Polynomial actions of unitary operators and idempotent ultrafilters*

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Abstract

Let p be an idempotent ultrafilter over $\mathbb{N}$. For a positive integer N, let $\mathcal{P}_{\leq N}$ denote the additive group of polynomials $P \in \mathbb{Z}[x]$ with $\deg P \leq N$ and $P(0) = 0$. Given a unitary operator U on a Hilbert space $\mathcal{H}$, we prove, for each $N \geq 1$, the existence of a unique decomposition $\mathcal{H} = \bigoplus_{r\geq 1} \mathcal{H}_r^{(N)}$ into closed, U-invariant subspaces such that

- for any polynomial $P \in \mathcal{P}_{\leq N}$, we have

$$p\text{-}\lim_{n\in\mathbb{N}} \left(U|_{\mathcal{H}_r^{(N)}}\right)^{P(n)} = 0_{\mathcal{H}_r^{(N)}} \text{ or } Id_{\mathcal{H}_r^{(N)}}, \text{ for each } r \geq 1;$$

- for each $r \neq s$ there exists $Q \in \mathcal{P}_{\leq N}$ such that

$$p\text{-}\lim_{n\in\mathbb{N}} \left(U|_{\mathcal{H}_r^{(N)}}\right)^{Q(n)} \neq p\text{-}\lim_{n\in\mathbb{N}} \left(U|_{\mathcal{H}_s^{(N)}}\right)^{Q(n)}.$$

In connection with this result we introduce the notion of rigidity group. Namely, a subgroup $G \subset \mathcal{P}_{\leq N}$ is called an N-rigidity group if there exist an idempotent ultrafilter p over $\mathbb{N}$ and a unitary operator U on a Hilbert space $\mathcal{H}$ such that

$$G = \{P \in \mathcal{P}_{\leq N} : p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = Id\} \tag{0.1}$$

and

$$p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)} = 0 \text{ for each } Q \in \mathcal{P}_{\leq N} \setminus G. \tag{0.2}$$

The main result of the paper states that a subgroup $G \subset \mathcal{P}_{\leq N}$ satisfying $\max\{\deg P : P \in G\} = N$ is an N-rigidity group if and only if G has finite index in $\mathcal{P}_{\leq N}$.

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Contents

Introduction

One of the goals of this paper is to establish a new Hilbert space decomposition theorem for polynomial actions of unitary operators which may be seen as a far reaching refinement of classical splitting results summarized in the following theorem.

Theorem 0.1. *Let $\mathcal{H}$ be a Hilbert space* [1] *and $U : \mathcal{H} \to \mathcal{H}$ a unitary operator (we write for short $U \in \mathcal{U}(\mathcal{H})$). Then*

$$\mathcal{H} = \mathcal{H}_{inv} \oplus \mathcal{H}_{erg}, \tag{0.3}$$

where $\mathcal{H}_{inv} = \{f \in \mathcal{H} : Uf = f\}$ *and* $\mathcal{H}_{erg} = \{f \in \mathcal{H} : \lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} U^n f = 0\}$;

$$\mathcal{H} = \mathcal{H}_{comp} \oplus \mathcal{H}_{wm}, \tag{0.4}$$

[1] We are tacitly assuming that $\mathcal{H}$ is separable. It is not hard to see that this assumption can be made without the loss of generality. The theorems in this paper which pertain to unitary operators on separable Hilbert spaces hold for non-separable spaces as well.

where

$$\mathcal{H}_{comp} = \{f \in \mathcal{H} : \overline{\{U^n f\}}_{n\in\mathbb{Z}} \textit{ is compact in the norm topology}\} = \overline{\text{span}}\{f \in \mathcal{H} : (\exists \lambda \in \mathbb{C})\ Uf = \lambda f\}$$

and

$$\mathcal{H}_{wm} = \{f \in \mathcal{H} :\ (\forall g \in \mathcal{H})\ \lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\langle U^n f, g\rangle| = 0\}.$$

The decomposition (0.3) in the above theorem is the classical *ergodic* Hilbert space decomposition which is behind the von Neumann's ergodic theorem (see, e.g. [22]). The more interesting decomposition (0.4) is a special case of Jacobs-Glicksberg-de Leeuw decomposition [30] which is connected with the notion of *weak mixing*.

Recall that a unitary operator $U \in \mathcal{U}(\mathcal{H})$ is called *weakly mixing* if it has no non-trivial eigenvectors, meaning that if for some $\lambda \in \mathbb{C}$ and $f \in \mathcal{H}$ one has $Uf = \lambda f$, then $f = 0$. The notion of weak mixing was introduced in [28] and has a multitude of equivalent formulations (see, e.g. [5] or [9]). An important class of weakly mixing operators has its origins in the theory of measure-preserving systems. Given a measure space $(X, \mathcal{B}, \mu)$ and an invertible measure-preserving transformation $T : X \to X$, define the unitary operator U_T on $L^2(X, \mathcal{B}, \mu)$ by the formula $U_T(f)(x) = f(Tx)$. The transformation T is called *weakly mixing* if U_T is a weakly mixing operator on the space $\mathcal{H} = L^2_0(X, \mathcal{B}, \mu) := \{f \in L^2(X, \mathcal{B}, \mu) :\ \int f\, d\mu = 0\}$.

It follows from (0.4) that a unitary operator $U \in \mathcal{U}(\mathcal{H})$ is weakly mixing if and only if for any $f, g \in \mathcal{H}$ one has $\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\langle U^n f, g\rangle| = 0$. This, in turn, implies that U is weakly mixing if and only if for any $f \in \mathcal{H}$ there exists a set $E \subset \mathbb{N} := \{1, 2, \ldots\}$ satisfying $d(E) := \lim_{N-M\to\infty} \frac{|E\cap[M,\ldots,N-1]|}{N-M} = 0$ such that $U^n f \to 0$ weakly, when $n \to \infty, n \notin E$ [2]. It is the presence of the exceptional set E which distinguishes between the notion of weak mixing and that of strong mixing (which is defined by the condition $U^n f \to 0$ weakly, when $n \to \infty$). One can show that a "generic" unitary operator is weakly but not strongly mixing (see for example [22], [31]). Moreover, the generic unitary operator is simultaneously weakly mixing and *rigid* (see, e.g. [7] or [31]), meaning that, there exists a sequence $n_k \to \infty$ such that for every $f \in \mathcal{H}$, $U^{n_k} f \to f$ (in $\mathcal{H}$), when $k \to \infty$. Since the exceptional set, along which a weakly mixing operator U is rigid, is of zero density, it does not affect the value of the Cesàro limits $\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\langle U^n f, g\rangle|$. Consequently, if one is interested in distinguishing between various classes of weakly mixing operators based on their rigidity properties, the Cesàro averages may not be so effective a tool and one may want to look for some alternative notions of convergence. We will see below

[2]If $\mathcal{H}$ is separable, one can actually show that $U \in \mathcal{U}(\mathcal{H})$ is weakly mixing if and only if there exists $E \subset \mathbb{N}$ with $d(E) = 0$ such that, for any $f \in \mathcal{H}$, $U^n f \to 0$ as $n \to \infty$, $n \notin E$.

that the notion of convergence along the idempotent ultrafilters provides a satisfactory alternative to Cesàro limits and seems to be especially useful for the study of behaviour of unitary operators along polynomials.

To provide an instructive glimpse into the effectiveness of idempotent ultrafilters, let us briefly discuss a polynomial generalization of the classical Khintchine's recurrence theorem which leads to interesting combinatorial applications. The celebrated Poincaré's recurrence theorem states that for any measure-preserving transformation T on a probability space $(X, \mathcal{B}, \mu)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n}A) > 0$. Khintchine's refinement [27] of Poincaré's theorem can be formulated as follows. Recall that a subset of $\mathbb{Z}$ (or of $\mathbb{N}$) is called *syndetic*, if it has bounded gaps.

Theorem 0.2. *Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ an invertible measure-preserving transformation. Assume that $A \in \mathcal{B}$, $\mu(A) > 0$. Then for any $\varepsilon > 0$ the set $\{n \in \mathbb{Z} : \mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon\}$ is syndetic.*

To prove Theorem 0.2 one can use von Neumann's ergodic theorem. Let $f = \mathbf{1}_A$ and let

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} U_T^n f = f^*,$$

where $f^* = \mathrm{proj}_{\mathcal{H}_{inv}} f$ is the orthogonal projection of f on the space of U_T-invariant functions. Then we have

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{-n}A) = \lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \int f \cdot U_T^n(f)\, d\mu$$

$$= \int f \cdot \mathrm{proj}_{\mathcal{H}_{inv}} f\, d\mu = \int \mathrm{proj}_{\mathcal{H}_{inv}} f \cdot \mathrm{proj}_{\mathcal{H}_{inv}} f\, d\mu \cdot \int \mathbf{1} \cdot \mathbf{1}\, d\mu$$

$$\geq \left(\int \mathrm{proj}_{\mathcal{H}_{inv}} f \cdot \mathbf{1}\, d\mu \right)^2 = \left(\int f\, d\mu \right)^2 = \mu(A)^2.$$

(We used the fact that $\mathrm{proj}_{\mathcal{H}_{inv}}$ is a self-adjoint operator and the Cauchy-Schwarz inequality) [3]. Consider now the following polynomial extension of Poincaré's theorem, obtained by Furstenberg [15].

Theorem 0.3. *Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ an invertible measure-preserving transformation. Assume that $A \in \mathcal{B}$, $\mu(A) > 0$. Then for any $\varepsilon > 0$ and any*

[3]This proof is essentially due to Hopf [26]. For a different proof of combinatorial nature, see [4], Section 5.

polynomial $P \in \mathbb{Z}[x]$ with $P(0)=0$, there are arbitrarily large n such that $\mu(A \cap T^{P(n)}A) > \mu(A)^2 - \varepsilon$ [4].

To prove Theorem 0.3, Furstenberg invokes the spectral theorem[5] and some classical results on uniform distribution. The polynomial recurrence theorem in question follows from the fact that for any $f \in \mathcal{H}$ and any unitary operator $U \in \mathcal{U}(\mathcal{H})$ the strong limit

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} U^{P(n)} f \text{ exists and, in addition,} \tag{0.5}$$

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{P(n)}A) =: c_A > 0. \tag{0.6}$$

While it is not known what is the "optimal" value of the constant c_A, one can provide examples where c_A is strictly smaller than $\mu(A)^2$. There is one more distinction between the polynomial result (0.5) and the von Neumann's ergodic theorem. Namely, while the limit of the "linear" Cesàro averages in von Neumann's theorem is an orthogonal projection, this is no longer the case for the polynomial averages in (0.5). To "fix" this situation, we will introduce the ultrafilter analogues of the limits (0.5) and (0.6) leading to a polynomial version of Khintchine's recurrence theorem which even in the case of linear polynomials gives more than Theorem 0.2.

We first briefly summarize some facts concerning the idempotent ultrafilters on $\mathbb{N}$. The reader will find more details in Section 1 and the references indicated there.

An ultrafilter p on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ satisfying (i) $\emptyset \notin p$, (ii) $\mathbb{N} \in p$, (iii) $A \in p$ and $A \subset B \subset \mathbb{N}$ implies $B \in p$, (iv) $A, B \in p$ implies $A \cap B \in p$ and (v) if $r \in \mathbb{N}$ and $\mathbb{N} = A_1 \cup \ldots \cup A_r$, then some $A_i \in p$. (In other words, an ultrafilter is a maximal filter.) The space of ultrafilters on $\mathbb{N}$ is denoted by $\beta\mathbb{N}$ (and is identified with the Stone-Čech compactification of $\mathbb{N}$). Any element $n \in \mathbb{N}$ can be identified with the ultrafilter $\{A \subset \mathbb{N} : n \in A\}$. Given $A \subset \mathbb{N}$, let $\overline{A} := \{p \in \beta\mathbb{N} : A \in p\}$. The family $\{\overline{A} : A \subset \mathbb{N}\}$ forms a basis for the open sets (and a basis for the closed sets as well) of $\beta\mathbb{N}$. The operation of addition on $\mathbb{N}$ can be extended to $\beta\mathbb{N}$ as follows. Given $p, q \in \beta\mathbb{N}$ and $A \subset \mathbb{N}$,

$$A \in p + q \Leftrightarrow \{n \in \mathbb{N} : A - n \in p\} \in q \text{ [6]}. \tag{0.7}$$

[4]Theorem 0.3 implies Sárközy's theorem stating that if $E \subset \mathbb{N}$ is a subset of positive upper Banach density: $d^*(E) := \limsup_{N-M\to\infty} \frac{|E \cap [M,N-1]|}{N-M} > 0$, and $P \in \mathbb{Z}[x]$ satisfies $P(0)=0$, then one can find infinitely many $n \in \mathbb{Z}$ such that, for some $x, y \in E$, $x - y = P(n)$.

[5]For a "geometric" proof avoiding the spectral theorem, see [3], [4].

[6]For $A \subset \mathbb{N}$ and $n \in \mathbb{N}$, the set $A - n$ is defined as $\{y \in \mathbb{N} : y + n \in A\}$.

Formula (0.7) makes $(\beta\mathbb{N}, +)$ a compact left topological semigroup [7]. By Ellis' lemma [14], any compact left topological semigroup has an idempotent. Note that whenever $p \in \beta\mathbb{N}$ is an idempotent, by (0.7), we have

$$A \in p \Leftrightarrow A \in p + p \Leftrightarrow \{n \in \mathbb{N} : \ A - n \in p\} \in p. \tag{0.8}$$

Let now X be a topological space. Given a sequence $(x_n) \subset X$ and an ultrafilter $p \in \beta\mathbb{N}$, we will write $p\text{-}\lim_{n\in\mathbb{N}} x_n = x$ if for any neighbourhood $U \ni x$, the set $\{n \in \mathbb{N} : x_n \in U\} \in p$. Then, whenever X is a compact Hausdorff space, $p\text{-}\lim_{n\in\mathbb{N}} x_n$ exists and is unique. Moreover, if $p = p + p$, then (0.8) implies

$$p\text{-}\lim_{n\in\mathbb{N}} x_n = p\text{-}\lim_{n\in\mathbb{N}} \left(p\text{-}\lim_{m\in\mathbb{N}} x_{n+m} \right). \tag{0.9}$$

An immediate application of the introduced concepts gives an ultrafilter analogue of the von Neumann's ergodic theorem (and also a natural analogue of the splitting (0.3) which we encountered in Theorem 0.1). Assume that $U \in \mathcal{U}(\mathcal{H})$, let $r > 0$, and let $f \in \mathcal{H}$ with $\|f\| = r$. The r-ball $X := \{g \in \mathcal{H} : \ \|g\| \leq r\}$, equipped with a metric d induced by the weak topology, is a U-invariant compact Hausdorff space. Let $p \in \beta\mathbb{N}$, $p + p = p$ and set $f^* := p\text{-}\lim_{n\in\mathbb{N}} U^n f$. Utilizing the formula (0.9), we have

$$\begin{aligned} f^* = p\text{-}\lim_{n\in\mathbb{N}} U^n f &= p\text{-}\lim_{n\in\mathbb{N}} \left(p\text{-}\lim_{m\in\mathbb{N}} U^{n+m} f \right) \\ = p\text{-}\lim_{n\in\mathbb{N}} U^n \left(p\text{-}\lim_{m\in\mathbb{N}} U^m f \right) &= p\text{-}\lim_{n\in\mathbb{N}} U^n f^*. \end{aligned}$$

It follows that for any $f \in \mathcal{H}$ and any idempotent $p \in \beta\mathbb{N}$, $f^* = p\text{-}\lim_{n\in\mathbb{N}} U^n f$ is a rigid vector[8]. Indeed, notice that, for each $\varepsilon > 0$, the set $\{n \in \mathbb{N} : \ d(U^n f, f) < \varepsilon\}$ is a member of p and hence is not empty. Therefore, we can find an increasing subsequence (n_i) such that $d(U^{n_i} f, f) \to 0$ which is equivalent to $U^{n_i} f \to f$ in $\mathcal{H}$. We remark in passing that it is not hard to show that any rigid vector is of the form $p\text{-}\lim_{n\in\mathbb{N}} U^n f$ for some idempotent $p \in \beta\mathbb{N}$ and $f \in \mathcal{H}$.

Let now Y be the unit ball in the space of bounded operators on $\mathcal{H}$. Then Y, equipped with a metric induced by the weak operator topology, becomes a compact semitopological

[7] Making $(\beta\mathbb{N}, +)$ a left topological semigroup means that for each $p \in \beta\mathbb{N}$ the function $\lambda_p(q) = p + q$ is continuous.

[8] A vector $f \in \mathcal{H}$ is called a *rigid vector* for U if for some increasing sequence $(n_k) \subset \mathbb{N}$ we have $U^{n_k} f \to f$ strongly. Note that in this particular case, the strong convergence is equivalent to $U^{n_k} f \to f$ weakly. An operator U is called *rigid along* (n_k) if $U^{n_k} \to Id$ strongly.

(i.e. left- and right topological) semigroup. Assume that $p \in \beta\mathbb{N}$, $p+p=p$ and let $W := p\text{-}\lim_{n\in\mathbb{N}} U^n$. It is easy to check that W is a self-adjoint idempotent and hence an orthogonal projection on the subspace of p-rigid vectors. We summarize this discussion in the following theorem.

Theorem 0.4. *Assume that $U \in \mathcal{U}(\mathcal{H})$ and let $p \in \beta\mathbb{N}$, $p+p=p$. Then $\mathcal{H} = \mathcal{H}_r \oplus \mathcal{H}_m$, where $\mathcal{H}_r = \{f \in \mathcal{H} : \ p\text{-}\lim_{n\in\mathbb{N}} U^n f = f\}$*[9] *and $\mathcal{H}_m = \{f \in \mathcal{H} : \ p\text{-}\lim_{n\in\mathbb{N}} U^n f = 0\}$.*

Remark 1. When dealing with p-limits, we usually use weak convergence. It is however worth noticing that the relation $p\text{-}\lim_{n\in\mathbb{N}} U^n f = f$ on the subspace $\mathcal{H}_r$ holds in the weak topology if and only if it holds in the strong topology, cf. footnote 8.

Denote by $\mathcal{P}$ the group of polynomials $P \in \mathbb{Z}[x]$ satisfying $P(0) = 0$. The following result represents a polynomial extension of Theorem 0.4.

Theorem 0.5 ([4]). *For each unitary operator $U \in \mathcal{U}(\mathcal{H})$, each $p \in \beta\mathbb{N}$, $p+p=p$, and each polynomial $P \in \mathcal{P}$,*

$$p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = \mathrm{proj}_{\mathcal{F}},$$

where $\mathcal{F}$ is a closed, U-invariant subspace of $\mathcal{H}$.

Theorem 0.5 allows one to derive a polynomial Khintchine-like theorem.

Corollary 0.6. *Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ an invertible measure-preserving transformation. Given $P \in \mathcal{P}$, for each $\varepsilon > 0$ and $A \in \mathcal{B}$, the set*

$$R_\varepsilon(A; P) := \{n \in \mathbb{Z} : \ \mu(A \cap T^{P(n)}A) \geq (\mu(A))^2 - \varepsilon\}$$

is an IP**-set* [10].

[9] Note that $\mathcal{H}_{comp} \subset \mathcal{H}_r$ because for each $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and each idempotent $p \in \beta\mathbb{N}$, we have $p\text{-}\lim_{n\in\mathbb{N}} \lambda^n = 1$, so if $Uf = \lambda f$, $p\text{-}\lim_{n\in\mathbb{N}} U^n f = f$.

[10] A set $S \subset \mathbb{N}$ is IP* if $S \in p$ for any idempotent $p \in \beta\mathbb{N}$. The notation reflects the fact that S is IP* if and only if S has a nontrivial intersection with any IP-set (see (1.2) in the next section for the definition of IP-set). The proof of Corollary 0.6 then goes as follows (cf. [6]). We have

$$a := p\text{-}\lim_{n\in\mathbb{N}} \mu(A \cap T^{-P(n)}A) = \langle \mathrm{proj}_{\mathcal{F}} \mathbf{1}_A, \mathbf{1}_A \rangle \geq (\langle \mathrm{proj}_{\mathcal{F}} \mathbf{1}_A, \mathbf{1} \rangle)^2 = (\mu(A))^2.$$

Therefore, for each idempotent $p \in \beta\mathbb{N}$, $R_\varepsilon(A; P) \cap \mathbb{N} \supset \{n \in \mathbb{N} : \ \left|\mu(A \cap T^{-P(n)}A) - a\right| < \varepsilon\}$, whence $R_\varepsilon(A; P) \cap \mathbb{N} \in p$. Thus $R_\varepsilon(A; P)$ is IP*.

It is not hard to show that any IP*-set is syndetic. On the other hand, not every syndetic set is IP*. (For example, $2\mathbb{N} + 1$ is syndetic but not IP*.) So, Corollary 0.6 forms a non-trivial extension of Theorem 0.2 in more than one respect.

For a more general form of the polynomial Khintchine theorem and some combinatorial applications, see [8].

Theorem 0.5 and Corollary 0.6 indicate that ergodic theorems along idempotent ultrafilters can be useful for ergodic-theoretical and combinatorial applications. But, as a matter of fact, studying the limits of the form $p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)}$ can also allow one to better understand the intricate properties of the unitary operators acting on a Hilbert space $\mathcal{H}$. This is of a special interest in case when U has continuous spectrum. To continue the line of juxtaposition of Cesàro limits with limits along ultrafilters, notice that if $U \in \mathcal{U}(\mathcal{H})$ is weakly mixing then for any non-constant polynomial $P \in \mathbb{Z}[x]$ one has [15] the strong limit

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} U^{P(n)} = 0.$$

Moreover, as shown in [3], once $\mathcal{H} = \mathcal{H}_{wm}$, we also have

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\langle U^{P(n)} f, g\rangle| = 0 \tag{0.10}$$

for each $f, g \in \mathcal{H}$ and each non-zero degree polynomial $P \in \mathbb{Z}[x]$. Note also that if (0.10) holds for some non-zero degree polynomial $Q \in \mathbb{Z}[x]$ then U must be weakly mixing and hence (0.10) holds for all non-zero degree polynomials $P \in \mathbb{Z}[x]$.

While the fact expressed by the formula (0.10) forms an important ingredient in the proofs of various polynomial recurrence theorems such as Theorem 0.3 and its far reaching extension, the polynomial Szemeredi theorem proved in [10], the Cesàro limits fail to discern the more subtle behaviour of weakly mixing operators along the idempotent ultrafilters [11].

It is not hard to see that $U \in \mathcal{U}(\mathcal{H})$ is weakly mixing if and only if for some idempotent $p \in \beta\mathbb{N}$, $p\text{-}\lim_{n\in\mathbb{N}} U^n = 0$. However, unlike the situation with Cesàro limits described by formula (0.10), the relation $p\text{-}\lim_{n\in\mathbb{N}} U^n = 0$ does not imply, in general, neither $p\text{-}\lim_{n\in\mathbb{N}} U^{2n} = 0$, nor, say, $p\text{-}\lim_{n\in\mathbb{N}} U^{n^2} = 0$. On the other hand, one can show that $p\text{-}\lim_{n\in\mathbb{N}} U^n = Id$ implies $p\text{-}\lim_{n\in\mathbb{N}} U^{kn} = Id$, for any $k \in \mathbb{N}$, and this is consistent (depending on a choice of U) with both $p\text{-}\lim_{n\in\mathbb{N}} U^{n^2} = 0$ and $p\text{-}\lim_{n\in\mathbb{N}} U^{n^2} = Id$. It turns out (see Corollary F below) that for any independent family $\{P_1, \ldots, P_m\} \subset \mathcal{P}$ [12]

[11] As it was mentioned in footnote 10 (and will be stressed many more times in the sequel), p −limits of various ergodic expressions are intrinsically connected with the behaviour of these expressions along IP-sets, which, in turn, are connected with important applications of ergodic theory to combinatorics (see [6], [8], [12], [16]).

[12] Independence here means that if for some $n_1, \ldots, n_m \in \mathbb{Z}$, $\sum_{i=1}^m n_i P_i = 0$, then $n_i = 0$ for each $i = 1, \ldots, m$.

and any choice of $E_i \in \{0, Id\}$, $i = 1, \ldots, m$, there exist $U \in \mathcal{U}(\mathcal{H})$ and an idempotent $p \in \beta\mathbb{N}$ such that $p\text{-}\lim_{n\in\mathbb{N}} U^{P_i(n)} = E_i$ for each $i = 1, \ldots, m$.

Let us fix $U \in \mathcal{U}(\mathcal{H})$ and an idempotent $p \in \beta\mathbb{N}$. It is not hard to show that

$$\{P \in \mathcal{P} : \, p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = Id\} \text{ is an additive subgroup of } \mathcal{P}.$$

One of the main results of this paper (Theorem E below) gives a complete characterization of this kind of groups[13], hereby contributing to the recently revived studies of the phenomenon of rigidity for weakly mixing operators (see [1], [2], [7], [19], [20]).

Not all idempotents and not all weakly mixing unitary operators are interesting when we study such p-limits. Indeed, if $p \in \beta\mathbb{N}$ is a minimal idempotent [14] and U is weakly mixing then $p\text{-}\lim_{n\in\mathbb{N}} U^{kn} = 0$ for each $k \geq 1$ [5]. Moreover, by Corollary B below, we will obtain $p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = 0$ for each positive degree polynomial $P \in \mathcal{P}$. We should also notice that if U is *mildly mixing* [17], [35], i.e. when U has no non-trivial rigid vectors, then $p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = 0$ for each positive degree polynomial $P \in \mathcal{P}$ and every idempotent $p \in \beta\mathbb{N}$ [15]. Therefore the problem of calculating p-limits along polynomial powers of U becomes interesting if U has non-trivial rigid vectors, in particular, if U itself is rigid.

We now pass to a description of main results of the paper.

Given $n \in \mathbb{N}$, let $\mathcal{P}_{\leq N}$ [16] denote the (additive) group of all polynomials $P \in \mathbb{Z}[x]$ with $\deg P \leq N$ satisfying $P(0) = 0$. Our first result (proved in Section 2) provides the following Hilbert space decomposition theorem for polynomial actions of unitary operators.

Theorem A. *For each $N \geq 1$, each idempotent $p \in \beta\mathbb{N}$ and each $U \in \mathcal{U}(\mathcal{H})$ there exists a unique decomposition*

$$\mathcal{H} = \bigoplus_{k\geq 1} \mathcal{H}_k^{(N)} \tag{0.11}$$

into U-invariant closed subspaces such that for each $P \in \mathcal{P}_{\leq N}$ and $k \geq 1$ we have

$$p\text{-}\lim_{n\in\mathbb{N}} \left(U|_{\mathcal{H}_k^{(N)}} \right)^{P(n)} = 0 \text{ or } Id \tag{0.12}$$

[13] This characterization problem is interesting only for unitary operators which are weakly mixing. Indeed, if $U \in \mathcal{U}(\mathcal{H})$ has discrete spectrum (that is, the space $\mathcal{H}$ is spanned by the eigenvectors of U), we have $p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = Id$ for each $P \in \mathcal{P}$ and each $p \in \beta\mathbb{N}$, $p = p + p$, see Proposition 2.7.

[14] An idempotent $p \in \beta\mathbb{N}$ is said to be *minimal* if it belongs to a minimal right ideal of $(\beta\mathbb{N}, +)$. U is weakly mixing if and only if $p\text{-}\lim_{n\in\mathbb{N}} U^n = 0$ for each minimal idempotent $p \in \beta\mathbb{N}$ [5]. See [5], [6] for the discussion of minimal idempotents and their applications to dynamics and combinatorics.

[15] The latter is not surprising. As we have already noticed, $p\text{-}\lim_{n\in\mathbb{N}} U^n = \mathrm{proj}_{\mathcal{H}_r}$.

[16] Similarly, $\mathcal{P}_{\geq N}$ denotes the set of polynomials $P \in \mathcal{P}$ satisfying $\deg P \geq N$.

and, moreover,

$$\text{(0.13)}\qquad \begin{array}{l} \textit{whenever } k \neq l, \textit{ there exists } Q \in \mathcal{P}_{\leq N} \\ \textit{such that } \ p\text{-}\lim_{n\in\mathbb{N}} \left(U|_{\mathcal{H}_k^{(N)}}\right)^{Q(n)} \neq p\text{-}\lim_{n\in\mathbb{N}} \left(U|_{\mathcal{H}_l^{(N)}}\right)^{Q(n)}. \end{array}$$

Furthermore, the decomposition (0.11) has the following property:

$$\text{(0.14)}\qquad \begin{array}{l} \textit{For any } k \geq 1, \textit{ if } Q \in \mathcal{P}_{\leq N} \textit{ is such that} \\ p\text{-}\lim_{n\in\mathbb{N}} \left(U|_{\mathcal{H}_k^{(N)}}\right)^{sQ(n)} = 0 \textit{ for each } s \in \mathbb{N}, \textit{then} \\ p\text{-}\lim_{n\in\mathbb{N}} \left(U|_{\mathcal{H}_k^{(N)}}\right)^{R(n)} = 0 \textit{ for each } R \in \mathcal{P}_{\geq \deg Q}. \end{array}$$

An important consequence of Theorem A is the following result.

Corollary B. *Assume that $P \in \mathbb{Z}[x]$, $P(0) = 0$ and $\deg P = N \geq 1$. Assume moreover that $p\text{-}\lim_{n\in\mathbb{N}} U^{lP(n)} = 0$ for all $l \geq 1$. Then $p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)} = 0$ for all $Q \in \mathcal{P}_{\geq N}$* [17].

Given a finite, positive Borel measure σ on the circle $\mathbb{S}^1$, we denote by $\widehat{\sigma}(n)$ its n-th Fourier coefficient: $\widehat{\sigma}(n) := \int_{\mathbb{S}^1} z^n \, d\sigma(z)$. The decomposition result given in Theorem A turns out to depend only on the maximal spectral type of U. Therefore, it yields a decomposition of any finite, positive Borel measure on the circle:

Corollary C. *Assume that σ is a probability Borel measure on $\mathbb{S}^1$. Let $N \in \mathbb{N}$ and $p \in \beta\mathbb{N}$, $p + p = p$. Then there exists a unique decomposition*

$$\text{(0.15)}\qquad \sigma = \sum_{k\geq 1} a_k \sigma_k^{(N)}$$

such that each $a_k > 0$, $\sum_{k\geq 1} a_k = 1$, each $\sigma_k^{(N)}$ is also a probability Borel measure on $\mathbb{S}^1$, $\sigma_k^{(N)} \perp \sigma_l^{(N)}$ whenever $k \neq l$, and, moreover, for each $Q \in \mathbb{Z}[x]$ of degree at most N and $k \geq 1$

$$\text{(0.16)}\qquad \begin{array}{l} p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}_k^{(N)}(Q(n)) = 0 \ \textit{ or} \\ p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}_k^{(N)}(Q(n) - Q(0)) = 1. \end{array}$$

[17]Note that in view of Theorem E below, in general, this assertion fails if we only assume that $p\text{-}\lim_{n\in\mathbb{N}} U^{lP(n)} = 0$ for all $1 \leq l \leq L_0$. Indeed, making use of Lemma 3.15 below, we can extend the cyclic group H generated by $(L_0 + 1)P$ to a subgroup $G \subset \mathcal{P}_{\leq N}$ (with $N = \deg P$) of finite index in $\mathcal{P}_{\leq N}$ so that $lP \notin G$ for $l = 1, \ldots, L_0$.

$$
\text{(0.17)}\qquad \begin{array}{l} \textit{If } k \neq l \textit{ then there exists } Q \in \mathbb{Z}[x] \textit{ of degree at most } N \\ \textit{such that } p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}_k^{(N)}(Q(n)) = 0 \textit{ and} \\ p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}_l^{(N)}(Q(n)-Q(0)) = 1 \textit{ or vice versa.} \end{array}
$$

Corollary C is complemented by the following result (which can be viewed as another form of Corollary B).

Corollary D. *Assume that σ is a continuous probability Borel measure on $\mathbb{S}^1$ and let $p \in \beta\mathbb{N}$, $p + p = p$.*
(i) If $p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}(ln + k) = 0$ for each $l \geq 1$ and $k \in \mathbb{Z}$, then

$$
p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}(Q(n)) = 0
$$

for each positive degree polynomial $Q \in \mathbb{Z}[x]$.
(ii) If, for some $P \in \mathcal{P}_{\leq N}$, we have $p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}(lP(n) + k) = 0$ for each $l \geq 1$ and $k \in \mathbb{Z}$, then

$$
p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}(Q(n)) = 0
$$

for each $Q \in \mathbb{Z}[x]$ of degree not smaller than the degree of P.

Motivated by the decomposition result given by Theorem A, we introduce the notion of rigidity group. Let $N \in \mathbb{N}$. A subgroup $G \subset \mathcal{P}_{\leq N}$ is called an N-*rigidity group* if there exist $p \in \beta\mathbb{N}$, $p + p = p$, and $U \in \mathcal{U}(\mathcal{H})$ such that

$$
G = \{P \in \mathcal{P}_{\leq N} :\ p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = Id\}
$$

and $p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)} = 0$ for each $Q \in \mathcal{P}_{\leq N} \setminus G$. The second main goal of the paper is to prove the following result.

Theorem E. *Assume that $G \subset \mathcal{P}_{\leq N}$ is a subgroup with $\max\{\deg P :\ P \in G\} = N$. Then G is an N-rigidity group if and only if G has finite index in $\mathcal{P}_{\leq N}$.*

Our strategy to prove Theorem E will be first to introduce in Section 3 the concept of N-*periodic* rigidity groups associated with G - these are subgroups of $\mathbb{Z}/k_1\mathbb{Z}\oplus\ldots\oplus\mathbb{Z}/k_N\mathbb{Z}$, where $k_j \geq 1$ is the smallest natural number k such that $kx^j \in G$, $j = 1, \ldots, N$ (N-rigidity groups will turn out to be the preimages of N-periodic rigidity groups via the natural map

$\mathbb{Z}^N \to \mathbb{Z}/k_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/k_N\mathbb{Z}$) [18]. The fact that $k_j \geq 1$ is well defined for each $j = 1, \ldots, N$ is not obvious and it is a consequence of the decomposition theorem (Theorem A)[19]. We then describe N-periodic rigidity groups as duals of quotients of $\mathbb{Z}/k_1\mathbb{Z} \times \ldots \times \mathbb{Z}/k_N\mathbb{Z}$ by the so called group couplings [20]. This will allow us to give a complete classification of N-periodic rigidity groups as groups $\widetilde{G} \subset \mathbb{Z}/k_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/k_N\mathbb{Z}$ which fulfill the following $(*)$-property: For each $r = 1, \ldots, N$

$$(*) \qquad \left.\begin{array}{l} (j_1, \ldots, j_{r-1}, j_r, j_{r+1}, \ldots, j_N) \in \widetilde{G} \\ (j_1, \ldots, j_{r-1}, j_r', j_{r+1}, \ldots, j_N) \in \widetilde{G} \end{array}\right\} \implies j_r = j_r'\ ^{21}.$$

In particular, to prove that a group $\widetilde{G}$ satisfying the $(*)$-property is an N-periodic rigidity group, we will construct a *weighted unitary operator* $U = V^T_{\Theta\circ\Phi}$ over an odometer T, where Φ is a cocycle taking values "up to a limit distribution" in a group coupling which will turn out to be the dual of $\widetilde{G}$ and Θ is a character of $\mathbb{Z}/k_1\mathbb{Z} \otimes \ldots \otimes \mathbb{Z}/k_N\mathbb{Z}$. For some $p \in \beta\mathbb{N}$, $p = p + p$, the N-periodic rigidity group of $V^T_{\Theta\circ\Phi}$ will be isomorphic to $\widetilde{G}$. The proof of Theorem E will then follow from some further algebraic considerations.

The following corollary of Theorem E confirms the possibility of independent behavior of p-limits for finite families of independent polynomials.

Corollary F. *Assume that $P_1, \ldots, P_N \in \mathcal{P}_{\leq N}$ are independent. Then, for any $s = 1, \ldots, N$, there exists an N-rigidity subgroup G containing $P_1, \ldots, P_s$ such that $P_{s+1}, \ldots, P_N \notin G$. That is, there exist $U \in \mathcal{U}(\mathcal{H})$ and $p \in \beta\mathbb{N}$, $p + p = p$, such that $p\text{-}\lim_{n\in\mathbb{N}} U^{P_i(n)} = Id$ for $i = 1, \ldots, s$ and $p\text{-}\lim_{n\in\mathbb{N}} U^{P_i(n)} = 0$ for $i = s+1, \ldots, N$.*

Finally, we will consider groups of *global rigidity*, that is, groups of the form $G = \{P \in \mathcal{P}_{\leq N} : p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = Id\}$ for some idempotent $p \in \beta\mathbb{N}$ and some $U \in \mathcal{U}(\mathcal{H})$. The following result will be proved in Section 3.

Theorem G. *Any subgroup G of $\mathcal{P}_{\leq N}$ is a group of global rigidity, that is, given a*

---

[18] The numbers $k_1, \ldots, k_N$ are determined by the fact that we consider $\{x, x^2, \ldots, x^N\}$ as the basis of $\mathcal{P}_{\leq N}$; if we choose a different basis in $\mathcal{P}_{\leq N}$, say $Q_1, \ldots, Q_N$, and replace x^j by Q_j we obtain another sequence of periods: $l_1, \ldots, l_N$, and a different N-periodic subgroup $\overline{G} \subset \mathbb{Z}/l_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/l_N\mathbb{Z}$ satisfying the $(*)$-property (see the definition of this property below), but G will still be equal to the preimage of $\overline{G}$ via the map $\mathbb{Z}^N \to \mathbb{Z}/l_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/l_N\mathbb{Z}$, see Section 3.

[19] The decomposition theorem, in turn, heavily depends on the p-limit version of the classical van der Corput lemma, see Lemma 2.3 below.

[20] A subgroup $K \subset \mathbb{Z}/k_1\mathbb{Z} \times \ldots \times \mathbb{Z}/k_N\mathbb{Z}$ is called a *group coupling* if it has full projection on each coordinate (cf. the notion of joining in ergodic theory).

[21] That is, $\widetilde{G}$ contains no nonzero element of the form $(0, \ldots, 0, j_r, 0, \ldots, 0)$.

subgroup $G \subset \mathcal{P}_{\leq N}$, there exist $U \in \mathcal{U}(\mathcal{H})$ and $p \in \beta\mathbb{N}$, $p+p=p$, such that

$$G = \{P \in \mathcal{P}_{\leq N} :\ p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = Id\}.$$

The structure of the paper is as follows. In Section 1 we collect some necessary facts and provide additional details on $\beta\mathbb{N}$ and p-limits. In Section 2 we prove Theorem A and derive Corollaries B, C and D. In Section 3 we characterize the class of groups $G \subset \mathcal{P}_{\leq N}$ satisfying the $(*)$ property in terms of N-periodic rigidity, or, equivalently, in terms of algebraic couplings (see Theorem 3.5). The main part of Section 3 is devoted to the proof of Theorem E. Section 3 will also contain the proofs of Corollary F which is a consequence of Theorem E, and of Theorem G which follows from Theorems A and E.

1 Ultrafilters, p-convergence and IP-limits

1.1 Idempotent ultrafilters and Hindman's theorem

In the previous section we have already introduced $\beta\mathbb{N}$, the space of ultrafilters on $\mathbb{N}$. In this section we will provide some additional discussion of results pertaining to ultrafilters which will be needed in the subsequent sections. The reader can find the missing details in [4], [5] and [6].

As we have already mentioned, given $p \in \beta\mathbb{N}$, the map $q \mapsto p+q$ is continuous. In other words, $(\beta\mathbb{N}, +)$ is a left topological compact Hausdorff semigroup. We also remark that the right translations are continuous only at principal ultrafilters.

In a general setting, when S is a left topological compact Hausdorff semigroup, the set of invertible elements of S is denoted by $I(S)$, and the set of idempotents of S is denoted by $E(S)$ (recall that $e \in E(S)$ means that $e \cdot e = e$). According to Ellis' lemma [14], under our assumptions on S, the set $E(S)$ is always non-empty. If $I(S) \neq \emptyset$ then $I(S) \cap E(S)$ is a singleton, namely the unit of S. When $S = \beta\mathbb{N}$, by $E(\beta\mathbb{N})$ we will always mean the set of idempotents of $(\beta\mathbb{N}, +)$.

Fix $k \in \mathbb{N}$ and $p \in E(\beta\mathbb{N})$. Since $\bigcup_{i=0}^{k-1}(k\mathbb{N}+i) = \mathbb{N}$ is a disjoint union, there is a unique i, $0 \leq i < k$, such that $k\mathbb{N}+i \in p$. Since $p+p=p$,

$$B := \{n \in \mathbb{N} :\ (k\mathbb{N}+i) - n \in p\} \in p.$$

It follows that $(k\mathbb{N}+i) \cap B \neq \emptyset$. Take $n \in (k\mathbb{N}+i) \cap B$. Then for some $r \in \mathbb{N}$, $n = kr+i$ and also $(k\mathbb{N}+i) - n \in p$. It follows immediately that $k\mathbb{N} \in p$. This gives us the following useful fact:

(1.1) If $p \in E(\beta\mathbb{N})$ then $k\mathbb{N} \in p$ for each $k \in \mathbb{N}$.

Let $p \in E(\beta\mathbb{N})$. By [25], Theorem 3.1 and [24], Theorem 3.3., each $A \in p$ must contain a set of the form

$$\mathrm{FS}((n_i)_{i\geq 1}) := \{n_{i_1} + n_{i_2} + \ldots + n_{i_k} : \, i_1 < i_2 < \ldots < i_k, k \geq 1\} \subset A. \quad (1.2)$$

[22]

In ergodic theory and topological dynamics the sets of the form $\mathrm{FS}((n_i)_{i\geq 1})$ are called IP-*sets*. It follows that any p-large set contains an IP-set. [23] On the other hand, given a sequence $(n_i)_{i\geq 1}$ one can find $p \in E(\beta\mathbb{N})$ for which

$$\mathrm{FS}(n_i)_{i\geq K} \in p \ \text{ for each } \ K \geq 1, \quad (1.3)$$

see e.g. [6], Theorem 2.5.

1.2 p- and IP-limits

Assume that X is a compact metric space with a metric d. Given $p \in \beta\mathbb{N}$, $(x_n) \subset X$ and $x \in X$ we define

$$p\text{-}\lim_{n\in\mathbb{N}} x_n = x$$

if for each open $U \ni x$ the set $\{n \in \mathbb{N} : \, x_n \in U\}$ is p-large. Since X is compact Hausdorff, a p-limit exists and is unique[24].

Now, if Y is another compact metric space and $f : X \to Y$ is continuous then

$$p\text{-}\lim_{n\in\mathbb{N}} f(x_n) = f\left(p\text{-}\lim_{n\in\mathbb{N}} x_n\right). \quad (1.4)$$

In view of (1.1), whenever p is an idempotent and $k \geq 1$,

$$\{n \in \mathbb{N} : \, x_n \in U\} \in p \Leftrightarrow \{n \in \mathbb{N} : \, x_n \in U\} \cap k\mathbb{N} \in p$$

$$\Leftrightarrow \{n \in \mathbb{N} : \, k|n \text{ and } \, x_n \in U\} \in p.$$

[22] In [25], Hindman proved that for any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, one of C_i contains an IP-set. The fact that any member of an idempotent ultrafilter has to contain an IP-set (Hindman's theorem) follows from [24], Theorem 3.3.

[23] It is perhaps worth mentioning that the result (1.1) immediately follows from Hindman's theorem. Indeed, among the sets $k\mathbb{N} + i$, $0 \leq i \leq k-1$, only $k\mathbb{N}$ can contain an IP-set.

[24] By the universality property of the Stone-Čech-compactification, if X is a compact Hausdorff space and $\underline{x} = (x_n)_{n\in\mathbb{N}} : \mathbb{N} \to X$, then there exists a unique continuous extension $\beta\underline{x} : \beta\mathbb{N} \to X$ of $\underline{x}$. The value of $\beta\underline{x}$ at $p \in \beta\mathbb{N}$ is given by the limit of the ultrafilter $\{A \subset X : \, \underline{x}^{-1}(A) \in p\}$ of subsets of X. Then $p\text{-}\lim_{n\in\mathbb{N}} x_n = \beta\underline{x}(p)$.

It follows that to check that $x \in X$ is the p-limit of a sequence (x_n), where $p \in E(\beta\mathbb{N})$, it is enough to deal with numbers which are multiples of a fixed $k \geq 1$. We write this as

$$p\text{-}\lim_{n\in\mathbb{N}} x_n = p\text{-}\lim_{k|n} x_n. \tag{1.5}$$

Remark 2. We would like to stress that in general $p\text{-}\lim_{n\in\mathbb{N}} x_n \neq p\text{-}\lim_{n\in\mathbb{N}} x_{kn}$. For example, we do not have equality for $k > 1$ when $x_n = U^n$, where U is a unitary operator on a Hilbert space $\mathcal{H}$ and X is the unit ball of the space of linear bounded operators on $\mathcal{H}$ equipped with the weak operator topology, see Theorem E (the group $G := \{mkx : m \in \mathbb{Z}\} \subset \mathcal{P}_1$ is of finite index in $\mathcal{P}_1$, hence G is a 1-rigidity group and $x \notin G$). In view of (1.5), it follows that it is not true, in general, that $p\text{-}\lim_{k|n} x_n = p\text{-}\lim_{n\in\mathbb{N}} x_{kn}$.

We now introduce a related notion of convergence, namely that of IP-convergence. The precise connection between these two notions is given by Lemma 1.1 below.

Let $\mathcal{F}$ denote the family of finite non-empty subsets of $\mathbb{N}$. Given an increasing sequence (n_i) of natural numbers, for each $\alpha \in \mathcal{F}$ we set

$$n_\alpha = \sum_{i\in\alpha} n_i.$$

Assume that $(x_n) \subset X$ and $x \in X$. Assume moreover that for each $\varepsilon > 0$ there exists $N \geq 1$ such that for each $\alpha \in \mathcal{F}$ satisfying $\min \alpha \geq N$ we have

$$d(x_{n_\alpha}, x) < \varepsilon.$$

Then we say that x *is the IP-limit of* $(x_{n_\alpha})_{\alpha\in\mathcal{F}}$ and write

$$\text{IP-}\lim x_{n_\alpha} = x.$$ [25]

Lemma 1.1. *(i) Assume that (n_i) is an increasing sequence of natural numbers. Then there exists $p \in E(\beta\mathbb{N})$ such that for each compact metric space (X, d) and a sequence $(x_n)_{n\geq 1} \subset X$ such that* IP-lim $x_{n_\alpha} = x$*, we have $x = p\text{-}\lim_{n\in\mathbb{N}} x_n$.*

(ii) Let (X, d) be a compact metric space and $(x_n)_{n\geq 1} \subset X$. Assume that $p \in E(\beta\mathbb{N})$ and $p\text{-}\lim_{n\in\mathbb{N}} x_n = x$. Then there exists an IP*-set* FS$((n_i)_{i\geq 1})$ *such that* IP-lim $x_{n_\alpha} = x$*.*

[25] Note that a necessary (but, in general, not sufficient) condition for the validity of $\text{IP-}\lim x_{n_\alpha} = x$ is that $x_{n_i} \to x$ when $i \to \infty$.

Proof. (i) In view of (1.3), there exists $p \in E(\beta\mathbb{N})$ such that $\mathrm{FS}((n_i)_{i\geq K}) \in p$ for each $K \geq 1$. It follows that for each $\alpha_0 \in \mathcal{F}$, $\{n_\alpha : \max \alpha_0 \leq \min \alpha\} \in p$. Now, by assumption, given $\varepsilon > 0$ there exists $\alpha_0 \in \mathcal{F}$ such that

$$\{n \in \mathbb{N} : d(x_n, x) < \varepsilon\} \supset \{n_\alpha : \max \alpha_0 \leq \min \alpha\} \in p$$

and therefore $\{n \in \mathbb{N} : d(x_n, x) < \varepsilon\} \in p$.

(ii) Denote $A_k = \{n \in \mathbb{N} : d(x_n, x) < \frac{1}{k}\}$. We have $A_k \in p$ for all $k \geq 1$. Since $p + p = p$ and each set A_k is infinite (by (1.2)), we can choose $n_1 < n_2 < \ldots$ such that $n_k \in B_k, B_k - n_k \in p$ for each $k \geq 1$, where

$$B_1 := A_1 \text{ and } B_k := A_k \cap (B_{k-1} - n_{k-1}) \cap \ldots \cap (B_1 - n_1) \text{ for each } k \geq 2.$$

It is now clear that for the IP-set $FS((n_i))$ we have $\mathrm{IP}\text{-}\lim x_{n_\alpha} = x$. □

We will also need the following fact.

Lemma 1.2 (see [4], Theorem 3.8)**.** *For each $p, q \in \beta\mathbb{N}$ and $(x_n) \subset X$*

$$(p+q)\text{-}\lim_{n\in\mathbb{N}} x_n = q\text{-}\lim_{k\in\mathbb{N}} \left(p\text{-}\lim_{l\in\mathbb{N}} x_{k+l} \right).$$

In particular, if $p \in E(\beta\mathbb{N})$, $p\text{-}\lim_{n\in\mathbb{N}} x_n = p\text{-}\lim_{k\in\mathbb{N}} (p\text{-}\lim_{l\in\mathbb{N}} x_{k+l})$.

1.3 p-limits in semitopological compactifications of $\mathbb{Z}$

Assume now that S is a compact metric semitopological (i.e. left- and right topological[26]) semigroup. In view of (1.4) and the continuity of left- and right- translations, it follows that whenever $(s_n)_{n\geq 1} \subset S$ and $u \in S$

$$(1.6) \qquad p\text{-}\lim_{n\in\mathbb{N}} (s_n u) = \left(p\text{-}\lim_{n\in\mathbb{N}} s_n \right) u \text{ and } p\text{-}\lim_{n\in\mathbb{N}} (u s_n) = u \left(p\text{-}\lim_{n\in\mathbb{N}} s_n \right).$$

Choose $s \in S$ and consider $s_n = s^n$, $n \geq 1$. By the universality property of $\beta\mathbb{N}$ (see footnote 24), it follows that the map $\beta\mathbb{N} \supset \mathbb{N} \ni n \mapsto s^n \in S$ has a unique extension to a surjective continuous semigroup homomorphism of $(\beta\mathbb{N}, +)$ onto $\overline{\{s^n : n \in \mathbb{N}\}}$, the extension being given by the formula $p \mapsto p\text{-}\lim_{n\in\mathbb{N}} s^n$ [27]. Since it is a homomorphism, we have the following.

[26] Recall that for $(\beta\mathbb{N}, +)$ the addition $+$ is left-continuous and is not right-continuous. So, $(\beta\mathbb{N}, +)$ is a left topological semigroup, but not a semitopological semigroup.

[27] Indeed, it follows from Lemma 1.2 (and (1.6)) that setting $s_n = s^n$ we obtain that

$$(p+q)\text{-}\lim_{n\in\mathbb{N}} s_n = q\text{-}\lim_{n\in\mathbb{N}} \left(p\text{-}\lim_{m\in\mathbb{N}} s_{n+m} \right) = q\text{-}\lim_{n\in\mathbb{N}} \left(p\text{-}\lim_{m\in\mathbb{N}} s_n \cdot s_m \right)$$

Lemma 1.3. *Assume that S is a compact metric semitopological semigroup. For each $p \in E(\beta\mathbb{N})$, $t := p\text{-}\lim_{n\in\mathbb{N}} s^n$ is an idempotent of S, i.e. $t \in E(S)$.*

From now on, we assume that

$$S \text{ is a metrizable semitopological compactification of } \mathbb{Z}. \tag{1.7}$$

In other words, we assume that S is a compact metric Abelian semitopological semigroup having a dense cyclic subgroup, that is, for some $u \in I(S)$, $S = \overline{\{u^n : n \in \mathbb{Z}\}}$. The unit of S will be denoted by 1. The following statement follows from Theorem 3.2 in [29].

$$\text{Multiplication is jointly continuous at each point } (i, s) \in I(S) \times S. \tag{1.8}$$

Lemma 1.4. *Assume that S is a compact metric Abelian semitopological semigroup satisfying (1.8). Let $p \in \beta\mathbb{N}$ and $p\text{-}\lim_{n\in\mathbb{N}} s_n = s$ and $p\text{-}\lim_{n\in\mathbb{N}} t_n = t$, where $t \in I(S)$. Then*

$$p\text{-}\lim_{n\in\mathbb{N}} s_n t_n = st. \tag{1.9}$$

If $p \in E(\beta\mathbb{N})$ and $u \in I(S)$ then $p\text{-}\lim_{n\in\mathbb{N}} u^n = 1$ and $p\text{-}\lim_{n\in\mathbb{N}} s_n u^n = s$.

Proof. According to (1.8), given $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(s', s) < \delta$ and $d(t', t) < \delta$ then $d(s't', ts) < \varepsilon$. It follows that

$$\{n \in \mathbb{N} : \ d(s_n t_n, st) < \varepsilon\} \supset \{n \in \mathbb{N} : \ d(t_n, t) < \delta\} \cap \{n \in \mathbb{N} : \ d(s_n, s) < \delta\}.$$

As this intersection is still a p-large set, so is the set $\{n \in \mathbb{N} : \ d(s_n t_n, st) < \varepsilon\}$ and (1.9) follows.

To complete the proof notice that if p is additionally an idempotent then, in view of Lemma 1.3, $p\text{-}\lim_{n\in\mathbb{N}} u^n = t$ is an idempotent which, by assumption, is in $I(S)$. Hence $t = 1$. Finally, using (1.9), $p\text{-}\lim_{n\in\mathbb{N}} s_n u^n = s \cdot 1 = s$. $\square$

Now we consider sequences of the form $(s^{P(n)})$, where $P \in \mathcal{P}$, and study their p-limits for $p \in E(\beta\mathbb{N})$.

We will need the following elementary observation on integer coefficient polynomials.

$$= q\text{-}\lim_{n\in\mathbb{N}} \left(s_n \cdot \left(p\text{-}\lim_{m\in\mathbb{N}} s_m \right) \right) = \left(q\text{-}\lim_{n\in\mathbb{N}} s_n \right) \cdot \left(p\text{-}\lim_{n\in\mathbb{N}} s_n \right).$$

Lemma 1.5. *Let $P \in \mathcal{P}$ be of degree $d \geq 1$. Then*

$$P(x+y) - P(x) = Q(x,y) + P(y), \tag{1.10}$$

where $Q(x,y) \in \mathbb{Z}[x,y]$ is a polynomial divisible by xy in the ring $\mathbb{Z}[x,y]$. The x-degree[28] *and the y-degree of $Q(x,y)$ are equal to $d-1$.* [29]

Lemma 1.6. *Assume that S satisfies (1.8) and and let $s \in I(S)$. Let $P \in \mathcal{P}$ be a polynomial of degree $d \geq 1$. Assume that for $j = 1, \dots, d-1$ there exists $r_j \geq 1$ such that*

$$p\text{-}\lim_{n\in\mathbb{N}} s^{r_j n^j} = 1.$$

Then $p\text{-}\lim_{n\in\mathbb{N}} s^{P(n)} \in E(S)$.

Proof. First, notice that in view of (1.8) and (1.9), we have

$$p\text{-}\lim_{n\in\mathbb{N}} s^{l r_j n^j} = 1 \tag{1.11}$$

for each $l \geq 1$ and $j = 1, \dots, d-1$.

By Lemmata 1.2 and 1.5

$$u := p\text{-}\lim_{n\in\mathbb{N}} s^{P(n)} = p\text{-}\lim_{n\in\mathbb{N}}(p\text{-}\lim_{m\in\mathbb{N}} s^{P(n+m)}) = p\text{-}\lim_{n\in\mathbb{N}} s^{P(n)}(p\text{-}\lim_{m\in\mathbb{N}}(s^{(Q(n,m)+P(m)}),$$

where $Q(x,y) \in \mathbb{Z}[x,y]$ is divisible by xy. Set $r = \mathrm{lcm}(r_1, \dots, r_{d-1})$. Using (1.5), Lemma 1.4 and (1.11), we obtain that

$$u = p\text{-}\lim_{r|n} s^{P(n)}(p\text{-}\lim_{m\in\mathbb{N}}(s^{(Q(n,m)} \cdot s^{P(m)}) = p\text{-}\lim_{r|n} s^{P(n)}(1\cdot u) = u^2.$$

$\square$

Lemma 1.6 applied in the case $d = 1$ shows that $p\text{-}\lim_{n\in\mathbb{N}} s^{an} \in E(S)$ for any $a \in \mathbb{Z}$.

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[28] We assume that the degree of 0 is equal to 0.

[29] The result is immediate for monomials, hence it holds for all polynomials $0 \neq P \in \mathcal{P}$ (we recall that for any $P \in \mathcal{P}$, we have $P(0) = 0$, so $\mathcal{P}$ does not contain non-trivial constant polynomials).

1.4 From ordinary convergence to IP- and p-limits

Let S be a compact metric Abelian semitopological semigroup with $I(S) \neq \emptyset$.

Assume that we have a countable collection of polynomials $P_i \in \mathcal{P}$, $i \geq 1$. Assume moreover that for some sequence $(q_n) \subset \mathbb{N}$ such that for each $i \geq 1$ we have $s^{P_i(q_n)} \to e_i$ in S, where $e_i \in E(S)$. Can we find $p \in E(\beta\mathbb{N})$ such that $p\text{-}\lim_{n\in\mathbb{N}} s^{P_i(n)} = e_i$ for each $i \geq 1$? The proposition below gives a list of conditions that will guarantee the positive answer to this question.

Proposition 1.7. *Fix $N \geq 1$ and assume that $0 \neq P_i \in \mathcal{P}_{\leq N}$, $i \geq 1$. Assume that S satisfies (1.8) and let $s \in I(S)$. Let (q_n) be an increasing sequence of natural numbers such that for some $r_j \geq 1$, $j = 1, \ldots, N-1$,*

$$s^{r_j q_n^j} \to 1. \tag{1.12}$$

Denote $r = \operatorname{lcm}(r_1, \ldots, r_{N-1})$ and assume in addition that

$$r | q_n \ \text{ for } \ n \geq n_0. \tag{1.13}$$

Finally, assume that for each $i \geq 1$

$$s^{P_i(q_n)} \to e_i \in E(S). \tag{1.14}$$

Then there exists $p \in E(\beta\mathbb{N})$ such that for each $i \geq 1$

$$p\text{-}\lim_{n\in\mathbb{N}} s^{P_i(n)} = e_i. \tag{1.15}$$

Proof. We divide the proof into two parts. In the first part we assume that $P_1 = P_2 = \ldots$ In the second part, using a diagonalization procedure and the first part of the proof, we complete the proof.

Part 1. We will first prove the result when the family $\{P_i : i \geq 1\}$ consists of one polynomial $P = P_1 = P_2 = \ldots$ and we set $e_1 = e \in E(S)$. Assume, without loss of generality, that $\deg P = N$ and write $P(x) = M_N x^N + \ldots + M_2 x^2 + M_1 x \in \mathbb{Z}[x]$. First, choose $k_1 > n_0$ so that

$$d\left(s^{P(q_{k_1})}, e\right) < \frac{1}{2^1},$$

$$d\left(s^{P(q_{k_1})}e, e\right) < \frac{1}{2^1},$$

which is possible by letting $k_1 \to \infty$ in (1.14) and using the semicontinuity of the multiplication (and the fact that $e \cdot e = e$).

Suppose that for some $w \geq 1$ the numbers $k_1 < k_2 < \ldots < k_w$ have already been selected so that for all choices $i_1 < \ldots < i_t$ with $i_1, \ldots, i_t \in \{1, \ldots, w\}$,

$$d\left(s^{P(q_{k_{i_1}} + \ldots + q_{k_{i_t}})}, e\right) < \frac{1}{2^{i_1}}, \tag{1.16}$$

$$d\left(s^{P(q_{k_{i_1}} + \ldots + q_{k_{i_t}})} e, e\right) < \frac{1}{2^{i_1}}. \tag{1.17}$$

We have now to select k_{w+1}. If we show that this choice depends only on the fact that k_{w+1} is sufficiently large then we are done because we deal with a bounded number of indices $i_1 < \ldots < i_t$.

Given $\alpha \subset \{1, \ldots, w\}$ (α may be empty), we set $a_\alpha = \sum_{j \in \alpha} q_{k_j}$ ($a_\emptyset = 0$). In view of Lemma 1.5, we obtain

$$s^{P(a_\alpha + q_{k_{w+1}})} = s^{P(a_\alpha)} s^{Q(a_\alpha, q_{k_{w+1}})} s^{P(q_{k_{w+1}})}.$$

The polynomials $R \in \mathcal{P}$, for which $s^{R(q_n)} \to 1$, form a subgroup of $\mathcal{P}$. Since $r | a_\alpha$, $Q(x, y)$ is divisible by x and has y-degree $N - 1$, it follows by (1.12) and (1.14) that

$$s^{Q(a_\alpha, q_{k_{w+1}})} \to 1$$

and

$$s^{Q(a_\alpha, q_{k_{w+1}})} s^{P(q_{k_{w+1}})} \to e,$$

$$s^{Q(a_\alpha, q_{k_{w+1}})} s^{P(q_{k_{w+1}})} e \to e$$

when $k_{w+1} \to \infty$. In view of (1.16) and (1.17), there exists $\varepsilon > 0$ such that

$$d(s^{P(a_\alpha)}, e) < \frac{1}{2^{\min \alpha}} - \varepsilon,$$

$$d(s^{P(a_\alpha)} e, e) < \frac{1}{2^{\min \alpha}} - \varepsilon$$

for each $\emptyset \neq \alpha \subset \{1, \ldots, w\}$. By the semicontinuity of multiplication, we can find $\delta > 0$ such that whenever $d(s', s'') < \delta$,

$$d(s^{P(a_\alpha)} s', s^{P(a_\alpha)} s'') < \varepsilon.$$

Select k_{w+1} large enough to satisfy

$$d(s^{Q(a_\alpha, q_{k_{w+1}})} s^{P(q_{k_{w+1}})}, e) < \delta$$

and

$$d(s^{Q(a_\alpha,q_{k_{w+1}})}s^{P(q_{k_{w+1}})}e,e)<\delta$$

for each $\emptyset\neq\alpha\subset\{1,\ldots,w\}$. Then

$$d(s^{P(a_\alpha+q_{k_{w+1}})},e)\leq d(s^{P(a_\alpha)}s^{Q(a_\alpha,q_{k_{w+1}})}s^{P(q_{k_{w+1}})},s^{P(q_{k_{w+1}})}e)$$

$$+d(s^{P(q_{k_{w+1}})}e,e)<\varepsilon+\left(\frac{1}{2^{\min\alpha}}-\varepsilon\right)=\frac{1}{2^{\min\alpha}}$$

and similarly

$$d(s^{P(a_\alpha+q_{k_{w+1}})}e,e)<\frac{1}{2^{\min\alpha}}.$$

for each $\emptyset\neq\alpha\subset\{1,\ldots,w\}$. We can also assume that k_{w+1} yields $d(s^{P(q_{k_{w+1}})},e)<\frac{1}{2^{w+1}}$ and $d(s^{P(q_{k_{w+1}})}e,e)<\frac{1}{2^{w+1}}$ which covers the case $\alpha=\emptyset$.

We have proved that our recurrence procedure can be continued. In view of (1.16), $\mathrm{IP}-\lim s^{P((q_{k_w})\alpha)}=e$. We use now Lemma 1.1 to complete the proof in the case of one-element family of polynomials.

Part 2. By Part 1 of the proof, we can select a subsequence $(q_{k_w^{(1)}})_{w\geq1}$ of $(q_n)_{n\geq1}$, so that

$$\mathrm{IP}\text{-}\lim s^{P_1((q_{k_w^{(1)}})\alpha)}=e_1$$

(that is, the IP-convergence holds along the IP-set $\mathrm{FS}((q_{k_w^{(1)}})_{w\geq1})$). At stage $j+1$ (using repeatedly Part 1 of the proof), we select a subsequence $(q_{k_w^{(j+1)}})_{w\geq1}$ of $(q_{k_w^{(j)}})_{w\geq1}$, so that

$$\mathrm{IP}\text{-}\lim s^{P_{j+1}((q_{k_w^{(j+1)}})\alpha)}=e_{j+1}$$

(that is, the IP-convergence holds along the sub-IP-set $\mathrm{FS}((q_{k_w^{(j+1)}})_{w\geq1})$ of $\mathrm{FS}((q_{k_w^{(j)}})_{w\geq1})$).

Now, for each $i\geq1$, we set $q_{m_i}=q_{k_i^{(i)}}$. It follows that for each $j\geq1$, up to a finite number of terms, $(q_{m_i})_{i\geq1}$ is a subsequence of $(q_{k_w^{(j)}})_{w\geq1}$. Therefore, whenever we have an IP-convergence along $\mathrm{FS}((q_{k_w^{(j)}})_{w\geq1})$, we also have the IP-convergence along $\mathrm{FS}((q_{m_i})_{i\geq1})$ (to the same limit). We conclude that for each $j\geq1$,

$$\mathrm{IP}\text{-}\lim s^{P_j((q_{m_i})\alpha)}=e_j$$

and the result follows from Lemma 1.1 (i). $\square$

Remark 3. We would like to emphasize that the assumption (1.12) in Proposition 1.7 seems to be crucial. Indeed, in the proof of Proposition 1.7, due to (1.12), we are able to choose a subsequence $(q_{k_n})_{n\geq1}$ so that $\mathrm{IP}\text{-}\lim s^{P_i(q_{k_n})}=e_i$ for each $i\geq1$ and then we obtain a "good" $p \in E(\beta\mathbb{N})$ using Lemma 1.1. However, in general, for a (q_n) growing rapidly to infinity, it is even possible to have $U \in \mathcal{U}(\mathcal{H})$ such that $U^{lq_n} \to 0$ for each $l \geq 1$ and $U^{q_n^2} \to Id$. Then, for any subsequence $(q_{k_n})_{n\geq 1}$ we have the same convergence but we cannot have the IP-convergence as this, by Lemma 1.1, yields $p \in E(\beta\mathbb{N})$ with $p\text{-}\lim_{n\in\mathbb{N}} U^{ln} = 0$ for each $l \geq 1$ and $p\text{-}\lim_{n\in\mathbb{N}} U^{n^2} = Id$, a contradiction with Corollary B.

2 p-polynomial decomposition of a Hilbert space

2.1 Some immediate consequences of Theorem A

Throughout this section we fix $p \in E(\beta\mathbb{N})$. Before we start the proof of Theorem A, let us make first some introductory remarks and derive some simple consequences of it.

Suppose that we have a decomposition (0.11) satisfying only (0.12). Then if for $k, l \in \mathbb{N}$ we have

$$p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_k})^{P(n)} = p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_l})^{P(n)}$$

for each polynomial $P \in \mathcal{P}_{\leq N}$ (call such k, l *equivalent*), then we can replace both subspaces $\mathcal{H}_k^{(N)}$ and $\mathcal{H}_l^{(N)}$ by one subspace $\mathcal{H}_k^{(N)} \oplus \mathcal{H}_l^{(N)}$ and still have a decomposition satisfying (0.12). By grouping up subspaces whose indices are equivalent, we achieve a decomposition (0.11) in which additionally to (0.12) we have (0.13).

Now, we claim that if a decomposition (0.11) satisfying (0.12) and (0.13) exists then it is unique. Indeed, it is enough to show that

(2.1) if $\mathcal{F}$ is a closed U-invariant subspace such that (0.12) is satisfied on it, then there is $k \geq 1$ such that $\mathcal{F} \subset \mathcal{H}_k^{(N)}$.

To prove this claim, suppose that for no $k \geq 1$, $\mathcal{F}$ is included in $\mathcal{H}_k^{(N)}$. It follows that for some $x \in \mathcal{F}$, $x = \sum_{i\geq 1} x_i$ with $x_i \in \mathcal{H}_i^{(N)}$ $(i \geq 1)$ there are $i_1 \neq i_2$ such that $x_{i_1} \neq 0 \neq x_{i_2}$. By interchanging the roles of x_{i_1} and x_{i_2}, if necessary, by (0.13), we can assume that there exists $Q \in \mathcal{P}_{\leq N}$ such that

$$p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)} x_{i_1} = 0,\ p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)} x_{i_2} = x_{i_2}.$$

On the other hand, we have either

$$p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)} x = x \ \text{ or } \ p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)} x = 0.$$

In the first case, we obtain

$$\sum_{i\geq 1} x_i = x = p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)}x = \sum_{i\geq 1} p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)}x_i,$$

where $p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)}x_i \in \mathcal{H}_i^{(N)}$, $i \geq 1$. In particular, $p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)}x_{i_k} = x_{i_k}$ for $k = 1, 2$ and we obtain contradiction. Similarly, we obtain a contradiction in the second case and therefore (2.1) follows.

The following result is a direct consequence of (2.1).

Corollary 2.1. *The subspaces $\mathcal{H}_k^{(N)}$ in the decomposition (0.11) are* spectral subspaces*, i.e. if $y \in \mathcal{H}$ and the spectral measure*[30] *of y is absolutely continuous with respect to the maximal spectral type of $U|_{\mathcal{H}_k^{(N)}}$ then $y \in \mathcal{H}_k^{(N)}$.*

Remark 4. It also follows from (2.1) that the decomposition in Theorem A is minimal in the sense that any decomposition satisfying (0.12) must be a refinement of the one of that theorem.

By the minimality of the decomposition in Theorem A, it follows that if $1 \leq M < N$ and

$$\mathcal{H} = \bigoplus_{l\geq 1} \mathcal{H}_l^{(M)} = \bigoplus_{k\geq 1} \mathcal{H}_k^{(N)}$$

are decompositions given by Theorem A for M and N respectively then for each $k \geq 1$ there exists a unique $l_k \geq 1$ such that $\mathcal{H}_k^{(N)} \subset \mathcal{H}_{l_k}^{(M)}$.

Corollary 2.2. *Assume that G is a finitely generated subgroup of the group of all polynomials $P \in \mathcal{P}$. Let $U \in \mathcal{U}(\mathcal{H})$. Then there exists a unique decomposition $\mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}_k^{(G)}$ into closed U-invariant subspaces such that (0.12) and (0.13) hold with $\mathcal{P}_{\leq N}$ replaced by G.*

Proof. Since G is finitely generated, $G \subset \mathcal{P}_{\leq N}$ for some $N \geq 1$. Apply Theorem A to obtain a decomposition for which (0.12) is satisfied when $\mathcal{P}_{\leq N}$ is replaced by G and then glue up subspaces which cannot be distinguished by polynomials belonging do G (similarly to the proof of Theorem A to obtain the required decomposition). □

Open problem: We have been unable to decide whether the decomposition described in Theorem A exists for arbitrary infinite family of polynomials in $\mathcal{P}$. We conjecture that the answer is negative already for the family of all monomials.

[30] The spectral measure of $y \in \mathcal{H}$ is the unique finite, positive Borel measure σ_y on $\mathbb{S}^1$ satisfying $\widehat{\sigma}_y(n) = \langle U^n y, y\rangle$ for all $n \in \mathbb{Z}$; see e.g. [18], Chapter 5.

2.2 Proof of Theorem A

Our proof is based on two results: Theorem 0.5 and the following lemma.

Lemma 2.3 ([13]). *Assume that $(x_n) \subset \mathcal{H}$ is bounded. If*

$$p\text{-}\lim_{h\in\mathbb{N}} \left(p\text{-}\lim_{n\in\mathbb{N}} \langle x_{n+h}, x_n \rangle \right) = 0$$

then $p\text{-}\lim_{n\in\mathbb{N}} x_n = 0$.

Let us start the proof with verifying the validity of Theorem A for $N = 1$. Denote by $\mathcal{H}_1$ the subspace uniquely determined by $p\text{-}\lim_{n\in\mathbb{N}} U^n = \mathrm{proj}_{\mathcal{H}_1}$ (see Theorem 0.5). Then $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with both subspaces being U-invariant. Moreover, $p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_0})^n = 0$ and $p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_1})^n = Id$. The latter equality implies

$$p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_1})^{kn} = Id \text{ for each } k \geq 1.$$

We now apply the same argument to $(U|_{\mathcal{H}_0})^2$. We obtain a decomposition $\mathcal{H}_0 = \mathcal{H}_{00} \oplus \mathcal{H}_{01}$ such that $p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{01}})^{2n} = Id$ and $p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{00}})^{2n} = 0$. It follows that

$$p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{01}})^n = 0, \; p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{01}})^{2n} = Id.$$

These two conditions determine the p-limits of $\left(U^{kn}|_{\mathcal{H}_{01}}\right)_n$ for each $k \geq 1$. Indeed, if $k = 2l$ then $p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{01}})^{2ln} = Id$ and if $k = 2l + 1$ then by Lemma 1.4

$$p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{01}})^{kn} = p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{01}})^{2ln+n} = Id \circ 0 = 0.$$

On H_{00} we have $p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{00}})^{in} = 0$ for $i = 1, 2$. Applying the above argument again to $(U|_{H_{00}})^3$, we obtain $\mathcal{H}_{00} = \mathcal{H}_{000} \oplus \mathcal{H}_{001}$ with

$$p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{001}})^{3ln} = Id, \; p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{001}})^{(3l+1)n} = 0$$

and

$$p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{001}})^{(3l+2)n} = 0$$

for each $l \geq 0$. On H_{000} we have $p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{H}_{000}})^{in} = 0$ for $i = 1, 2, 3$. Continuing in this fashion, we will obtain a sequence

$$\mathcal{H}_0 \supset \ldots \supset \mathcal{H}_{0^k 1} \supset \mathcal{H}_{0^{k+1} 1} \supset \ldots$$

of nested closed U-invariant subspaces, where 0^k1 stands for $\underbrace{0\dots0}_{k}1$, $k\geq 1$. Let $\mathcal{H}_\infty = \left(\bigoplus_{k\geq 0}\mathcal{H}_{0^k1}\right)^\perp$. By construction, $U|_{\mathcal{H}_\infty}$ is totally p-mixing[31] on $\mathcal{H}_\infty$ (indeed, if $p\text{-}\lim_{n\in\mathbb{N}}\left(U|_{\mathcal{H}_\infty}\right)^n = \mathrm{proj}_{\mathcal{F}}$ then $\mathcal{F}\subset\mathcal{H}_1$ whence $\mathcal{F}=\{0\}$; if $p\text{-}\lim_{n\in\mathbb{N}}\left(U|_{\mathcal{H}_\infty}\right)^n = 0$ and if $p\text{-}\lim_{n\in\mathbb{N}}\left(U|_{\mathcal{H}_\infty}\right)^{2n} = \mathrm{proj}_{\mathcal{F}}$ then $\mathcal{F}\subset\mathcal{H}_{01}$ whence $\mathcal{F}=\{0\}$, etc.). We have proved the following.

Lemma 2.4. *If $U\in\mathcal{U}(\mathcal{H})$ and $p\in E(\beta\mathbb{N})$ then there exists a decomposition*

$$\mathcal{H} = \mathcal{H}_\infty \oplus \bigoplus_{k=0}^{\infty}\mathcal{H}_{0^k1}$$

into U-invariant subspaces such that U is totally p-mixing on $\mathcal{H}_\infty$ and

$$p\text{-}\lim_{n\in\mathbb{N}}\left(U|_{\mathcal{H}_{0^k1}}\right)^{rn} = \begin{cases} Id & \text{if} \quad (k+1)|r \\ 0 & \quad\quad \text{otherwise.}\end{cases}$$

The following result is needed to complete the inductive proof of the case $N=1$.

Lemma 2.5. *Let $p\in E(\beta\mathbb{N})$. Assume that $U\in\mathcal{U}(\mathcal{H})$ and $k\geq 1$. Assume moreover that for some $r_j\in\mathbb{N}$*

$$p\text{-}\lim_{n\in\mathbb{N}} U^{r_jn^j} = Id \tag{2.2}$$

for $j=1,\dots,k-1$ and

$$p\text{-}\lim_{n\in\mathbb{N}} U^{ln^k} = 0 \;\text{ for all }\; l\geq 1. \tag{2.3}$$

Then

$$p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = 0 \tag{2.4}$$

for all polynomials $P\in\mathcal{P}_{\geq k}$.

Proof. Let $y\in\mathcal{H}$ and let $P\in\mathcal{P}$ be a polynomial of degree $k+1$. By Lemma 1.5, we have

$$\langle U^{P(n+h)}y, U^{P(n)}y\rangle = \langle U^{Q(n,h)}y, U^{-P(h)}y\rangle,$$

where the x-degree of Q is k and xy divides $Q(x,y)$. Therefore, in view of (2.2) and (2.3), whenever we fix $h\in\mathbb{N}$ such that $\mathrm{lcm}(r_1,\dots,r_{k-1})|h$ then

$$p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n,h)}y = 0.$$

[31] A unitary operator $V\in\mathcal{U}(\mathcal{G})$ is called *totally p-mixing* if $p\text{-}\lim_{n\in\mathbb{N}} V^{ln} = 0$ for each $l\geq 1$.

Hence, using (1.5), we obtain

$$p\text{-}\lim_{h\in\mathbb{N}}\left(p\text{-}\lim_{n\in\mathbb{N}}\langle U^{P(n+h)}y, U^{P(n)}y\rangle\right) = p\text{-}\lim_{h\in\mathbb{N}}\left(p\text{-}\lim_{n\in\mathbb{N}}\langle U^{Q(n,h)}y, U^{-P(h)}y\rangle\right) =$$

$$p\text{-}\lim_{\mathrm{lcm}(r_1,\ldots,r_{k-1})|h}\left(p\text{-}\lim_{n\in\mathbb{N}}\langle U^{Q(n,h)}y, U^{-P(h)}y\rangle\right) = 0.$$

By Lemma 2.3, $p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)}y = 0$. Since $y \in \mathcal{H}$ was arbitrary, $p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = 0$ for all polynomials $P \in \mathcal{P}$ of degree $k+1$.

To obtain the result for all polynomials $P \in \mathcal{P}_{\geq k+1}$, we use induction and Lemma 2.3. Indeed, if P has degree $d \geq k+2$ then

$$p\text{-}\lim_{h\in\mathbb{N}}\left(p\text{-}\lim_{n\in\mathbb{N}}\langle U^{P(n+h)-P(n)}y, y\rangle\right) =$$

$$p\text{-}\lim_{h\in\mathbb{N}}\left(p\text{-}\lim_{n\in\mathbb{N}}\langle U^{Q(n,h)}y, U^{-P(h)}y\rangle\right) = 0$$

because the x-degree of $Q(x,h)$ is $d-1$ and by the induction assumption

$$p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n,h)} = 0.$$

We have proved (2.4) for all $P \in \mathcal{P}_{\geq k+1}$.

To complete the proof, it remains to prove (2.4) for polynomials of degree k. Suppose that for some $Q(x) = s_1x + s_2x^2 + \ldots + s_kx^k \in \mathcal{P}$ with $s_k \geq 1$ we have $p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)} = \mathrm{proj}_{\mathcal{F}}$ with $\mathcal{F} \neq \{0\}$. Since (2.2) and (2.3) hold for $U|_{\mathcal{F}}$, by replacing $\mathcal{H}$ by $\mathcal{F}$ if necessary, we can assume that

$$p\text{-}\lim_{n\in\mathbb{N}} U^{Q(n)} = Id. \tag{2.5}$$

Then $p\text{-}\lim_{n\in\mathbb{N}} U^{rQ(n)} = Id$, where $r = \mathrm{lcm}(r_1,\ldots,r_{k-1})$. By putting together (2.5) and (2.2) we obtain that $p\text{-}\lim_{n\in\mathbb{N}} U^{rs_kn^k} = Id$, contrary to (2.3). □

The proof of Theorem A for $N = 1$ is now completed by invoking Lemmas 2.4 and 2.5.

Assume now that Theorem A holds for some $N \geq 1$. We will prove it for $N+1$. To this end pick a subspace $\mathcal{H}_k^{(N)}$. Two cases can arise.

Case 1. Assume there exists $Q \in \mathcal{P}_{\leq N}$ such that $p\text{-}\lim_{n\in\mathbb{N}} \left(U|_{\mathcal{H}_k^{(N)}}\right)^{sQ(n)} = 0$ for each $s \geq 1$. In this case we apply the induction assumption (the property (0.14)) to conclude that $\mathcal{H}_k^{(N)}$ will also be one of the subspaces in the decomposition (0.11) with N replaced by $N+1$ (indeed, $p\text{-}\lim_{n\in\mathbb{N}} \left(U|_{\mathcal{H}_k^{(N)}}\right)^{R(n)} = 0$ for all $R \in \mathcal{P}_{\geq N+1}$).

Case 2. There exist $a_1, \ldots, a_N \geq 1$ for which we have

$$\begin{aligned} &p\text{-}\lim_{n\in\mathbb{N}} \left(U^{in^s}|_{\mathcal{H}_k^{(N)}}\right) = 0 \ \text{ for each } \ i = 1, \ldots, a_s - 1, \text{ and} \\ &p\text{-}\lim_{n\in\mathbb{N}} \left(U^{a_s n^s}|_{\mathcal{H}_k^{(N)}}\right) = Id \end{aligned} \tag{2.6}$$

for $s = 1, \ldots, N$.

We now repeat verbatim the proof of Lemma 2.4 for $U = U|_{\mathcal{H}_k^{(N)}}$ and for the family $\{nx^{N+1} : n \geq 1\}$ instead of the family of linear polynomials. This yields the decomposition

$$\mathcal{H}_k^{(N)} = \mathcal{F}_\infty \oplus \bigoplus_{l=0}^{\infty} \mathcal{F}_{0^l 1}, \tag{2.7}$$

where

$$p\text{-}\lim_{n\in\mathbb{N}} \left(U^{in^{N+1}}|_{\mathcal{F}_\infty}\right) = 0 \ \text{ for each } \ i \geq 1 \tag{2.8}$$

and

$$\begin{aligned} &p\text{-}\lim_{n\in\mathbb{N}} \left(U^{jn^{N+1}}|_{\mathcal{F}_{0^l 1}}\right) = 0 \ \text{ for each } \ j = 1, \ldots, l, \\ &p\text{-}\lim_{n\in\mathbb{N}} \left(U^{(l+1)n^{N+1}}|_{\mathcal{F}_{0^l 1}}\right) = Id. \end{aligned} \tag{2.9}$$

In view of Lemma 2.5, the space $\mathcal{F}_\infty$ will be one of the subspaces $\mathcal{H}_r^{(N+1)}$ in the decomposition (0.11) for $N+1$. The subspaces $\mathcal{F}_{0^l 1}$ are (in general) not yet elements of the decomposition (0.11) for $N+1$ because we know only the p-limits for all $P \in \mathcal{P}_{\leq N}$ and all monomials jn^{N+1} for $j \geq 1$ (the latter follows by the same argument as in the proof of Lemma 2.4). Take $Q(x) = b_1 x + \ldots + b_N x^N + b_{N+1} x^{N+1}$, where

$$0 \leq b_i < a_i \text{ for } i = 1, \ldots, N \text{ and } 1 \leq b_{N+1} < l+1. \tag{2.10}$$

Using Lemma 2.4, we obtain a decomposition $\mathcal{F}_{0^l 1} = \mathcal{G}_0 \oplus \mathcal{G}_1$ such that

$$p\text{-}\lim_{n\in\mathbb{N}} \left(U^{Q(n)}|_{\mathcal{G}_0}\right) = 0, \ p\text{-}\lim_{n\in\mathbb{N}} \left(U^{Q(n)}|_{\mathcal{G}_1}\right) = Id. \tag{2.11}$$

Then we take another b_i's satisfying (2.10) and apply the same reasoning to $\mathcal{G}_0$ and $\mathcal{G}_1$ separately. Since the condition (2.10) defines only a finite family of polynomials, we end up with a (finite) decomposition

$$\mathcal{F}_{0^l 1} = \mathcal{J}_1 \oplus \ldots \oplus \mathcal{J}_K$$

having the property that for each $Q \in \mathcal{P}_{\leq N+1}$ whose coefficients satisfy (2.10), we have $p\text{-}\lim_{n\in\mathbb{N}} \left(U^{Q(n)}|_{\mathcal{J}_q}\right) = 0$ or Id, $q = 1, \ldots, K$. But, using (2.6) and (2.9), the same conclusion holds for all $Q \in \mathcal{P}_{\leq N+1}$, and we take the subspaces $\mathcal{J}_q$ as elements of the decomposition (0.11) for $N+1$.

To complete the proof of Theorem A (the property (0.14) has still to be shown) we need to show that on subspaces $\mathcal{J}_q$ for no polynomial $Q \in \mathcal{P}_{\leq N+1}$ we have $p\text{-}\lim_{n\in\mathbb{N}} \left(U^{mQ(n)}|_{\mathcal{J}_q}\right) = 0$ for all $m \geq 1$. This is however clear, if $L := \mathrm{lcm}(a_1, \ldots, a_N, l{+}1)$ then $p\text{-}\lim_{n\in\mathbb{N}} \left(U^{LQ(n)}|_{\mathcal{J}_q}\right) = Id$.

In this way we have completed the proof of Theorem A. As an immediate consequence of it (see (0.14)), we obtain Corollary B.

Remark 5. Our proof of Theorem A was based on Theorem 0.5. However it could be organized in another way. Lemma 1.6 is sufficient in order to perform the first stage of the construction (see Lemma 2.4). Then an application of Lemma 1.6 to the restrictions of the operator U to any of the spaces $\mathcal{H}_{0^{k_1}}$ shows that $p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)}|_{\mathcal{H}_{0^{k_1}}}$ is an orthogonal projection for any polynomial $P \in \mathcal{P}$ of degree 2. Continuing this way, we will be able to prove Theorem A using only Lemma 1.6. Note that Theorem 0.5 follows from Theorem A.

2.3 p-polynomial properties of Fourier transforms of measures

We have already noted that the subspaces that appear in (0.11) in Theorem A are spectral (see Corollary 2.1). This suggests that the assertion of that theorem depends only on the maximal spectral type of $U \in \mathcal{U}(\mathcal{H})$. Therefore from this theorem one can derive some harmonic analysis consequences concerning measures on the circle. To this end, we consider a special case of Theorem A in which $\mathcal{H} = L^2(\mathbb{S}^1, \sigma)$ and $U = V_\sigma$, $V_\sigma(f)(z) = zf(z)$ with σ a probability Borel measure on $\mathbb{S}^1$.

Lemma 2.6. *Assume that $P \in \mathcal{P}_{\leq N}$. Under the above notation, we have:*
(i) $p\text{-}\lim_{n\in\mathbb{N}} V_\sigma^{P(n)} = Id$ if and only if $p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}(P(n)) = 1$.
(ii) $p\text{-}\lim_{n\in\mathbb{N}} V_\sigma^{P(n)} = 0$ if and only if $p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}(P(n)+k) = 0$ for each $k \in \mathbb{Z}$.

Proof. For (i) just notice that by a convexity argument[32]

$$p\text{-}\lim_{n\in\mathbb{N}} \int_{\mathbb{S}^1} z^{P(n)}\, d\sigma(z) = 1 \text{ if and only if } p\text{-}\lim_{n\in\mathbb{N}} z^{P(n)} = 1$$

in $L^2(\mathbb{S}^1, \sigma)$. For (ii), use the elementary fact that $p\text{-}\lim_{n\in\mathbb{N}} V_\sigma^{P(n)} = 0$ if and only if for each $r, s \in \mathbb{Z}$

$$p\text{-}\lim_{n\in\mathbb{N}} \left\langle V_\sigma^{P(n)} z^r, z^s \right\rangle = 0.$$

□

Remark 6. Note that the condition $p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}(P(n)+k) = 0$ for each $k \in \mathbb{Z}$ and each $P \in \mathcal{P}_{\leq N}$ is equivalent to saying that $p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}(Q(n)) = 0$ for each $Q \in \mathbb{Z}[x]$ of degree at most N.

Proof of Corollary C. In view of Wiener's lemma,[33] each of subspaces $\mathcal{H}_k^{(N)}$ in Theorem A is of the form $\mathcal{H}_k^{(N)} = \mathbf{1}_{A_k^{(N)}} L^2(\mathbb{S}^1, \sigma)$, where $A_k^{(N)} \cap A_l^{(N)} = \emptyset$ (modulo σ) whenever $k \neq l$. In other words, we have proved the existence of decomposition of σ into the following mutually singular terms:

$$\sigma = \sum_{k\geq 1} \sigma|_{A_k^{(N)}}.$$

We set $\sigma_k^{(N)} = \frac{1}{\sigma(A_k^{(N)})}\sigma|_{A_k^{(N)}}$ and $a_k = \sigma(A_k^{(N)})$. To complete the proof of Corollary C it is enough to apply Theorem A and Lemma 2.6 (see also Remark 6). □

The decomposition result given by Corollary C does not depend on the fact that σ is continuous or not. In case of discrete measures however the following results holds.

---

[32] If $g : \mathbb{S}^1 \to \mathbb{S}^1$ is measurable and $|\int_{\mathbb{S}^1} g\, d\sigma - 1| < \varepsilon$ then we have

$$\left|\sigma(A)\cdot\left(\frac{1}{\sigma(A)}\int_A g\, d\sigma - 1\right) + \sigma(A^c)\cdot\left(\frac{1}{\sigma(A^c)}\int_{A^c} g\, d\sigma - 1\right)\right| < \varepsilon,$$

where $A := \{z \in \mathbb{S}^1 : |g(z) - 1| \geq \delta\} = \{z \in \mathbb{S}^1 : \mathrm{Re}(g(z)) \leq 1 - \delta^2/2\}$. Now, the real parts of both numbers $\frac{1}{\sigma(A^c)}\int_{A^c} g\, d\sigma - 1$ and $\frac{1}{\sigma(A)}\int_A g\, d\sigma - 1$ are non-positive, whence $\left|\sigma(A)\cdot\left(\frac{1}{\sigma(A)}\int_A g\, d\sigma - 1\right)\right| < \varepsilon$ and since $\mathrm{Re}(\frac{1}{\sigma(A)}\int_A g\, d\sigma) < 1 - \delta^2/2$, we have $\sigma(A) \leq \varepsilon/(\delta^2/2)$.

[33] Wiener's lemma says that whenever $\mathcal{F} \subset L^2(\mathbb{S}^1, \sigma)$ is a closed V_σ-invariant subspace then there exists a Borel subset $A \subset \mathbb{S}^1$ such that $\mathcal{F} = \mathbf{1}_A L^2(\mathbb{S}^1, \sigma)$ [32].

Proposition 2.7. *Assume that σ is a probability Borel measure on $\mathbb{S}^1$. Then the following conditions are equivalent:*
(i) σ is atomic.
(ii) $p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}(n) = 1$ for each $p \in E(\beta\mathbb{N})$.
(iii) $p\text{-}\lim_{n\in\mathbb{N}} \widehat{\sigma}(P(n)) = 1$ for each $P \in \mathcal{P}$ and $p \in E(\beta\mathbb{N})$.

Proof. To obtain the proof that (i) implies (ii), first notice that if $p\text{-}\lim_{n\in\mathbb{N}} U^{P(n)} = 0$ (for a non-zero $P \in \mathcal{P}$) for some $U \in \mathcal{U}(\mathcal{H})$ then U has continuous spectrum; indeed, in view of Lemma 1.1 (ii) and footnote 25, $p\text{-}\lim_{n\in\mathbb{N}}\langle U^{P(n)}x, x\rangle = 0$ implies $\lim_{s\to\infty}\langle U^{k_s}x, x\rangle = 0$ for an ordinary subsequence $(k_s)_{s\geq 1} \subset \mathbb{N}$ which is sufficient to conclude that the spectral measure σ_x is continuous.

Let us prove now that (ii) implies (i). Suppose that U has partly continuous spectrum. Then $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_c$ where $\mathcal{H}_c \neq \{0\}$ and U has continuous spectrum on $\mathcal{H}_c$ and discrete spectrum on $\mathcal{H}_d$. It follows that there exists an increasing sequence $(m_i) \subset \mathbb{N}$ of density 1 such that $(U|_{\mathcal{H}_c})^{m_i} \to 0$ weakly. Since (m_i) is of density one, we can find an increasing subsequence (n_k) of it such that $\mathrm{FS}((n_k))$ is contained in $\{m_i : i \geq 1\}$[34]. Now,

$$\mathrm{IP}\text{-}\lim\,(U|_{\mathcal{H}_c})^{n_\alpha} = 0.$$

Hence, by Lemma 1.1, there exists $p \in E(\beta\mathbb{N})$ such that $p\text{-}\lim_{n\in\mathbb{N}}(U|_{\mathcal{H}_c})^n = 0$ and therefore we cannot have $p\text{-}\lim_{n\in\mathbb{N}} U^n = Id$, a contradiction. □

Proof of Corollary D. Let us prove (i) (the proof of (ii) is similar). By Lemma 2.6, $p\text{-}\lim_{n\in\mathbb{N}} V_\sigma^{ln} = 0$ for each $l \geq 1$. In view of Corollary B, $p\text{-}\lim_{n\in\mathbb{N}} V_\sigma^{P(n)} = 0$ for each $0 \neq P \in \mathcal{P}$. The result now follows directly from Lemma 2.6. □

3 Classification of N-rigidity groups

The main goal of this section is to prove Theorem E.

3.1 The notion of N-rigidity group

Motivated by the properties of the decomposition (0.11) in Theorem A, we will now introduce and study N-rigidity groups.

---

[34]This fact is well known. As a matter of fact, each subset A of $\mathbb{N}$ containing intervals of arbitrary lengths includes an IP-set: if $n_1, \ldots, n_k$ have already been selected, choose $n_{k+1} \in A$ so that all sums $n_{i_1} + \ldots + n_{i_s} + n_{k+1}$ are in A for each $1 \leq i_1 < \ldots < i_s \leq k$.

We fix $N \geq 1$. Assume that $U \in \mathcal{U}(\mathcal{H})$ and $p \in E(\beta\mathbb{N})$. Let $\{0\} \neq \mathcal{J} \subset \mathcal{H}$ be a closed U-invariant subspace such that (0.12) holds for it. Let us now consider

$$G(p, N, U, \mathcal{J}) := \{P \in \mathcal{P}_{\leq N} : \ p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{P(n)} = Id\}. \tag{3.1}$$

It is not hard to see that $G(p, N, U, \mathcal{J})$ is a subgroup of $\mathcal{P}_{\leq N}$. Let N' be the maximum degree of elements in $G(p, N, U, \mathcal{J})$. We claim that on $\mathcal{J}$, for each $r = 1, \ldots, N'$, there exists (a unique) integer $k_r \geq 1$ such that

$$\begin{array}{l} p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{jn^r} = 0 \ \text{ for } \ j = 1, \ldots, k_r - 1 \\ \text{and } \ p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{k_r n^r} = Id. \end{array} \tag{3.2}$$

Indeed, otherwise for some $1 \leq j \leq N'$, $p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{ln^j} = 0$ for all $l \geq 1$ and, by Corollary B, we have $p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{P(n)} = 0$ for all $P \in \mathcal{P}_{\geq j}$. In other words

$$N' = \max \left\{ 1 \leq m \leq N : \begin{array}{l} \text{for each } r = 1, \ldots, m \text{ there exists } k_r \geq 1 \text{ such that} \\ p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{jn^r} = 0 \ \text{ for } \ j = 1, \ldots, k_r - 1 \\ \text{and } \ p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{k_r n^r} = Id. \end{array} \right\}$$

Now, fix $1 \leq m \leq N'$ and let $\{Q_1, \ldots, Q_m\}$ be an arbitrary set of generators for $\mathcal{P}_{\leq m}$. Since $Q_j(x) = \sum_{s=1}^{m} a_{j,s} x^s$, $j = 1, \ldots, m$,

$$p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{bQ_j(n)} = Id,$$

where $b = \mathrm{lcm}(k_1, \ldots, k_m)$. Hence there are (unique) integers $l_1, \ldots, l_m \geq 1$ such that $p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{jQ_r(n)} = 0$ for $j = 1, \ldots, l_r - 1$ and $p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{l_r Q_r(n)} = Id$ for $r = 1, \ldots, m$. It follows that

$$N' = \max \left\{ 1 \leq m \leq N : \begin{array}{l} \text{for each } r = 1, \ldots, m \\ \text{there exists } l_r \geq 1 \text{ such that} \\ p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{jQ_r} = 0 \\ \text{for } \ j = 1, \ldots, l_r - 1 \\ \text{and } \ p\text{-}\lim_{n \in \mathbb{N}} (U|_{\mathcal{J}})^{l_r Q_r} = Id. \end{array} \right\} \tag{3.3}$$

Clearly,

$$G(p, N, U, \mathcal{J}) = G(p, N, U, \mathcal{H}_t^{(N)}), \tag{3.4}$$

where $t \geq 1$ is unique so that $\mathcal{J} \subset \mathcal{H}_t^{(N)}$ (see Theorem A, formula (0.11) and Section 2.1).

In view of (3.3), we will assume in what follows that $N' = N$ (if $N' < N$, the corresponding group has already been introduced above as $G(p, N', U, \mathcal{J})$). Let $P(x) = \sum_{r=1}^{N} a_r x^r \in \mathbb{Z}[x]$. Then there exist integers $0 \le j_r < k_r$ (see (3.2) for the definition of k_r), $m_r \in \mathbb{Z}$ such that $a_r = j_r + m_r k_r$ for $r = 1, \dots, N$. By (3.2) and Lemma 1.4

$$p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{J}})^{P(n)} = p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{J}})^{j_1 n + j_2 n^2 + \ldots + j_N n^N}.$$

For $k \ge 1$, denote $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ and let π_k stand for the natural homomorphism $\mathbb{Z} \to \mathbb{Z}_k$. Set

$$\widetilde{G}(p, N, U, \mathcal{J}) := \left\{ \begin{array}{l} \pi_{k_1} \times \ldots \times \pi_{k_N}(j_1, \dots, j_N) \in \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N} : \\ 0 \le j_s < k_s,\ s = 1, \dots, N \text{ and} \\ p\text{-}\lim_{n\in\mathbb{N}} (U|_{\mathcal{J}})^{j_1 n + \ldots + j_N n^N} = Id \end{array} \right\}. \tag{3.5}$$

It follows from Lemma 1.4 that $\widetilde{G}(p, N, U, \mathcal{J})$ is a subgroup of $\mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$. We also have

$$G(p, N, U, \mathcal{J}) = (\pi_{k_1} \times \ldots \times \pi_{k_N})^{-1} (\widetilde{G}(p, N, U, \mathcal{J}))\ ^{35}. \tag{3.6}$$

Assume that $G \subset \mathcal{P}_{\le N}$ satisfies

$$\max\{\deg P :\ P \in G\} = N.$$

Then G is called an N-*rigidity group* (or, sometimes, *rigidity group* if no confusion arises) if there are $p \in E(\beta\mathbb{N})$ and $U \in \mathcal{U}(\mathcal{H})$ and $\{0\} \ne \mathcal{J} \subset \mathcal{H}$ such that $G = G(p, N, U, \mathcal{J})$. Given $p \in E(\beta\mathbb{N})$, the groups of the form $G = G(p, N, U, \mathcal{J})$ are called (p, N)-*rigidity groups*.

Then $U|_{\mathcal{J}}$ satisfies (3.2) and the vector $(k_1, \dots, k_N) \in \mathbb{N}^N$ is called a *period* of G. Other periods of G will be obtained in the same way by choosing a different basis in $\mathcal{P}_{\le N}$, see Remark 7 below. By (3.2), the group $\widetilde{G} = \widetilde{G}(p, N, U, \mathcal{J})$ satisfies the $(*)$-property (see Introduction): For each $r = 1, \dots, N$

$$\begin{array}{l} (j_1, \dots, j_{r-1}, j_r, j_{r+1}, \dots, j_N) \in \widetilde{G} \\ (j_1, \dots, j_{r-1}, j_r', j_{r+1}, \dots, j_N) \in \widetilde{G} \end{array} \implies j_r = j_r'. \tag{$*$}$$

Note that $\widetilde{G}$ satisfies $(*)$ if and only if $(0, \dots, 0, j_r, 0, \dots, 0) \in \widetilde{G}$ implies $j_r = 0$.

[35] If $\mathcal{J} = \mathcal{H}$ we will simply write $G(p, N, U)$ and $\widetilde{G}(p, N, U)$.

Remark 7. The definition of $\widetilde{G} = \widetilde{G}(p, N, U, \mathcal{J})$ depends on the choice $x, x^2, \ldots, x^N$ as generators in $\mathcal{P}_{\leq N}$. If we identify $\mathcal{P}_{\leq N}$ with $\mathbb{Z}^N$ then (3.6) can be written in a more suggestive form

$$G(p, N, U, \mathcal{J}) = (\pi_{k_1} \times \ldots \times \pi_{k_N})^{-1} (\pi_{k_1} \times \ldots \times \pi_{k_N}(G(p, N, U, \mathcal{J}))).$$

Note however that if we take a different set of generators of $\mathcal{P}_{\leq N}$, say $Q_1, \ldots, Q_N$, then we obtain integers $l_1, \ldots, l_N \geq 1$ as in (3.3) and, for $G = G(p, N, U, \mathcal{J})$, we can define the $\widetilde{G}$ using π_{l_j}, where the identification of $\mathcal{P}_{\leq N}$ with $\mathbb{Z}^N$ is given by

$$Q_j \mapsto (\underbrace{0, \ldots, 0}_{j-1}, 1, \underbrace{0, \ldots, 0}_{N-j}).$$

Then (3.6) is also true. The vector $(l_1, \ldots, l_N) \in \mathbb{N}^N$ is said to be a *period* of G. Note however that since (3.3) holds for any period, the crucial $(*)$-property does not depend on the choice of isomorphism between $\mathcal{P}_{\leq N}$ and $\mathbb{Z}^N$.

In view of (3.6) and of Remark 7, $G(p, N, U, \mathcal{J})$ is entirely determined by a choice of a period and by the corresponding to it group $\widetilde{G}(p, N, U, \mathcal{J})$. The latter group will be called an *N-periodic rigidity group* of U. Periods and N-periodic rigidity groups depend on a choice of generators in $\mathcal{P}_{\leq N}$. However, as it was explained in Remark 7, all N-periodic rigidity groups obtained from an N-rigidity group $G = G(p, N, U, \mathcal{J})$ satisfy the $(*)$-property.

Remark 8. Any natural number, including 1, can appear as one of the entries of a period of an N-periodic rigidity group. Moreover, a (p, N)-rigidity group G equals $\mathcal{P}_{\leq N}$ if and only if it has a period equal to $(1, \ldots, 1)$ (and then, there is only one period for it).

3.2 Algebraic characterization of groups satisfying the $(*)$-property

A subgroup $K \subset \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$ is called an *algebraic coupling*[36] if it has the full projection on each coordinate. Our aim is to show that a group $\widetilde{G} \subset \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$ satisfies the $(*)$-property if and only if it annihilates[37] an algebraic coupling contained in $\mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$, see Theorem 3.5 below.

To simplify the notation we will write G instead of $\widetilde{G}$. We have already identified $\mathbb{Z}_k$ with $\mathbb{Z}/k\mathbb{Z}$ and elements $z + k\mathbb{Z}$ of $\mathbb{Z}/k\mathbb{Z}$ will be denoted by $\overline{z} \in \mathbb{Z}_k$.

[36]The concept of algebraic coupling is analogous to the notion of joining in ergodic theory [18].

[37]To speak about annihilators of subgroups of a locally compact Abelian group V, we need a bilinear form defined on $\widehat{V} \times V$, where $\widehat{V}$, the Pontriagin's *dual* of V, denotes the group of continuous group homomorphisms from V to $\mathbb{S}^1$, see (3.7) below.

Let $V = \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$ and let $\overline{\pi}_j : V \to \mathbb{Z}_{k_j}$ denote the projection onto the jth coordinate. Then, up to some identification, $\widehat{V} = \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$.

Assume that we are given injective characters $\chi_j : \mathbb{Z}_{k_j} \to \mathbb{S}^1$, $j = 1, \ldots, N$. Let $\xi : \widehat{V} \times V \to \mathbb{S}^1$ be a $\mathbb{Z}$-bilinear map (the Abelian group $\mathbb{S}^1$ is a $\mathbb{Z}$-module) defined by the formula

$$\xi((c_1, \ldots, c_N), (d_1, \ldots, d_N)) = \chi_1(c_1 d_1) \cdot \ldots \cdot \chi_N(c_N d_N) \tag{3.7}$$

for any $(c_1, \ldots, c_N) \in \widehat{V}$, $(d_1, \ldots, d_N) \in V$, where we write $c_j d_j$ for the multiplication in the ring $\mathbb{Z}_{k_j}$.

Given a subset $A \subset V$, the *annihilator* $A^\perp$ of A (with respect to ξ) is defined by

$$A^\perp := \{c \in \widehat{V} : \xi(c, A) = \{1\}\} \text{ [38]}. \tag{3.8}$$

The following lemma is classical.

Lemma 3.1. *The set $A^\perp$ is a subgroup of $\widehat{V}$ and $A \subset (A^\perp)^\perp$ for any $A \subset V$.*

Our present goal is to prove the following result.

Proposition 3.2. *Assume we are given injective characters $\chi_j : \mathbb{Z}_{k_j} \to \mathbb{S}^1$, $j = 1, \ldots, N$ and $\xi : \widehat{V} \times V \to \mathbb{S}^1$ is defined by the formula (3.7), where $V = \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$. Let $G \subset V$ be a subgroup of V. Then*

(i) $G = (G^\perp)^\perp$.

(ii) *The group G has the $(*)$-property if and only if $K := G^\perp$ is an algebraic coupling of $\mathbb{Z}_{k_1}, \ldots, \mathbb{Z}_{k_N}$.*

In order to prove (i) above, it will be convenient to introduce a more general framework. Assume that V is a locally compact Abelian group and define a bilinear form by

$$\langle \cdot, \cdot \rangle : \widehat{V} \times V \to \mathbb{S}^1, \ \ \langle \phi, v \rangle = \phi(v).$$

The above form allows one to define the concept of annihilator of a set both in V as well as in $\widehat{V}$ [39]. We then have one more classical observation (cf. [34]).

Lemma 3.3. *For each closed subgroup $G \subset V$, $G = (G^\perp)^\perp$.*

[38]The annihilator of a subset $B \subset \widehat{V}$ is defined similarly.

[39]Classically, if G is a closed subgroup of V, $G^\perp$ has a natural identification with $\widehat{(V/G)}$.

Proof. By definition, $G \subset (G^\perp)^\perp$. On the other hand, if $v \notin G$ then there exists $\phi \in \widehat{V}$ such that $\phi(G) = \{1\}$ and $\phi(v) \neq 1$. It follows that $\phi \in G^\perp$ and $\phi(v) \neq 1$ implies $v \notin (G^\perp)^\perp$. □

Now, suppose that α and β are continuous automorphisms of V and $\widehat{V}$ respectively. Set

$$\xi : \widehat{V} \times V \to \mathbb{S}^1, \ \ \xi(\phi, v) = \langle \beta(\phi), \alpha(v) \rangle.$$

We obtain the following extension of Lemma 3.3.

Lemma 3.4. *The assertion of Lemma 3.3 holds when $\perp$ is defined, using (3.8), with respect to ξ.*

Proof. The proof repeats the argument from the proof of Lemma 3.3 (take $v \notin G$, choose ϕ so that $\beta(\phi)(\alpha(G)) = \{1\}$ and $\beta(\phi)(\alpha(v)) \neq 1$). □

The claim (i) of Proposition 3.2 now follows immediately from (3.7).

Proof of (ii) in Proposition 3.2. Assume that $K = G^\perp$ is an algebraic coupling of $\mathbb{Z}_{k_1}, \ldots, \mathbb{Z}_{k_N}$ and suppose that $g = (g_1, 0, \ldots, 0) \in G$. For any $c_1 \in \mathbb{Z}_{k_1}$ there exists $(c_2, \ldots, c_N) \in \mathbb{Z}_{k_2} \times \ldots \times \mathbb{Z}_{k_N}$ such that $c := (c_1, c_2, \ldots, c_N) \in K$. Then

$$1 = \xi(c, g) = \chi_1(c_1 g_1).$$

Since c_1 is arbitrary and χ_1 is an injective character, $g_1 = 0$. Repeating the argument for the remaining coordinates proves that G has the $(*)$-property.

Conversely, assume that K is not an algebraic coupling of $\mathbb{Z}_{k_1}, \ldots, \mathbb{Z}_{k_N}$. Assume that $\overline{\pi}_1(K) \neq \mathbb{Z}_{k_1}$ and let c_1 be a generator of $\overline{\pi}_1(K)$. There is an integer z such that $zc_1 = 0$ in $\mathbb{Z}_{k_1}$ and $\overline{z} \neq 0$ in $\mathbb{Z}_{k_1}$. Then $0 \neq g := (\overline{z}, 0, \ldots, 0) \in K^\perp = G$ (the latter equality follows from (i)), thus, G does not have the $(*)$-property. □

Let us consider the special case $N = 2$. It follows from the $(*)$-property that $G \subset \mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2}$ is the graph of an isomorphism between subgroups $\mathbb{Z}_s^{(1)} \subset \mathbb{Z}_{k_1}$ and $\mathbb{Z}_s^{(2)} \subset \mathbb{Z}_{k_2}$, where both $\mathbb{Z}_s^{(1)}$ and $\mathbb{Z}_s^{(2)}$ are isomorphic to $\mathbb{Z}_s$ (for some $s | k_i$, $i = 1, 2$).

In the general case, take $G \subset \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$ satisfying the property $(*)$. Denote by G_1 the projection of G on the first $N-1$ coordinates, i.e. on $\mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_{N-1}}$. By the property $(*)$, there exists a group homomorphism $w : G_1 \to \mathbb{Z}_{k_N}$ such that

$$G = \{(g_1, w(g_1)) \in \left(\mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_{N-1}}\right) \oplus \mathbb{Z}_{k_N} : \ g_1 \in G_1\}. \tag{3.9}$$

Note that if $g_1 = (0, \ldots, 0) \in \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_{N-1}}$ then $w(g_1) = 0$. It easily follows that G satisfies the $(*)$-property if and only if the kernel of w (which is a subgroup of

$G_1 \subset \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_{N-1}}$) also satisfies the property $(*)$. Then, we can repeat the same argument with G replaced by $\ker w$ to go one step down.

3.3 Constructions – measure-theoretic preparations

Proposition 3.2 provides a full algebraic description of groups satisfying the $(*)$-property as groups which annihilate algebraic couplings of finite cyclic groups.

Let us summarize the notational agreements from the previous section. We will be writing V as $\mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$. By χ_r we denote the natural embedding character $\overline{1} \mapsto e^{2\pi i/k_r}$ of $\mathbb{Z}_{k_r}$ in the group

$$\Sigma_{k_r} := \{e^{2\pi i u/k_r} : \ u = 0, 1, \ldots, k_r - 1\}.$$

Notice that $\mathbb{Z}_{k_r}$ has the natural structure of $\mathbb{Z}_{k_r}$-module. Therefore, if $(j_1, \ldots, j_N) \in \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$ then the formula

$$j_1 \times \ldots \times j_N (c_1, \ldots, c_N) = (j_1 c_1, \ldots, j_N c_N), \ \ (c_1, \ldots, c_N) \in \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N},$$

defines an endomorphism $j_1 \times \ldots \times j_N$ of $\mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$ and the form ξ from the previous section can be written as

$$\xi((c_1, \ldots, c_N), (j_1, \ldots, j_N)) = \chi_1 \otimes \ldots \otimes \chi_N (j_1 \times \ldots \times j_N (c_1, \ldots, c_N)).$$

We have now the following result.

Theorem 3.5. *Assume that $G \subset \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$ is a subgroup. Then the following assertions are equivalent:*

(a) G is an N-periodic rigidity group.

(b) G satisfies the $(*)$-property.

(c) $G = K^\perp$ for some algebraic coupling K of $\mathbb{Z}_{k_1}, \ldots, \mathbb{Z}_{k_N}$.

We have already seen that (a) implies (b) and that (b) and (c) are equivalent. It remains to show that (c) implies (a) and for that we need a relevant construction.

Let us note in passing the following immediate consequence of Theorem 3.5.

Corollary 3.6. *For any choice of $(k_1, \ldots, k_N) \in \mathbb{N}^N$, the trivial group is an N-periodic rigidity group (with respect to the period $(k_1, \ldots, k_N)$). In other words, for some $U \in \mathcal{U}(\mathcal{H})$ and $p \in E(\beta\mathbb{N})$, if $P(x) = a_1 x + \ldots + a_N x^N$ with $a_i \in \mathbb{Z}$ $(i = 1, \ldots, N)$ then*

$$p\text{-}\lim_{n \in \mathbb{N}} U^{P(n)} = \begin{cases} Id & \text{if} \quad a_i \equiv 0 \mod k_i, \ i = 1, \ldots, N, \\ 0 & \text{otherwise.} \end{cases}$$

We will now introduce the main probabilistic tools used in the constructions which will be carried over in the next subsection. These constructions will complete the proof of Theorem 3.5, that is, given an algebraic coupling $K \subset \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$, we will find $U \in \mathcal{U}(\mathcal{H})$ and $p \in E(\beta\mathbb{N})$ such that $K^{\perp} = \widetilde{G}(p, N, U)$.

Assume that $(X, \mathcal{B}, \mu)$ is a probability Borel space and let $K \subset \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$ be an algebraic coupling. Assume that $Y = (Y_1, \ldots, Y_N) : X \to \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$ is measurable with $Y(X) = K$ and

$$Y_*(\mu) = \lambda_K \text{ [40]}. \tag{3.10}$$

Given $(j_1, \ldots, j_N) \in \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$, set

$$K_{j_1,\ldots,j_N} := j_1 \times \ldots \times j_N(K) = \{(j_1c_1, \ldots, j_Nc_N) : (c_1, \ldots, c_N) \in K\}.$$

Then $K_{j_1,\ldots,j_N} \subset \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$ is a subgroup and if we denote $Y_{j_1,\ldots,j_N} = (j_1Y_1, \ldots, j_NY_N)$ then (in view of (3.10))

$$\left(Y_{j_1,\ldots,j_N}\right)_*(\mu) = \lambda_{K_{j_1,\ldots,j_N}}.$$

Assume that X is a compact monothetic metric group $\mu = \lambda_X$ the normalized Haar measure on the σ-algebra $\mathcal{B}$ of Borel subsets of X. Let $Tx = x + x_0$, where x_0 is such that $\{nx_0 : n \in \mathbb{Z}\}$ is dense in X. Assume that $\xi : X \to \mathbb{S}^1$ is measurable.

Let $V_\xi^T \in \mathcal{U}\left(L^2(X, \mathcal{B}, \mu)\right)$, $V_\xi^T(f) = \xi \cdot f \circ T$, thus $(V_\xi^T)^n(f) = \xi^{(n)} \cdot f \circ T^n$ for $n \in \mathbb{Z}$, where

$$\xi^{(n)}(x) = \begin{cases} \xi(x) \cdot \ldots \cdot \xi(T^{n-1}x) & \text{if } \ n > 0 \\ 1 & \text{if } \ n = 0 \\ (\xi(T^n x) \cdot \ldots \cdot \xi(T^{-1}x))^{-1} & \text{if } \ n < 0. \end{cases}$$

Then, the cocycle identity $\xi^{(m+n)}(x) = \xi^{(m)}(x) \cdot \xi^{(n)}(T^m x)$ holds. Often, ξ itself is called a cocycle. By the same token, we can define cocycles taking values in general compact Abelian groups.

If we take $f \in L^2(X, \mathcal{B}, \mu)$ to be a character of X then, for each $n \in \mathbb{Z}$, we have

$$\left\langle \left(V_\xi^T\right)^n f, f \right\rangle = \int_X \xi^{(n)}(x) f(T^n x)\overline{f(x)}\, d\mu(x) =$$

$$f(nx_0) \int_X \xi^{(n)}(x)|f(x)|^2\, d\mu(x) = f(x_0)^n \int_X \xi^{(n)}(x)\, d\mu(x) = f(x_0)^n \left\langle \left(V_\xi^T\right)^n 1, 1 \right\rangle.$$

[40] If Z is a random variable on $(\Omega, \mathcal{F}, P)$ taking values in a Borel space $(\Sigma, \mathcal{A})$ then by Z_* or $Z_*(P)$ we denote the *distribution* of Z: $Z_*(A) := P(Z^{-1}(A))$ for $A \in \mathcal{A}$. Whenever K is a compact group, λ_K stands for its (normalized) Haar measure.

It follows that if (n_t) is a strictly increasing sequence of natural numbers then

$$\text{(3.11)} \qquad \text{if } \int_X \xi^{(n_t)}(x)\,d\mu(x) \to 0 \text{ then } \left(V_\xi^T\right)^{n_t} \to 0 \text{ weakly.}$$

If, moreover, $n_t x_0 \to 0$ in X then, by the same argument as above, we have

$$\text{(3.12)} \qquad \xi^{(n_t)} \to 1 \text{ in measure implies } \left(V_\xi^T\right)^{n_t} \to Id \text{ strongly.}$$

We recall that by footnote 32, $\xi^{(n_t)} \to 1$ in measure if and only if $\int_X \xi^{(n_t)}\,d\mu \to 1$.

Assume now that $\varphi : X \to H$ is a cocycle, where H is a compact Abelian metrizable group. Assume also that $\chi \in \widehat{H}$.

Proposition 3.7. *Assume that (n_t) is a strictly increasing sequence of natural numbers and $n_t x_0 \to 0$ (in X). Assume moreover that*

$$\left(\chi \circ \varphi^{(n_t)}\right)_*(\mu) \to \lambda_F$$

where $F \subset \mathbb{S}^1$ is a closed subgroup.[41] *Then $\left(V_{\chi\circ\varphi}^T\right)^{n_t} \to 0$ if $F \neq \{1\}$ and $\left(V_{\chi\circ\varphi}^T\right)^{n_t} \to Id$ if $F = \{1\}$.*

Proof. Since

$$\int_X (\chi\circ\varphi)^{(n_t)}\,d\mu = \int_X \chi\circ\varphi^{(n_t)}\,d\mu = \int_{\mathbb{S}^1} z\,d\left(\chi\circ\varphi^{(n_t)}\right)_*(\mu) \to \int_{\mathbb{S}^1} z\,d\lambda_F,$$

the result follows directly from (3.11) and (3.12). □

Note that, in particular, whenever $L \subset H$ is a (non-trivial) compact subgroup,

$$\text{(3.13)} \qquad (\varphi^{(n_t)})_* \to \lambda_L \text{ implies } \int_X \chi(\varphi^{(n_t)}(x))\,d\mu(x) \to 0$$

for each character $\chi \in \widehat{H}$, $\chi(L) \neq \{1\}$. This, together with (3.11), yields a criterion for a weak convergence to zero of our special weighted operators $(V_{\chi\circ\varphi}^T)^{n_t}$ along any strictly increasing sequence (n_t) of natural numbers.

Denote $\Theta = \chi_1 \otimes \ldots \otimes \chi_N$ and suppose that we are given a cocycle $\Phi := (\varphi_1, \ldots, \varphi_N) : X \to \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$ (over the (n_t)-odometer T, see Section 3.4 below). We intend to study the weighted operator $U = V_{\Theta\circ\Phi}^T$. We are interested in computing the weak limits of $(U^{j_1 n_t + j_2 n_t^2 + \ldots + j_N n_t^N})_{t\geq 1}$ for $(j_1, \ldots, j_N) \in \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$. Using the cocycle identity, we have

$$\int_X \Theta\left(\varphi_1^{(j_1 n_t + j_2 n_t^2 + \ldots + j_N n_t^N)}, \ldots, \varphi_N^{(j_1 n_t + j_2 n_t^2 + \ldots + j_N n_t^N)}\right)\,d\mu$$

[41] That is, either $F = \mathbb{S}^1$ or F is finite.

$$= \int_X \Theta \Big(\varphi_1^{(j_1 n_t)} + \varphi_1^{(j_2 n_t^2)} \circ T^{j_1 n_t} + \ldots + \varphi_1^{(j_N n_t^N)} \circ T^{j_1 n_t + j_2 n_t^2 + \ldots + j_{N-1} n_t^{N-1}} ,$$
$$\varphi_2^{(j_2 n_t^2)} + \varphi_2^{(j_1 n_t)} \circ T^{j_2 n_t^2} + \ldots + \varphi_2^{(j_N n_t^N)} \circ T^{j_1 n_t + j_2 n_t^2 + \ldots + j_{N-1} n_t^{N-1}}, \ldots,$$
$$\varphi_N^{(j_N n_t^N)} + \varphi_N^{(j_1 n_t)} \circ T^{j_N n_t^N} + \ldots + \varphi_N^{(j_{N-1} n_t^{N-1})} \circ T^{j_1 n_t + \ldots + j_{N-2} n_t^{N-2} + j_N n_t^N} \Big) \, d\mu.$$

Our basic assumption on the cocycles $\varphi_1, \ldots, \varphi_N$ is the following: For each $r = 1, \ldots, N$, whenever $s \in \{1, \ldots, N\} \setminus \{r\}$,

$$\varphi_r^{(n_t^s)} \to 0 \text{ in measure as } t \to \infty. \tag{3.14}$$

It follows that whenever $r = 1, \ldots, N$, $s \in \{1, \ldots, N\} \setminus \{r\}$ and $i = 1, \ldots, k_s - 1$,

$$\varphi_r^{(i n_t^s)} \to 0 \text{ in measure}$$

and hence

$$\varphi_r^{(i n_t^s)} \circ T^k \to 0 \text{ in measure as } t \to \infty$$

for each $k \in \mathbb{Z}$. It follows that under the assumption (3.14) (and tacitly assuming that one of the limits below exists), we have

$$\begin{aligned} &\lim_{t\to\infty} \int_X \Theta\Big(\varphi_1^{(j_1 n_t + j_2 n_t^2 + \ldots + j_N n_t^N)}, \ldots, \varphi_N^{(j_1 n_t + j_2 n_t^2 + \ldots + j_N n_t^N)} \Big) \, d\mu = \\ &\lim_{t\to\infty} \int_X \Theta \Big(\varphi_1^{(j_1 n_t)}, \varphi_2^{(j_2 n_t^2)}, \ldots, \varphi_N^{(j_N n_t^N)} \Big) \, d\mu. \end{aligned} \tag{3.15}$$

Therefore, by Proposition 3.7, the weak limit of $(U^{j_1 n_t + j_2 n_t^2 + \ldots + j_N n_t^m N})_{t \geq 1}$ (exists and) depends only on

$$\lim_{t\to\infty} \int_X \Theta \Big(\varphi_1^{(j_1 n_t)}, \varphi_2^{(j_2 n_t^2)}, \ldots, \varphi_N^{(j_N n_t^N)} \Big) \, d\mu =$$
$$\lim_{t\to\infty} \int_X \chi_1 \Big(\varphi_1^{(j_1 n_t)} \Big) \cdot \chi_2 \Big(\varphi^{(j_2 n_t^2)} \Big) \cdot \ldots \cdot \chi_N \Big(\varphi_N^{(j_N n_t^N)} \Big) \, d\mu.$$

Another characteristic feature of the forthcoming construction is that for each $r = 1, \ldots, N$ and $i = 1, \ldots, k_r$

$$\mu \left(\left\{ x \in X : \ \varphi_r^{(i n_t^r)}(x) \neq i \varphi_r^{(n_t^r)}(x) \right\} \right) \to 0. \tag{3.16}$$

This implies

$$\begin{aligned} &\lim_{t\to\infty} \int_X \Theta\Big(\varphi_1^{(j_1 n_t + j_2 n_t^2 + \ldots + j_N n_t^N)}, \ldots, \varphi_N^{(j_1 n_t + j_2 n_t^2 + \ldots + j_N n_t^N)} \Big) \, d\mu \\ &= \lim_{t\to\infty} \int_X \chi_1 \otimes \ldots \otimes \chi_N \Big(j_1 \times \ldots \times j_N (\varphi_1^{(n_t)}, \ldots, \varphi_N^{(n_t^N)}) \Big) \, d\mu \end{aligned} \tag{3.17}$$

and since φ_r takes values in $\mathbb{Z}_{k_r}$ (by taking $i = k_r$ in (3.16)), we will have

$$\mu\left(\left\{x \in X :\ \varphi_r^{(k_r n_t^r)}(x) \neq 0\right\}\right) \to 0. \tag{3.18}$$

It follows from (3.18) that

$$\int_X \chi_r\left(\varphi_r^{(k_r n_t^r)}\right)\, d\mu \to 1. \tag{3.19}$$

Moreover, our construction of $\Phi = (\varphi_1, \ldots, \varphi_N)$ will ensure that for each $(j_1, \ldots, j_N) \in \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$, we will have

$$\begin{aligned} &\lim_{t\to\infty} \int_X \chi_1 \otimes \ldots \otimes \chi_N \left(j_1 \times \ldots \times j_N(\varphi_1^{(n_t)}, \ldots, \varphi_N^{(n_t^N)})\right) d\mu \\ &= \begin{cases} 1 & \text{if} \quad (j_1, \ldots, j_N) \in G \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.20}$$

Since for each $i = 1, \ldots, k_r - 1$, $(0, \ldots, 0, \overset{r}{i}, 0, \ldots, 0) \notin G$ (in view of the $(*)$-property, we have the following:

$$\int_X \chi_r\left(\varphi_r^{(i n_t^r)}\right) d\mu \to 0 \ \text{ for } \ r = 1, \ldots, N \text{ and } i = 1, \ldots, k_r - 1. \tag{3.21}$$

The key property of Φ will be that the assumption (3.22) of the lemma below will be satisfied.

Lemma 3.8. *Assume that*

$$\Phi = (\varphi_1, \ldots, \varphi_N) : X \to \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$$

satisfies

$$\left(\varphi_1^{(n_t)}, \ldots, \varphi_N^{(n_t^N)}\right)_* (\mu) \to \lambda_K, \tag{3.22}$$

where $K = G^\perp$. Then (3.20) holds.

Proof. We have

$$\begin{gathered} \left(\chi_1 \otimes \ldots \otimes \chi_N \circ (j_1 \times \ldots \times j_N)\left(\varphi_1^{(n_t)}, \ldots, \varphi_N^{(n_t^N)}\right)\right)_* (\mu) = \\ \left(\chi_1 \otimes \ldots \otimes \chi_N \circ (j_1 \times \ldots \times j_N)\right)_* \left(\left(\varphi_1^{(n_t)}, \ldots, \varphi_N^{(n_t^N)}\right)_* (\mu)\right) \to \\ \left(\chi_1 \otimes \ldots \otimes \chi_N \circ (j_1 \times \ldots \times j_N)\right)_* (\lambda_K). \end{gathered}$$

The measure $\left(\chi_1 \otimes \ldots \otimes \chi_N \circ (j_1 \times \ldots \times j_N)\right)_* (\lambda_K)$ is the Haar measure of a finite group $F \subset \mathbb{S}^1$. By Proposition 3.2, $F = \{1\}$ if and only if $(j_1, \ldots, j_N) \in G$. The result follows from Proposition 3.7 (cf. (3.13)). $\square$

Now, (3.17), (3.21) and (3.19) imply that for $r = 1, \ldots, N$

$$U^{in_t^r} \to 0 \ \text{ for } \ i = 1, \ldots, k_r - 1; \tag{3.23}$$

$$U^{k_r n_t^r} \to Id. \tag{3.24}$$

Moreover, (3.20) implies that

$$U^{j_1 n_t + \ldots + j_N n_t^N} \to \begin{cases} Id & \text{if} \qquad\qquad (j_1, \ldots, j_N) \in G \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

We will always assume that

$$\operatorname{lcm}(k_1, \ldots, k_N) | n_t \ \text{ for all } t \geq t_0 \tag{3.26}$$

(cf. (1.13)). Finally, apply Proposition 1.7 to conclude that (3.23), (3.24) and (3.25) imply the existence of $p \in E(\beta\mathbb{N})$ such that for $r = 1, \ldots, N$

$$p\text{-}\lim_{n \in \mathbb{N}} U^{in^r} = 0 \ \text{ for } \ i = 1, \ldots, k_r - 1, \tag{3.27}$$

$$p\text{-}\lim_{n \in \mathbb{N}} U^{k_r n^r} = Id \tag{3.28}$$

and

$$p\text{-}\lim_{n \in \mathbb{N}} U^{j_1 n + \ldots + j_N n^N} = \begin{cases} Id & \text{if} \qquad\qquad (j_1, \ldots, j_N) \in G \\ 0 & \text{otherwise.} \end{cases} \tag{3.29}$$

Therefore $G = G(p, N, U)$ and the proof of Theorem 3.5 is complete.

3.4 Main construction

Assume that $(n_t)_{t \geq 1}$ is an increasing sequence of natural numbers with $n_t | n_{t+1}$ for $t \geq 1$. In other words

$$n_{t+1} = \rho_{t+1} n_t, \tag{3.30}$$

where $n_0 = 1$ and the natural numbers ρ_{t+1} satisfy $\rho_{t+1} \geq 2$ for $t \geq 0$. If we denote $\rho_0 = 1$ then for each $t \geq 1$, $n_t = \Pi_{i=0}^t \rho_i$. Set

$$X = \Pi_{t=1}^{\infty} \mathbb{Z}_{\rho_t}.$$

We will view X as a compact group (in the product topology) with the group law given by the coordinate addition with carrying the remainder to the right. Let μ denote the normalized Haar measure on X. Denote

$$\overline{1} = (1, 0, 0, \ldots)$$

and notice that X is a monothetic group since the set $\{n \cdot \overline{1} : n \in \mathbb{Z}\}$ is dense in X. We set $Tx = x + \overline{1}$ for $x \in X$ and call T an *odometer* or, more precisely, the (n_t)-*odometer*.

Let $D_0^t = \{x \in X : x_1 = \ldots = x_t = 0\}$. Then the sets $T^i D_0^t =: D_i^t$ for $i = 0, \ldots, n_t - 1$ are pairwise disjoint, $\bigcup_{i=0}^{n_t-1} D_i^t = X$ and $T^{n_t} D_0^t = D_0^t$. In this way we obtain a sequence $\mathcal{D}^t := (D_0^t, \ldots, D_{n_t-1}^t)$, $t \geq 1$, of partitions of X.

Moreover, since $D_0^{t+1} \subset D_0^t$, the partition $\mathcal{D}^{t+1}$ refines $\mathcal{D}^t$, see Figure 1.

For $k = 0, \ldots, \rho_{t+1} - 1$, we denote $C_k^{t+1} = D_{kn_t}^{t+1} \cup \ldots \cup D_{kn_t+n_t-1}^{t+1}$ - the kth column of $\mathcal{D}^t$.

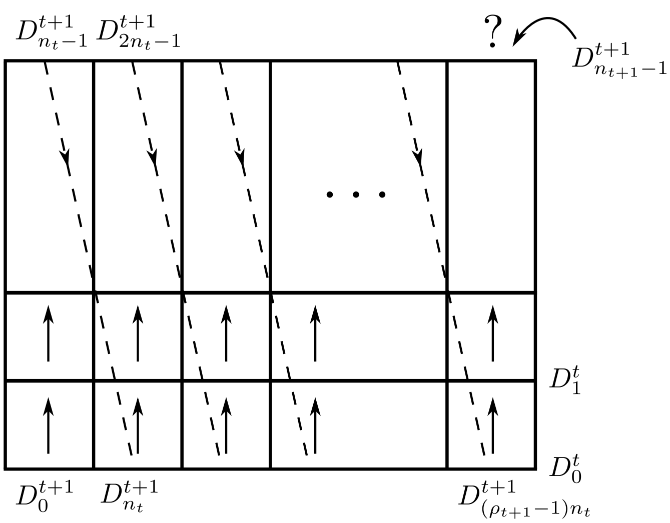

Figure 1: At stage $t + 1$ the dynamics of T is defined on the complement of $D_{n_{t+1}-1}^{t+1}$.

Let H be an Abelian group. Given the (n_t)-odometer T we will be working with so called *Morse cocycles* (see e.g. [21]) $\varphi : X \setminus \{-\overline{1}\} \to H$.

The map φ is determined by *input data*, namely, by a collection of sequences $(b_j^{(t)})_{j=0}^{\rho_t-2}$, $t \geq 1$. To define φ, first, we define auxiliary maps ψ_t defined partially on X (see formulas (3.31) and (3.32) below) with values in H and then we set

$$\varphi(x) = \psi_t(x)$$

for t such that x belongs to the domain of ψ_t (Lemma 3.9 (a) yields the correctness of this definition).

Assume that for any $t \in \mathbb{N}$ we are given a finite sequence $b_0^{(t)}, \dots, b_{\rho_t-2}^{(t)} \in H$. By induction on $t \geq 1$, we will now define:

- a sequence $a_0^{(t)}, \dots, a_{n_t-2}^{(t)} \in H$,
- values $\psi_t(x) \in H$ for $x \in X \setminus D_{n_{t+1}-1}^{t+1}$.

For $t = 1$ we set: $a_j^{(1)} = b_j^{(1)}$ for $j = 0, \dots, \rho_1 - 2$ and

$$\psi_1(x) = \begin{cases} a_i^{(1)} & x \in D_i^1, i = 0, \dots, n_1 - 2 \\ b_j^{(2)} & x \in D_{(j+1)n_1-1}^2,\ j = 0, \dots, \rho_2 - 2. \end{cases} \tag{3.31}$$

If $t \geq 1$ and $a_j^{(t)}$ and ψ_t are already defined, we set:
$a_{jn_t+s}^{(t+1)} = a_s^{(t)}$ for $s = 0, \dots, n_t - 2$, $j = 0, \dots, \rho_{t+1} - 1$,
$a_{jn_t+n_t-1}^{(t+1)} = b_j^{(t+1)}$ for $j = 0, \dots, \rho_{t+1} - 2$,

$$\psi_{t+1}(x) = \begin{cases} a_i^{(t+1)} & x \in D_i^{t+1}, i = 0, \dots, n_{t+1} - 2 \\ b_j^{(t+2)} & x \in D_{(j+1)n_{t+1}-1}^{t+2},\ j = 0, \dots, \rho_{t+2} - 2. \end{cases} \tag{3.32}$$

We denote $\Sigma_t = a_0^{(t)} + \ldots + a_{n_t-2}^{(t)}$ for $t \geq 1$. This construction can be visualized at Figure 2.

Lemma 3.9. *Under the notation above:*
(a) *ψ_t equals the restriction of ψ_{t+1} to $X \setminus D_{n_{t+1}-1}^{t+1}$.*
(b) *Let $i \geq 1$, $0 \leq u \leq \rho_{t+1} - i - 1$. Then*

$$\psi_t^{(in_t)}(x) := \psi_t(x) + \ldots + \psi_t(T^{in_t-1}x) = i\Sigma_t + (b_u^{(t+1)} + \ldots + b_{u+i-1}^{(t+1)})$$

for $x \in C_u^{t+1}$.
(c) *If $b_0^{(t)} + \ldots + b_{\rho_t-2}^{(t)} = 0$, for any $t \geq 1$ then $\Sigma_t = 0$ for any $t \geq 1$.*

Proof. (a) Let $x \in D_k^{t+1}$ for some $k \leq n_{t+1} - 2$. Write $k = jn_t + r$, $0 \leq r \leq n_t - 1$, $0 \leq j \leq \rho_{t+1} - 1$. Then $D_k^{t+1} \subset D_r^t$. If $r \leq n_t - 2$ then

$$\psi_t(x) = a_r^{(t)} = a_{jn_t+r}^{(t+1)} = \psi_{t+1}(x).$$

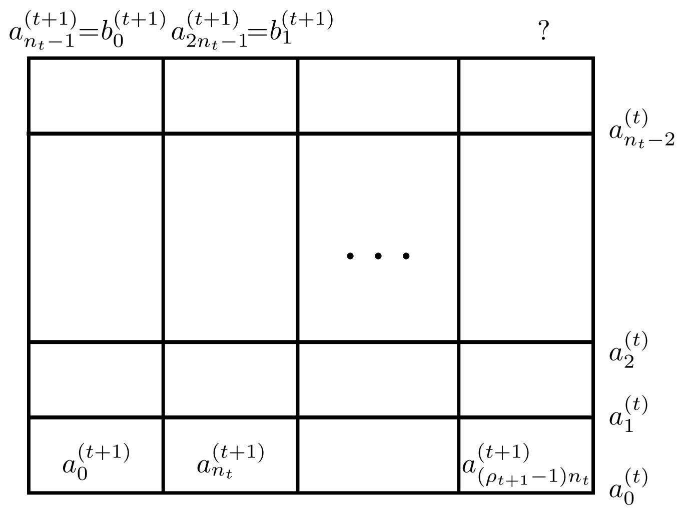

Figure 2: At stage $t+1$ of the construction, we have $a^{(t+1)}_{jn_t+s} = \varphi|_{D^{t+1}_{jn_t+s}} = \varphi|_{D^t_s} = a^{(t)}_s$ for $j = 0, 1, \ldots, \rho_{t+1}-1$ and $s = 0, 1, \ldots, n_t - 2$. We define φ on $D^{t+1}_{jn_t-1}$, $j = 1, 2, \ldots, \rho_{t+1}-1$ by setting $a^{(t+1)}_{jn_t-1} = b^{(t+1)}_{j-1}$; φ remains undefined on $D^{t+1}_{n_{t+1}-1}$.

If $r = n_t - 1$ then $j \leq \rho_{t+1} - 2$ and

$$\psi_t(x) = b^{(t+1)}_j = a^{(t+1)}_{jn_t+n_t-1} = \psi_{t+1}(x).$$

(b) follows directly from the definition of $\psi^{(in_t)}_t(\cdot)$ (see Figure 3 for an explanation how to compute $\psi^{(n_t)}_t(x)$).

(c) follows by induction:

$$\Sigma_{t+1} = a^{(t+1)}_0 + \ldots + a^{(t+1)}_{n_{t+1}-2} =$$
$$\sum_{k=0}^{n_t-2} a^{(t+1)}_k + a^{(t+1)}_{n_t-1} + \sum_{k=0}^{n_t-2} a^{(t+1)}_{k+n_t} + a^{(t+1)}_{2n_t-1} + \ldots + a^{(t+1)}_{(\rho_{t+1}-1)n_t-1} + \sum_{k=0}^{n_t-2} a^{(t+1)}_{k+(\rho_{t+1}-1)n_t} =$$
$$\rho_{t+1}\Sigma_t + a^{(t+1)}_{n_t-1} + a^{(t+1)}_{2n_t-1} + \ldots + a^{(t+1)}_{(\rho_{t+1}-1)n_t-1} = \rho_{t+1}\Sigma_t + b^{(t)}_0 + \ldots + b^{(t)}_{\rho_{t+1}-2}.$$

Recall that $K = G^{\perp}$ is an algebraic coupling of $\mathbb{Z}_{k_1}, \ldots, \mathbb{Z}_{k_N}$. Our aim is to define a Morse cocycle $\Phi = (\varphi_1, ..., \varphi_N) : X \setminus \{-\overline{1}\} \to \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_N}$ so that (3.14), (3.16) and (3.20) are satisfied.

We will additionally assume now that $n_1 > 1$ and

$$\rho_{t+1} = n_t^{N+1}\rho'_{t+1} \tag{3.33}$$

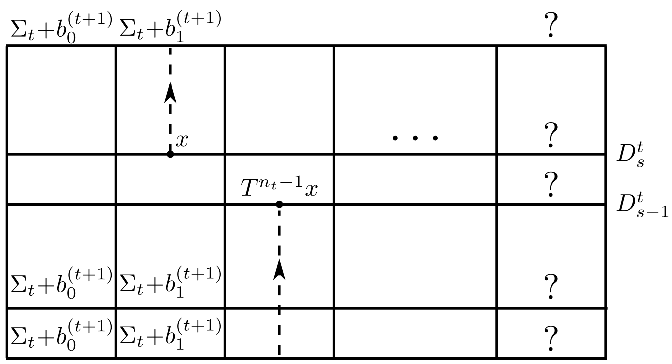

Figure 3: The values taken by $\varphi^{(n_t)}$ viewed at stage $t+1$; $\varphi^{(n_t)}$ is constant on each of the first $\rho_{t+1}-1$ columns of $\mathcal{D}^t$. If $x \in D_{jn_t+s}^{t+1}$ ($j = 0, 1, \dots, \rho_{t+1}-2$, $s = 0, 1, \dots, n_t-1$) then $\varphi^{(n_t)}(x) = \sum_{i=0}^{n_t-1} \varphi(T^i x) = a_s^{(t)} + \dots + a_{n_t-2}^{(t)} + b_j^{(t+1)} + a_0^{(t)} + \dots + a_{s-1}^{(t)} = \Sigma_t + b_j^{(t+1)}$.

for $t \geq 1$ [42].

For each $t \geq 1$ choose a sequence $(c_{1,i,t}, c_{2,i,t}, \dots, c_{N,i,t})_{i=0}^{n_t-1} \subset K$ such that

$$\frac{1}{n_t} \left|\{0 \leq i \leq n_t - 1 : (c_{1,i,t}, \dots, c_{N,i,t}) = (c_1, \dots, c_N)\}\right| \to \frac{1}{|K|}$$

for every $(c_1, \dots, c_N) \in K$.

Given a real number x we denote by $[x]$ the greatest integer less than or equal to x.

We define the input data for determining $\varphi_l : X \setminus \{-\overline{1}\} \to \mathbb{Z}_{k_l}$ to be the sequences $b_{0,l}^{(t)}, \dots, b_{\rho_t-2,l}^{(t)}$ for $t \in \mathbb{N}$ as follows.

For $l > 1$:

$$b_{j,l}^{(t+1)} = \begin{cases} c_{l,s,t} & \text{if } j = sn_t^N \rho'_{t+1} + rn_t^{l-1} \text{ for some } 0 \leq s \leq n_t - 1, 0 \leq r \leq n_t^{N+1-l}\rho'_{t+1} - 1, \\ -n_t c_{l,[\frac{s-1}{n_t^{N-l}\rho'_{t+1}}],t} & \text{if } j = sn_t^l - 1 \text{ for some } 1 \leq s \leq n_t^{N+1-l}\rho'_{t+1} - 1, \\ -n_t c_{l,n_t-1,t} & \text{if } j = n_t^{N+1}\rho'_{t+1} - 2, \\ 0 & \text{otherwise} \end{cases}$$

[42] To proceed with the construction in which Φ satisfies the conditions (3.14), (3.16) and (3.20), it is sufficient to take $\rho_{t+1} = n_t^{N+1}$ for all $t \geq 1$. We introduced the factor ρ'_{t+1} because we need in Section 3.5 the condition that n_t is divisible by a given number for t large enough, see (3.26) and (3.34) below. Setting $\rho'_{t+1} = t$ for all $t \geq 1$ guarantees this condition.

and for $l = 1$:

$$b_{j,1}^{(t+1)} = \begin{cases} c_{1,\left[\frac{j}{n_t^N \rho'_{t+1}}\right],t} & \text{if } n_t \text{ does not divide } j+1 \text{ and } j \neq n_t^{N+1}\rho'_{t+1} - 2, \\ -(n_t - 1)c_{1,\left[\frac{j}{n_t^N \rho'_{t+1}}\right],t} & \text{if } n_t | j+1, \\ -(n_t - 2)c_{1,n_t-1,t} & \text{if } j = n_t^{N+1}\rho'_{t+1} - 2. \end{cases}$$

We give an example of the sequences defined this way in the table below. In this table $N = 3$, $n_t = 3$, $\rho'_{t+1} = 1$. We omit the indices t and $t+1$ and we mark only the nonzero terms. To save space we write c_l^j instead of $c_{l,j,t}$.

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
$b_{j,3}$	c_3^0									c_3^0									c_3^0								$-3c_3^0$
$b_{j,2}$	c_2^0			c_2^0			c_2^0		$-3c_2^0$	c_2^0			c_2^0			c_2^0		$-3c_2^0$	c_2^0			c_2^0			c_2^0		$-3c_2^0$
$b_{j,1}$	c_1^0	c_1^0	$-2c_1^0$	c_1^0	c_1^0	$-2c_1^0$	c_1^0	c_1^0	$-2c_1^0$	c_1^0	c_1^0	$-2c_1^0$	c_1^0	c_1^0	$-2c_1^0$	c_1^0	c_1^0	$-2c_1^0$	c_1^0	c_1^0	$-2c_1^0$	c_1^0	c_1^0	$-2c_1^0$	c_1^0	c_1^0	$-2c_1^0$

j	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
$b_{j,3}$	c_3^1									c_3^1									c_3^1								$-3c_3^1$
$b_{j,2}$	c_2^1			c_2^1			c_2^1		$-3c_2^1$	c_2^1			c_2^1			c_2^1		$-3c_2^1$	c_2^1			c_2^1			c_2^1		$-3c_2^1$
$b_{j,1}$	c_1^1	c_1^1	$-2c_1^1$	c_1^1	c_1^1	$-2c_1^1$	c_1^1	c_1^1	$-2c_1^1$	c_1^1	c_1^1	$-2c_1^1$	c_1^1	c_1^1	$-2c_1^1$	c_1^1	c_1^1	$-2c_1^1$	c_1^1	c_1^1	$-2c_1^1$	c_1^1	c_1^1	$-2c_1^1$	c_1^1	c_1^1	$-2c_1^1$

j	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
$b_{j,3}$	c_3^2									c_3^2									c_3^2							$-3c_3^2$
$b_{j,2}$	c_2^2			c_2^2			c_2^2		$-3c_2^2$	c_2^2			c_2^2			c_2^2		$-3c_2^2$	c_2^2			c_2^2			c_2^2	$-3c_2^2$
$b_{j,1}$	c_1^2	c_1^2	$-2c_1^2$	c_1^2	c_1^2	$-2c_1^2$	c_1^2	c_1^2	$-2c_1^2$	c_1^2	c_1^2	$-2c_1^2$	c_1^2	c_1^2	$-2c_1^2$	c_1^2	c_1^2	$-2c_1^2$	c_1^2	c_1^2	$-2c_1^2$	c_1^2	c_1^2	$-2c_1^2$	c_1^2	$-c_1^2$

In order to obtain the sequences in case $\rho'_{t+1} > 1$ one should repeat each of the three "patterns" above ρ'_{t+1} times. By a pattern we mean a part of the table containing c_l^i with fixed i. Of course, the last pattern has to be treated in a slightly modified way because of the lack of the column $\rho_{t+1} - 1$.

For each $l = 1, \ldots, N$, let $\varphi_l : X \setminus \{-\bar{1}\} \to \mathbb{Z}_{k_l}$ be defined according to the above rule by the input data $(b_{j,l}^{(t)})_{j=0}^{\rho_t - 2}$, $t \geq 1$.

It is clear that the sequences $(b_{j,l}^{(t+1)})_{j=0,\ldots,\rho_{t+1}-2}$ satisfy the assumption of the condition (c) in Lemma 3.9. Thus, thanks to Lemma 3.9 (b), we calculate $\varphi_l^{(in_t^k)}(x)$ for $x \in X$ as follows. Assume that x belongs to the sth column C_s^{t+1} and $s \leq \rho_{t+1} - in_t^{k-1} - 1$. Then, by Lemma 3.9 (b), $\varphi_l^{(in_t^k)}(x) = \sum_{j=s}^{s+in_t^{k-1}-1} b_{j,l}^{(t+1)}$, that is, we sum up the in_t^{k-1} consecutive elements in the lth row of the table (i.e. in the row $b_{j,l}^{(t+1)}$, starting from the column s).

We are mainly interested in the "average" behaviour of $(\varphi_1^{(n_t)}(x), \varphi_2^{(n_t^2)}(x), \ldots, \varphi_N^{(n_t^N)}(x))$. In the following table we present the values of $(\varphi_1^{(n_t)}(x), \varphi_2^{(n_t^2)}(x), \varphi_3^{(n_t^3)}(x))$ in the case considered above. The column marked by j contains the values for $x \in C_j^{t+1}$.

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
$\varphi_3^{(n_t^3)}$	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	c_3^0	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$\varphi_2^{(n_t^2)}$	c_2^0	c_2^0	c_2^0	c_2^0	c_2^0	c_2^0	$\star$	$\star$	$\star$	c_2^0	c_2^0	c_2^0	c_2^0	c_2^0	c_2^0	$\star$	$\star$	$\star$	c_2^0	c_2^0	c_2^0	c_2^0	c_2^0	c_2^0	$\star$	$\star$	$\star$
$\varphi_1^{(n_t)}$	c_1^0	c_1^0	$\star$	c_1^0	c_1^0	$\star$	c_1^0	c_1^0	$\star$	c_1^0	c_1^0	$\star$	c_1^0	c_1^0	$\star$	c_1^0	c_1^0	$\star$	c_1^0	c_1^0	$\star$	c_1^0	c_1^0	$\star$	c_1^0	c_1^0	$\star$

j	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53
$\varphi_3^{(n_t^3)}$	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	c_3^1	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$\varphi_2^{(n_t^2)}$	c_2^1	c_2^1	c_2^1	c_2^1	c_2^1	c_2^1	$\star$	$\star$	$\star$	c_2^1	c_2^1	c_2^1	c_2^1	c_2^1	c_2^1	$\star$	$\star$	$\star$	c_2^1	c_2^1	c_2^1	c_2^1	c_2^1	c_2^1	$\star$	$\star$	$\star$
$\varphi_1^{(n_t)}$	c_1^1	c_1^1	$\star$	c_1^1	c_1^1	$\star$	c_1^1	c_1^1	$\star$	c_1^1	c_1^1	$\star$	c_1^1	c_1^1	$\star$	c_1^1	c_1^1	$\star$	c_1^1	c_1^1	$\star$	c_1^1	c_1^1	$\star$	c_1^1	c_1^1	$\star$

j	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79
$\varphi_3^{(n_t^3)}$	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	c_3^2	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
$\varphi_2^{(n_t^2)}$	c_2^2	c_2^2	c_2^2	c_2^2	c_2^2	c_2^2	$\star$	$\star$	$\star$	c_2^2	c_2^2	c_2^2	c_2^2	c_2^2	c_2^2	$\star$	$\star$	$\star$	c_2^2	c_2^2	c_2^2	c_2^2	c_2^2	$\star$	$\star$	$\star$
$\varphi_1^{(n_t)}$	c_1^2	c_1^2	$\star$	c_1^2	c_1^2	$\star$	c_1^2	c_1^2	$\star$	c_1^2	c_1^2	$\star$	c_1^2	c_1^2	$\star$	c_1^2	c_1^2	$\star$	c_1^2	c_1^2	$\star$	c_1^2	c_1^2	$\star$	c_1^2	$\star$

We do not give explicitly the values in the places marked by $\star$ because the frequency of $\star$'s in a row of the table tends to 0 as $t \to \infty$[43].

Similarly, we analyze the values of $\varphi_k^{(n_t^l)}(x)$ for $k \neq l$ and we note that in such a case the frequency of zeros tends to 1 as $t \to \infty$.

We summarize our construction in the following proposition.

Proposition 3.10. (a) *Given* $l \in \{1, \ldots, N\}$ *and* $i \in \mathbb{N}$,

$$\mu\big(\{x \in X : \ \varphi_l^{(in_t^l)}(x) = i\varphi_l^{(n_t^l)}(x)\}\big) \to 1 \ \ as \ \ t \to \infty.$$

(b) *Given* $\underline{c} \in K$,

$$\mu\big(\{x \in X : \ (\varphi_1^{(n_t)}(x), \varphi_2^{(n_t^2)}(x), \ldots, \varphi_N^{(n_t^N)}(x)) = \underline{c}\}\big) \to \frac{1}{|K|} \ \ as \ \ t \to \infty.$$

(c) *If* $k \neq l$ *then*

$$\mu\big(\{x \in X : \varphi_k^{(n_t^l)}(x) = 0\}\big) \to 1 \ \ as \ \ t \to \infty.$$

$\square$

Thus $\Phi = (\varphi_1, \ldots, \varphi_N)$ constructed above has the desired properties (3.14), (3.16) and (3.20).

[43] This is guaranteed by the condition $n_t^N \rho'_{t+1}/\rho_{t+1} \to 0$, see (3.33).

3.5 How many constructions can be done over the same odometer?

Assume that $N \geq 2$. Suppose that we have a sequence of periods $\underline{k}^{(i)} = (k_1^{(i)}, \ldots, k_N^{(i)})$, $i \geq 1$. Let $K_i \subset \mathbb{Z}_{k_1^{(i)}} \times \ldots \times \mathbb{Z}_{k_N^{(i)}}$ be algebraic couplings and suppose that we want to realize $G_i := K_i^{\perp}$, $i \geq 1$, as (N,p)-periodic rigidity groups for a common $p \in E(\beta\mathbb{N})$ (with the "$\perp$" depending on $i \geq 1$, see Section 3.3). To this end assume that T is the odometer determined by the sequence $(n_t)_{t\geq 1}$ satisfying for each $i \geq 1$ (cf. (3.26))

$$\mathrm{lcm}(k_1^{(i)}, \ldots, k_N^{(i)}) | n_t \text{ for all } t \geq t_i. \tag{3.34}$$

Moreover, we assume that (3.33) holds. Then we see that the construction of the relevant cocycle (taking values in $\mathbb{Z}_{k_1^{(i)}} \times \ldots \times \mathbb{Z}_{k_N^{(i)}}$) in Section 3.4 can be carried out over this fixed T for each $i \geq 1$. This yields a sequence of natural numbers $(n_t)_{t\geq 1}$ and a sequence of unitary operators $(U_i)_{i\geq 1}$ acting on Hilbert spaces $\mathcal{H}_i$, such that for each $i \geq 1$

$$\lim_{t\to\infty} U_i^{j_1 n_t + \ldots + j_N n_t^N} = \begin{cases} Id & \text{if} \qquad (j_1, \ldots, j_N) \in K_i^{\perp} \\ 0 & \text{otherwise.} \end{cases} \tag{3.35}$$

Apply now Part 2 of the proof of Proposition 1.7 to conclude that we can find a subsequence (m_s) of (n_t) such that the convergence (3.35) can be replaced by IP-convergence along $\mathrm{FS}((m_s))$ for each $i \geq 1$. Using Lemma 1.1, we have proved the following result.

Proposition 3.11. *Assume that $N \geq 2$. Assume moreover that $G_i \subset \mathbb{Z}_{k_1^{(i)}} \oplus \ldots \oplus \mathbb{Z}_{k_N^{(i)}}$ is a subgroup satisfying the $(*)$-property, $i \geq 1$. Then there exists $p \in E(\beta\mathbb{N})$ such that G_i is an (N,p)-periodic rigidity group for each $i \geq 1$.*

3.6 N-rigidity groups are subgroups of $\mathcal{P}_N$ of finite index (proof of Theorem E)

Recall from Section 3.1 that N-rigidity groups are preimages of N-periodic rigidity groups. Now, we give a purely algebraic description of rigidity groups as subgroups of $\mathbb{Z}^N$, see also Remark 7.

For $i = 1, \ldots, N$, we denote by e_i the ith standard basis vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ in $\mathbb{Z}^N$. Given a sequence $\underline{k} = (k_1, \ldots, k_N)$ of natural numbers, we denote by

$$\pi_{\underline{k}} : \mathbb{Z}^N \to \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}$$

the canonical projection $\pi_{k_1} \times \ldots \times \pi_{k_N}$. In view of Theorem 3.5 and (3.6), we obtain the following result.

Proposition 3.12. *A subgroup G of $\mathbb{Z}^N$ is an N-rigidity group if and only if there exists a sequence $\underline{k} = (k_1, \ldots, k_N)$ of natural numbers such that:*

(a) *G has the $(*)$-property, that is, for any $i = 1, \ldots, N$ and any element $g = (g_1, \ldots, g_N) \in G$: if $k_j | g_j$ for each $j \neq i$, then $k_i | g_i$,*

(b) *$G = \pi_{\underline{k}}^{-1}(\pi_{\underline{k}}(G))$.*

It follows that the problem of full description of rigidity groups contained in $\mathcal{P}_{\leq N}$ is reduced to the description of subgroups of $\mathbb{Z}^N$ satisfying (a) and (b) in Proposition 3.12.

Proof of Theorem E. If G is an N-rigidity group then, by (b), there is a group isomorphism

$$\mathbb{Z}^N / G \cong (\mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_N}) / \pi_{\underline{k}}(G)$$

which implies that the index of G in $\mathbb{Z}^N$ is finite.

To prove the converse, assume that G has finite index in $\mathbb{Z}^N$ and, for $i = 1, \ldots, N$, let k_i be the smallest natural number such that $k_i e_i \in G$. We show that the conditions (a) and (b) are satisfied with $\underline{k} = (k_1, \ldots, k_N)$.

(a) Assume that $g = (g_1, \ldots, g_N) \in G$ and let $k_j | g_j$ for each $j \neq 1$. It follows that

$$(0, g_2, \ldots, g_N) \in k_1\mathbb{Z} \oplus \ldots \oplus k_N\mathbb{Z} = \ker(\pi_{\underline{k}}).$$

Moreover, $k_1\mathbb{Z} \oplus \ldots \oplus k_N\mathbb{Z} \subset G$ by the definition of $k_1, \ldots, k_N$. Then

$$G \ni g - (0, g_2, \ldots, g_N) = (g_1, 0, \ldots, 0) = g_1 e_1.$$

It follows that k_1 divides g_1 by the choice of k_1. The same argument works when the index 1 is replaced by any other $i \in \{1, \ldots, N\}$.

(b) Let $x \in \pi_{\underline{k}}^{-1}(\pi_{\underline{k}}(G))$. Then $x - g \in \ker(\pi_{\underline{k}})$ for some $g \in G$. But $\ker(\pi_{\underline{k}}) \subset G$, thus $x \in G$.

We have completed the proof of Theorem E. $\square$

Corollary 3.13. *If G is a rigidity subgroup of $\mathbb{Z}^N$ and $\sigma : \mathbb{Z}^N \to \mathbb{Z}^N$ is an injective $\mathbb{Z}$-endomorphism then $\sigma(G)$ is also a rigidity subgroup of $\mathbb{Z}^N$. In particular, this holds if σ is a $\mathbb{Z}$-automorphism of $\mathbb{Z}^N$.*

Corollary 3.14. *An intersection of finitely many rigidity subgroups of $\mathbb{Z}^N$ is a rigidity subgroup. A subgroup H of a rigidity group G is a rigidity group if and only if H is of finite index in G.*

We now pass to the proof of Corollary F. We start with a "separation" lemma.

Lemma 3.15. *Assume that H is a subgroup of $\mathbb{Z}^N$ and $Q_1, \ldots, Q_t \in \mathbb{Z}^N \setminus H$. Then there exists a rigidity subgroup $G \subset \mathbb{Z}^N$ such that $H \subset G$ and $Q_1, \ldots, Q_t \notin G$.*

Proof. In view of Corollary 3.14, we can assume that $t = 1$. Let $P_1, \ldots, P_s$ be a $\mathbb{Z}$-basis of H. Then $s \leq N$ and there exist $P_{s+1}, \ldots, P_N \in \mathbb{Z}^N$ such that the set $\{P_1, \ldots, P_N\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}^N$. This means that $P_1, \ldots, P_N$ are independent over $\mathbb{Q}$ (and over $\mathbb{Z}$ as well) and the subgroup generated by $P_1, \ldots, P_N$ has finite index in $\mathbb{Z}^N$.

For $M \in \mathbb{N}$ we let $G^{(M)}$ denote the group generated by

$$P_1, \ldots, P_s, MP_{s+1}, \ldots, MP_N.$$

Clearly, each $G^{(M)}$ has finite index in $\mathbb{Z}^N$, thus it is a rigidity subgroup, and contains H. It is enough to prove that there exists M such that $Q_1 \notin G^{(M)}$.

Suppose that $Q_1 \in G^{(1)}$. Then there exist $a_1, \ldots, a_N \in \mathbb{Z}$ such that

$$Q_1 = a_1P_1 + \ldots + a_sP_s + a_{s+1}P_{s+1} + \ldots + a_NP_N.$$

Since $Q_1 \notin H$, at least one of $a_{s+1}, \ldots, a_N$ is nonzero.

Let $\overline{M} \in \mathbb{N}$ be a number greater than the maximum of the absolute values of $a_{s+1}, ..., a_N$. If $Q_1 \in G^{(\overline{M})}$ then there exist $b_1, ..., b_N \in \mathbb{Z}$ such that

$$Q_1 = b_1P_1 + \ldots + b_sP_s + b_{s+1}\overline{M}P_{s+1} + \ldots + b_N\overline{M}P_N.$$

By the choice of $\overline{M}$, the sequences $(a_{s+1}, \ldots, a_N)$ and $(b_{s+1}\overline{M}, \ldots, b_N\overline{M})$ are different and we get a contradiction with the independence of $P_1, \ldots, P_N$. ☐

Proof of Corollary F. Either apply Lemma 3.15 to $H = \mathbb{Z}P_1 + \ldots + \mathbb{Z}P_s$ or observe that $P_{s+1}, \ldots, P_N$ do not belong to the subgroup generated by $P_1, \ldots, P_s, 2P_{s+1}, \ldots, 2P_N$ and this subgroup has finite index in $\mathbb{Z}^N$. Now, apply Theorem E. ☐

3.7 Every finitely generated group of polynomials is a group of global rigidity (proof of Theorem G)

Assume that $G \subset \mathcal{P}_{\leq N}$ is an arbitrary subgroup such that in G we can find a polynomial of degree N.

Lemma 3.16. *There exists a sequence $G_i \subset \mathcal{P}_{\leq N}$, $i \geq 1$, of subgroups of finite index in $\mathcal{P}_{\leq N}$ such that $G = \bigcap_{i \geq 1} G_i$.*

Proof. Let us write $\mathcal{P}_{\leq N} \setminus G = \{R_1, R_2, \ldots\}$. Using Lemma 3.15, for each $i \geq 1$, we can find $G_i \subset \mathcal{P}_{\leq N}$ of finite index such that $G \subset G_i$ and $R_i \notin G_i$. Clearly, $G = \bigcap_{i \geq 1} G_i$. $\square$

Assume that $\underline{k}^{(i)} = (k_1^{(i)}, \ldots, k_N^{(i)})$ is the period of G_i, see Section 3.1. Then apply Proposition 3.11 (to $\pi_{\underline{k}^{(i)}}(G_i)$) to obtain that there exist $p \in E(\beta\mathbb{N})$ and $U_i \in \mathcal{U}(\mathcal{H}_i)$, $i \geq 1$, such that

$$p\text{-}\lim_{n \in \mathbb{N}} U_i^{P(n)} = Id$$

for $P \in G_i$ and $p\text{-}\lim_{n \in \mathbb{N}} U_i^{R(n)} = 0$ for the remaining $R \in \mathcal{P}_{\leq N} \setminus G_i$, $i \geq 1$. Set $U = U_1 \oplus U_2 \oplus \ldots$ We have

$$p\text{-}\lim_{n \in \mathbb{N}} U^{P(n)} = Id \text{ if and only if } P \in \bigcap_{i \geq 1} G_i. \tag{3.36}$$

Taking into account Lemma 3.16 and (3.36) we have completed the proof of Theorem G.

References

[1] J. Aaronson, M. Hosseini, M. Lemańczyk, *IP-rigidity and eigenvalue groups*, Ergodic Theory Dynam. Systems, published online doi:10.1017/etds2012.190.

[2] T. Adams, *Tower multiplexing and slow weak mixing*, arXiv 1301.0791.

[3] V. Bergelson, *Weakly mixing PET*, Ergodic Theory Dynam. Systems **7** (1987), 337-349.

[4] V. Bergelson, *Ergodic Ramsey Theory – an Update*, Ergodic Theory of $\mathbb{Z}^d$-actions, London Math. Soc., Lecture Notes **228** (1996), 1-61.

[5] V. Bergelson, *Minimal idempotents and ergodic Ramsey theory*, Topics in Dynamics and Ergodic Theory 8-39, London Math. Soc. Lecture Notes Series **310**, Cambridge Univ. Press, 2003.

[6] V. Bergelson, *Ultrafilters, IP sets, Dynamics, and Combinatorial Number Theory*, Contemporary Math. **530** (2010), 23-47.

[7] V. Bergelson, A. del Junco, M. Lemańczyk, J. Rosenblatt, *Rigidity and non-recurrence along sequences*, Ergodic Theory Dynam. Systems, published online doi: 10.1017/etds2013.

[8] V. Bergelson, H. Furstenberg, R. McCutcheon, *IP sets and polynomial recurrence*, Ergodoc Theory Dynam. Systems **16** (1996), 963-974.

[9] V. Bergelson, A. Gorodnik, *Weakly mixing group actions: a brief survey and an example*, Modern Dynamical Systems and Applications, 3-25, Cambridge Univ. Press, Cambridge 2004.

[10] V. Bergelson, A. Leibman, *Polynomial extensions of van der Waerden's and Szemerédi's theorems*, J. Amer. Math. Soc. **9** (1996), 725-753.

[11] V. Bergelson, R. McCutcheon, *Uniformity in polynomial Szemerédi theorem*, Ergodic theory of $\mathbb{Z}^d$-actions (Warwick 1993-1994), 273-296, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 1996.

[12] V. Bergelson, R. McCutcheon, *An ergodic IP polynomial Szemerédi theorem*, Memoirs Amer. Math. Soc. **146**, 2000.

[13] V. Bergelson, R. McCutcheon, *Central sets and a non-commutative Roth theorem*, Amer. J. Math. **129** (2007), 1251-1275.

[14] R. Ellis, *Distal transformations groups*, Pacific J. Math. **8** (1958), 401-405.

[15] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, New Jersey, 1981.

[16] H. Furstenberg, Y. Katznelson, *An ergodic Szemerédi theorem for IP-systems and combinatorial theory*, J. Anal. Math. **45** (1985), 117-168.

[17] H. Furstenberg, B. Weiss, *The finite multipliers of infinite ergodic transformations*, Lecture Notes in Math. **668** (1978), 127-132.

[18] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs **101**, AMS, Providence, RI, 2003.

[19] S. Grivaux, *IP-Dirichlet measures and IP-rigid dynamical systems: an approach via generalized Riesz products*, Studia Math. **215** (2013), 237-259.

[20] S. Grivaux, T. Eisner, *Hilbertian Jamison sequences and rigid dynamical systems*, J. Functional Analysis **261** (2011), 2013-2052.

[21] M. Guenais, *Morse cocycles and simple Lebesgue spectrum*, Ergodic Theory Dynam. Systems **19** (1999), 437–446.

[22] P.R. Halmos, *Lectures on Ergodic Theory*, Chelsea 1956.

[23] H. Helson, *Cocycles on the circle*, J. Operator Theory **16** (1986), 189-199.

[24] N. Hindman, *The existence of certain ultrafilters on* $\mathbb{N}$ *and a conjecture of Graham and Rothschild*, Proc. Amer. Math. Soc. **36** (1972), 341-346.

[25] N. Hindman, *Finite sums from sequences within cells of a partion of* $\mathbb{N}$, J. Comb. Theory Ser. A. **17** (1974), 1-11.

[26] E. Hopf, *Ergodentheorie*, Springer, 1937.

[27] A.Y. Khintchine, *Eine Verschärfung des Poincaréschen Wiederkehrsatzes*, Comp. Math. **1** (1934), 177-179.

[28] B.O. Koopman, J. von Neumann, *Dynamical systems of continuous spectra*, Proc. Nat. Acad. Sci. USA **18** (1932), 255-263.

[29] J.D. Lawson, *Joint continuity in semitopological semigroups*, Illinois J.Math. **18** (1974), 275-285.

[30] U. Krengel, *Ergodic Theorems*, Walter de Gryuter & Co., Berlin, 1985.

[31] M.G. Nadkarni, *Spectral Theory of Dynamical Systems*, Hindustan Book Agency, New Delhi, 1998.

[32] W. Parry, *Topics in Ergodic Theory*, Cambridge Tracts in Mathematics, 75 Cambridge University Press, Cambridge-New York, 1981.

[33] K. Petersen, *Ergodic Theory*, Cambridge Univ. Press, Cambridge 1981.

[34] W. Rudin, *Fourier Analysis on Groups*, 1962.

[35] K. Schmidt, P. Walters, *Mildly mixing actions of locally compact groups*, Proc. London Math. Soc. **45** (1982), 506-518.

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