, g_2 \in G$ and $z \in Z$. Two cocycles $c, c' : G \times Z \rightarrow K$ are *cohomologous* if there is a Borel map $d : Z \rightarrow K$ such that $d(gz) c(g, z) d(z)^{-1} = c'(g, z)$ for all $g \in G$ and $z \in Z$. Given a cocycle c , we can define an action $T^c = (T_g^c)_{g \in G}$ of G on $Z \times K$ by setting

$$T_g^c(z, k) = (T_g z, c(g, z) k).$$

It is called the c -skew product extension of T . If c is cohomologous to another cocycle c' of T then the actions T^c and $T^{c'}$ of G are isomorphic. If T preserves a σ -finite measure μ then T^c preserves the σ -finite measure $\mu \times \lambda_K$, where λ_K denotes a left Haar measure on K . If T^c is ergodic then c is called *ergodic*.

Let N be a closed subgroup of G . Let the two groups be unimodular. Fix a Borel cross-section $s : G/N \rightarrow G$ of the natural projection $G \rightarrow G/N$ such that $s(N) = 1$. Then a cocycle $h_s : G \times G/N \rightarrow N$ of the natural σ -finite (Haar) measure preserving G -action on G/N by rotations is well defined by the formula $h_s(g, x) := s(gx)^{-1} g s(x)$. The cohomology class of h_s does not depend on the choice of s . Below we will call h_s a *choice-cocycle*. It is easy to see that the h_s -skew product extension is isomorphic to the action of G on itself by left translations.

Remark 2.0. Given a cocycle $c : G \times G/N \rightarrow K$, a continuous group homomorphism $\phi : N \rightarrow K$ is well defined by $\phi(n) := c(n, N)$. Consider a mapping $d : G/N \ni x \mapsto d(x) := c(s(x), N) \in K$. Then we have

$$\begin{aligned} c(g_1, g_2 N) &= c(g_1 g_2, N) c(g_2, N)^{-1} \\ &= c(s(g_1 g_2 N) h_s(g_1 g_2, N), N) c(s(g_2 N) h_s(g_2, N), N)^{-1} \\ &= d(g_1 g_2 N) \phi(h_s(g_1 g_2, N) h_s(g_2, N)^{-1}) d(g_2 N)^{-1} \\ &= d(g_1 g_2 N) \phi(h_s(g_1, g_2 N)) d(g_2 N)^{-1}. \end{aligned}$$

In other words, c is cohomologous to the cocycle $\phi \circ h_s$. Thus, the cohomology class of c is determined completely by ϕ (or, more precisely, by the conjugacy class of ϕ).

Remark 2.1. If K is Abelian then the essential range⁴ $E(c)$ of c is the closure of the subgroup $\phi(N)$ in K . Indeed, this follows from the fact that h_s is ergodic (in fact, the corresponding h_s -skew product is transitive).

Let $\Gamma_1 \supset \Gamma_2 \supset \dots$ be a nested sequence of lattices in G . Then we consider a projective sequence of homogeneous G -spaces

$$(2-1) \quad G/\Gamma_1 \leftarrow G/\Gamma_2 \leftarrow \dots$$

All arrows are G -equivariant and onto. Denote by X the projective limit of this sequence. Then X is a locally compact second countable G -space. Indeed, G/Γ_1 is locally compact and every arrow in (2-1) is a finite-to-one mapping.⁵ The G -action is minimal and uniquely ergodic. The only invariant probability measure μ on X is the projective limit of the probability Haar measures on G/Γ_n .

Definition 2.2. We call the dynamical system (X, μ, G) a G -odometer⁶. If Γ_n is normal in Γ_1 for each n then we call the corresponding G -odometer *normal*.

We note that X is compact if and only if each Γ_n is *uniform*, i.e. co-compact in G .

To show a ‘product structure’ on X we denote by $s_j : \Gamma_{j-1}/\Gamma_j \rightarrow \Gamma_{j-1}$ a Borel cross-section of the natural projection $\Gamma_{j-1} \rightarrow Y_j := \Gamma_{j-1}/\Gamma_j$ such that $s_j(\Gamma_j) = 1$. For consistency of the notation we set $\Gamma_0 = G$. Let $h_j : \Gamma_{j-1} \times Y_j \ni (g, y) \mapsto h_j(g, y) \in \Gamma_j$ stand for the corresponding choice-cocycle. Then $gs_j(y) = s_j(gy)h_j(g, y)$ for all $g \in \Gamma_{j-1}$ and $y \in Y_j$. Denote by ψ_j a mapping

$$G/\Gamma_1 \times \Gamma_1/\Gamma_2 \times \dots \times \Gamma_{j-1}/\Gamma_j \ni (y_1, \dots, y_j) \mapsto s_1(y_1)s_2(y_2) \dots s_j(y_j)\Gamma_j \in G/\Gamma_j$$

for $j > 1$ and $\psi_1 := \text{id}$. It is easy to see that each ψ_j is a Borel isomorphism. We obtain the following infinite commutative diagram

$$\begin{array}{ccccccc} G/\Gamma_1 & \longleftarrow & G/\Gamma_2 & \longleftarrow & G/\Gamma_3 & \longleftarrow & \dots \\ \psi_1 \uparrow & & \psi_2 \uparrow & & \psi_3 \uparrow & & \dots, \\ Y_1 & \longleftarrow & Y_1 \times Y_2 & \longleftarrow & Y_1 \times Y_2 \times Y_3 & \longleftarrow & \dots \end{array}$$

where the horizontal arrows in the lower lines are natural projections. Therefore the sequence of mappings $(\psi_j^{-1})_{j=1}^\infty$ generates a Borel isomorphism ψ of $X := \text{proj lim}_j G/\Gamma_j$ onto $Y := Y_1 \times Y_2 \times \dots$. Since X is a G -space, Y is also a G -space. We thus obtain the following proposition.

Proposition 2.3. *The G -space X is Borel isomorphic to the infinite Cartesian product*

$$Y = G/\Gamma_1 \times \Gamma_1/\Gamma_2 \times \Gamma_2/\Gamma_3 \times \dots$$

in such a way that

$$g \cdot (y_1, y_2, y_3, \dots) = (gy_1, h_1(g, y_1)y_2, h_2(h_1(g, y_1), y_2)y_3, \dots)$$

⁴For the definition of the essential range of a cocycle we refer to [Sc].

⁵We note that Γ_n is of finite index in Γ_1 .

⁶Thus the G -odometers are ergodic. Cf. [CoPe], where the case of discrete and finitely generated G was under consideration.

for all $(y_1, y_2, \dots) \in Y$ and $g \in G$.

We recall that given a lattice Γ in G and a measure preserving action $V = (V_\gamma)_{\gamma \in \Gamma}$ of Γ on a standard probability space (Z, κ) , there is a natural *induced* measure preserving G -action [Ma1] on the product space $(G/\Gamma \times Z, \lambda \times \kappa)$:

$$g(x, z) := (gx, V_{h_s(g, x)}z),$$

where λ is the probability Haar measure on G/Γ and $h_s : G \times G/\Gamma \rightarrow \Gamma$ is the 1-cocycle corresponding to a cross-section $s : G/\Gamma \rightarrow G$. We denote this action by $\text{Ind}_\Gamma^G(V)$. The isomorphism class of $\text{Ind}_\Gamma^G(V)$ does not depend on a particular choice of s .

Let K be a compact metric group and let $\alpha : \Gamma \times Z \rightarrow K$ be a cocycle of V . Then we can define an *induced* cocycle $\tilde{\alpha} : G \times (G/\Gamma \times Z) \rightarrow K$ of $\text{Ind}_\Gamma^G(V)$ by setting

$$\tilde{\alpha}(g, (x, z)) := \alpha(h_s(g, x), z).$$

The next lemma is about basic properties of induced cocycles. They follow easily from the definitions and we state this lemma without proof.

Lemma 2.4.

- (i) $\text{Ind}_\Gamma^G(V^\alpha) = (\text{Ind}_\Gamma^G(V))^{\tilde{\alpha}}$. In particular, $\tilde{\alpha}$ is ergodic if and only if α is ergodic.
- (ii) Two cocycles α, β of V are cohomologous if and only if the induced cocycles $\tilde{\alpha}$ and $\tilde{\beta}$ of $\text{Ind}_\Gamma^G(V)$ are cohomologous.
- (iii) For each cocycle δ of $\text{Ind}_\Gamma^G(V)$, there exists a cocycle α of V such that δ is cohomologous to $\tilde{\alpha}$.
- (iv) Let \mathfrak{F} be a factor of V and let $\tilde{\mathfrak{F}} := \text{Ind}_\Gamma^G(\mathfrak{F})$ denote the corresponding factor of $\text{Ind}_\Gamma^G(V)$. Then $\tilde{\alpha}$ is $\tilde{\mathfrak{F}}$ -measurable if and only if α is \mathfrak{F} -measurable.

The next statement follows easily from Proposition 2.3.

Corollary 2.5. *Let T be a G -odometer associated with a sequence $\Gamma_1 \supset \Gamma_2 \supset \dots$ of lattices in G . Let V denote the Γ_1 -odometer associated with the sequence $\Gamma_2 \supset \Gamma_3 \supset \dots$ of lattices in Γ_1 . Then $\text{Ind}_{\Gamma_1}^G(V) = T$.*

We now find some conditions under which a G -odometer is free. Let X denote the space of an odometer T . Every point $x \in X$ can be written as an infinite sequence $x = (g_1\Gamma_1, g_2\Gamma_2, \dots)$ for some $g_1 \in G$, $g_2 \in g_1\Gamma_1$, $g_3 \in g_2\Gamma_2$, \dots . Denote by S_x the stability group of T at x .

Proposition 2.6. *Let $x = (g\Gamma_1, g\gamma_1\Gamma_2, g\gamma_1\gamma_2\Gamma_3, \dots) \in X$ for some $g \in H_3(\mathbb{R})$ and $\gamma_n \in \Gamma_n$, $n \in \mathbb{N}$. Then $S_x = g\left(\bigcap_{n=1}^{\infty} \gamma_1 \cdots \gamma_{n-1} \Gamma_n \gamma_{n-1}^{-1} \cdots \gamma_1^{-1}\right)g^{-1}$. Therefore T is free if and only if*

$$\bigcap_{n=1}^{\infty} \gamma_1 \cdots \gamma_{n-1} \Gamma_n \gamma_{n-1}^{-1} \cdots \gamma_1^{-1} = \{1\}$$

for a.e. x , i.e. for a.e. sequence $(\gamma_1\Gamma_2, \gamma_2\Gamma_3, \dots) \in \prod_{n \in \mathbb{N}} \Gamma_n/\Gamma_{n+1}$, where the infinite product space is endowed with the infinite product measure obtained from equi-distributions on the factors. Hence T is free if and only if the Γ_1 -odometer

associated with the sequence $\Gamma_2 \supset \Gamma_3 \supset \dots$ is free. The restriction of T to the center of G is free if and only if the intersection of the center with the group $\bigcap_{n=1}^{\infty} \Gamma_n$ is trivial.

Proof. Since $T_h x = (hg\Gamma_1, hg\gamma_1\Gamma_2, hg\gamma_1\gamma_2\Gamma_3, \dots)$, it follows that $T_h x = x$ if and only if $h \in \bigcap_{n=1}^{\infty} g\gamma_1 \cdots \gamma_{n-1} \Gamma_n \gamma_{n-1}^{-1} \cdots \gamma_1^{-1} g^{-1}$. \square

Corollary 2.7. *Let (X, μ, T) be the odometer associated with a sequence $\Gamma_1 \supset \Gamma_2 \supset \dots$ of lattices in G .*

- (i) *If Γ_n is normal in Γ_1 for each n , then T is free if and only if $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$.*
- (ii) *Let $\tilde{\Gamma}_n := \bigcap_{\gamma \in \Gamma_1} \gamma \Gamma_n \gamma^{-1}$. Then $\tilde{\Gamma}_n$ is a subgroup of finite index in Γ_n . It is normal in Γ_1 . Moreover, $\Gamma_1 \supset \tilde{\Gamma}_2 \supset \tilde{\Gamma}_3 \supset \dots$. Denote by $(\tilde{X}, \tilde{\mu}, \tilde{T})$ the corresponding normal G -odometer. We call it the normal cover of (X, μ, T) . The sequence of natural G -equivariant maps $G/\tilde{\Gamma}_n \rightarrow G/\Gamma_n$ generates a factor map $(\tilde{X}, \tilde{\mu}, \tilde{T}) \rightarrow (X, \mu, T)$. Thus every odometer is a factor of a normal odometer. Moreover, if $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$ then $\bigcap_{n=1}^{\infty} \tilde{\Gamma}_n \subset \bigcap_{n=1}^{\infty} \Gamma_n$ and the corresponding normal cover is free⁷. Furthermore, $\bigcap_{n=1}^{\infty} \tilde{\Gamma}_n \cap \{c(t) \mid t \in \mathbb{R}\} = \bigcap_{n=1}^{\infty} \Gamma_n \cap \{c(t) \mid t \in \mathbb{R}\}$.*

Remark 2.8. If there is a Følner sequence $(F_n)_{n=1}^{\infty}$ in G such that $G = \bigsqcup_{\gamma \in \Gamma_n} F_{n-1}\gamma$ and $F_n = \bigsqcup_{\gamma \in C_n} F_{n-1}\gamma$ for some finite subset $C_n \subset \Gamma_n$ and each $n > 0$ then the corresponding G -odometer is the (C, F) -action associated with $(F_{n-1}, C_n)_{n=1}^{\infty}$ (see [Da3]). Every (C, F) -action is free. We note that $\Gamma_n = C_n \Gamma_{n+1}$ and $c\Gamma_{n+1} \cap c'\Gamma_{n+1} = \emptyset$ whenever $c \neq c' \in C_n$. Hence $\#C_n$ is the index of Γ_{n+1} in Γ_n .

3. FREENESS FOR HEISENBERG ODOMETERS

In this section we consider in detail the problem of when $H_3(\mathbb{R})$ -odometers are free. Some criteria of the freeness are found. A couple of counterexamples are also given here.

Theorem 3.1. *Let T be the $H_3(\mathbb{R})$ -odometer associated with a sequence $\Gamma_1 \supset \Gamma_2 \supset \dots$ of lattices in $H_3(\mathbb{R})$. Then T is free if and only if $\{c(t) \mid t \in \mathbb{R}\} \cap \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$.*

Proof. Let (X, \mathfrak{B}, μ) denote the (standard probability) space of T . Suppose that $\{c(t) \mid t \in \mathbb{R}\} \cap \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. It follows from Proposition 2.6 that the restriction of T to the center of $H_3(\mathbb{R})$ is free. Let \mathfrak{S} stand for the space of closed subgroups in $H_3(\mathbb{R})$. It is a standard Borel space when endowed with the Fell topology [Fe]. Denote by S_x the stability group of T at a point $x \in X$. It follows from Proposition 2.6 that S_x is countable. The map $X \ni x \mapsto S_x \in \mathfrak{S}$ is Borel [AGH]. It is easy to see that

$$(3-1) \quad S_{T_g x} = gS_x g^{-1} \quad \text{for all } g \in H_3(\mathbb{R}) \quad \text{and } x \in X.$$

We let $\tilde{S}_x := S_x \cap \{c(t) \mid t \in \mathbb{R}\}$. It follows from (3-1) that the Borel map $X \ni x \mapsto \tilde{S}_x \in \mathfrak{S}$ is invariant under T and hence constant almost everywhere. Since the restriction of T to the center of $H_3(\mathbb{R})$ is free, we obtain that \tilde{S}_x is trivial

⁷See also [CoPe] for an analogous claim for a subclass of discrete G .

for almost all x . According to [GoSi] and [CRa]⁸, there is $S \in \mathfrak{S}$ and a Borel map $\phi : X \rightarrow G$ such that

$$(3-2) \quad \phi(x)S_x\phi(x)^{-1} = S \quad \text{for almost all } x.$$

Since S is countable and closed and $S \cap \{c(t) \mid t \in \mathbb{R}\} = \{1\}$, it follows that either S is trivial or there are a countable subgroup D in \mathbb{R} , a homomorphism $\varphi : D \rightarrow \mathbb{R}$ and three reals $\theta_1, \theta_2, \theta_3$ such that $\theta_1^2 + \theta_2^2 \neq 0$ and⁹

$$S = \{c(\varphi(d)\theta_3)b(d\theta_2)a(d\theta_1) \mid d \in D\}.$$

Suppose that S is non-trivial. We note that (3-2) defines ϕ up to multiplying with an element of the normalizer $N(S)$ of S in $H_3(\mathbb{R})$. We recall that $N(S) := \{g \in H_3(\mathbb{R}) \mid gSg^{-1} = S\}$. It is straightforward to verify that

$$N(S) = \{c(t_1\theta_3)b(t_2\theta_2)a(t_2\theta_1) \mid t_1, t_2 \in \mathbb{R}\}.$$

Indeed, an element $c(r_3)b(r_2)a(r_1)$ of $H_3(\mathbb{R})$ belongs to $N(S)$ if and only if the commutator $[c(r_3)b(r_2)a(r_1), c(\varphi(d)\theta_3)b(d\theta_2)a(d\theta_1)]$ belongs to S for all $d \in D$. Thus $c(r_1d\theta_2 - r_2d\theta_1) \in S$. This is possible only if $r_1\theta_2 - r_2\theta_1 = 0$, i.e. the vector $(r_1, r_2) \in \mathbb{R}^2$ belongs to the line generated by (θ_1, θ_2) , as desired. We note that $N(S)$ is a closed normal subgroup of $H_3(\mathbb{R})$ and $H_3(\mathbb{R})/N(S) = \mathbb{R}$. Moreover, the natural projection can be written in the following way

$$(3-3) \quad H_3(\mathbb{R})/N(S) \ni gN(S) = b(-t\theta_1)a(t\theta_2)N(S) \mapsto t \in \mathbb{R},$$

where the real t is uniquely determined by the coset $gN(S)$. We now let $\tilde{\phi}(x) := \phi(x)N(S)$. Then $\tilde{\phi}$ is a well defined (mod 0) Borel map from X to $H_3(\mathbb{R})/N(S) = \mathbb{R}$. Substituting (3-2) into (3-1) we deduce that

$$\phi(T_gx)^{-1}S\phi(T_gx) = g^{-1}\phi(x)^{-1}S\phi(x)g$$

and hence

$$(3-4) \quad \phi(x)g\phi(T_gx)^{-1} \in N(S)$$

for a.a. x . Therefore $-\tilde{\phi}(x) + \tilde{\phi}(T_gx) \in gN(S)$ for a.a. x for each $g \in H_3(\mathbb{R})$. It follows that $\tilde{\phi}$ is invariant under $(T_g)_{g \in N(S)}$. Denote by $q : (X, \mu) \rightarrow (Y, \nu)$ the $(T_g)_{g \in N(S)}$ -ergodic decomposition. Then there are an ergodic flow $V = (V_t)_{t \in \mathbb{R}}$ on (Y, ν) and a function $\psi : Y \rightarrow \mathbb{R}$ such that $V_t \circ q = q \circ T_{b(-t\theta_1)a(t\theta_2)}$, $\tilde{\phi} = \psi \circ q$ and $-\psi + \psi \circ V_t = t$ for all $t \in \mathbb{R}$. The latter equality follows from (3-3) and (3-4). Of course, such a function ψ does not exist. Thus we obtain a contradiction. Hence S is trivial, i.e. T is free.

The converse follows from Proposition 2.6 and from the obvious remark that if T is free then the restriction of T to the center of $H_3(\mathbb{R})$ is also free. \square

The next claim follows from Proposition 2.6, Theorem 3.1 and Corollary 2.7(ii).

⁸The stability groups of an ergodic probability preserving action of a simple connected nilpotent Lie group are conjugate (in a Borel way) over a conull subset.

⁹We note that D is not necessarily closed in \mathbb{R} . Consider, for instance, the closed subgroup $S = \{c(n)a(m\tau + n) \mid m, n \in \mathbb{Z}\}$ in $H_3(\mathbb{R})$, where τ is an irrational number.

Corollary 3.2.

- (i) A Heisenberg odometer is free if and only if its restriction to the center is free.
- (ii) A Heisenberg odometer is free if and only if its normal cover is free.

It follows from the next example that the condition $\{c(t) \mid t \in \mathbb{R}\} \cap \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$ does not imply that $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$.

Example 3.3. Let $\Gamma_n := \{c(n!i_3)b(n!i_2)a(i_1) \mid i_1, i_2, i_3 \in \mathbb{Z}\}$. Then Γ_n is a lattice in $H_3(\mathbb{R})$, $\Gamma_1 \supset \Gamma_2 \supset \dots$ and $\{c(t) \mid t \in \mathbb{R}\} \cap \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. On the other hand, $\bigcap_{n=1}^{\infty} \Gamma_n = \{a(i_1) \mid i_1 \in \mathbb{Z}\}$.

Consider now the case of normal Heisenberg odometers.

Proposition 3.4. *If T is a normal free Heisenberg odometer then $\bigcap_{n=1}^{\infty} p(\Gamma_n) = \{0\}$.*

Proof. Suppose that $\bigcap_{n=1}^{\infty} p(\Gamma_n) \neq \{0\}$. Then there is $g \in H_3(\mathbb{R}) \setminus \{c(t) \mid t \in \mathbb{R}\}$ and a sequence $(r_n)_{n=1}^{\infty}$, $r_n \in \mathbb{R}$, such that $gc(r_n) \in \Gamma_n$ for all $n \in \mathbb{N}$. Since g does not belong to the center of $H_3(\mathbb{R})$, there is γ in Γ_1 such that g and γ do not commute (see Corollary 1.4). Therefore $\gamma g \gamma^{-1} = gc(t)$ for some $t \neq 0$. This yields $gc(r_n)c(t) = \gamma gc(r_n)\gamma^{-1} \in \gamma \Gamma_n \gamma^{-1} = \Gamma_n$ and hence $c(t) \in \Gamma_n$ for all n . This contradicts to the fact that $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. \square

It may seem that the condition $\bigcap_{n=1}^{\infty} p(\Gamma_n) = \{0\}$ is satisfied for every odometer with $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. In the next example we show that this is not true. Hence the corresponding odometer is not normal.

Example 3.5. Let $k_1 := 1$ and $k_n := k_{n-1}(k_{n-1} + 1)$. Let

$$\Gamma_n := \{c(k_n^2 i_3 + k_n i_1)b(k_n^2 i_2)a(i_1) \mid i_1, i_2, i_3 \in \mathbb{Z}\}.$$

Then Γ_n is a lattice in $H_3(\mathbb{R})$, $\Gamma_1 \supset \Gamma_2 \supset \dots$ and $\bigcap_{n=1}^{\infty} \Gamma_n = \{1\}$. On the other hand, $p(\Gamma_n) = \begin{pmatrix} 1 & 0 \\ 0 & k_n^2 \end{pmatrix} \mathbb{Z}^2$ and hence $\bigcap_{n=1}^{\infty} p(\Gamma_n) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbb{Z}^2 \neq \{0\}$.

4. SPECTRAL ANALYSIS FOR HOMOGENEOUS ACTIONS OF $H_3(\mathbb{R})$ ON NIL-MANIFOLDS

In this section we review

- (a) Kirillov's orbit method to get a description of the unitary dual for the Heisenberg group [Ki] and
- (b) the Howe-Richardson formula for the multiplicities of the irreducible components in spectral decomposition of homogeneous $H_3(\mathbb{R})$ -spaces [Ho], [Ri].

Given a lattice Γ in $H_3(\mathbb{R})$, the quotient space $H_3(\mathbb{R})/\Gamma$ is called a *nil-manifold*. Consider the quasi-regular representation U of $H_3(\mathbb{R})$ in $L^2(H_3(\mathbb{R})/\Gamma)$, i.e. the Koopman representation of $H_3(\mathbb{R})$ corresponding to the natural action of this group by rotations on $H_3(\mathbb{R})/\Gamma$. Since Γ is uniform, it follows that there is a countable family Θ of irreducible unitary representations of $H_3(\mathbb{R})$ and positive integers m_W , $W \in \Theta$, such that

$$(4-1) \quad U = \bigoplus_{W \in \Theta} \bigoplus_{j=1}^{m_W} W.$$

The family Θ (or, rather their unitary equivalence classes) and the map $\Theta \ni W \mapsto m_W$ are defined uniquely by U . Our purpose is to describe them explicitly. Partition Θ into two subsets Θ_1 and Θ_∞ consisting of the 1-dimensional representations and the ∞ -dimensional representations of $H_3(\mathbb{R})$ respectively.

If $\pi_{\alpha,\beta} \in \Theta_1$ for some $(\alpha, \beta) \in \mathbb{R}^2$, then there is $f \in L^2(H_3(\mathbb{R})/\Gamma)$ such that $f \circ T_g^{-1} = e^{2\pi i(\alpha t_1 + \beta t_2)} f$ for all $g \in \mathbb{H}_3(\mathbb{R})$, where $g = c(t_3)b(t_2)a(t_1)$. In particular, f is invariant under the action of the center. We recall (see the footnote No 3) that the natural quotient map

$$H_3(\mathbb{R})/\Gamma \rightarrow H_3(\mathbb{R})/p^{-1}(p(\Gamma)) = \mathbb{R}^2/p(\Gamma)$$

is the $(T_{c(t)})_{t \in \mathbb{R}}$ -ergodic decomposition. Hence f is constant on the fibers of this quotient map. Moreover, f is an eigenfunction for the \mathbb{R}^2 -action by rotations on $\mathbb{R}^2/p(\Gamma)$. The converse is also true. Indeed, if $A \in \text{GL}_2(\mathbb{R})$ then $(A^*)^{-1}\mathbb{Z}^2$ are the eigenvalues of the \mathbb{R}^2 -actions by rotations on $\mathbb{R}^2/A(\mathbb{Z}^2)$. If $p(\Gamma) = A(\mathbb{Z}^2)$ for some matrix $A \in \text{GL}_2(\mathbb{R})$ then we denote by $p(\Gamma)^*$ the *dual lattice* $(A^*)^{-1}\mathbb{Z}^2$ in \mathbb{R}^2 . It is easy to see that the dual lattice is well defined, i.e. does not depend on the choice of A . We now obtain that

$$\Theta_1 = \{\pi_{\alpha,\beta} \mid (\alpha, \beta) \in p(\Gamma)^*\}.$$

Since T is ergodic (it is even transitive), it follows that the corresponding multiplicities are all 1, i.e. $m_W = 1$ whenever $W \in \Theta_1$.

Now let us calculate Θ_∞ . Fix a $p(\Gamma)$ -fundamental parallelepiped Δ in \mathbb{R}^2 . Consider the map

$$\Psi : H_3(\mathbb{R})/\Gamma \ni g\Gamma \mapsto ((t_1, t_2) + p(\Gamma), t_3 + \xi_\Gamma \mathbb{Z}) \in H_3(\mathbb{R})/p^{-1}(p(\Gamma)) \times \mathbb{R}/\xi_\Gamma \mathbb{Z},$$

where the reals t_1, t_2, t_3 are determined by the formula $g \in c(t_3)b(t_2)a(t_1)\Gamma$ with $t_1, t_2 \in \Delta$. This map conjugates the $H_3(\mathbb{R})$ -action on the nilmanifold $X := H_3(\mathbb{R})/\Gamma$ with the $\mathbb{R}/\xi_\Gamma \mathbb{Z}$ -group extension of the $H_3(\mathbb{R})$ -action on the quotient space $Y := H_3(\mathbb{R})/p^{-1}(p(\Gamma))$. We let $\eta_n(w) = e^{2\pi i n w / \xi_\Gamma}$, $n \in \mathbb{R}$, $w \in [0, \xi_\Gamma) = \mathbb{R}/\xi_\Gamma \mathbb{Z}$. Since the group characters $\{\eta_n \mid n \in \mathbb{Z}\}$ form an orthonormal basis in $L^2(\mathbb{R}/\xi_\Gamma \mathbb{Z})$, we obtain a decomposition of $L^2(X)$ into the following orthogonal sum

$$(4-2) \quad L^2(X) = \bigoplus_{n \in \mathbb{Z}} L^2(Y) \otimes \eta_n.$$

It is easy to see that every subspace $L^2(Y) \otimes \eta_n$ is invariant under U . We note that if $\Psi(g\Gamma) = (y, t_3 + \xi_\Gamma \mathbb{Z})$ for some $g \in H_3(\mathbb{R})$, $y \in Y$ and $t_3 \in \mathbb{R}$ then $\Psi(c(t)g\Gamma) = (y, t + t_3 + \xi_\Gamma \mathbb{Z})$ for all $t \in \mathbb{R}$. This implies that

$$U(c(t))(f \otimes \eta_n) = e^{2\pi i n t / \xi_\Gamma} f \otimes \eta_n$$

for all $f \in L^2(Y)$, $n \in \mathbb{Z}$. Hence $n/\xi_\Gamma \in \Theta_\infty$ and

$$(4-3) \quad U \upharpoonright (L^2(Y) \otimes \eta_n) \subset \bigoplus_1^{m_{\pi_n/\xi_\Gamma}} \pi_{n/\xi_\Gamma}$$

if $n \neq 0$. Of course, $U \upharpoonright (L^2(Y) \otimes 1) = \bigoplus_{W \in \Theta_1} W$. Therefore we deduce from (4-1), (4-2) and (4-3) that

$$\Theta_\infty = \{\pi_\gamma \mid \gamma \in \xi_\Gamma^{-1}\mathbb{Z}, \gamma \neq 0\}$$

and

$$U \upharpoonright (L^2(Y) \otimes \eta_n) = \bigoplus_1^{m_{\pi_n/\xi_\Gamma}} \pi_{n/\xi_\Gamma}$$

if $n \neq 0$. Thus we have computed the spectrum of U .

Theorem 4.1. *We have $\Theta = \{\pi_{\alpha,\beta} \mid (\alpha,\beta) \in p(\Gamma)^*\} \cup \{\pi_\gamma \mid \gamma \in \xi_\Gamma^{-1}\mathbb{Z}, \gamma \neq 0\}$.*

Our next aim is to compute the spectral multiplicities m_W when $W \in \Theta_\infty$. For that we will apply Kirillov's orbit method [Ki] and the Howe-Richardson multiplicity formula (see [Ho], [Ri] and an earlier work [Mo]). In view of Proposition 1.3 we may assume without loss of generality that

$$\Gamma = \left\{ \left(\begin{array}{ccc} 1 & l & \frac{n}{k} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{array} \right) \mid l, m, n \in \mathbb{Z} \right\}$$

for some $k \in \mathbb{N}$ which is fixed. By Theorem 4.1, $\pi_\gamma \in \Theta$ if and only if $\gamma = kn$ with $n \in \mathbb{Z} \setminus \{0\}$. Thus our purpose is to calculate the multiplicity $m_{\pi_{kn}}$ for each $n \in \mathbb{Z} \setminus \{0\}$.

The Lie algebra of $H_3(\mathbb{R})$ is

$$\mathfrak{h}_3(\mathbb{R}) := \left\{ \left(\begin{array}{ccc} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{array} \right) \mid a, b, c \in \mathbb{R} \right\}.$$

We endow it with the natural topology. Then the exponential map $\exp : \mathfrak{h}_3(\mathbb{R}) \rightarrow H_3(\mathbb{R})$ is a homeomorphism. We choose a special *strong Malcev* basis

$$X_1 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 := \begin{pmatrix} 0 & 0 & \frac{1}{k} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $\mathfrak{h}_3(\mathbb{R})$. Fix an ideal $\mathfrak{m} := \mathbb{R}X_3 + \mathbb{R}X_2$ of $\mathfrak{h}_3(\mathbb{R})$. Then $\exp(\mathfrak{m}) = H_{2,b}$. We consider the dual space $\mathfrak{h}_3(\mathbb{R})^*$ to $\mathfrak{h}_3(\mathbb{R})$ as the space of lower triangular 3×3 -matrices (modulo the subspace of upper triangular matrices including the diagonal). The canonical duality form is given via the trace of the product, i.e.

$$\mathfrak{h}_3(\mathbb{R}) \times \mathfrak{h}_3(\mathbb{R})^* \ni \left(\left(\begin{array}{ccc} 0 & d_1 & d_3 \\ 0 & 0 & d_2 \\ 0 & 0 & 0 \end{array} \right), \left(\begin{array}{ccc} * & * & * \\ z_1 & * & * \\ z_3 & z_2 & * \end{array} \right) \right) \mapsto z_1 d_1 + z_2 d_2 + z_3 d_3.$$

We note that then

$$X_1^* = \begin{pmatrix} * & * & * \\ 1 & * & * \\ 0 & 0 & * \end{pmatrix}, \quad X_2^* = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 1 & * \end{pmatrix}, \quad X_3^* = \begin{pmatrix} * & * & * \\ 0 & * & * \\ k & 0 & * \end{pmatrix}$$

is the dual basis for $\mathfrak{h}_3(\mathbb{R})^*$. Then $\mathfrak{m}^* = \mathbb{R}X_2^* + \mathbb{R}X_3^* \subset \mathfrak{h}_3(\mathbb{R})^*$. It is not difficult now to compute explicitly the co-adjoint action Ad^* of $H_3(\mathbb{R})$ on $\mathfrak{h}_3(\mathbb{R})^*$:

$$\text{Ad}^*(c(t_3)b(t_2)a(t_1)) \begin{pmatrix} * & * & * \\ z_1 & * & * \\ z_3 & z_2 & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ z_1 + t_2 z_3 & * & * \\ z_3 & z_2 - t_1 z_3 & * \end{pmatrix}.$$

Hence the $\text{Ad}^*(H_3(\mathbb{R}))$ -orbit $\Omega(l)$ of a linear functional $l = \begin{pmatrix} * & * & * \\ z_1 & * & * \\ z_3 & z_2 & * \end{pmatrix} \in \mathfrak{h}_3(\mathbb{R})^*$

with $z_3 \neq 0$ is $l + \mathbb{R}X_2^* + \mathbb{R}X_1^*$. Recall that according to Kirillov's orbit method [Ki], $\widehat{H_3(\mathbb{R})}$ is in a one-to-one correspondence with $\text{Ad}^*(H_3(\mathbb{R}))$ -orbits in $\mathfrak{h}_3(\mathbb{R})^*$.¹⁰ More precisely, the irreducible unitary representation of $H_3(\mathbb{R})$ corresponding to $\Omega(l)$ is induced from the following 1-dimensional representation (character) τ of $\exp(\mathfrak{m}) = H_{2,b}$:

$$\tau(\exp X) := e^{2\pi i \langle X, l \rangle}, \quad X \in \mathfrak{m}.$$

It is straightforward to verify that $\text{Ind}_{H_{2,b}}^{H_3(\mathbb{R})} \tau$ of $H_3(\mathbb{R})$ is unitarily equivalent to π_{z_3} .

We now explain the Howe-Richardson formula for $m_{\pi_{kn}}$. For that we let

$$\mathcal{W} := \{l \in \mathbb{Z}X_3^* + \mathbb{Z}X_2^* \mid l \upharpoonright (\mathbb{R}X_3) \neq 0\}.$$

To put it another way,

$$\mathcal{W} = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ kn & m & * \end{pmatrix} \mid n, m \in \mathbb{Z}, n \neq 0 \right\}.$$

Given $l \in \mathfrak{m}^* \subset \mathfrak{h}_3(\mathbb{R})^*$ and $g \in H_3(\mathbb{R})$, we let $\text{Ad}_0^*(g)l := \text{Ad}^*(g)l \upharpoonright \mathfrak{m} \in \mathfrak{m}^*$. We now see that if $\gamma = c(t_3)b(t_2)a(t_1) \in \Gamma$ then

$$\text{Ad}_0^*(\gamma) \begin{pmatrix} * & * & * \\ 0 & * & * \\ kn & m & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ kn & m - t_1 nk & * \end{pmatrix}.$$

Since $t_1 \in \mathbb{Z}$, it follows that Ad_0^* is an action of Γ on \mathcal{W} . By [Ri] and [Ho], $m_{\pi_{z_3}}$ equals the number of $\text{Ad}_0^*(\Gamma)$ -orbits in the intersection $\Omega(l) \cap \mathcal{W}$. Hence

$$m_{\pi_{z_3}} = \begin{cases} |n|k & \text{if } z_3 = kn, n \in \mathbb{Z} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

We deduce from this formula, Proposition 1.3 and Theorem 4.1 the following theorem.

Theorem 4.2. *Let Γ be an arbitrary lattice in $H_3(\mathbb{R})$. Then*

$$m_{\pi_\gamma} = \begin{cases} |n|k_\Gamma & \text{if } \gamma = n\xi_\Gamma^{-1}, n \in \mathbb{Z} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$U = \bigoplus_{(\alpha, \beta) \in p(\Gamma)^*} \pi_{\alpha, \beta} \oplus \bigoplus_{0 \neq n \in \mathbb{Z}} \bigoplus_1^{|n|k_\Gamma} \pi_{n\xi_\Gamma^{-1}}.$$

In the next statement we show that the homogeneous actions of the Heisenberg group on nilmanifolds are isospectral.

¹⁰The case $z = 0$ corresponds to the family of 1-dimensional representations.

Corollary 4.3. *Let Γ and Γ' be two lattices in $H_3(\mathbb{R})$. Denote by T and T' the corresponding measure preserving actions of $H_3(\mathbb{R})$ on the homogeneous spaces $H_3(\mathbb{R})/\Gamma$ and $H_3(\mathbb{R})/\Gamma'$ respectively. The following are equivalent:*

- (i) *T and T' are isomorphic.*
- (ii) *$p(\Gamma) = p(\Gamma')$ and $k_\Gamma = k_{\Gamma'}$.*
- (iii) *$p(\Gamma) = p(\Gamma')$ and $\xi_\Gamma = \xi_{\Gamma'}$.*
- (iv) *The Koopman representations of $H_3(\mathbb{R})$ generated by T and T' are unitarily equivalent.*
- (v) *T and T' have the same spectrum (i.e. the same maximal spectral type).*

Proof. It is well known (and easy to see) that T and T' are isomorphic if and only if Γ and Γ' are conjugate. Therefore (i) and (ii) are equivalent according to the second claim of Proposition 1.3. It follows from Remark 1.6 that (ii) is equivalent (iii). Of course, (i) implies (iv). It follows from Theorem 4.2 that (iv) implies (iii). The equivalence of (iv) and (v) follows also from Theorem 4.2. \square

5. HEISENBERG ODOMETERS

In this section we describe the spectral decomposition of Heisenberg odometers into irreducible components. A system of algebraic invariants (consisting of two countable Abelian groups with a relation between them) is associated to every $H_3(\mathbb{R})$ -odometer. It is shown that this system of invariants is complete for a certain f-isomorphism relation (to be defined below) on the class of Heisenberg odometers. We also show that the f-isomorphism is weaker than the isomorphism, i.e. the class of $H_3(\mathbb{R})$ -odometers is not isospectral.

Let $\Gamma_1 \supset \Gamma_2 \supset \dots$ be a nested sequence of lattices in $H_3(\mathbb{R})$. Denote by (X, μ, T) the associated $H_3(\mathbb{R})$ -odometer. Let (Y, ν) stand for the space of $(T_{c(t)})_{t \in \mathbb{R}}$ -ergodic components and let $f : X \rightarrow Y$ stand for the corresponding projection. Then an \mathbb{R}^2 -action $V = (V_{t_1, t_2})_{(t_1, t_2) \in \mathbb{R}^2}$ is well defined by the formula $V_{t_1, t_2} f(x) := f(T_{b(t_2)a(t_1)} x)$. We call it the *underlying \mathbb{R}^2 -odometer*.¹¹

Definition 5.1. We say that T is non-degenerate if one of the following equivalent conditions is satisfied:

- (i) The underlying \mathbb{R}^2 -odometer is non-transitive.
- (ii) The subgroup $\bigcup_{j=1}^{\infty} p(\Gamma_j)^*$ is not closed in \mathbb{R}^2 .
- (iii) The sequence $(p(\Gamma_j))_{j=1}^{\infty}$ does not stabilize, i.e. for each $j > 0$ there is $j_1 > j$ such that $p(\Gamma_j) \neq p(\Gamma_{j_1})$.

Theorem 5.2. *Let U stand for the Koopman unitary representation of $H_3(\mathbb{R})$ generated by a Heisenberg odometer T .*

- (i) *If T is non-degenerate then*

$$U = \bigoplus_{(\alpha, \beta) \in \bigcup_{j=1}^{\infty} p(\Gamma_j)^*} \pi_{\alpha, \beta} \oplus \bigoplus_{0 \neq \gamma \in \bigcup_{j=1}^{\infty} \xi_{\Gamma_j}^{-1} \mathbb{Z}} \bigoplus_1^{\infty} \pi_{\gamma}.$$

- (ii) *If there is $l > 0$ such that $p(\Gamma_j) = p(\Gamma_l)$ for all $j \geq l$ then*

$$U = \bigoplus_{(\alpha, \beta) \in p(\Gamma_l)^*} \pi_{\alpha, \beta} \oplus \bigoplus_{0 \neq \gamma \in \bigcup_{j=l}^{\infty} \xi_{\Gamma_j}^{-1} \mathbb{Z}} \bigoplus_1^{m(\gamma)} \pi_{\gamma},$$

¹¹It is the \mathbb{R}^2 -odometer associated with the sequence $(p(\Gamma_j))_{j=1}^{\infty}$ of lattices $p(\Gamma_j)$ in \mathbb{R}^2 .

where $m(\gamma) := |\gamma| \xi_{\Gamma_j} k_{\Gamma_j}$ for each $\gamma \in \xi_{\Gamma_j}^{-1} \mathbb{Z}$, $j \geq l$.

Proof. (i) Consider the following nested family

$$L^2(H_3(\mathbb{R})/\Gamma_1) \subset L^2(H_3(\mathbb{R})/\Gamma_2) \subset \dots$$

of U -invariant subspaces in $L^2(X)$. Their union $\bigcup_{j=1}^{\infty} L^2(H_3(\mathbb{R})/\Gamma_j)$ is dense in $L^2(X)$. Therefore Theorem 4.2 yields that

$$U = \lim_{j \rightarrow \infty} \left(\bigoplus_{(\alpha, \beta) \in p(\Gamma_j)^*} \pi_{\alpha, \beta} \oplus \bigoplus_{0 \neq n \in \mathbb{Z}} \bigoplus_1^{|n|k_{\Gamma_j}} \pi_{n\xi_{\Gamma_j}^{-1}} \right).$$

Thus we see that U decomposes into a direct sum of 1-dimensional and infinite dimensional irreducible unitary representations. Of course, the multiplicity of every 1-dimensional irreducible sub-representation $\pi_{\alpha, \beta}$ of U is 1. Our purpose now is to compute the multiplicity of each infinite dimensional irreducible sub-representation π_{γ} , $0 \neq \gamma \in \bigcup_{j=1}^{\infty} \xi_{\Gamma_j}^{-1} \mathbb{Z}$, of U . There are $j \in \mathbb{N}$ and $n \in \mathbb{Z} \setminus \{0\}$ such that

$$\gamma = \frac{n}{\xi_{\Gamma_j}} = \frac{nq}{\xi_{\Gamma_{j+1}}}, \text{ where } q := \frac{\xi_{\Gamma_{j+1}}}{\xi_{\Gamma_j}}.$$

Then the multiplicity of π_{γ} in $U \upharpoonright L^2(H_3(\mathbb{R})/\Gamma_j)$ is $|n|k_{\Gamma_j}$ and the multiplicity of π_{γ} in $U \upharpoonright L^2(H_3(\mathbb{R})/\Gamma_{j+1})$ is $|n|qk_{\Gamma_{j+1}}$. Let $p(\Gamma_j) = A_j(\mathbb{Z}^2)$ for some matrix $A_j \in GL_2(\mathbb{R})$, $j \in \mathbb{N}$. It follows from Remark 1.6 that

$$\frac{k_{\Gamma_{j+1}}}{k_{\Gamma_j}} = \frac{|\det A_{j+1}|}{|\det A_j|} \cdot \frac{\xi_{\Gamma_j}}{\xi_{\Gamma_{j+1}}}.$$

Then

$$|n|qk_{\Gamma_{j+1}} = |n| \frac{\xi_{\Gamma_{j+1}}}{\xi_{\Gamma_j}} k_{\Gamma_j} \frac{|\det A_{j+1}|}{|\det A_j|} \cdot \frac{\xi_{\Gamma_j}}{\xi_{\Gamma_{j+1}}} = |n|k_{\Gamma_j} \frac{|\det A_{j+1}|}{|\det A_j|}.$$

Since T is non-degenerated, we may assume without loss of generality that $p(\Gamma_j) \neq p(\Gamma_{j+1})$ and hence $\frac{|\det A_{j+1}|}{|\det A_j|} \geq 2$ for each j . Hence by Theorem 4.2, the multiplicity of π_{γ} in $U \upharpoonright L^2(H_3(\mathbb{R})/\Gamma_{j+1})$ is at least two times greater than the multiplicity of π_{γ} in $U \upharpoonright L^2(H_3(\mathbb{R})/\Gamma_j)$. It follows that the multiplicity of π_{γ} in U is infinite.

(ii) is proved in a similar way. It is enough to note that

$$\frac{k_{\Gamma_{j+1}}}{k_{\Gamma_j}} = \frac{\xi_{\Gamma_j}}{\xi_{\Gamma_{j+1}}}$$

for all $j \geq l$ and hence $m(\gamma)$ is well defined. \square

Definition 5.3. A subgroup S in \mathbb{R}^m is *off-rational* if its closure \overline{S} is co-compact in \mathbb{R}^m and there are a subgroup $Q \subset \mathbb{Q}^m$ and a matrix $A \in GL_m(\mathbb{R})$ such that $S = AQ$.

We now associate to S an off-rational subgroup $\tau(S)$ in \mathbb{R} . Since S is off-rational, there is a sequence of matrices $A_j \in GL_m(\mathbb{R}) \cap M_m(\mathbb{Z})$ such that $A_1^{-1} \mathbb{Z}^m \subset$

$A_2^{-1}\mathbb{Z}^m \subset \dots$ and $\bigcup_{j=1}^{\infty} A_j^{-1}\mathbb{Z}^m = Q$ and hence $S = \bigcup_{j=1}^{\infty} AA_j^{-1}\mathbb{Z}^m$.¹² Consider now a sequence of subgroups

$$\frac{\det A}{\det A_1}\mathbb{Z} \subset \frac{\det A}{\det A_2}\mathbb{Z} \subset \dots$$

in \mathbb{R} . Then $\tau(S) := \bigcup_{j=1}^{\infty} \frac{\det A}{\det A_j}\mathbb{Z}$ is a dense off-rational subgroup of \mathbb{R} . It is straightforward to verify that $\tau(S)$ does not depend on the choice of the sequence $(A_j)_{j=1}^{\infty}$.

Suppose we are given a sequence $\Gamma = (\Gamma_j)_{j=1}^{\infty}$ of lattices $\Gamma_1 \supset \Gamma_2 \supset \dots$ in $H_3(\mathbb{R})$. Then $S_{\Gamma} := \bigcup_{j=1}^{\infty} p(\Gamma_j)^*$ is an off-rational subgroup of \mathbb{R}^2 and $\xi_{\Gamma} := \bigcup_{j=1}^{\infty} \xi_{\Gamma_j}^{-1}\mathbb{Z}$ is an off-rational subgroup in \mathbb{R} . If $\{c(t) \mid t \in \mathbb{R}\} \cap \bigcap_{j=1}^{\infty} \Gamma_j = \{1\}$ then ξ_{Γ} is dense in \mathbb{R} . Thus, in view of Theorem 3.1, if T is free then ξ_{Γ} is dense in \mathbb{R} .

Proposition 5.4. *We have $\tau(S_{\Gamma}) \supset \xi_{\Gamma}$.*

Proof. Let $p(\Gamma_j) = A_j(\mathbb{Z}^2)$ for some $A_j \in GL_2(\mathbb{R})$, $j \in \mathbb{N}$. Since $k_{\Gamma_j} = \frac{|\det A_j|}{\xi_{\Gamma_j}}$ by Remark 1.6, we obtain that $\frac{1}{\det A_j}\mathbb{Z} \supset \frac{1}{\xi_{\Gamma_j}}\mathbb{Z}$ for each j . \square

Proposition 5.5. *Given an off-rational subgroup S in \mathbb{R}^2 and an off-rational subgroup ξ in \mathbb{R} such that $\tau(S) \supset \xi$, there is a sequence Γ of lattices $\Gamma_1 \supset \Gamma_2 \supset \dots$ in $H_3(\mathbb{Z})$ such that $S_{\Gamma} = S$ and $\xi_{\Gamma} = \xi$. If S is dense then $\bigcap_{j=1}^{\infty} p(\Gamma_j) = \{0\}$. If, in addition, ξ is dense in \mathbb{R} then $\bigcap_{j=1}^{\infty} \Gamma_j = \{1\}$.*

Proof. Let us choose for each $j > 0$, a matrix $A_j \in GL_2(\mathbb{R})$ with $\det A_j > 0$ and a positive real ζ_j such that $(A_1^*)^{-1}\mathbb{Z}^2 \subset (A_2^*)^{-1}\mathbb{Z}^2 \subset \dots$, $\bigcup_{j=1}^{\infty} (A_j^*)^{-1}\mathbb{Z}^2 = S$, $\zeta_1^{-1}\mathbb{Z} \subset \zeta_2^{-1}\mathbb{Z} \subset \dots$ and $\bigcup_{j=1}^{\infty} \zeta_j^{-1}\mathbb{Z} = \xi$. We then have $A_1\mathbb{Z}^2 \supset A_2\mathbb{Z}^2 \supset \dots$ and $\zeta_1\mathbb{Z} \supset \zeta_2\mathbb{Z} \supset \dots$. Since $\tau(S) \supset \xi$, we can assume without loss of generality (passing to a subsequence of $(A_j)_{j=1}^{\infty}$ if necessary) that $\det(A_j^*)^{-1}|\zeta_j^{-1}$ or, equivalently, $\zeta_j \mid \det A_j$ for each j . We now let

$$\Gamma_j := \{c(t_3)b(t_2)a(t_1) \mid (t_1, t_2) \in A_j(\mathbb{Z}^2), t_3 \in \zeta_j\mathbb{Z}\}.$$

Then Γ_j is a lattice in $H_3(\mathbb{R})$ with $\xi_{\Gamma_j} = \zeta_j$, $p(\Gamma_j) = A_j(\mathbb{Z}^2)$ and $k_{\Gamma_j} = \det A_j/\zeta_j$. We let $\Gamma := (\Gamma_j)_{j=1}^{\infty}$. Of course, $\Gamma_1 \supset \Gamma_2 \supset \dots$, $\xi_{\Gamma} = \xi$ and $S_{\Gamma} = S$.

If S is dense in \mathbb{R}^2 , it follows that $\bigcap_{j=1}^{\infty} p(\Gamma_j) = \{0\}$. If ξ is dense in \mathbb{R} , it follows that $\bigcap_{j=1}^{\infty} \zeta_j\mathbb{Z} = \{0\}$ and therefore $\{c(t) \mid t \in \mathbb{R}\} \cap \bigcap_{j=1}^{\infty} \Gamma_j = \{1\}$. It follows that $\bigcap_{j=1}^{\infty} \Gamma_j = \{1\}$. \square

Definition 5.6. Two $H_3(\mathbb{R})$ -odometers T and T' are called *f-isomorphic* if they are associated with some sequences $(\Gamma_j)_{j=1}^{\infty}$ and $(\Gamma'_j)_{j=1}^{\infty}$ (respectively) of lattices in $H_3(\mathbb{R})$ such that Γ_j and Γ'_j are conjugate in $H_3(\mathbb{R})$ for each j .

Theorem 5.7. *Let $\Gamma = (\Gamma_j)_{j=1}^{\infty}$ and $\Gamma' = (\Gamma'_j)_{j=1}^{\infty}$ be two sequences of lattices in $H_3(\mathbb{R})$ such that $\Gamma_1 \supset \Gamma_2 \supset \dots$ and $\Gamma'_1 \supset \Gamma'_2 \supset \dots$. Let T denote the odometer associated to Γ and let T' denote the odometer associated to Γ' . Then T and T' are f-isomorphic if and only if $S_{\Gamma} = S_{\Gamma'}$ and $\xi_{\Gamma} = \xi_{\Gamma'}$.*

¹²We note that A_j and A are defined up to right multiplication with a matrix from $GL_m(\mathbb{Z})$.

Proof. Of course, if T and T' are f-equivalent then $S_\Gamma = S_{\Gamma'}$ and $\xi_\Gamma = \xi_{\Gamma'}$. Let us prove the converse. We first set $j_1 := 1$. Since $S_\Gamma = S_{\Gamma'}$ and $\xi_\Gamma = \xi_{\Gamma'}$, there are $l_1 > 1$ and $j_2 > j_1$ such that $p(\Gamma_{j_1})^* \subset p(\Gamma'_{l_1})^* \subset p(\Gamma_{j_2})^*$ and $\xi_{\Gamma_{j_1}}^{-1}\mathbb{Z} \subset \xi_{\Gamma'_{l_1}}^{-1}\mathbb{Z} \subset \xi_{\Gamma_{j_2}}^{-1}\mathbb{Z}$.¹³ Hence $p(\Gamma_{j_1}) \supset p(\Gamma'_{l_1}) \supset p(\Gamma_{j_2})$, $\xi_{\Gamma_{j_1}} | \xi_{\Gamma'_{l_1}}$ and $\xi_{\Gamma'_{l_1}} | \xi_{\Gamma_{j_2}}$. We now let $\tilde{\Gamma}_{j_1} := \Gamma_{j_1} \cap p^{-1}(p(\Gamma'_{l_1}))$. Then $\tilde{\Gamma}_{j_1}$ is a lattice in $H_3(\mathbb{R})$,¹⁴ $\tilde{\Gamma}_{j_1} \supset \Gamma_{j_2}$, $p(\tilde{\Gamma}_{j_1}) = p(\Gamma'_{l_1})$ and $\xi_{\tilde{\Gamma}_{j_1}} = \xi_{\Gamma_{j_1}}$. According to Lemma 1.5 there is a lattice $\hat{\Gamma}_{j_1}$ such that $\tilde{\Gamma}_{j_1} \supset \hat{\Gamma}_{j_1} \supset \Gamma_{j_2}$ with $p(\hat{\Gamma}_{j_1}) = p(\Gamma'_{l_1})$ and $\xi_{\hat{\Gamma}_{j_1}} = \xi_{\Gamma'_{l_1}}$. It follows from Proposition 1.3 and Remark 1.6 that $\hat{\Gamma}_{j_1} = q_1 \Gamma'_{l_1} q_1^{-1}$ for some element $q_1 \in H_3(\mathbb{R})$. In the next step we argue in a similar way to find integers $j_3 > j_2$ and $l_2 > l_1$, a lattice $\hat{\Gamma}_{j_2} \subset H_3(\mathbb{R})$ and an element $q_2 \in H_3(\mathbb{R})$ such that $\Gamma_{j_2} \supset \hat{\Gamma}_{j_2} \supset \Gamma_{j_3}$ and $\hat{\Gamma}_{j_2} = q_2 \Gamma'_{l_2} q_2^{-1}$. Continuing this infinitely many times we obtain a sequence $(\hat{\Gamma}_{j_s})_{s=1}^\infty$ of lattices in $H_3(\mathbb{R})$, a subsequence $(\Gamma'_{l_s})_{s=1}^\infty$ of Γ' and a sequence $(q_s)_{s=1}^\infty$ of elements in $H_3(\mathbb{R})$ such that

$$(5-1) \quad \begin{aligned} \Gamma_{j_1} \supset \hat{\Gamma}_{j_1} \supset \Gamma_{j_2} \supset \hat{\Gamma}_{j_2} \supset \cdots \text{ and} \\ \hat{\Gamma}_{j_s} = q_s \Gamma'_{l_s} q_s^{-1} \text{ with } q_s \in H_3(\mathbb{R}) \text{ for each } s > 1. \end{aligned}$$

It follows from (5-1) that T is isomorphic to the $H_3(\mathbb{R})$ -odometer associated with $(\hat{\Gamma}_{j_s})_{s=1}^\infty$. Of course, T' is isomorphic to the $H_3(\mathbb{R})$ -odometer associated with $(\Gamma'_{l_s})_{s=1}^\infty$. Hence T and T' are f-isomorphic. \square

Corollary 5.8. *The Heisenberg odometers T and T' are f-isomorphic if and only if the Koopman unitary representations of $H_3(\mathbb{R})$ associated with them are unitarily equivalent.*

Proof. Let T be associated with a sequence Γ of nested lattices in $H_3(\mathbb{R})$ and let T' be associated with a sequence Γ' of nested lattices in $H_3(\mathbb{R})$. It follows from Theorem 5.2 that if the unitary representations of $H_3(\mathbb{R})$ associated with T and T' are unitarily equivalent then $S_\Gamma = S_{\Gamma'}$ and $\xi_\Gamma = \xi_{\Gamma'}$. Applying Theorem 5.7 we conclude that T and T' are f-isomorphic. The converse is obvious. \square

We now consider Cartesian products of Heisenberg odometers. First of all we recall the Clebsch-Gordan decompositions for the tensor products of two irreducible unitary representations of $H_3(\mathbb{R})$ (see [Ki]):

$$\begin{aligned} \pi_{\alpha,\beta} \otimes \pi_{\alpha',\beta'} &= \pi_{\alpha+\alpha',\beta+\beta'}, \quad \text{for all } \alpha, \beta, \alpha', \beta', \\ \pi_\gamma \otimes \pi_{\gamma'} &= \bigoplus_1^\infty \pi_{\gamma+\gamma'} \quad \text{whenever } \gamma \neq -\gamma', \\ \pi_\gamma \otimes \pi_{-\gamma} &= \int_{\mathbb{R}^2} \pi_{\alpha,\beta} d\alpha d\beta \quad \text{for all } \gamma \\ \pi_\gamma \otimes \pi_{\alpha,\beta} &= \pi_\gamma \quad \text{for all } \alpha, \beta, \gamma. \end{aligned}$$

¹³This is true because all the groups $p(\Gamma_j)^*$, $p(\Gamma'_j)^*$, $\xi_{\Gamma_j}^{-1}\mathbb{Z}$, $\xi_{\Gamma'_j}^{-1}\mathbb{Z}$, $j \in \mathbb{N}$, are finitely generated.

¹⁴ $\tilde{\Gamma}_{j_1}$ and $\tilde{\Gamma}'_{j_1}$ have the same p -fibers over $p(\Gamma'_{l_1})$.

Proposition 5.9. *Let T and T' be two Heisenberg odometers associated with the nested sequences of lattices $\Gamma = (\Gamma_j)_{j=1}^\infty$ and $\Gamma' = (\Gamma'_j)_{j=1}^\infty$ in $H_3(\mathbb{R})$ respectively. Then*

- (i) $T \times T'$ is ergodic if and only if $S_\Gamma \cap S_{\Gamma'} = \{0\}$.
- (ii) $T \times T'$ is ergodic and has discrete maximal spectral type if and only if $S_\Gamma \cap S_{\Gamma'} = \{0\}$ and $\xi_\Gamma \cap \xi_{\Gamma'} = \{0\}$. In this case the Koopman unitary representation $U_{T \times T'}$ of $H_3(\mathbb{R})$ decomposes into irreducible representations as follows

$$(5-2) \quad U_{T \times T'} = \bigoplus_{(\alpha, \beta) \in S_\Gamma + S_{\Gamma'}} \pi_{\alpha, \beta} \oplus \bigoplus_{0 \neq \gamma \in \xi_\Gamma + \xi_{\Gamma'}} \bigoplus_1^\infty \pi_\gamma.$$

- (iii) $T \times T'$ is not isomorphic to any Heisenberg odometer.¹⁵

Proof. (i) and (ii) follow immediately from the Clebsch-Gordan decompositions, Theorem 4.1 and the fact that the invariant vectors of the Koopman unitary representation of $H_3(\mathbb{R})$ form the subspace corresponding to the component $\pi_{0,0}$.

(iii) Suppose that $T \times T'$ is a Heisenberg odometer. Then it is ergodic and has discrete maximal spectral type. Hence by (ii), $\xi_\Gamma \cap \xi_{\Gamma'} = \{0\}$. This implies that the sum $\xi_\Gamma + \xi_{\Gamma'}$ is not off-rational.¹⁶ On the other hand, by (5-2) and Theorem 4.1, $\xi_\Gamma + \xi_{\Gamma'} = \xi_{\Gamma''}$ for a nested sequence Γ'' of lattices in $H_3(\mathbb{R})$ that determines the Heisenberg odometer $T \times T'$. Since $\xi_{\Gamma''}$ is always off-rational, we get a contradiction. \square

Our next purpose is to show that Heisenberg odometers (even the normal ones) are not isospectral.

Let T be the Heisenberg odometer associated with a decreasing sequence $(\Gamma_j)_{j+1}^\infty$ of lattices in $H_3(\mathbb{Z})$. Let (X, μ) be the space of T and fix $x \in X$. Then $x = (g_j \Gamma_j)_{j \in \mathbb{Z}}$ with $g_j \Gamma_j = g_{j+1} \Gamma_j$ for all $j \in \mathbb{N}$. Of course, we have $g_1 \Gamma_1 g_1^{-1} \supset g_2 \Gamma_2 g_2^{-1} \supset \dots$. Let $T^{(x)}$ denote the odometer associated with the sequence $(g_j \Gamma_j g_j^{-1})_{j=1}^\infty$ of lattices in $H_3(\mathbb{R})$. Denote by $X^{(x)}$ the space of this odometer. Then the map

$$i^{(x)} : X \ni (h_1 \Gamma_1, h_2 \Gamma_2, \dots) \mapsto (h_1 g_1^{-1} \cdot g_1 \Gamma_1 g_1^{-1}, h_2 g_2^{-1} \cdot g_2 \Gamma_2 g_2^{-1}, \dots) \in X^{(x)}$$

is a topological conjugacy of T with $T^{(x)}$ such that $i^{(x)} x = (g_j \Gamma_j g_j^{-1})_{j \in \mathbb{N}} \in X^{(x)}$. Below we call $i^{(x)}$ a *changing the origin in X* .

Lemma 5.10. *Let T and T' be $H_3(\mathbb{R})$ -odometers associated with the sequences $(\Gamma_j)_{j \in \mathbb{N}}$ and $(\Gamma'_j)_{j \in \mathbb{N}}$ respectively. If T' is a factor (in the measure theoretic sense) of T then there are sequences $(g_j)_{j \in \mathbb{N}}$ and $(g'_j)_{j \in \mathbb{N}}$ in $H_3(\mathbb{R})$ with $g_j \Gamma_j = g_{j+1} \Gamma_j$ and $g'_j \Gamma'_j = g'_{j+1} \Gamma'_j$ for all $j \in \mathbb{N}$ such that for each $n > 0$, there is $m > 0$ with $g_m \Gamma_m g_m^{-1} \subset g'_n \Gamma'_n (g'_n)^{-1}$.*

Proof. Let (X, μ) and (X', μ') be the spaces of T and T' . Let $\phi : X \rightarrow X'$ be the Borel factor map intertwining T with T' . Denote by p_n the canonical projection of

¹⁵As follows from the proof below, $T \times T'$ even is not spectrally equivalent to any Heisenberg odometer.

¹⁶Indeed, if $\theta_1 Q_1 \cap \theta_2 Q_2 = 0$ for some non-zero reals θ_1, θ_2 and nontrivial subgroups $Q_1, Q_2 \subset \mathbb{Q}$ then $\theta_1/\theta_2 \notin \mathbb{Q}$. If, furthermore, $\theta_1 Q_1 + \theta_2 Q_2 = \theta_3 Q_3$ for a non-zero $\theta_3 \in \mathbb{R}$ and nontrivial subgroup $Q_3 \subset \mathbb{Q}$ then we get that θ_3/θ_2 is rational and irrational at the same time.

X onto $H_3(\mathbb{R})/\Gamma_n$ and denote by q_n the canonical projection of X' onto $H_3(\mathbb{R})/\Gamma'_n$. By Lusin's theorem, for each $n \in \mathbb{N}$, there is a closed subset $X_n \subset X$ with $\mu(X_n) > 1 - 2^{-25n}$ such that the map $q_n \circ \phi$ is continuous on X_n . For each n , let us fix three elements $\gamma_{n,1}, \gamma_{n,2}$ and $\gamma_{n,3}$ that generate Γ_n . It follows from the pointwise ergodic theorem that for each $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$ there is a subset $Y_n \subset X_n$ of "large" measure such that for each $x \in Y_n$, there exists a limit

$$\lim_{N \rightarrow \infty} \frac{\#\{j \in \{1, \dots, N\} \mid \gamma_{n,i}^j x \in X_n\}}{N} > 0.9.$$

The "large" means that the series $\sum_{n=1}^{\infty} (1 - \mu(Y_n))$ converges. Hence by the Borel-Cantelli lemma, the intersection $\bigcap_{n>M} (X_n \cap Y_n)$ is non-empty for some $M \in \mathbb{N}$. Without loss of generality we may assume that $M = 1$. Changing the origin in X , i.e. replacing T with an isomorphic odometer, we may assume that

$$e := (\Gamma_1, \Gamma_2, \dots) \in \bigcap_{n>M} (X_n \cap Y_n).$$

In a similar way, changing the origin in X' we may assume that

$$\phi(e) = e' := (\Gamma'_1, \Gamma'_2, \dots).$$

Under these assumptions we will show that for each $n > 0$ there is $m > 0$ with $\Gamma_m \subset \Gamma'_n$.

Given n , we fix a small open neighborhood \tilde{U} of Γ'_n in $H_3(\mathbb{Z})/\Gamma'_n$. Then $U := q_n^{-1}(\tilde{U})$ is an open neighborhood of e' in X' . Therefore there is an open neighborhood V of e in X such that

$$(5-3) \quad \phi^{-1}(U) \supset V \cap X_n \neq \emptyset.$$

By definition of the topology on X , we may assume without loss of generality that there are $m > 0$ and an open neighborhood \tilde{V} of Γ_m in $H_3(\mathbb{R})/\Gamma_m$ such that $V = p_m^{-1}(\tilde{V})$. For each $j \in \mathbb{N}$ and $i \in \{1, 2, 3\}$, we have $\gamma_{m,i}^j \Gamma_m = \Gamma_m$. Hence $\gamma_{m,i}^j e \in V$. Let $J_N := \{j \in \{1, \dots, N\} \mid \gamma_{m,i}^j e \in X_n\}$. Then for each $j \in J_N$, we have $\gamma_{m,i}^j e \in V \cap X_n$. It follows now from (5-3) and the fact that ϕ is equivariant that

$$\gamma_{m,i}^j e' = \gamma_{m,i}^j \phi(e) = \phi(\gamma_{m,i}^j e) \in U.$$

Hence $\gamma_{m,i}^j \Gamma'_n \in \tilde{U}$ for each $j \in J_N$. Since $\#(J_N)/N > 0.9$ for all large N and \tilde{U} is small, it is easy to deduce that $\gamma_{m,i} \in \Gamma'_n$. Indeed, $d := p(\gamma_{m,i})p(\Gamma'_n) \in \mathbb{R}^2/p(\Gamma'_n)$. As usual, p stands for the standard projection of $H_3(\mathbb{R})$ onto \mathbb{R}^2 . Take a weak limit point κ of the sequence $\frac{1}{N} \sum_{j=1}^N \delta_{\gamma_{m,i}^j \Gamma'_n}$ of measures on $H_3(\mathbb{R})/\Gamma'_n$. The natural projection

$$v : H_3(\mathbb{R})/\Gamma'_n \rightarrow H_3(\mathbb{R})/p^{-1}(p(\Gamma'_n)) = \mathbb{T}^2$$

maps κ to the limit of the sequence $\frac{1}{N} \sum_{j=1}^N \delta_{d^j}$. This limit is the Haar measure, say λ_A , of the compact monothetic subgroup A of \mathbb{T}^2 generated by the element d . Since we assumed that \tilde{U} is small, the projection $v(\tilde{U})$ is also a small (in diameter) neighborhood of the unit in \mathbb{T}^2 . We thus obtain that $\lambda_A(v(\tilde{U})) > 0.8$. (Of course,

we may assume without loss of generality that the boundary of $v(\tilde{U})$ is λ_A -null.) This is only possible if A is trivial, i.e. $\gamma_{m,i} \in c(t)\Gamma'_n$ for some $t \in [0, \xi_{\Gamma'_n})$. Thus κ is supported by the circle $\mathbb{T} = v^{-1}(p^{-1}(p(\Gamma'_n)))$, which is the fiber of the fibration v over the unit of \mathbb{T}^2 . Since the intersection of \tilde{U} with this fiber is small in diameter, we apply a similar reasoning as above to conclude that $t = 0$.

Therefore $\Gamma_m \subset \Gamma'_n$. \square

We now provide an example of f-isomorphic but non-isomorphic odometers.

Example 5.11(cf. [Li-Ug, Example 4.9]). Fix a sequence of natural numbers $k_1 < k_2 < \dots$ such that $k_1 = 1$ and $k_n(k_n + 1) = k_{n+1}$ for each n . Let

$$\begin{aligned}\Gamma_n &:= \{c(k_n j_3)b(k_n j_2)a(k_n j_1) \mid j_1, j_2, j_3 \in \mathbb{Z}\} \text{ and} \\ \Gamma'_n &:= \{c(k_n j_3 + j_1)b(k_n j_2)a(k_n j_1) \mid j_1, j_2, j_3 \in \mathbb{Z}\},\end{aligned}$$

$n \in \mathbb{N}$. It is easy to see that $H_3(\mathbb{Z}) = \Gamma_1 \supset \Gamma_2 \supset \dots$ and $H_3(\mathbb{Z}) = \Gamma'_1 \supset \Gamma'_2 \supset \dots$. Moreover, Γ_n and Γ'_n are both normal in $H_3(\mathbb{Z})$ for each $n \in \mathbb{N}$. Denote by T and T' the $H_3(\mathbb{R})$ -odometers associated with the sequences $(\Gamma_n)_{n=1}^\infty$ and $(\Gamma'_n)_{n=1}^\infty$ respectively. They are normal. Since $b(1/k_n)\Gamma'_n b(-1/k_n) = \Gamma_n$ for each $n \in \mathbb{N}$, we obtain that T and T' are f-isomorphic. If they were isomorphic then by Lemma 5.10, there is $h \in H_3(\mathbb{R})$ such that for each $n > 0$, there exists $m > 0$ with $\Gamma'_m \subset h\Gamma_n h^{-1}$. However $\xi_{\Gamma'_m} = 1$ for each m but $\xi_{h\Gamma_n h^{-1}} \rightarrow \infty$ as $n \rightarrow \infty$, a contradiction. We also examine certain ‘‘symmetry’’ property for T and T' . Recall that a measure preserving action R of $H_3(\mathbb{R})$ is called *symmetric* [Da4] if R is isomorphic to $R \circ \theta$, where θ denotes the flip in $H_3(\mathbb{R})$. Since $\theta(H_3(\mathbb{Z})) = H_3(\mathbb{Z})$, the symmetric $H_3(\mathbb{Z})$ -actions are defined in a similar way. It is easy to see that T is symmetric because $\theta(\Gamma_n) = \Gamma_n$ for each j . It is straightforward to verify that

$$\theta(\Gamma'_n) = a(-1/k_n)\Gamma_n a(1/k_n) = \{c(k_n j_3 + j_2)b(k_n j_2)a(k_n j_1) \mid j_1, j_2, j_3 \in \mathbb{Z}\}$$

for each n . We now show that T' is asymmetric. If T' and $T' \circ \theta$ were isomorphic then by Lemma 5.10, there is $h \in H_3(\mathbb{R})$ such that for each $n > 0$, there exists $m > 0$ with $\Gamma'_m \subset h\theta(\Gamma'_n)h^{-1}$. However it is easy to see that

$$\inf\{t \in \mathbb{R} \mid c(t)a(r) \in \Gamma'_m \text{ for some } r \in \mathbb{R}'\} = 1$$

while

$$\inf\{t \in \mathbb{R} \mid c(t)a(r) \in \theta(\Gamma'_n) \text{ for some } r \in \mathbb{R}'\} \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

a contradiction. Hence T' is not symmetric. We also note that T' is f-isomorphic to $T' \circ \theta$ because $\theta(\Gamma'_n) = a(-1/k_n)b(1/k_n)\Gamma'_n b(-1/k_n)a(1/k_n)$. Hence the Koopman representations $U_{T'}$ and $U_{T' \circ \theta}$ are unitarily equivalent.

We conclude this section with a ‘‘coordinate-free’’ definition of Heisenberg odometers.

Proposition 5.12. *Let T be an ergodic action of $H_3(\mathbb{R})$ on a standard probability space (X, \mathfrak{B}, μ) . Let $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \dots$ be a sequence of factors of T such that the union $\bigcup_{j=1}^\infty \mathfrak{F}_j$ is dense in \mathfrak{B} and the restriction $T \upharpoonright \mathfrak{F}_j$ is isomorphic to a transitive $H_3(\mathbb{R})$ -action on the 3-dimensional torus. Then T is a Heisenberg odometer.*

Proof. Since the dynamical system $T \upharpoonright \mathfrak{F}_j$ is transitive, it is isomorphic to the homogeneous space $H_3(\mathbb{R})/\Gamma_j$ for a co-compact subgroup $\Gamma_j \subset H_3(\mathbb{R})$. Since $H_3(\mathbb{R})/\Gamma_j$ is isomorphic to the 3-dimensional torus, Γ_j is a lattice. Thus the dynamical system (X, μ, T) is the projective limit of the following sequence of $H_3(\mathbb{R})$ -homogeneous spaces

$$H_3(\mathbb{R})/\Gamma_1 \xleftarrow{q_1} H_3(\mathbb{R})/\Gamma_2 \xleftarrow{q_2} \dots$$

Hence there is a sequence $(g_n)_{n=1}^\infty$ of elements in $H_3(\mathbb{R})$ such that $\Gamma_j \supset g_j^{-1}\Gamma_{j+1}g_j$ and $q_j(g\Gamma_{j+1}) = gg_j\Gamma_j$ for each j . We now let

$$\Gamma'_j := g_1^{-1}g_2^{-1} \cdots g_{j-1}^{-1}\Gamma_j g_{j-1} \cdots g_2 g_1$$

and define an equivariant isomorphism $r_j : H_3(\mathbb{R})/\Gamma_j \rightarrow H_3(\mathbb{R})/\Gamma'_j$ by setting $r_j(g\Gamma_j) := gg_{j-1} \cdots g_2 g_1 \Gamma'_j$. Then we obtain a nested sequence $\Gamma'_1 \supset \Gamma'_2 \supset \dots$ of lattices in $H_3(\mathbb{R})$. Moreover, the following infinite diagram commutes

$$\begin{array}{ccccc} H_3(\mathbb{R})/\Gamma_1 & \xleftarrow{q_1} & H_3(\mathbb{R})/\Gamma_2 & \xleftarrow{q_2} & \dots \\ r_1 \downarrow & & r_1 \downarrow & & \\ H_3(\mathbb{R})/\Gamma'_1 & \longleftarrow & H_3(\mathbb{R})/\Gamma'_2 & \longleftarrow & \dots, \end{array}$$

where the horizontal arrows in the bottom line denote the natural projections. Hence T is isomorphic to the Heisenberg odometer associated with the sequence $(\Gamma'_j)_{j=1}^\infty$. \square

6. JOINING STRUCTURE OF HEISENBERG ODOMETERS

In this section we describe the 2-fold ergodic self-joinings, as dynamical systems, for all Heisenberg odometers. We recall that a 2-fold self-joining of a G -action $T = (T_g)_{g \in G}$ on a standard probability space (X, \mathfrak{B}, μ) is a measure λ on the product space $(X \times X, \mathfrak{B} \otimes \mathfrak{B})$ such that λ is invariant under the diagonal action $(T_g \times T_g)_{g \in G}$ and the projections of λ to each of the two coordinates are μ . The corresponding dynamical system $(X \times X, \lambda, (T_g \times T_g)_{g \in G})$ is also called a 2-fold self-joining of T . A 2-fold self-joining of T is called off-diagonal if it is supported on the graph of a transformation of (X, μ) commuting with T . In this section we show that 2-fold self-joinings of Heisenberg odometers need not be Heisenberg odometers themselves, even in the transitive case. In particular, there is an ergodic non-transitive 2-fold self-joining of a transitive Heisenberg odometer. This is in sharp contrast with the actions with pure point spectrum. Indeed, each transitive action T of a locally compact second countable group G with pure point spectrum can be represented as an action by rotations (via an onto group homomorphism from G to K) on a homogeneous space K/H , where K is a compact second countable group and H is a subgroup of K [Ma3]. It is a standard argument (see [dJRu] for example) that each ergodic 2-fold self-joining of T considered as a dynamical system is a factor of an ergodic 2-fold self-joining of the transitive G -action by rotations on K . All such self-joinings are off-diagonal, isomorphic to the original G -action on K , and hence transitive themselves. Since each factor of a transitive system is transitive, it follows that each ergodic 2-fold self-joining of a transitive action with pure point spectrum is transitive. Similar reasoning shows that the

family of G -actions with pure point spectrum is also closed under the operation of taking ergodic 2-fold self-joinings.¹⁷

(A) Self-joinings of transitive Heisenberg odometers. Let Γ be a lattice in $H_3(\mathbb{R})$. In view of Proposition 1.3, we may assume without loss of generality that $\Gamma = \{c(n/k)b(m)a(l) \mid n, m, l \in \mathbb{Z}\}$ for some $k \in \mathbb{N}$. Every element $g \in H_3(\mathbb{R})$ can be written uniquely as $g = c(t_3)b(t_2)a(t_1)\gamma$ for some $\gamma \in \Gamma$ and $0 \leq t_3 < 1/k$, $0 \leq t_2 < 1$ and $0 \leq t_1 < 1$. Then the quotient space $H_3(\mathbb{R})/\Gamma$ is a 3-torus

$$\mathbb{T}^3 = \{(t_1, t_2, t_3) \mid 0 \leq t_1 < 1, 0 \leq t_2 < 1 \text{ and } 0 \leq t_3 < 1/k\}.$$

Our purpose now is to describe the ergodic 2-fold self-joinings of the quotient (transitive) action $T = (T_g)_{g \in H_3(\mathbb{R})}$ of $H_3(\mathbb{R})$ on this space. We write it in skew product form as follows:

$$(6-1) \quad T_g(y, z) = (p(g) \cdot y, \alpha(g, y) + z),$$

where $(y, z) \in Y \times Z := (\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{R}/k^{-1}\mathbb{Z})$, the symbol “ \cdot ” denotes the usual action of \mathbb{R}^2 on Y by rotations and $\alpha : H_3(\mathbb{R}) \times Y \rightarrow Z$ is the corresponding cocycle. Let $\tilde{\Gamma} := p^{-1}(p(\Gamma)) = \{c(t)b(m)a(n) \mid t \in \mathbb{R}, n, m \in \mathbb{Z}\}$. Since $Y = H_3(\mathbb{R})/\tilde{\Gamma}$, the map $s : Y \ni (y_1, y_2) \mapsto b(y_2)a(y_1) \in H_3(\mathbb{R})$, $0 \leq y_1, y_2 < 1$, is a cross-section of the quotient map $H_3(\mathbb{R}) \rightarrow H_3(\mathbb{R})/\tilde{\Gamma}$. Denote by $h_s : H_3(\mathbb{R}) \times Y \rightarrow \tilde{\Gamma}$ the corresponding choice-cocycle. A direct calculation shows that

$$\begin{aligned} h_s(c(t_3), y) &= c(t_3), \\ h_s(b(t_2), y) &= b([t_2 + y_2])c(-y_1[t_2 + y_2]), \\ h_s(a(t_1), y) &= a([t_1 + y_1])c(t_1 y_2) \end{aligned}$$

for all $t_1, t_2, t_3 \in \mathbb{R}$ and $y = (y_1, y_2)$, $0 \leq y_1, y_2 < 1$, where “[\cdot]” denotes the integer part. By Remark 2.0 (in which $N = \tilde{\Gamma}$), we may assume without loss of generality that $\alpha = \phi \circ h_s$ for a group homomorphism $\phi : \tilde{\Gamma} \rightarrow Z$. We now compute ϕ . For $\tilde{\gamma} = c(t)b(m)a(n)$ with $t \in \mathbb{R}$ and $n, m \in \mathbb{Z}$, we have $T_{\tilde{\gamma}}\Gamma = c(t)\Gamma$. Applying (6-1), we obtain

$$T_{\tilde{\gamma}}(\tilde{\Gamma}, 0) = (\tilde{\Gamma}, \alpha(\tilde{\gamma}, \tilde{\Gamma})) = (\tilde{\Gamma}, t + k^{-1}\mathbb{Z}).$$

Thus, we obtain that $\phi(c(t)b(m)a(n)) = t + k^{-1}\mathbb{Z}$. This yields

$$\begin{aligned} \alpha(c(t_3), y) &= t_3 + k^{-1}\mathbb{Z}, \\ \alpha(b(t_2), y) &= -y_1[t_2 + y_2] + k^{-1}\mathbb{Z}, \\ \alpha(a(t_1), y) &= t_1 y_2 + k^{-1}\mathbb{Z} \end{aligned}$$

for all $t_1, t_2, t_3 \in \mathbb{R}$ and $y = (y_1, y_2)$, $0 \leq y_1, y_2 < 1$. For $d = (d_1, d_2) \in Y$, $0 \leq d_1, d_2 < 1$, we denote by $\alpha \circ d$ the cocycle

$$H_3(\mathbb{R}) \times Y \ni (g, z) \mapsto \alpha \circ d(g, y) := \alpha(g, d + y) \in Z.$$

¹⁷For each G -action T with pure point spectrum there are a compact group K , a closed subgroup $H \subset K$ and a continuous homomorphism $\phi : G \rightarrow K$ with dense range in K such that $T_g k H = \phi(g) k H$ for all $g \in G$ and $k \in K$ [Ma3]. Hence, by [dJRu], every ergodic 2-fold self-joining of T is a factor of an ergodic 2-fold self-joining of the G -action by rotations (via ϕ) on K . The latter joining is off-diagonal; it is isomorphic as a dynamical system to the G -action by rotations (via ϕ) on K . It remains to note that a factor of each action with pure point spectrum is an action with pure point spectrum.

By Remark 2.0, we can represent the cocycle

$$\alpha \times \alpha \circ d : H_3(\mathbb{R}) \times Y \rightarrow Z \times Z$$

as $\psi \circ h_s$, where ψ is a continuous group homomorphism from $\tilde{\Gamma}$ to $Z \times Z$. To compute ψ , we first note that

$$h_s \circ d(g, y) = h_s(g, y + d) = s(p(g) \cdot (y + d))^{-1} g s(y + d).$$

Substituting $y := \tilde{\Gamma}$ and $g := c(t)b(m)a(n)$ we obtain that

$$h_s \circ d(g, \tilde{\Gamma}) = s(d + \tilde{\Gamma})^{-1} g s(d + \tilde{\Gamma}) = c(nd_2 - md_1)g.$$

This formula, the formula for ϕ and Remark 2.0 yield that

$$(6-2) \quad \psi(c(t)b(m)a(n)) = (t + k^{-1}\mathbb{Z}, t + nd_2 - md_1 + k^{-1}\mathbb{Z}).$$

If d is aperiodic, i.e. d_1 or d_2 is irrational, then the subgroup $\psi(\tilde{\Gamma})$ is dense in $Z \times Z$. It now follows from Remark 2.1 the essential range $E(\alpha \times \alpha \circ d)$ of the cocycle $\alpha \times \alpha \circ d$ is $Z \times Z$. We note however that $\psi(\tilde{\Gamma}) \neq Z \times Z$. On the other hand, if d is periodic, i.e. d_1 and d_2 are both rational, then there is a positive integer $q = q(d)$ such that

$$(6-3) \quad E(\alpha \times \alpha \circ d) = \psi(\tilde{\Gamma}) = \bigcup_{j=0}^{q-1} \{(t + k^{-1}\mathbb{Z}, t + j/(qk) + k^{-1}\mathbb{Z} \mid 0 \leq t < 1\}.$$

We denote this subgroup of $Z \times Z$ by D_q .

Let Δ_d denote the measure on $Y \times Y$ sitting on the subset $\{(y, d+y) \mid y \in Y\}$ and projecting on the Haar measure on Y along each of the two coordinate projections. Given a closed subgroup Λ in $Z \times Z$, we denote by λ_Λ the Haar measure on Λ . We consider it as a measure on $Z \times Z$. Given $z \in Z$, we denote by $\lambda_\Lambda \circ z$ the image of λ_Λ under the rotation $Z \times Z \ni (z_1, z_2) \mapsto (z_1, z_2 + z) \in Z \times Z$.

Proposition 6.1. *The set $J_2^e(T)$ of all ergodic 2-fold self-joinings of T is the union of two families as follows:¹⁸*

$$J_2^e(T) = \{\Delta_d \times \lambda_{Z \times Z} \mid d \text{ is aperiodic}\} \cup \{\Delta_d \times \lambda_{D_{q(d)}} \circ z \mid d \text{ is periodic, } z \in Z\}.$$

Every joining from the first family is a non-transitive dynamical system and every joining from the second family is a transitive dynamical system.

Proof. Take $\kappa \in J_2^e(T)$. Represent T in the skew product form (6-1). Then κ is a measure on the space $Y \times Z \times Y \times Z$. Project κ onto $Y \times Y$. The projection is an ergodic 2-fold self-joining on the transitive action of the 2-torus Y on itself. Hence there is $d \in Y$ such that this projection is Δ_d . It is a standard argument that the dynamical system $((T_g \times T_g)_{g \in H_3(\mathbb{R})}, \kappa)$ can now be represented as the skew product over the same (as T) base Y but with another cocycle $\alpha \times \alpha \circ d$. It remains to apply the argument preceding the statement of Proposition 6.1 to obtain the first assertion of the proposition.

Let V denote the action of $\tilde{\Gamma}$ on the compact group $\overline{\psi(\tilde{\Gamma})}$ by translations via ψ , i.e. an element $\gamma \in \tilde{\Gamma}$ acts as the translation by $\psi(\gamma)$. It is easy to see that the action $((T_g \times T_g)_{g \in H_3(\mathbb{R})}, \kappa)$ is isomorphic to $\text{Ind}_{\tilde{\Gamma}}^{H_3(\mathbb{R})}(V)$, the second claim of the proposition follows from the fact whether or not $\psi(\tilde{\Gamma})$ is closed and the following lemma. \square

¹⁸Provided that a special representation of T as a skew product is chosen as above.

Lemma 6.2. *Let G be a locally compact second countable group and let H be a closed subgroup of G . A measure preserving action V of H is transitive if and only if the induced measure preserving action $\text{Ind}_H^G(V)$ of G is transitive.*

Proof. If V is non-transitive then, obviously, $\text{Ind}_H^G(V)$ is non-transitive. Suppose now that V is transitive. Denote by N a closed subgroup of H such that V is isomorphic to the natural H -action on the homogeneous space H/N . Then V is isomorphic to the induced action $\text{Ind}_N^H(I)$, where I denotes the trivial action of N on a singleton. We now have (see [Ma1]) that

$$\text{Ind}_H^G(V) = \text{Ind}_H^G(\text{Ind}_N^H(I)) = \text{Ind}_N^G(I),$$

i.e. $\text{Ind}_H^G(V)$ is isomorphic to the natural G -action on G/N . \square

Remark 6.3. We note that if an action of an Abelian locally compact group is transitive then every ergodic 2-fold self-joining of this action is also transitive (and isomorphic to the original action). As follows from Proposition 6.1, this is no longer true for the transitive actions of the Heisenberg group.

We are going to show now that an ergodic 2-fold self-joining of a transitive odometer can be rather “far from” the Heisenberg odometers. Though its maximal spectral type is discrete, the subgroup of 1-dimensional components of the spectrum is not off-rational.

Example 6.4. Consider a particular case when $k = 1$, i.e. $\Gamma = H_3(\mathbb{Z})$. Suppose that d_1 is irrational (see (6-2)). Then the action $R = (R_g)_{g \in H_3(\mathbb{R})}$ of $H_3(\mathbb{R})$ defined on $Y \times Z \times Z$ by

$$(6-4) \quad R_g(y, z) := (p(g) \cdot y, z + \psi(h_s(g, y))), \quad y \in Y, z \in Z \times Z,$$

is an ergodic 2-fold self-joining of T . Denote by U_R the Koopman unitary representations of $H_3(\mathbb{R})$ generated by R . We are going to decompose U_R into irreducible components. It follows from (6-4) that

$$L^2(H_3(\mathbb{R})/\Gamma) = \bigoplus_{n,m=1}^{\infty} L^2(Y) \otimes \eta_n \otimes \eta_m,$$

and each subspace $L^2(Y) \otimes \eta_n \otimes \eta_m$ is U_R -invariant, where $\eta_n(t) := \exp(2\pi int)$, $t \in [0, 1)$, $n \in \mathbb{Z}$. Since

$$R_{c(t)}(y, z) = (y, z + (t, t) + \mathbb{Z}^2) \quad \text{for all } t \in \mathbb{R},$$

It follows that $U_R(c(t)) \upharpoonright (L^2(Y) \otimes \eta_n \otimes \eta_m) = \exp(2\pi it(n+m))\text{Id}$. This yields

$$(6-5) \quad U_R = \bigoplus_{\gamma \in \mathbb{Z}} \bigoplus_{r=1}^{\infty} \pi_{\gamma} \oplus \bigoplus_{k \in \mathbb{Z}} U_R \upharpoonright (L^2(Y) \otimes \eta_k \otimes \eta_{-k})$$

We note that the space $\mathcal{H}_1 := \bigoplus_{k \in \mathbb{Z}} L^2(Y) \otimes \eta_k \otimes \eta_{-k}$ is the subspace of vectors fixed by $(U_R(c(t)))_{t \in \mathbb{R}}$. It is straightforward to verify that $\mathcal{H}_1 = L^2(Y) \otimes \mathcal{T}$, where

$$\mathcal{T} := \{v \in L^2(Z \times Z) \mid v(z_1, z_2) = v(z_1 + z, z_2 + z) \text{ at a.a. } z_1, z_2 \text{ for each } z \in Z\}.$$

Hence $\mathcal{T} = L^2(\mathfrak{F})$, where \mathfrak{F} is the sub- σ -algebra of sets in $Z \times Z$ measurable with respect to the map $Z \times Z \ni (z_1, z_2) \mapsto z_1 - z_2 \in Z$. Thus, the σ -algebra $\mathfrak{B}_Y \otimes \mathfrak{F}$ is a factor of R such that $L^2(\mathfrak{B}_Y \otimes \mathfrak{F}) = \mathcal{H}_1$. In view of (6-4), the restriction $\widehat{R} = (\widehat{R}_g)_{g \in G}$ of R to this factor can be written as

$$\widehat{R}_g(y, z) = (p(g) \cdot y, z + \widetilde{\psi}(h_s(g, y))),$$

where $\widetilde{\psi} : \widetilde{\gamma} \rightarrow Z$ is the difference of the first and the second coordinate of $\widetilde{\psi}$, i.e., by (6-2), $\widetilde{\psi}(c(t)b(m)a(n)) = md_1 - nd_2 + \mathbb{Z}$, $t \in \mathbb{R}$, $n, m \in \mathbb{R}$. This means that \widehat{R} is induced from the the following action of $\widetilde{\Gamma}$ on Z by rotations

$$\widetilde{\Gamma} \times Z \ni (c(t)b(m)a(n), z) \mapsto z + md_1 - nd_2 + \mathbb{Z}.$$

This action has a pure point spectrum and hence \widehat{R} has pure point spectrum. It is straightforward to verify that

$$U_R \upharpoonright \mathcal{H}_1 = \bigoplus_{\alpha, \beta \in \Omega_1} \pi_{\alpha, \beta},$$

where $\Omega_1 = \mathbb{Z}^2 + \begin{pmatrix} d_2 & 0 \\ 0 & -d_1 \end{pmatrix} \mathbb{Z}^2$. Since d_1 is irrational, Ω_1 is not off-rational. In view of (6-5),

$$U_R = \bigoplus_{\gamma \in \mathbb{Z}} \bigoplus_{r=1}^{\infty} \pi_{\gamma} \oplus \bigoplus_{\alpha, \beta \in \Omega_1} \pi_{\alpha, \beta}.$$

Since Ω_1 is not off-rational, R is not isomorphic to any Heisenberg odometer.¹⁹

(B) Self-joinings of general Heisenberg odometers. We now consider a Heisenberg odometer T associated to a sequence of lattices $\Gamma_1 \supset \Gamma_2 \supset \dots$ in $H_3(\mathbb{R})$. Then every ergodic 2-fold self-joining of T is the inverse limit of a sequence of ergodic 2-fold self-joinings of the transitive actions of $H_3(\mathbb{R})$ on the homogeneous spaces $H_3(\mathbb{R})/\Gamma_j$, $j \in \mathbb{N}$, viewed as factors of T . Using this simple observation and Proposition 6.1 we can explicitly describe all ergodic 2-fold self-joinings of T .

The T -action can be represented as a skew product. The space of this action is the product $Y \times Z$ of two compact Abelian groups $Y := \text{proj lim}_{j \rightarrow \infty} \mathbb{R}^2/p(\Gamma_j)$ and $Z := \text{proj lim}_{j \rightarrow \infty} Z_j$, where $Z_j := \mathbb{R}/\xi_{\Gamma_j}\mathbb{Z}$. The factor-space Y is the factor of T generated by all proper functions of T ; the restriction of T to this factor has pure point spectrum. The corresponding cocycle of this factor with values in Z is ergodic.²⁰ Given $d \in Y$, we denote by Δ_d the image of the Haar measure on Y under the map $Y \ni y \mapsto (y, y + d) \in Y \times Y$. Every element $d \in Y$ is a sequence $(d_j)_{j \in \mathbb{N}}$ of elements $d_j \in \mathbb{R}^2/p(\Gamma_j)$ such that d_{j+1} maps to d_j under the natural projection $\mathbb{R}^2/p(\Gamma_{j+1}) \rightarrow \mathbb{R}^2/p(\Gamma_j)$ for each j . In a similar way, every element $z \in Z$ is a sequence $(z_j)_{j \in \mathbb{N}}$ of elements $z_j \in Z_j$ such that z_{j+1} maps to z_j under the natural projection $Z_{j+1} \rightarrow Z_j$ for each j . If d_j is periodic then we denote by D_j be the closed subgroup of $Z_j \times Z_j$ associated with d_j in the way described in the part (A) of this section (see (6-3)). We note that D_j contains the diagonal of $Z_j \times Z_j$

¹⁹In fact, we have shown that R is not spectrally equivalent to any Heisenberg odometer.

²⁰See Section 7 for details.

as a subgroup of finite index. Moreover, D_{j+1} maps onto D_j under the natural projection $Z_{j+1} \rightarrow Z_j$ for each j . Hence a projective limit $D_d := \text{projlim}_{j \rightarrow \infty} Z_j$ is well defined. It is a closed subgroup of Z . As in Proposition 6.1 we will use the following notation. Given a closed subgroup Λ of $Z \times Z$, let λ_Λ stand for the Haar measure on Λ . Given $z \in Z$, let $\lambda_\Lambda \circ z$ denote the image of λ_Λ viewed as a measure on $Z \times Z$ under the rotation $(z_1, z_2) \mapsto (z_1, z_2 + z)$ of $Z \times Z$.

Theorem 6.5. *The set $J_2^c(T)$ of all ergodic 2-fold self-joinings of T is the union of the following two families:²¹*

$$J_2^c(T) = \{ \Delta_d \times \lambda_{Z \times Z} \mid d = (d_j)_{j \in \mathbb{N}} \text{ with } d_j \text{ aperiodic for each } j \} \\ \cup \{ \Delta_d \times \lambda_{D_d} \circ z \mid d = (d_j)_{j \in \mathbb{N}} \text{ with } d_j \text{ periodic for each } j \text{ and } z \in Z \}.$$

7. ON SPECTRAL DETERMINACY OF HEISENBERG ODOMETERS

(A) The case of transitive odometers. Let T be an ergodic action of $H_3(\mathbb{R})$ on a standard probability space (X, μ) . Denote by U_T the corresponding Koopman unitary representation of $H_3(\mathbb{R})$.

Theorem 7.1. *If U_T is unitarily equivalent to the Koopman unitary representation generated by the action Q of $H_3(\mathbb{R})$ by translations on $H_3(\mathbb{R})/\Gamma$ for a lattice Γ in $H_3(\mathbb{R})$ then T is isomorphic to Q .*

Proof. Without loss of generality we may assume that there is $k \geq 1$ such that $\Gamma = \{c(i_3/k)b(i_2)a(i_1) \mid i_1, i_2, i_3 \in \mathbb{Z}\}$. By Theorem 4.1,

$$(7-1) \quad U_T = \bigoplus_{(\alpha, \beta) \in \mathbb{Z}^2} \pi_{\alpha, \beta} \oplus \bigoplus_{0 \neq n \in \mathbb{Z}} \bigoplus_{j=1}^{|n|} \pi_{nk}$$

up to the unitary equivalence. Denote by \mathcal{H} the subspace of $L^2(X, \mu)$ where the unitary representation $\bigoplus_{(\alpha, \beta) \in \mathbb{Z}^2} \pi_{\alpha, \beta}$ is defined. Since each unitary representation $\pi_{\alpha, \beta}$ is 1-dimensional, it follows from the classical von Neumann theory of Abelian actions with pure point spectrum that there is a measure preserving factor map $q : (X, \mu) \rightarrow (\mathbb{R}^2/\mathbb{Z}^2, \lambda_{\mathbb{R}^2/\mathbb{Z}^2})$ intertwining, for each $g \in H_3(\mathbb{R})$, the transformation T_g with the rotation on $\mathbb{R}^2/\mathbb{Z}^2$ by $p(g)$ and such that $\mathcal{H} = \{f \circ q \mid f \in L^2(\mathbb{R}^2/\mathbb{Z}^2)\}$. Since $\pi_{nk}(c(t))v = e^{2\pi i n k t} v$ for each vector v of the Hilbert space where π_{nk} is defined, it follows from (7-1) that \mathcal{H} is the subspace of vectors fixed by the unitary 1-parameter group $\{U_T(c(t)) \mid t \in \mathbb{R}\}$. Therefore by the von Neumann mean ergodic theorem, $\frac{1}{2N} \int_{-N}^N U_T(c(t)) dt$ converges strongly as $N \rightarrow \infty$ to the orthogonal projection to \mathcal{H} which is the mathematical expectation to the sub- σ -algebra of $\{T_{c(t)} \mid t \in \mathbb{R}\}$ -invariant subsets. Thus q is the $\{T_{c(t)} \mid t \in \mathbb{R}\}$ -ergodic decomposition and $(\mathbb{R}^2/\mathbb{Z}^2, \lambda_{\mathbb{R}^2/\mathbb{Z}^2})$ is the space of $\{T_{c(t)} \mid t \in \mathbb{R}\}$ -ergodic components. We note that $U_T(c(k^{-1})) = \text{Id}$. Moreover, if $0 < s < k^{-1}$ then the unitary operator $U_T(c(s))$ has no non-trivial fixed vectors in the orthocomplement to \mathcal{H} in $L^2(X)$. Hence the restriction of $\{T_{c(t)} \mid t \in \mathbb{R}\}$ to almost every ergodic component is isomorphic to the periodic flow with the smallest period k^{-1} . Thus (X, μ) splits into the product

$$(X, \mu) = (\mathbb{R}^2/\mathbb{Z}^2, \lambda_{\mathbb{R}^2/\mathbb{Z}^2}) \times (\mathbb{R}/k^{-1}\mathbb{Z}, \lambda_{\mathbb{R}/k^{-1}\mathbb{Z}})$$

²¹Provided that some special cocycle representing T as a skew product over its maximal factor with pure point spectrum is chosen.

such that $q(y, z) = y$ and

$$(7-2) \quad T_{c(t)}(y, z) = (y, t + z)$$

for a.a. $y \in \mathbb{R}^2/\mathbb{Z}^2$ and $z \in \mathbb{R}/k^{-1}\mathbb{Z}$. Thus there exists a Borel cocycle

$$\delta : H_3(\mathbb{R}) \times \mathbb{R}^2/\mathbb{Z}^2 \rightarrow \mathbb{R}/k^{-1}\mathbb{Z}$$

such that

$$T_g(y, z) = (p(g) \cdot y, \delta(g, y) + z) \quad \text{for a.a. } (y, z) \in X.$$

It follows that T is a transitive action on a 3-dimensional torus. Hence there is a lattice Γ' in $H_3(\mathbb{R})$ such that T is isomorphic to a homogeneous $H_3(\mathbb{R})$ -action on $H_3(\mathbb{R})/\Gamma'$. Since the 1-dimensional spectrum of T is $\{\pi_{\alpha, \beta} \mid (\alpha, \beta) \in \mathbb{Z}^2\}$ we deduce from Theorem 4.2 that $p(\Gamma') = \mathbb{Z}^2$. It follows from (7-2) that $\xi_{\Gamma'} = k^{-1}$. Thus $p(\Gamma) = p(\Gamma')$ and $\xi_\Gamma = \xi_{\Gamma'}$. Hence by Corollary 4.3, Γ and Γ' are conjugate, i.e. T and Q are isomorphic. \square

Remark 7.2.

- (1) In fact, a stronger claim has been proved: if the maximal spectral types of T and Q are equivalent then T and Q are isomorphic.
- (2) As a byproduct, we have discovered the following interesting fact: let T be a transitive Heisenberg odometer and let U_T denote the corresponding Koopman unitary representation of $H_3(\mathbb{R})$. If V is a unitary representation of $H_3(\mathbb{R})$ with the same maximal spectral type as U_T but with a different multiplicity function, then V is not unitarily equivalent to the Koopman unitary representation of $H_3(\mathbb{R})$ generated by any probability preserving action of $H_3(\mathbb{R})$.

(B) The general case. Here we prove the main result of this section: the $H_3(\mathbb{R})$ -odometers are not spectrally determined.

Lemma 7.3. *Let $\xi = \bigcup_{j=1}^{\infty} d_j^{-1}\mathbb{Z}$ be an off-rational subgroup of \mathbb{R} and let $S = \bigcup_{j=1}^{\infty} A_j^{-1}\mathbb{Z}^2$ be an off-rational subgroup of \mathbb{R}^2 . We set $Z := \text{proj lim}_{j \rightarrow \infty} \mathbb{R}/d_j\mathbb{Z}$ and $Y := \text{proj lim}_{j \rightarrow \infty} \mathbb{R}^2/A_j^*\mathbb{Z}^2$. Denote by λ_Z and λ_Y the Haar measure on Z and Y respectively. Denote by $\bar{r}_j : Z \rightarrow \mathbb{R}/d_j^{-1}\mathbb{Z}$ and $\bar{q}_j : Y \rightarrow \mathbb{R}^2/A_j^*\mathbb{Z}^2$ the canonical projections. Let $\vartheta : \mathbb{R} \rightarrow Z$ and $\varphi : \mathbb{R}^2 \rightarrow Y$ be continuous homomorphisms defined by $\bar{r}_j(\vartheta(t)) = t + d_j^{-1}\mathbb{Z}$ and $\bar{q}_j(\varphi(c)) = c + A_j^*\mathbb{Z}^2$ for all $t \in \mathbb{R}$, $c \in \mathbb{R}^2$, $j \in \mathbb{N}$. Let T be an ergodic measure preserving action on a probability space (X, μ) . Suppose that the associated Koopman unitary representation U_T of $H_3(\mathbb{R})$ decomposes into irreducibles as*

$$(7-4) \quad U_T = \bigoplus_{(\alpha, \beta) \in S} \pi_{\alpha, \beta} \oplus \bigoplus_{\gamma \in \xi} \bigoplus_{l=1}^{m_\gamma} \pi_\gamma$$

for a spectral multiplicity map $m : \xi \ni \gamma \mapsto m_\gamma \in \mathbb{N} \cup \{\infty\}$. Then (X, μ) is isomorphic to the product $(Y \times Z, \lambda_Y \times \lambda_Z)$ in such a way that

$$T_g(y, z) = (D_g y, \alpha(g, y) + z),$$

where

$$D : H_3(\mathbb{R}) \times Y \ni (g, y) \mapsto D_g y := \varphi(p(g)) + y \in Y$$

is an ergodic action of $H_3(\mathbb{R})$ on Y with pure point spectrum and

$$\alpha : H_3(\mathbb{R}) \times Y \ni (g, y) \mapsto \alpha(g, y) \in Z$$

is an ergodic cocycle of D such that $\alpha(c(t), y) = \vartheta(t)$ for all $t \in \mathbb{R}$ and $y \in Y$.

Proof. We have a decomposition of $L^2(X, \mu)$ corresponding to (7-4):

$$L^2(X, \mu) = \bigoplus_{(\alpha, \beta) \in S} \mathcal{H}_{\alpha, \beta} \oplus \bigoplus_{\gamma \in \xi} \bigoplus_{l=1}^{m_\gamma} \mathcal{H}_{\gamma, l}.$$

We now define a unitary representation $W = (W(z))_{z \in Z}$ of Z in $L^2(X, \mu)$ by setting

$$W(z)v = \begin{cases} v & \text{if } v \in \mathcal{H}_{\alpha, \beta}, (\alpha, \beta) \in S \text{ and} \\ e^{2\pi i \gamma \bar{r}_j(z)} v & \text{if } v \in \mathcal{H}_{\gamma, l}, \gamma \in \xi, 1 \leq l \leq m_\gamma. \end{cases}$$

We consider here $\bar{r}_j(z)$ as a real number from the segment $[0, d_j]$. Then W is continuous in the weak operator topology. Moreover,

$$(7-5) \quad W(\vartheta(t)) = U_T(c(t)) \quad \text{for all } t \in \mathbb{R}.$$

Since the group $\text{Aut}(X, \mu)$ of all μ -preserving transformations of X is a closed subgroup the unitary group $\mathcal{U}(L^2(X, \mu))$ and the subgroup $\vartheta(\mathbb{R})$ is dense in Z , it follows that $W(z) \in \text{Aut}(X, \mu)$ for all $z \in Z$. Thus, $\{W(z) \mid z \in Z\}$ is a compact group of transformations commuting with the action T . Therefore there are a probability space (Y, ν) and an action $D = (D_g)_{g \in H_3(\mathbb{R})}$ of $H_3(\mathbb{R})$ on it such that (X, μ) is isomorphic to the product $(Y \times Z, \nu \times \lambda_Z)$ such that

$$T_g(y, z) = (D_g y, \alpha(g, y) + z) \quad \text{and} \quad W(z')(y, z) = (y, z + z')$$

for all $g \in H_3(\mathbb{R})$ and $z' \in Z$, where $\alpha : H_3(\mathbb{R}) \times Y \rightarrow Z$ is a measurable cocycle of D (see, e.g. [dJRu]). It follows from (7-5) that $\alpha(c(t), y) = \vartheta(t)$ for all $t \in \mathbb{R}$. Since

$$\begin{aligned} \bigoplus_{(\alpha, \beta) \in S} \mathcal{H}_{\alpha, \beta} &= \{v \in L^2(X, \mu) \mid U_T(c(t))v = v \text{ for all } t \in \mathbb{R}\} \\ &= \{v \in L^2(X, \mu) \mid W(z)v = v \text{ for all } z \in Z\}, \end{aligned}$$

it follows that $L^2(Y) = \bigoplus_{(\alpha, \beta) \in S} \mathcal{H}_{\alpha, \beta}$ and the Koopman unitary representation of $H_3(\mathbb{R})$ generated by D decomposes into irreducible ones as $U_D = \bigoplus_{(\alpha, \beta) \in S} \pi_{\alpha, \beta}$. Therefore without loss of generality we may assume that Y is a compact Abelian group dual to S and $D_g y = \varphi(p(g)) + y$ for all $y \in Y$ and $g \in H_3(\mathbb{R})$. \square

Now we prove a converse to Lemma 7.3.

Lemma 7.4. *Let S, ξ, Y, Z and D be as in Lemma 7.3.*

(i) *Given a cocycle β of D with values in Z such that*

$$(7-6) \quad \beta(c(t), y) = \vartheta(t) \quad \text{for all } t \in \mathbb{R},$$

there is a map $m : \xi \ni \gamma \mapsto m_\gamma \in \mathbb{N} \cup \{\infty\}$ such that the Koopman unitary representation U_{D^β} of $H_3(\mathbb{R})$ generated by the β -skew product extension D^β of D is unitarily equivalent to (7-4).

(ii) *Given two cocycles β, β' of D with values in Z satisfying (7-6), the skew product extensions D^β and $D^{\beta'}$ of D are isomorphic if and only if there exist an element $y_0 \in Y$ and a Borel map $f : Y \rightarrow Z$ such that*

$$\beta(g, y + y_0) = f(y) + \beta'(g, y) - f(\varphi(p(g)) + y)$$

for each $g \in H_3(\mathbb{R})$ at a.a. $y \in Y$.

Proof. (i) We first note that ξ is isomorphic to the dual \widehat{Z} of Z . Hence we can identify an element γ of ξ with a character $\xi_\gamma \in L^2(Z, \lambda_Z)$. Then we obtain a decomposition of $L^2(Y \times Z, \lambda_Y \times \lambda_Z)$ into an orthogonal sum of U_{D^β} -invariant subspaces:

$$L^2(Y \times Z, \lambda_Y \times \lambda_Z) = \bigoplus_{\gamma \in \xi} L^2(Y, \lambda_Y) \otimes \xi_\gamma.$$

It follows from the property of β that

$$U_{D^\beta}(c(t))(v \otimes \xi_\gamma) = \xi_\gamma(\vartheta(t))v \otimes \xi_\gamma = e^{2\pi i \gamma t} v \otimes \xi_\gamma, \quad v \in L^2(Y, \lambda_Y).$$

Therefore $U_{D^\beta} \upharpoonright (L^2(Y, \lambda_Y) \otimes \xi_\gamma) = \bigoplus_1^{m_\gamma} \pi_\gamma$ for some multiplicity $m_\gamma \in \mathbb{N} \cup \{\infty\}$ for each $\gamma \in \xi, \gamma \neq 0$. Of course, $U_{D^\beta} \upharpoonright (L^2(Y, \lambda_Y) \otimes 1) = \bigoplus_{(\alpha, \beta) \in S} \mathcal{H}_{\alpha, \beta}$. It follows that (7-4) holds, as desired.

(ii) Since D^β and $D^{\beta'}$ are isomorphic, it follows from a rather standard description of the centralizer of an ergodic compact group extension of an action by rotations on a compact Abelian group (see, e.g., [Da1]) that there exist an element $y_0 \in Y$, a continuous automorphism θ of Z and a Borel map $f : Y \rightarrow Z$ such that

$$\beta(g, y + y_0) = f(y) + \theta(\beta'(g, y)) - f(\varphi(p(g)) + y)$$

for each $g \in H_3(\mathbb{R})$ at a.a. $y \in Y$. Now (7-6) yields that $\theta(\vartheta(t)) = \vartheta(t)$ for all $t \in \mathbb{R}$. Since ϑ has a dense range in Z , it follows that θ is the identity, as desired. \square

Consider now the case where T is the $H_3(\mathbb{R})$ -odometer associated with a sequence $\Gamma_1 \supset \Gamma_2 \supset \dots$ of lattices in $H_3(\mathbb{R})$. Denote by X the space of T . We set $Y_j := \mathbb{R}^2/p(\Gamma_j)$ and $Z_j := \mathbb{R}/\xi_{\Gamma_j}\mathbb{Z}$. Recall that the homogeneous space $H_3(\mathbb{R})/\Gamma_j$ is isomorphic to the product $Y_j \times Z_j$ in such a way that the natural $H_3(\mathbb{R})$ -action has a skew product form as follows

$$(7-7) \quad g \cdot (y, z) := (p(g) \cdot y, \alpha_j(g, y) + z)$$

for all $(y, z) \in Y_j \times Z_j$ and $g \in H_3(\mathbb{R})$, where $\alpha_j : H_3(\mathbb{R}) \times Y_j \rightarrow \mathbb{R}/\xi_{\Gamma_j}\mathbb{Z}$ is a cocycle satisfying

$$(7-8) \quad \alpha_j(c(t), y) = t + \xi_{\Gamma_j}\mathbb{Z}$$

The natural projection $H_3(\mathbb{R})/\Gamma_j \rightarrow H_3(\mathbb{R})/\Gamma_{j-1}$ corresponds to the mapping

$$Y_j \times Z_j \ni (y, z) \mapsto (q_j(y), r_j(z + f_j(y))) \in Y_{j-1} \times Z_{j-1},$$

where $q_j : Y_j \rightarrow Y_{j-1}$ and $r_j : Z_j \rightarrow Z_{j-1}$ are the natural projections and $f_j : Y_j \rightarrow Z_j$ is a Borel map.²² It is easy to see that the $H_3(\mathbb{R})$ -equivariance of the map $H_3(\mathbb{R})/\Gamma_j \rightarrow H_3(\mathbb{R})/\Gamma_{j-1}$ is equivalent to the following property

$$\alpha_{j-1}(g, q_j(y)) = r_j(f_j(p(g) \cdot y) + \alpha_j(g, y) - f_j(y))$$

for all $g \in G$ and $y \in Y_j$. Therefore, replacing α_j with a cohomologous cocycle, we may assume without loss of generality that

$$(7-9) \quad \alpha_{j-1}(g, q_j(y)) = r_j(\alpha_j(g, y))$$

for every j . Of course, this cohomological change will not affect property (7-8). We now set $Y := \text{proj lim}_{j \rightarrow \infty} Y_j$ and $Z := \text{proj lim}_{j \rightarrow \infty} Z_j$ in the category of compact Abelian groups furnished with the Haar measures. Denote by $\bar{q}_j : Y \rightarrow Y_j$ and $\bar{r}_j : Z \rightarrow Z_j$ the canonical projections. We note that Z and Y are the same objects as in Lemmata 7.3 and 7.4 if we set $\xi := \bigcup_{j=1}^{\infty} \xi_{\Gamma_j}^{-1} \mathbb{Z}$ and $S := \bigcup_{j=1}^{\infty} p(\Gamma_j)^*$. Recall that D is an ergodic action of $H_3(\mathbb{R})$ on Y with pure point spectrum S (see the statement of Lemma 7.3). In view of (7-9), a cocycle $\alpha : H_3(\mathbb{R}) \times Y \rightarrow Z$ of D is well defined by

$$(7-10) \quad \bar{r}_j(\alpha(g, y)) = \alpha_j(g, \bar{q}_j(y)), \quad j \in \mathbb{N}.$$

It follows from (7-8) that $\alpha(c(t), y) = \vartheta(t)$ for all $t \in \mathbb{R}$ and $y \in Y$. Since $X = \text{proj lim}_{j \rightarrow \infty} H_3(\mathbb{R})/\Gamma_j$, we obtain that X is measure theoretically isomorphic to $Y \times Z$ in such a way that T is the α -skew product extension D^α of D .

Definition 7.5. Let Y , Z and D be as above. A Borel cocycle α of D with values in Z is called *finitary* if there are cocycles α_j of the natural action of $H_3(\mathbb{R})$ on Y_j with values in Z_j such that (7-10) is satisfied.

Remark 7.6. For $y_0 \in Y$, we define a Z -valued cocycle β of D by setting $\beta(g, y) := \alpha(g, y + y_0)$. If α is finitary then so is β .

We now state one of the main results of this section.

Theorem 7.7. *Let S be an off-rational subgroup of \mathbb{R}^2 and let ξ be an off-rational subgroup of \mathbb{R} such that $\tau(S) \supset \xi$. Fix a nested sequence $\Gamma = (\Gamma_j)_{j=1}^{\infty}$ of lattices in $H_3(\mathbb{R})$ with $S_\Gamma = S$ and $\xi_\Gamma = \xi^{23}$. Let T be an ergodic action of $H_3(\mathbb{R})$ on a standard probability space. Suppose that (7-4) holds for the associated Koopman unitary representation U_T of $H_3(\mathbb{R})$. As in Lemma 7.3, represent T as a skew product extension D^α of D for a cocycle α of D with values in Z . Then T is an $H_3(\mathbb{R})$ -odometer (which is f -isomorphic to the $H_3(\mathbb{R})$ -odometer associated with Γ) if and only if α is cohomologous to a finitary cocycle.*

²²This follows from the fact that the projection $H_3(\mathbb{R})/\Gamma_j \rightarrow H_3(\mathbb{R})/\Gamma_{j-1}$ intertwines the actions of the center of $H_3(\mathbb{R})$ on these homogeneous spaces while Y_j is the space of ergodic components for the action of the center.

²³It exists by Proposition 5.5.

Proof. (\Rightarrow) If T is a Heisenberg odometer then (as was shown above) there exists a representation T as $D^{\alpha'}$ with a finitary cocycle α' . By Lemma 7.4(ii), α is cohomologous to a ‘rotation of α' by an element of Y ’. As was noted in Remark 7.6, the class of finitary cocycles is invariant under such rotations. Thus, α is cohomologous to a finitary cocycle.

(\Leftarrow) Conversely, suppose that α is finitary. Then a sequence of cocycles $\alpha_j : H_3(\mathbb{R}) \times Y_j \rightarrow Z_j$ is well defined by (7-10). Since $\alpha(c(t), y) = \varphi(t)$ for all $t \in \mathbb{R}$ and $y \in Y$, it follows from (7-10) that α_j satisfies (7-8) for each $j \in \mathbb{N}$. This yields that the α_j -skew product extension $H_3(\mathbb{R})$ (see (7-7)) is transitive on $Y_j \times Z_j$. Then by Proposition 5.12, T is a Heisenberg odometer. \square

Corollary 7.8. *Under the notation of Theorem 7.7, if S is closed in \mathbb{R}^2 then T is an $H_3(\mathbb{R})$ -odometer. This means that the subclass of degenerate Heisenberg odometers is spectrally determined.*

We now consider the problem of spectral determinacy within the entire class Heisenberg odometers.

Corollary 7.9. *Let T be a non-degenerate Heisenberg odometer associated with a sequence $\Gamma = (\Gamma_j)_{j=1}^{\infty}$ of lattices in $H_3(\mathbb{R})$. Then there is an ergodic action R of $H_3(\mathbb{R})$ such that*

- R has the same maximal spectral type as T but
- R is not isomorphic to T (and hence to any $H_3(\mathbb{R})$ -odometer).

Proof. By Lemma 7.3 and Theorem 7.7, we can represent T as D^{β} for a Z -valued finitary cocycle β of D satisfying (7-6). Let $\alpha : \mathbb{R}^2 \times Y \rightarrow Z$ be a cocycle of the underlying \mathbb{R}^2 -odometer. By analogy with Definition 7.5, we call α *finitary* if

$$\bar{r}_j(\alpha(s, y)) = \alpha_j(s, \bar{q}_j(y)), \quad s \in \mathbb{R}^2, j \in \mathbb{N}$$

for a sequence of cocycles $\alpha_j : \mathbb{R}^2 \times Y_j \rightarrow Z_j$. Since T is non-degenerated, the underlying \mathbb{R}^2 -odometer is not transitive. Therefore there is a cocycle α of it which is not cohomologous to a finitary one. Then we consider a Z -valued cocycle

$$\beta'(g, y) := \beta(g, y) + \alpha(p(g), y)$$

of D . Of course, β' satisfies (7-6). Therefore by Lemma 7.4(i), the maximal spectral type of the β' -skew product extension $D^{\beta'}$ of D coincides with the maximal spectral type of T . However β' is not finitary.²⁴ Therefore by Theorem 7.7, $D^{\beta'}$ is not isomorphic to an odometer. \square

To construct an explicit counter-example to the spectral determinacy of non-degenerate Heisenberg odometers we first restate Theorem 7.7 in a more suitable (for this purpose) form. Let $(\Gamma_j)_{j=1}^{\infty}$, D , Z and α be the same objects as in Lemma 7.3 and Theorem 7.7. We need some more notation. Let D' denote the $p(\Gamma_1)$ -odometer associated with the sequence $p(\Gamma_1) \supset p(\Gamma_2) \supset \cdots$ of cofinite subgroups in $p(\Gamma_1)$. Let $Y' := \text{projlim}_{j \rightarrow \infty} p(\Gamma_1)/p(\Gamma_j)$ be the space of D' . We note that Y' is a compact Abelian group (furnished with Haar measure). Let $\tilde{\Gamma}_1 := p^{-1}(p(\Gamma_1))$. Denote by $\tilde{D} = (\tilde{D}_\gamma)_{\gamma \in \tilde{\Gamma}_1}$ the following $\tilde{\Gamma}_1$ -action on Y' :

$$\tilde{D}_\gamma y' := p(\gamma) \cdot y'.$$

²⁴This is because β is finitary and α is not finitary.

Then one can deduce easily from Corollary 2.5 that $D = \text{Ind}_{\tilde{\Gamma}_1}^{H_3(\mathbb{R})}(\tilde{D})$. Therefore by Lemma 2.4(iii), there is a Z -valued cocycle β of \tilde{D} such that α is cohomologous to the induced cocycle $\tilde{\beta}$ of D . Moreover, by Lemma 2.4(ii), the cohomology class of β is defined uniquely by a cohomology class of α . Let \mathfrak{Y}_j denote the factor of D generated by the canonical projection $\bar{p}_j : Y \rightarrow Y_j$ and let $\tilde{\mathfrak{Y}}_j$ denote the factor of \tilde{D} generated by the canonical projection $\bar{p}'_j : Y' \rightarrow p(\Gamma_1)/p(\Gamma_j)$. It is straightforward to verify that \mathfrak{Y}_j is induced by $\tilde{\mathfrak{Y}}_j$. Therefore Lemma 2.4(ii), (iv) yield that α is cohomologous to a finitary cocycle if and only if β is cohomologous to a finitary cocycle, say β' , of \tilde{D} , i.e. the map

$$Y' \ni y' \mapsto \bar{r}_j(\beta'(\gamma, y')) \in Z_j$$

is $\tilde{\mathfrak{Y}}_j$ -measurable for each $\gamma \in \tilde{\Gamma}_1$ and $j \in \mathbb{N}$. Thus we obtain the following corollary from Theorem 7.7.

Corollary 7.10. *Using the notation of Theorem 7.7, T is an $H_3(\mathbb{R})$ -odometer (associated with Γ) if and only if β is cohomologous to a finitary cocycle.*

We are now ready to construct an action of $H_3(\mathbb{R})$ which is unitarily equivalent to a Heisenberg odometer but not isomorphic to it.

Example 7.11. Let p_1, p_2, \dots be an infinite sequence of pairwise different primes, $p_1 = 1$. We set

$$\Gamma_j := \{c(n)b(p_1 \cdots p_j m)a(p_1 \cdots p_j l) \mid n, m, l \in \mathbb{Z}\}.$$

Then $\Gamma := (\Gamma_j)_{j=1}^\infty$ is a decreasing sequence of lattices in $H_3(\mathbb{Z})$. Denote by T the associated $H_3(\mathbb{R})$ -odometer. As in Lemma 7.3, represent T as a skew product extension $T = D^\alpha$ for a cocycle α of D with values in Z . We note that in this case $Z = Z_j = \mathbb{R}/\mathbb{Z}$ for each $j \in \mathbb{N}$. By Theorem 7.7, α is (up to a cohomological change) finitary. It is straightforward to verify that the following holds

- (i) $Y' = \text{proj} \lim_{j \rightarrow \infty} \mathbb{Z}^2/p_1 \cdots p_j \mathbb{Z}^2$,
- (ii) $\tilde{\Gamma}_1 = \{c(t)b(m)a(l) \mid t \in \mathbb{R}, m, l \in \mathbb{Z}\}$ and
- (iii) α is cohomologous to the cocycle induced from the cocycle β of \tilde{D} given by $\beta(\gamma, y') = t + \mathbb{Z}$ if $\gamma = c(t)b(m)a(n) \in \tilde{\Gamma}_1$.

Thus β is finitary. It follows from (i) and the mutual coprimeness of p_j , $j \in \mathbb{N}$, that the dynamical system (Y', D') is isomorphic to the infinite product $Y' = \bigotimes_{j \in \mathbb{N}} \mathbb{Z}^2/p_j \mathbb{Z}^2$ such that

$$D'_\gamma(y_1, y_2, \dots) = (\gamma_1 + y_1, \gamma_2 + y_2, \dots)$$

for all $\gamma \in p(\Gamma_1) = \mathbb{Z}^2$ and $y_j \in \mathbb{Z}^2/p_j \mathbb{Z}^2$, where $\gamma_j := \gamma + p_j \mathbb{Z}^2$, $j \in \mathbb{N}$. Partition \mathbb{N} into two infinite disjoint subsets J_1 and J_2 and let $Y'_1 := \bigotimes_{j \in J_1} \mathbb{Z}^2/p_j \mathbb{Z}^2$ and $Y'_2 := \bigotimes_{j \in J_2} \mathbb{Z}^2/p_j \mathbb{Z}^2$. Then Y'_1 and Y'_2 are natural (coordinate) factors of D' . Denote the restriction of D' to these factors by D^1 and D^2 respectively. We see that D^1 and D^2 are both free and D' is naturally isomorphic to the Cartesian product $D^1 \times D^2$.

We now note that the group Y' acting on itself by rotations is the centralizer $C(D')$ of D' . Since D' has pure point spectrum, $\tilde{\mathfrak{Y}}_j$ is the σ -algebra of subsets fixed

by a compact subgroup of Y' . We can find a family of elements $(S_j)_{j \in \mathbb{N}}$ in $C(D')$ such that the following two properties are satisfied:

- (iv) $\tilde{\mathfrak{Y}}_j$ is fixed by S_j , i.e. S_j is trivial on $\tilde{\mathfrak{Y}}_j$.
- (v) If we represent S_j as the product $S_j = S_{1,j} \times S_{2,j}$ with $S_{1,j} \in C(D^1)$ and $S_{2,j} \in C(D^2)$ then the ‘joint’ action

$$\mathbb{Z}^2 \times \bigoplus_{j=1}^{\infty} \mathbb{Z} \ni (\gamma, n_1, n_2, \dots) \mapsto D_\gamma^1 S_{1,1}^{n_1} S_{1,2}^{n_2} \dots$$

of the group $\mathbb{Z}^2 \times \bigoplus_{j=1}^{\infty} \mathbb{Z}$ on Y'_1 is free.

By an auxiliary Lemma 7.12 (see the end of this section), there is a cocycle ω of D^1 with values in Z such that the cocycle $\bar{\omega} := \bigotimes_{j \in J_1} \omega \circ S_{1,j}$ with values in Z^{J_1} is ergodic and, moreover, the product \mathbb{Z}^2 -action $(D^1)^{\bar{\omega}} \times D^2$ on $Y'_1 \times Z^{J_1} \times Y_2$ is ergodic. In other words, the Z^{J_1} -valued cocycle $\bar{\omega} \otimes 1$ of D' (recall that $D' = D^1 \times D^2$) is ergodic.

Claim A. For any j , the cocycle $\omega \otimes 1$ of D' is not cohomologous to a \mathfrak{Y}_j -measurable cocycle. Indeed, if $\omega \otimes 1$ is cohomologous to a \mathfrak{Y}_j -measurable cocycle then in view of (iv), $\omega \otimes 1$ is cohomologous to the cocycle $(\omega \otimes 1) \circ S_j$. Therefore the Z^2 -valued cocycle $\omega \otimes 1 \times (\omega \otimes 1) \circ S_j$ of D' is not ergodic. However it is easy to see that

$$\omega \otimes 1 \times (\omega \otimes 1) \circ S_j = (\omega \times \omega \circ S_{1,j}) \otimes 1$$

and the skew product $(D')^{(\omega \times \omega \circ S_{1,j}) \otimes 1}$ is a factor of the skew product $(D')^{\bar{\omega} \otimes 1}$ which is ergodic by the choice of ω . We get a contradiction which proves Claim A.

We now define a Z -valued cocycle κ of \tilde{D} by setting

$$\kappa(\gamma, y') := \beta(\gamma, y') + (\omega \otimes 1)(p(\gamma), y')$$

for each $\gamma \in \tilde{\Gamma}_1$ and $y' \in Y'$. Denote by $\tilde{\alpha}$ the Z -valued cocycle of D induced from κ . Finally, we let $\tilde{T} := D^{\tilde{\alpha}}$. Then \tilde{T} is an ergodic $H_3(\mathbb{R})$ -action.

It follows from Claim A that κ is not cohomologous to a finitary cocycle. Now Corollary 7.10 yields that \tilde{T} is not an $H_3(\mathbb{R})$ -odometer. On the other hand, by Lemma 7.4(i), the maximal spectral type of the Koopman unitary representation $U_{\tilde{T}}$ coincides with the maximal spectral type of U_T . Since $\xi_\Gamma = \mathbb{Z}$ and π_n is contained in U_T with infinite multiplicity for each $n \neq 0$ by Theorem 5.2(i), to prove that T and \tilde{T} are unitarily equivalent it remains to show that the multiplicity π_n in $U_{\tilde{T}}$ is also infinite for each $n \neq 0$.

Denote by U the Koopman unitary representation of $\tilde{\Gamma}_1$ generated by the skew product \tilde{D}^κ . The space of U is $L^2(Y' \times Z)$. It decomposes into an orthogonal sum

$$L^2(Y' \times Z) = \bigoplus_{\eta \in \widehat{Y'_2}} \bigoplus_{n \in \mathbb{Z}} L^2(Y'_1) \otimes \eta \otimes \chi_n$$

of U -invariant subspaces $L^2(Y'_1) \otimes \eta \otimes \chi_n$.²⁵ Since the Koopman unitary representation $U_{\tilde{T}}$ is induced from U , we obtain the following decomposition of $L^2(Y \times Z)$

²⁵The subspace $L^2(Y'_1) \otimes \eta \otimes \chi_n$ is invariant under U because $\omega \otimes 1$ as a function of $(\gamma, y') = (\gamma, y'_1, y'_2) \in \tilde{\Gamma}_1 \times Y'_1 \times Y'_2$ does not depend on y'_2 while β does not depend on y' at all.

into $U_{\widehat{T}}$ -invariant subspaces:

$$L^2(Y \times Z) = \bigoplus_{\eta \in \widehat{Y_2'}} \bigoplus_{n \in \mathbb{Z}} L^2(\mathbb{R}^2/\mathbb{Z}^2) \otimes L^2(Y_1') \otimes \eta \otimes \chi_n.$$

Moreover, it is straightforward to verify that $U_{\widehat{T}}(c(t))v = e^{2\pi int}v$ for each vector $v \in L^2(\mathbb{R}^2/\mathbb{Z}^2) \otimes L^2(Y_1') \otimes \eta \otimes \chi_n$. Therefore the restriction of $U_{\widehat{T}}$ to the subspace $L^2(\mathbb{R}^2/\mathbb{Z}^2) \otimes L^2(Y_1') \otimes \eta \otimes \chi_n$ is a multiple of π_n for each $n \neq 0$ and $\eta \in \widehat{Y_2'}$. Since the group $\widehat{Y_2'}$ is infinite, the multiplicity of π_n in $U_{\widehat{T}}$ is also infinite, as desired.

We now state and prove an auxiliary lemma used in Example 7.11.

Lemma 7.12. *Let K be a compact metric group. Let $R = (R_g)_{g \in G}$ and $Q = (Q_g)_{g \in G}$ be two ergodic actions of an infinite countable discrete Abelian group G such that R is free and the Cartesian product $R \times Q$ is ergodic. Fix a countable infinite Abelian subgroup A in the centralizer $C(R)$ of R such that $A \cap \{R_g \mid g \in G\} = \{\text{Id}\}$. Then there is a cocycle ω of R with values in K such that the K^A -valued cocycle $\overline{\omega} := \bigotimes_{a \in A} \omega \circ a$ of R is ergodic and the Cartesian product $R^{\overline{\omega}} \times Q$ is ergodic.*

Proof. To prove this lemma we will use the orbit theory of amenable actions and their cocycles (see [Sc], [BeG], [GoSi], [Da1]). Let (X, μ) denote the space of R . We denote by $\mathcal{R} \subset X \times X$ the R -orbit equivalence relation. Recall that the *full group* $[\mathcal{R}]$ of \mathcal{R} is the group of all μ -preserving invertible transformations F of X such that Fx belongs to the \mathcal{R} -class of x for a.a. $x \in X$. Choose

- (i) a weakly mixing free action $V = (V_g)_{g \in G}$ of G such that the orbit equivalence relation of V is \mathcal{R} and
- (ii) a cocycle τ of V with values in K such that the skew product V^τ is weakly mixing.

Let $V^A = (V^A(g))_{g \in G}$ denote the product G -action on the product space (X^A, μ^A) , i.e. $(V_g^A x_b) := V_g x_b$ for each $x = (x_b)_{b \in A} \in X^A$ and $g \in G$. We consider also a shift action $S = (S_a)_{a \in A}$ of A on X^A :

$$(S_a x)_b := x_{b-a}, \quad a, b \in A, \quad x = (x_b)_{b \in A} \in X^A.$$

Of course,

- (iii) $S_a \in C(V^A)$ and $S_a \notin \{V_b^A \mid b \in A\}$ for each $a \in A \setminus \{0\}$.

We define a cocycle τ^A of V^A with values in K by setting $\tau^A(g, x) := \tau(g, x_0)$ for all $g \in G$ and $x \in X^A$. Consider the following cocycle $\overline{\tau^A}$ of V^A with values in K^A :

$$\overline{\tau^A} = \bigotimes_{a \in A} \tau^A \circ S_a^{-1}.$$

We claim that $\overline{\tau^A}$ is ergodic. For that we will show that the $\overline{\tau^A}$ -skew product extension of V^A is ergodic. Take $g \in G$, $x = (x_b)_{b \in A} \in X^A$ and $y = (y_b)_{b \in A} \in K^A$. Then $(x, y) = (x_a, y_a)_{a \in A} \in (X \times K)^A$ and

$$\begin{aligned} ((V^A)_{g}^{\overline{\tau^A}}(x, y))_a &= (V_g^A x, \overline{\tau^A}(g, x) + y)_a = (V_g x_a, \tau^A(g, S_a^{-1} x) + y_a) \\ &= (V_g x_a, \tau(g, x_a) + y_a) = V_g^\tau(x_a, y_a). \end{aligned}$$

Thus, $(V^A)^{\overline{\tau^A}} = (V^\tau)^A$, where $(V^\tau)^A := \bigotimes_{a \in A} V^\tau$. Since the skew product V^τ is weakly mixing, $(V^A)^{\overline{\tau^A}}$ is ergodic, i.e. $\overline{\tau^A}$ is ergodic, as claimed.

Denote by \mathcal{V} the V^A -orbit equivalence relation on X^A . It is standard to verify that (iii) implies that $\{S_a \mid a \in A\} \cap [\mathcal{V}] = \{\text{Id}\}$. In a similar way, we deduce from the condition of the lemma that $A \cap [\mathcal{R}] = \{\text{Id}\}$. Therefore we may apply the main result from [BeG]: there is a measure preserving isomorphism $L : X \rightarrow X^A$ and a map $A \ni a \mapsto \gamma_a \in [\mathcal{V}]$ such that $(L \times L)(\mathcal{R}) = \mathcal{V}$ and $LaL^{-1} = \gamma_a S_a^{-1}$ for all $a \in A$.

We note that τ^A extends naturally to $[\mathcal{V}]$. Indeed, for each $\gamma \in [\mathcal{V}]$, there is a unique (mod 0) partition $X^A = \bigsqcup_{g \in G} Z_g$ of X^A into Borel subsets Z_g , $g \in G$, such that $\gamma z = (V^A)_g z$ whenever $z \in Z_g$. We now set $\tau^A(\gamma, z) := \tau^A(g, z)$ whenever $z \in Z_g$. Since V^A is free, $\tau^A(\gamma, \cdot)$ is well defined. It is easy to verify that

$$\tau^A(\gamma_1 \gamma_2, z) = \tau^A(\gamma_1, \gamma_2 z) \tau^A(\gamma_2, z)$$

at a.e. z for all $\gamma_1, \gamma_2 \in [\mathcal{V}]$. We now define a cocycle ω of R with values in K by setting

$$\omega(g, x) := \tau^A(LR_g L^{-1}, Lx) \quad \text{for all } g \in G \text{ and } x \in X.$$

For each $a \in A$, we have

$$\begin{aligned} \omega \circ a(g, x) &= \omega(g, ax) \\ &= \tau^A(LR_g L^{-1}, Lax) \\ &= \tau^A(LR_g L^{-1}, \gamma_a S_a^{-1} Lx) \\ &= f(R_g x) + \tau^A(S_a^{-1} LR_g L^{-1} S_a, S_a^{-1} Lx) - f(x), \end{aligned}$$

where $f(x) := \tau^A(\gamma_a, S_a^{-1} Lx)$. Thus, $\omega \circ a$ is cohomologous to $\tau^A \circ (S_a^{-1} L) = (\tau^A \circ S_a^{-1}) \circ L$. This implies that the cocycle $\overline{\omega}$ of R , given by

$$\overline{\omega}(g, x) := \bigotimes_{a \in A} \omega(g, ax)$$

for all $x \in X$, $g \in G$, is cohomologous to the cocycle

$$\left(\bigotimes_{a \in A} \tau^A \circ S_a^{-1} \right) \circ L = \overline{\tau^A} \circ L.$$

Since $\overline{\tau^A}$ is ergodic, $\overline{\omega}$ is ergodic too. Thus the first claim of the lemma is proved.

We now show how to deduce the second claim of the lemma from the first one. Given an ergodic action W of G , we denote by $\Lambda_d(W)$ the discrete spectrum of W . Then $\Lambda_d(W)$ is a countable subgroup of \widehat{G} . It is known that $R^{\overline{\omega}} \times Q$ is ergodic if and only if $\Lambda_d(R^{\overline{\omega}}) \cap \Lambda_d(Q) = \{1\}$. Of course, $\Lambda_d(R) \subset \Lambda_d(R^{\overline{\omega}})$. Since $R \times Q$ is ergodic, $\Lambda_d(R) \cap \Lambda_d(Q) = \{1\}$. Therefore to prove that $R^{\overline{\omega}} \times Q$ is ergodic it suffices to show that $\Lambda_d(R) = \Lambda_d(R^{\overline{\omega}})$. Suppose that there is $\kappa \in \Lambda_d(R^{\overline{\omega}}) \setminus \Lambda_d(R)$. Then for some character $\chi \in \widehat{K^A}$, the \mathbb{T} -valued cocycle $\chi \circ \overline{\omega}$ of R is cohomologous to the cocycle $G \times X \ni (g, x) \mapsto \kappa(g) \in \mathbb{T}$. Since $\widehat{K^A} = \bigoplus_{a \in A} \widehat{K}$, there is a finite non-empty subset $A_0 \subset A$ and a map $A_0 \ni a \mapsto \chi_a \in \widehat{K} \setminus \{1\}$ such that

$\chi(k) = \prod_{a \in A_0} \chi_a(k_a)$ for each $k = (k_a)_{a \in A} \in K^A$. Since A is infinite, there is $b \in A$ such that $A_0 \cap (b + A_0) = \emptyset$. For each subset $B \subset A$, we define by $\bar{\omega}_B$ the cocycle $\bigotimes_{a \in B} \omega \circ a$ of R with values in K^B . Since $\bar{\omega}$ is ergodic, $\bar{\omega}_B$ is also ergodic. We note that

$$\bar{\omega}_{A_0 \cup (b+A_0)} = \bar{\omega}_{A_0} \times \bar{\omega}_{b+A_0} = \bar{\omega}_{A_0} \times \bar{\omega}_{A_0} \circ S_b.$$

Therefore the \mathbb{T}^2 -valued cocycle

$$\chi \circ \bar{\omega}_{A_0} \times \chi \circ \bar{\omega}_{A_0} \circ S_b = \chi \circ \bar{\omega} \times \chi \circ \bar{\omega} \circ S_b$$

of R is also ergodic. However the later cocycle is cohomologous to the cocycle

$$G \times X \ni (g, x) \mapsto (\kappa(g), \kappa(g))$$

taking values in the diagonal of the torus \mathbb{T}^2 . Hence this cocycle is not ergodic, a contradiction. \square

8. ON $H_3(\mathbb{Z})$ -ODOMETERS

Let $\Gamma_1 \supset \Gamma_2 \supset \dots$ be a decreasing sequence of lattices (i.e. cofinite subgroups) in $H_3(\mathbb{Z})$. Denote by $T = (T_g)_{g \in H_3(\mathbb{Z})}$ the associated $H_3(\mathbb{Z})$ -odometer. Let (X, μ) be the space of this odometer. We call T *normal* if Γ_j is normal in $H_3(\mathbb{Z})$ for each j .²⁶

If T is normal then X is a compact totally disconnected group and μ is the normalized Haar measure on X . Indeed, we obtain a sequence

$$H_3(\mathbb{Z})/\Gamma_1 \leftarrow H_3(\mathbb{Z})/\Gamma_2 \leftarrow \dots$$

of finite groups $H_3(\mathbb{Z})/\Gamma_j$ and canonical onto homomorphisms such that $X = \text{projlim}_{j \rightarrow \infty} H_3(\mathbb{Z})/\Gamma_j$. Moreover, a group homomorphism $\varphi : H_3(\mathbb{Z}) \rightarrow X$ is well defined by the formula $\varphi(g) = (\varphi(g)_j)_{j=1}^\infty$, where $\varphi(g)_j := g\Gamma_j$. Of course, $\varphi(H_3(\mathbb{Z}))$ is dense in X . It is easy to see that $T_g x = \varphi(g)x$ for all $g \in H_3(\mathbb{Z})$ and $x \in X$. Hence T has a pure point spectrum in the sense of [Ma3]. Moreover, T is normal in the sense of [Zi]. Hence [Fa, Corollary 2] yields the following.

Corollary 8.1. *The normal $H_3(\mathbb{Z})$ -odometers are isospectral.*

We also note that every ergodic 2-fold self-joining of a normal $H_3(\mathbb{Z})$ -odometer T is off-diagonal. Hence T is 2-fold simple [dJRu].

Let L_j denote the left regular representation of $H_3(\mathbb{R})/\Gamma_j$. Let \mathcal{I}_j stand for the unitary dual of $H_3(\mathbb{R})/\Gamma_j$.²⁷ It is well known that (up to the unitary equivalence) $L_j = \bigoplus_{\tau \in \mathcal{I}_j} \bigoplus_1^{d_\tau} \tau$, where d_τ is the dimension of τ . In particular, $\#(H_3(\mathbb{R})/\Gamma_j) =$

²⁶There is a difference in nomenclature used in our paper and [Li-Ug]. By an $H_3(\mathbb{Z})$ -odometer the authors of [Li-Ug] mean a free normal $H_3(\mathbb{Z})$ -odometer. An $H_3(\mathbb{Z})$ -odometer satisfying the condition $\bigcap_{j=1}^\infty \Gamma_n = \{1\}$ is called a $H_3(\mathbb{Z})$ -*subodometer* there. The $H_3(\mathbb{Z})$ -odometers which do not satisfy this condition were not considered there. We prefer to use the term “normal” to make it consistent with the well known *normality* concept introduced by R. Zimmer in [Zi] (see also [Fa]).

²⁷Since we do not distinguish between unitarily equivalent representation, we consider elements of \mathcal{I}_j as irreducible unitary representations of $H_3(\mathbb{R})/\Gamma_j$ rather than their unitary equivalence classes.

$\sum_{\tau \in \mathcal{I}_j} d_\tau^2$. Moreover, $\#\mathcal{I}_j$ equals the cardinality of the set of conjugacy classes in $H_3(\mathbb{R})/\Gamma_j$. Denote by U_T the Koopman unitary representation of $H_3(\mathbb{Z})$ generated by T . The canonical projection $X \rightarrow H_3(\mathbb{R})/\Gamma_j$ generates an embedding $L^2(H_3(\mathbb{R})/\Gamma_j) \subset X$. Therefore we obtain an increasing sequence

$$L^2(H_3(\mathbb{R})/\Gamma_1) \subset L^2(H_3(\mathbb{R})/\Gamma_2) \subset \cdots$$

of U_T -invariant subspaces whose union is dense in $L^2(X)$ and such that the restriction $U_T \upharpoonright L^2(H_3(\mathbb{R})/\Gamma_j)$ is unitarily equivalent to $L_j \circ p_j$, where $p_j : H_3(\mathbb{R}) \rightarrow H_3(\mathbb{R})/\Gamma_j$ is the canonical projection. This implies that

$$\{\tau \circ p_1 \mid \tau \in \mathcal{I}_1\} \subset \{\tau \circ p_2 \mid \tau \in \mathcal{I}_2\} \subset \cdots$$

and we obtain the following decomposition of U_T into the orthogonal sum of irreducible finite dimensional unitary representations of $H_3(\mathbb{Z})$.

Theorem 8.2. *Let $\mathcal{I}_T := \bigcup_{j \in \mathbb{N}} \{\tau \circ p_j \mid \tau \in \mathcal{I}_j\}$ and d_ι is the dimension of ι . Then we have*

$$U_T = \bigoplus_{\iota \in \mathcal{I}_T} \bigoplus_1^{d_\iota} \iota.$$

An explicit computation of \mathcal{I}_T in terms of the sequence $(\Gamma_j)_{j=1}^\infty$ was done in [Li-Ug].

Corollary 8.3. *Two normal $H_3(\mathbb{Z})$ -odometers T and R are (measure theoretically) isomorphic if and only if $\mathcal{I}_T = \mathcal{I}_R$.*

We now provide an example of non-isomorphic normal $H_3(\mathbb{Z})$ -odometers V and V' such that the Koopman representations of $H_3(\mathbb{R})$ generated by the $H_3(\mathbb{R})$ -odometers associated with the same sequences of lattices (as V and V' respectively) are unitarily equivalent.²⁸ We proceed it with a lemma which was utilized in [Li-Ug] with reference to [CoPe]. We provide an alternative proof.

Lemma 8.4. *Let V be the $H_3(\mathbb{Z})$ -odometer associated with a sequence $(\Gamma_j)_{j=1}^\infty$ of normal lattices in $H_3(\mathbb{Z})$ such that $\bigcap_{j=1}^\infty \Gamma_j = \{1\}$. Let Σ be a lattice in $H_3(\mathbb{Z})$ such that the homogeneous space $H_3(\mathbb{Z})/\Sigma$ is a factor, say \mathfrak{F} , of V . Then there is $j_0 > 0$ such that $\Sigma \supset \Gamma_{j_0}$.*

Proof. Since V is 2-fold simple, we apply the Veech theorem [dJRu]: there is a compact subgroup K in the centralizer $C(V)$ of V such that \mathfrak{F} is the σ -algebra $\text{Fix } K$ of subsets fixed by K . Since \mathfrak{F} is finite, K is open. We recall that the space of V is a compact group $X = \text{proj lim}_{j \rightarrow \infty} H_3(\mathbb{Z})/\Gamma_j$. Hence the centralizer $C(V)$ is isomorphic to X acting on itself by right translations.²⁹ Let Λ_j be the kernel of the canonical projection $q_j : X \rightarrow H_3(\mathbb{Z})/\Gamma_j$. Then Λ_j is an open subgroup in X , $\Lambda_1 \supset \Lambda_2 \supset \cdots$ and $\bigcap_{j=1}^\infty \Lambda_j = \{1\}$. Therefore there is $j_0 > 0$ such that $K \supset \Lambda_{j_0}$. Hence $\mathfrak{F} \subset \text{Fix } \Lambda_{j_0}$. However $\text{Fix } \Lambda_{j_0}$ is exactly the factor of V determined by q_{j_0} . Thus we obtain that $H_3(\mathbb{Z})/\Sigma$ is a factor of $H_3(\mathbb{Z})/\Gamma_{j_0}$. Since Γ_{j_0} is normal in $H_3(\mathbb{Z})$, it follows that $\Sigma \supset \Gamma_{j_0}$. \square

²⁸We note that the Koopman representations of $H_3(\mathbb{Z})$ generated by V and V' are not unitarily equivalent by Corollary 8.1. Thus we obtain an example of unitarily non-equivalent Koopman unitary representations U_V and $U_{V'}$ of $H_3(\mathbb{Z})$ with pure point spectrum such that the induced unitary representations $\text{Ind}_{H_3(\mathbb{Z})}^{H_3(\mathbb{R})}(U_V)$ and $\text{Ind}_{H_3(\mathbb{Z})}^{H_3(\mathbb{R})}(U_{V'})$ of $H_3(\mathbb{R})$ are unitarily equivalent.

²⁹We recall that V acts on X by left translations.

Example 8.5 (cf. [Li–Ug, Example 4.9]). Let $(\Gamma_n)_{n=1}^\infty$ and $(\Gamma'_n)_{n=1}^\infty$ be as in Example 5.11. Denote by V and V' the $H_3(\mathbb{Z})$ -odometers associated with the sequences $(\Gamma_n)_{n=1}^\infty$ and $(\Gamma'_n)_{n=1}^\infty$ respectively. They are normal. It was shown in [Li–Ug, Example 4.9] that T and T' are not isomorphic.³⁰ Indeed, otherwise in view of Lemma 8.4, there is $n > 0$ such that $\Gamma'_n \subset \Gamma_2$. However we see that $c(1)a(k_n) \in \Gamma'_n$ but $c(1)a(k_n) \notin \Gamma_2$, a contradiction. On the other hand, $b(1/k_n)\Gamma'_n b(-1/k_n) = \Gamma_n$ for each $n \in \mathbb{N}$. Hence the $H_3(\mathbb{R})$ -odometers T and T' associated with $(\Gamma_n)_{n=1}^\infty$ and $(\Gamma'_n)_{n=1}^\infty$ are f-isomorphic. By Corollary 5.8, the Koopman unitary representations of $H_3(\mathbb{R})$ generated by them are unitarily equivalent. We also examine the “symmetry” property for V and V' . Since $\theta(H_3(\mathbb{Z})) = H_3(\mathbb{Z})$, the symmetric $H_3(\mathbb{Z})$ -actions are defined in a similar way as for the $H_3(\mathbb{R})$ -actions. It is easy to see that V is symmetric. It is straightforward to verify that

$$\theta(\Gamma'_n) = a(-1/k_n)\Gamma_n a(1/k_n) = \{c(k_n j_3 + j_2)b(k_n j_2)a(k_n j_1) \mid j_1, j_2, j_3 \in \mathbb{Z}\}$$

for each n . If V' and $V' \circ \theta$ were isomorphic then by Lemma 8.4, there is $n > 0$ such that $\theta(\Gamma'_n) \subset \Gamma'_2$. However we see that $c(k_n + 1)b(k_n) \in \theta(\Gamma'_n)$ but $c(k_n + 1)b(k_n) \notin \theta(\Gamma'_2)$. This contradiction yields that V' is not symmetric.

9. CONCLUDING REMARKS AND OPEN PROBLEMS

In view of Corollary 8.1, we ask the following natural question.

Problem 9.1. Whether the non-normal $H_3(\mathbb{Z})$ -odometers are isospectral?

Let G be a locally compact second countable group and let Γ be a lattice in G .

Definition 9.2. We call two probability preserving actions V and V' of Γ *flow equivalent* if the induced G -actions $\text{Ind}_\Gamma^G(V)$ and $\text{Ind}_\Gamma^G(V')$ are isomorphic.

In the case $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$ this definition corresponds to the classical concept of flow equivalence in topological dynamics. As was observed, e.g. in [DaLe, Proposition 1.3], if G is Abelian then two ergodic actions of Γ are flow equivalent if and only if they are isomorphic.

Problem 9.3. Whether two flow equivalent ergodic actions of $H_3(\mathbb{Z})$ are isomorphic? Whether two flow equivalent $H_3(\mathbb{Z})$ -odometers are isomorphic?

In this connection we make the following remark.

Remark 9.4. Let V and V' be two ergodic probability preserving actions of $H_3(\mathbb{Z})$. Suppose that they are flow equivalent. Let (Z, κ) be the space of $V = (V_\gamma)_{\gamma \in H_3(\mathbb{Z})}$ and let (Z', κ') be the space of $V' = (V'_\gamma)_{\gamma \in H_3(\mathbb{Z})}$. Denote the homogeneous space $H_3(\mathbb{R})/H_3(\mathbb{Z})$ by Y . The translation of $y \in Y$ by $g \in H_3(\mathbb{R})$ is denoted by $g * y$. Let λ stand for the Haar measure on Y . Then $(Y \times Z, \lambda \times \kappa)$ is the space of $\text{Ind}_{H_3(\mathbb{Z})}^{H_3(\mathbb{R})}(V)$ and $(Y \times Z', \lambda \times \kappa')$ is the space of $\text{Ind}_{H_3(\mathbb{Z})}^{H_3(\mathbb{R})}(V')$. Thus Y is a factor of both V and V' . Let R be a measure preserving isomorphism of $Y \times Z$ onto $Y \times Z'$ that conjugates $\text{Ind}_{H_3(\mathbb{Z})}^{H_3(\mathbb{R})}(V)$ with $\text{Ind}_{H_3(\mathbb{Z})}^{H_3(\mathbb{R})}(V')$. Suppose, in addition, that

$$(9-1) \quad R \text{ maps the factor } Y \text{ of } V \text{ to the factor } Y \text{ of } V'.$$

³⁰This fact follows also from Example 5.11 because if V and V' are isomorphic then the induced $H_3(\mathbb{R})$ -actions are also isomorphic. However these induced actions are the $H_3(\mathbb{R})$ -odometers associated with $(\Gamma_n)_{n=1}^\infty$ and $(\Gamma'_n)_{n=1}^\infty$ respectively. As was shown in Example 5.11, these odometers are not isomorphic.

Then there exists a measurable field $Y \ni y \mapsto R_y$ of isomorphisms R_y from (Z, κ) to (Z', κ') and a transformation Q of Y commuting with the $H_3(\mathbb{R})$ -action on Y such that $R(y, z) = (Qy, R_y z)$ for a.a. $(y, z) \in Y \times Z$. By [Da2, Lemma 2.1], Q is a translation by an element of the normalizer of $H_3(\mathbb{Z})$ in $H_3(\mathbb{R})$. A straightforward verification shows that this normalizer is $\{c(t)b(m)a(n) \mid t \in \mathbb{R}, n, m \in \mathbb{Z}\}$. Thus there is $t_0 \in \mathbb{R}$ such that $Qy = c(t_0) * y$ for λ -a.a. $y \in Y$. Since R conjugates the induced $H_3(\mathbb{R})$ -actions, we obtain that

$$(c(t_0)g * y, R_{g*y}V_{h_s(g,y)}z) = (gc(t_0) * y, V'_{h_s(g,c(t_0)*y)}R_y z)$$

for $(\lambda \times \kappa)$ -a.a. $(y, z) \in Y \times Z$ for each $g \in H_3(\mathbb{R})$. Here $h_s : H_3(\mathbb{R}) \times Y \rightarrow H_3(\mathbb{Z})$ is the choice cocycle corresponding to a Borel cross-section $s : Y \rightarrow H_3(\mathbb{R})$. By Fubini's theorem, there is $y_0 \in Y$ such that

$$(9-2) \quad R_{y_0}V_{h_s(g,y_0)}z = V'_{h_s(g,c(t_0)*y_0)}R_{y_0}z$$

for κ -a.a. $z \in Z$ and all g from the stability group $S_{y_0} := \{g \in H_3(\mathbb{R}) \mid g * y_0 = y_0\}$ of y_0 . We used here that S_{y_0} is countable (in fact, it is conjugate to $H_3(\mathbb{Z})$) to guarantee that the " κ -a.a. $z \in Z$ " does not depend on $g \in S_{y_0}$. Next, we have

$$\begin{aligned} h_s(g, c(t_0) * y_0) &= s(gc(t_0) * y_0)^{-1}gs(c(t_0) * y_0) \\ &= s(c(t_0) * y_0)^{-1}gs(c(t_0) * y_0) \\ &= r_0^{-1}s(y_0)^{-1}gs(y_0)r_0 \\ &= r_0^{-1}h_s(g, y_0)r_0 \end{aligned}$$

for some $r_0 \in H_3(\mathbb{Z})$ and all $g \in S_{y_0}$. We used the fact that

$$s(c(t_0) * y_0) \in c(t_0)s(y_0)H_3(\mathbb{Z}).$$

Now (9-2) implies that

$$R_{y_0}V_{h_s(g,y_0)}R_{y_0}^{-1} = V'_{r_0^{-1}h_s(g,y_0)r_0} = (V'_{r_0})^{-1}V'_{h_s(g,y_0)}V'_{r_0}.$$

Since $\{h_s(g, y_0) \mid g \in S_{y_0}\} = H_3(\mathbb{Z})$, it follows that V and V' are isomorphic.

Thus Problem 9.3 reduces to the following question:

Problem 9.5. Suppose that $\text{Ind}_{H_3(\mathbb{Z})}^{H_3(\mathbb{R})}(V)$ and $\text{Ind}_{H_3(\mathbb{Z})}^{H_3(\mathbb{R})}(V')$ are isomorphic. Is there an isomorphism satisfying (9-1)?

Let V and V' be the $H_3(\mathbb{Z})$ -odometers associated with sequences $\Gamma_1 \supset \Gamma_2 \supset \dots$ and $\Gamma'_1 \supset \Gamma'_2 \supset \dots$ of lattices in $H_3(\mathbb{Z})$ respectively. Let T and T' be the $H_3(\mathbb{R})$ -odometers associated with the same sequences of lattices (considered as lattices in $H_3(\mathbb{R})$). It follows from Corollary 2.5 that V and V' are flow equivalent if and only if T and T' are isomorphic. The concept of f-isomorphism for the $H_3(\mathbb{R})$ -odometers motivated the following definition.

Definition 9.6. Let T and T' be two ergodic actions of a locally compact second countable group G on standard probability spaces (X, \mathfrak{B}, μ) and $(X', \mathfrak{B}', \mu')$ respectively. We say that T and T' are *f-isomorphic* if there are increasing sequences $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \dots$ and $\mathfrak{F}'_1 \subset \mathfrak{F}'_2 \subset \dots$ of factors of T and T' respectively such that

$\bigvee_{j=1}^{\infty} \mathfrak{F}_j = \mathfrak{B}$, $\bigvee_{j=1}^{\infty} \mathfrak{F}'_j = \mathfrak{B}'$ and $T \upharpoonright \mathfrak{F}_j$ is isomorphic to $T' \upharpoonright \mathfrak{F}'_j$ for each $j \in \mathbb{N}$. We say that T and T' are *F-isomorphic* if there is a finite sequence R_1, \dots, R_n of ergodic G -actions such that $R_1 = T$, $R_n = T'$ and R_j is f-isomorphic to R_{j+1} for each $j = 1, \dots, n-1$. Of course, f-isomorphism is a symmetric relation on the set of ergodic G -actions and F-isomorphism is the smallest equivalence relation majorizing f-isomorphism.

We note that F-isomorphism is weaker than isomorphism.

Example 9.7. Let T be a G -action with MSJ (see [dJRu]). Let (X, \mathfrak{B}, μ) be the space of T . Denote by $T \odot T$ the symmetric factor of the Cartesian product $T \times T$, i.e. the restriction of $T \times T$ to the σ -algebra of subsets invariant under the involution $(x, y) \mapsto (y, x)$. Then it is easy to verify that the G -actions $R := T \times T \times \dots$ and $R' := (T \odot T) \times T \times T \times \dots$ are f-isomorphic. Indeed, let $\mathfrak{F}_n := \mathfrak{B}^{\otimes n} \otimes (\mathfrak{B} \odot \mathfrak{B}) \otimes \mathfrak{N} \otimes \mathfrak{N} \otimes \dots$ and $\mathfrak{F}'_n := (\mathfrak{B} \odot \mathfrak{B}) \otimes \mathfrak{B}^{\otimes n} \otimes \mathfrak{N} \otimes \mathfrak{N} \otimes \dots$. Then $(\mathfrak{F}_n)_{n=1}^{\infty}$ is an increasing sequence of factors of R with $\bigvee_{n=1}^{\infty} \mathfrak{F}_n$ being the entire σ -algebra of R ; $(\mathfrak{F}'_n)_{n=1}^{\infty}$ is an increasing sequence of factors of R' with $\bigvee_{n=1}^{\infty} \mathfrak{F}'_n$ being the entire σ -algebra of R' ; and $R \upharpoonright \mathfrak{F}_n$ is isomorphic to $R' \upharpoonright \mathfrak{F}'_n$ for each n . On the other hand, R and R' are not isomorphic (see [dJRu]).

Problem 9.8.

- (1) Give an example of two ergodic G -actions which are F-isomorphic but not f-isomorphic.
- (2) We note that in Example 9.7 the actions R and R' are weakly isomorphic. Are there F-isomorphic G -actions which are not weakly isomorphic?
- (3) Are there F-isomorphic G -actions which are not Markov quasi-equivalent?³¹

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³¹See [FPS] for the definitions.

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