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# Approximately transitive dynamical systems and simple spectrum

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**Abstract.** For some countable discrete torsion Abelian groups we give examples of their finite measure-preserving actions which have simple spectrum and no approximate transitivity property.

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**1. Introduction.** The property of approximate transitivity (AT) has been introduced to ergodic theory by Connes and Woods [5] in 1985 in connection with their measure-theoretic characterization of ITPFI hyperfinite factors in the theory of von Neumann algebras. Since then, approximately transitive dynamical systems have been objects of study in several papers, e.g., [1,9,12,13,15,16,22]. All systems with the AT property have zero entropy [5,8] and it is only recently [1,9] that explicit examples of zero entropy systems without the AT property have been discovered.

In view of the form of its definition (see below) the AT property seems to be related to spectral properties, more precisely to the spectral multiplicity of the measure-preserving system. Indeed, it was in the 1990's when J.-P. Thouvenot, using some generic type arguments, observed that the AT property implies the existence of a cyclic vector in the $L^1$-space of the underlying dynamical system. Moreover, a modification of the definition of the AT property (in which we replace $L^1$ norms by $L^2$ norms [16]) implies simplicity of the $L^2$ spectrum. However, in general it is still an interesting open problem whether the original AT property implies simplicity of the $L^2$ spectrum,

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i.e., simplicity of the spectrum of the classical Koopman representation of the dynamical system. Moreover, as noted in [9], the other implication was not known. The aim of this note is to answer this latter question—we give examples of systems which are not AT but have simple $L^2$ spectrum.

One further motivation for this note is the problem of a measure-theoretic characterization of the simplicity of the $L^2$ spectrum. This problem has a long history in ergodic theory, see for example [20,21] and the references therein, and our examples also give the answer to a further open question (stated explicitly in case of $\mathbb{Z}$-actions in [11] as well as in [9]): namely we give a negative answer to the question of whether the class of funny rank one systems [10,19] coincides with the class of systems with simple $L^2$ spectrum; indeed a funny rank one system enjoys the AT property (see below).

We would like however to emphasize that in the note we do not consider $\mathbb{Z}$-actions which are the most popular objects of study in ergodic theory, and therefore the problems we mentioned above remain open for the class of $\mathbb{Z}$-actions (and actions of many other groups). The systems which are considered below are actions of countable discrete Abelian groups with torsion. More precisely $G = \bigoplus_{n=0}^{\infty} \mathbb{Z}/p_n\mathbb{Z}$ where $(p_n)_{n\geq 0}$ is an increasing sequence of prime numbers. We will then deal with the so-called Morse extensions given by some special $\mathbb{Z}/2\mathbb{Z}$-valued cocycles over a discrete spectrum action of $G$ studied by M. Guenais in [14] (in that paper she proved that the resulting $G$-actions have Haar component of multiplicity one in the $L^2$ spectrum). All such systems have simple spectrum. We will show that for each sufficiently fast choice of parameters in Guenais' construction we can apply the criterion for being a non-AT system formulated in [1] (this criterion is an elaborated version of the method already used in [9]). More precisely, we will prove the following.

**Theorem 1.1.** *Assume that $G = \bigoplus_{n=0}^{\infty} \mathbb{Z}/p_n\mathbb{Z}$ with $(p_n)_{n\geq 0}$ an increasing sequence of prime numbers and $p_n \geq 5^{2(n+1)}, n \geq 0$. Then there exists a simple $L^2$ spectrum action of $G$ without the AT property.*

Following Connes and Woods [5] we now recall the definition of the AT property.

Let $G$ be a countable discrete Abelian group. Assume that this group acts as measure preserving maps: $g \mapsto T_g \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, where $(X, \mathcal{B}, \mu)$ is a standard probability Borel space. The action $(X, \mathcal{B}, \mu, T)$ with $T = (T_g)_{g\in G}$ (or simply $T$) is called AT if for an arbitrary family of nonnegative functions $f_1, \ldots, f_l \in L^1_+(X, \mathcal{B}, \mu), l \geq 2$, and any $\varepsilon > 0$, there exist a positive integer $s$, elements $g_1, \ldots, g_s \in G$, real numbers $\lambda_{j,k} > 0, j = 1, \ldots, l, k = 1, \ldots, s$ and $f \in L^1_+(X, \mathcal{B}, \mu)$ such that

$$\left\| f_j - \sum_{k=1}^{s} \lambda_{j,k} f \circ T_{g_k} \right\|_1 < \varepsilon, \quad 1 \leq j \leq l. \tag{1.1}$$

Recall now that $T$ has the *funny rank one* property [10,19] if for every $A \in \mathcal{B}$ and $\varepsilon > 0$ we can find $F \in \mathcal{B}, g_1, \ldots, g_N \in G$ such that the family $\mathcal{R} = \{T_{g_1}F, \ldots, T_{g_N}F\}$, called a *funny Rokhlin tower*, consists of sets which are disjoint and for some $J \subset \{1, \ldots, N\}$

$$\mu\left(A\triangle\bigcup_{i\in J}T_{g_i}F\right)<\varepsilon. \tag{1.2}$$

Alternatively, $T$ is of funny rank one if and only if there exists a sequence $(\mathcal{R}_n)_{n\geq 1}$ of funny towers with bases $F_n$ such that each set $A\in\mathcal{B}$ can be $\varepsilon$-approximated [as in (1.2)] by the union of some levels for each $\mathcal{R}_n$ whenever $n\geq n_\varepsilon$. From this it easily follows that each system which is of funny rank one is AT. Indeed, the set of functions which are constant on levels of $\mathcal{R}_n$ for some $n\geq 1$ (and zero outside $\bigcup\mathcal{R}_n$) are dense in $L^1$, and the set of those which are additionally positive is dense in $L^1_+$; it follows that given $f_1,\ldots,f_l\in L^1_+(X,\mu)$ and $\varepsilon>0$ it is enough to take $f=\mathbb{1}_{F_n}$ [in (1.1)] for $n\geq 1$ large enough.

The action $T$ of $G$ on $(X,\mathcal{B},\mu)$ induces a (continuous) unitary representation, called a *Koopman representation*, of $G$ in the space $L^2(X,\mathcal{B},\mu)$ given by $U_{T_g}f=f\circ T_g$, $f\in L^2(X,\mathcal{B},\mu)$ and $g\in G$. We recall that a Koopman representation is said to have *simple spectrum* if for some $f\in L^2(X,\mathcal{B},\mu)$, $L^2(X,\mathcal{B},\mu)=\overline{\operatorname{span}}\{f\circ T_g:\ g\in G\}$. Given $f\in L^2(X,\mu)$ we define its *spectral measure* $\sigma_f$ (or in a more precise notation, $\sigma_{U_T,f}$) to be a finite Borel measure on the dual $\widehat{G}$ of $G$ determined by

$$\widehat{\sigma}_{U_T,f}(g)=\widehat{\sigma}_f(g):=\int\limits_{\widehat{G}}\chi(g)\,d\sigma_f(\chi)=\int\limits_X f\circ T_g\cdot\overline{f}\,d\mu,\ \text{ for all } g\in G.$$

See [20, 21] for more information on the spectral theory of $G$-actions.

**2. A criterion for a system to be non-AT.** Let $G$ be a countable discrete Abelian group which we assume to act on a probability standard Borel space $(X,\mathcal{B},\mu)$ as measure-preserving maps: $g\mapsto T_g$. Assume that $\mathcal{P}=\{P_0,P_1\}$ is a partition of $X$ (with $P_0\in\mathcal{B}$). Through its $\mathcal{P}$-names every point $x\in X$ can be now coded: $x\mapsto\pi(x)=(x_g)_{g\in G}$ where

$$x_g=\begin{cases}0 & \text{if } T_g(x)\in P_0\\ 1 & \text{if not.}\end{cases}$$

Let $\Lambda$ be a finite subset of $G$. By a *funny word on the alphabet* $\{0,1\}$ *based on* $\Lambda$ we mean a sequence $(W_g)_{g\in\Lambda}$ with $W_g\in\{0,1\}$, $g\in\Lambda$. For any two funny words $W,W'$ based on the same set $\Lambda\subset G$ their *Hamming distance* is given by

$$\overline{d}_\Lambda(W,W')=\frac{1}{|\Lambda|}|\{g\in\Lambda:\ W_g\neq W'_g\}|.$$

As noted in [1] we have the following extension of Dooley–Quas' [9] necessary condition for a system $T=(T_g)_{g\in G}$ to have the AT property.

**Proposition 2.1.** [9] *Let $(X,\mathcal{B},\mu,T)$ be an AT dynamical system. Then for any $\varepsilon>0$ there exist a finite set $\Lambda\subset G$ and a funny word $W$ based on $\Lambda$ such that*

$$|\Lambda|\mu(\{x\in X:\ \overline{d}_\Lambda(\pi(x)|_\Lambda,W)<\varepsilon\})>1-\varepsilon.$$

The contrapositive of Proposition 2.1 gives a criterion for a system to be non-AT. Some further work has been done in [1] to formulate a condition stronger than the negation of the assertion in Proposition 2.1 and which may be applied to many systems (the criterion obtained this way is an elaborated version of the method already used in [9]). We now present this criterion (Proposition 2.2 below) in its generality needed for this note.

A probability Borel measure $\rho$ defined on $\widehat{G}$ is called a *strong Blum–Hanson measure* (SBH measure shortly) if the following holds

$$\limsup_{k\to+\infty} \sup_{\Theta\subset G, |\Theta|=k, (\eta_\theta)_{\theta\in\Theta}\in\{1,-1\}^k} \int_{\widehat{G}} \left| \frac{1}{\sqrt{k}} \sum_{\theta\in\Theta} \eta_\theta \cdot \chi(\theta) \right|^2 d\rho(\chi) < 1+\varepsilon_0,$$

for some $\varepsilon_0$ in $[0,1)$. Clearly the Haar measure $m_{\widehat{G}}$ of $\widehat{G}$ is an SBH measure; more generally, every absolutely continuous probability measure with density $d \in L^1(\widehat{G}, m_{\widehat{G}})$ satisfying $\|d\|_\infty < 1+\varepsilon_0$ for some $\varepsilon_0 \in [0,1)$, is an SBH measure. It is shown in [1] that each SBH measure is a Rajchman measure (i.e., its Fourier transform vanishes at infinity [4]).

**Proposition 2.2.** *Assume that a dynamical system $(X, \mathcal{B}, \mu, T)$ is ergodic and that there exists a partition $\mathcal{P} = \{P_0, P_1\}$ with the following properties:*

(i) *There exists $S \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ commuting with all elements $T_g, g \in G$, such that $SP_0 = P_1$.*

(ii) *The spectral measure $\sigma_{\mathbb{1}_{P_0} - \mathbb{1}_{P_1}}$ is an SBH measure.*

*Then the system $T = (T_g)_{g\in G}$ is not AT.*

*Proof.* (The proof is similar to the proof of Proposition 3.4 from [1]; we provide it for sake of completeness.) Let $W$ be a funny word based on a subset $\Theta = \{g_1, g_2, \dots, g_k\} \subset G$ and define $\Lambda_W$ on $X$ by

$$\Lambda_W(x) = \frac{1}{k} \sum_{j=1}^{k} A_j^W(x), \quad x \in X$$

where $A_j^W$ is defined as

$$A_j^W(x) = \begin{cases} 1 & \text{if } W_{g_j} = x_{g_j} \\ -1 & \text{if not} \end{cases}$$

(recall that $(x_g) = \pi(x)$ is the $\mathcal{P}$-name of $x$). We claim that the distribution $\Lambda_W^*$ of $\Lambda_W$ is symmetric. Indeed, we have $\pi(x) = 1 - \pi(Sx)$ and therefore

$$\Lambda_W(x) = -\Lambda_W(Sx)$$

and since $S$ is measure-preserving the symmetry of $\Lambda_W^*$ follows.
Observe that

$$A_j^W(x) = (-1)^{W_{g_j}} (\mathbb{1}_{P_0} - \mathbb{1}_{P_1})(T_{g_j}x) \tag{2.1}$$

and that

$$\Lambda_W(x) = 1 - 2\bar{d}_\Theta(W, \pi(x)|_\Theta). \tag{2.2}$$

In view of (2.2), the symmetry of $\Lambda_W^*$ and the Tchebychev inequality, for $0 < \varepsilon < 1/2$, we obtain that

$$\begin{aligned}\mu(\{x \in X :\ \overline{d}_\Theta(W, \pi(x)|_\Theta) < \varepsilon\}) &= \mu(\{x \in X :\ \Lambda_W(x) > 1 - 2\varepsilon\}) \\ &= \frac{1}{2}\mu(\{x \in X :\ |\Lambda_W(x)| > 1 - 2\varepsilon\}) \\ &\leq \frac{1}{2(1-2\varepsilon)^2}||\Lambda_W||_2^2. \end{aligned} \tag{2.3}$$

But, in view of (2.1) and the Spectral Theorem

$$\begin{aligned}\|\Lambda_W\|_2^2 &= \int_X \left|\frac{1}{k}\sum_{j=1}^{k} A_j^W\right|^2 d\mu \\ &= \frac{1}{k^2}\sum_{i,j=1}^{k}\int_X (-1)^{W_{g_i}}\left(\mathbb{1}_{P_0} - \mathbb{1}_{P_1}\right)(T_{g_i}x) \\ &\quad \cdot (-1)^{W_{g_j}}\left(\mathbb{1}_{P_0} - \mathbb{1}_{P_1}\right)(T_{g_j}x)\, d\mu(x) \\ &= \frac{1}{k^2}\sum_{i,j=1}^{k}(-1)^{W_{g_i}+W_{g_j}}\hat{\sigma}_{\mathbb{1}_{P_0}-\mathbb{1}_{P_1}}(g_i - g_j) \\ &= \frac{1}{k}\int_{\widehat{G}}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{k}(-1)^{W_{g_i}}\chi(g_i)\right|^2 d\sigma_{\mathbb{1}_{P_0}-\mathbb{1}_{P_1}}(\chi).\end{aligned}$$

It follows that for $k$ large enough we have

$$\|\Lambda_W\|_2^2 < \frac{1}{k}(1 + \varepsilon_0). \tag{2.4}$$

Combining (2.3) and (2.4) we obtain that

$$k\ \mu(\{x \in X :\ \overline{d}_\Theta(W, \pi(x)|_\Theta) < \varepsilon\}) \leq \frac{1+\varepsilon_0}{2(1-2\varepsilon)^2}$$

and since $\frac{1+\varepsilon_0}{2(1-2\varepsilon)^2} < 1 - \varepsilon$ for $\varepsilon > 0$ small enough and $W$ was arbitrary, this contradicts Proposition 2.1. □

**3. Countable discrete Abelian group action with simple spectrum and without the AT property.** In this section we shall present the proof of Theorem 1.1.

**3.1. Cocycles for discrete spectrum actions of countable discrete Abelian groups.** For completeness, we now briefly present an extension of the classical spectral theory of compact group extensions of $\mathbb{Z}$-actions to group actions.

Let $G$ be a countable discrete Abelian group acting on a standard probability Borel space $(X, \mathcal{B}, \mu)$ as measure-preserving maps: $g \mapsto T_g \in \mathrm{Aut}(X, \mathcal{B}, \mu)$. Assume that $\Gamma$ is a compact metric Abelian group (written multiplicatively) with Haar measure $m_\Gamma$. A *cocycle* associated to $T = (T_g)_{g\in G}$ with values in $\Gamma$ is a measurable function $\varphi : X \times G \to \Gamma$ which satisfies

$$\varphi(x, g + g') = \varphi(x, g)\varphi(T_g x, g') \tag{3.1}$$

for a.e. $x \in X$ and all $g, g' \in G$. We will also write $\varphi_g(x)$ instead of $\varphi(x, g)$. Then the $\Gamma$-*extension of* $T$ *associated to* $\varphi$ is the $G$-action $T_\varphi = ((T_\varphi)_g)_{g\in G}$ defined on $(X \times \Gamma, \mu \otimes m_\Gamma)$ by

$$(T_\varphi)_g : X \times \Gamma \to X \times \Gamma, \ \ (x, \gamma) \mapsto (T_g x, \varphi_g(x)\gamma).$$

The space $L^2(X \times \Gamma, \mu \otimes m_\Gamma)$ can be decomposed as

$$L^2(X \times \Gamma, \mu \otimes m_\Gamma) = \bigoplus_{\chi\in\widehat{\Gamma}} L_\chi, \tag{3.2}$$

where each of the subspaces $L_\chi := \{f \bigotimes \chi : f \in L^2(X, \mu)\}$ is $(U_{T_\varphi})_g$-invariant, and the restriction of the Koopman representation $U_{T_\varphi}$ of $G$ to $L_\chi$ is, via the map

$$f \otimes \chi \mapsto f, \ f \in L^2(X, \mu), \tag{3.3}$$

unitarily equivalent to the $G$-representation $V_{\varphi,T,\chi} = (V_{\varphi,T,\chi,g})_{g\in G}$ given by

$$V_{\varphi,T,\chi,g} : L^2(X, \mu) \to L^2(X, \mu), \ \ V_{\varphi,T,\chi,g}(f)(x) = \chi(\varphi_g(x))f(T_g x), x \in X.$$

It follows that the spectral properties of $T_\varphi$ are determined by the spectral properties of $V_{\varphi,T,\chi}, \chi \in \widehat{\Gamma}$.

Assume now that $\Gamma = \mathbb{Z}_2 = \{1, -1\}$. Set

$$S : X \times \mathbb{Z}_2 \to X \times \mathbb{Z}_2, \ \ S(x, \varepsilon) = (x, -\varepsilon)$$

and notice that $S$ commutes with all automorphisms $(T_\varphi)_g, g \in G$, i.e., $S$ is in the centralizer of $T_\varphi$. In this case the decomposition (3.2) is given by $L^2(X \times \mathbb{Z}_2, \mu \otimes m_{\mathbb{Z}_2}) = L_0 \oplus L_1$ where $L_0 = \{F \in L^2(X \times \mathbb{Z}_2, \mu \otimes m_{\mathbb{Z}_2}) : F \circ S = F\}$ and $L_1 = \{F \in L^2(X \times \mathbb{Z}_2, \mu \otimes m_{\mathbb{Z}_2}) : \ F \circ S = -F\}$. Thus the restriction of the Koopman representation $U_{T_\varphi}$ on $L_1$ is unitarily equivalent to the representation $V = (V_g)_{g\in G}$ on $L^2(X, \mu)$ given by

$$(V_g(f))(x) = \varphi_g(x)f(T_g x), \quad \text{for all } x \in X, \ g \in G.$$

Let $\mathcal{P} = \{P_0, P_1\}$ be the partition of $X \times \mathbb{Z}_2$ given by $P_0 = X \times \{1\}$, $P_1 = X \times \{-1\}$. Then $\mathbb{1}_{P_0} - \mathbb{1}_{P_1} \in L_1$, in fact

$$\mathbb{1}_{P_0} - \mathbb{1}_{P_1} = \mathbb{1}_X \otimes \left(\mathbb{1}_{\{1\}} - \mathbb{1}_{\{-1\}}\right) = \mathbb{1}_X \otimes \chi$$

where $\chi$ is the only non-trivial character of $\mathbb{Z}_2$; in particular $\|\mathbb{1}_{P_0} - \mathbb{1}_{P_1}\|_{L^2} = 1$, so its spectral measure is a probability measure. In view of (3.3) we have

$$\begin{aligned}\widehat{\sigma}_{U_{T_\varphi},\mathbb{1}_{P_0}-\mathbb{1}_{P_1}}(g) &= \widehat{\sigma}_{V_{\varphi,T,\chi},\mathbb{1}_X}(g)\\ &= \langle V_{\varphi,T,\chi,g}\mathbb{1}_X, \mathbb{1}_X\rangle_{L^2(X,\mu)} = \int_X \chi(\varphi_g(x))\, d\mu(x)\\ &= \int_X \varphi_g(x)\, d\mu(x).\end{aligned}$$

Moreover, $SP_0 = P_1$. In this way we have proved the following.

**Lemma 3.1.** *For any ergodic* $G$*-action* $T = (T_g)_{g\in G}$ *and its* $\mathbb{Z}_2$*-extension* $T_\varphi$ *for the partition* $\mathcal{P} = \{P_0, P_1\}$ *defined above we have*:

(i) *The element* $S$ *defined above is in* $\mathrm{Aut}(X \times \mathbb{Z}_2, \mu \otimes m_{\mathbb{Z}_2})$ *and belongs to the centralizer of* $U_{T_\varphi}$.
(ii) *The spectral measure* $\sigma_{\mathbb{1}_{P_0}-\mathbb{1}_{P_1}} = \sigma_{U_{T_\varphi},\mathbb{1}_{P_0}-\mathbb{1}_{P_1}}$ *satisfies*

$$\widehat{\sigma}_{\mathbb{1}_{P_0}-\mathbb{1}_{P_1}}(g) = \int_X \varphi_g(x)\, d\mu(x) \ \text{ for all } g \in G. \tag{3.4}$$

Comparing the above lemma with Proposition 2.2 we can see that the only thing which is missing is the fact that in general the measure $\sigma_0 = \sigma_{U_{T_\varphi},\mathbb{1}_{P_0}-\mathbb{1}_{P_1}}$ is not SBH, and to achieve non-AT property we will have to control the Fourier transform of $\sigma_0$ to obtain absolute continuity and a relevant boundedness of the density.

Recall also that if in addition the spectrum of the $G$-action $T$ is discrete then by an obvious extension of Helson's analysis from [17] we obtain that the spectral type of $V$ enjoys the purity law: it is either discrete, or continuous and purely singular, or equivalent to Haar measure $m_{\widehat{G}}$.

**3.2. Morse cocycles and Guenais constructions.** We now present Guenais construction [14]. Assume that $(p_n)_{n\geq 0}$ is an increasing sequence of prime numbers and let $G = \oplus_{n\geq 0}\mathbb{Z}/p_n\mathbb{Z}$. Consider the action of $G$ on its dual $X = \prod_{n=0}^{+\infty}\mathbb{Z}/p_n\mathbb{Z}$ by translations $T_g x = x + g$, where the space $X$ is equipped with Haar measure $\mu = m_X$ (which is the product of counting measures $m_{\mathbb{Z}/p_n\mathbb{Z}}$ on coordinates). The resulting $G$-action has discrete spectrum. Moreover, the action $T = (T_g)_{g\in G}$ is of funny rank one. To see this, set

$$F_n = \{x \in X : x_0 = \cdots = x_{n-1} = 0\},\ G_n = \bigoplus_{k=0}^{n-1}\mathbb{Z}/p_n\mathbb{Z}, \quad n \geq 0.$$

The sequence $(\mathcal{R}_n)$ of funny Rokhlin towers [see (1.2)] is given by $(T_gF_n)_{g\in G_n}$ where $F_0 = X, G_0 = \{0\}$. Note that the union of levels of each such Rokhlin tower fills up the whole space.

As before we consider $\Gamma = \mathbb{Z}_2$. By a *Morse cocycle* we mean a cocycle which is constant on all levels of the funny Rokhlin towers $(T_gF_n)_{g\in G_n}, n \geq 1$, in the sense made precise below.

Take an arbitrary sequence of $(\varepsilon_n(j))_{0\leq j\leq p_n}, n \geq 0)$ with values in $\mathbb{Z}_2$ in which $\varepsilon_n(0) = 1$. Then define

$$\varphi(x, g) = \varphi_g(x) = \prod_{n=0}^{\infty}\varepsilon_n(x_n)\varepsilon_n(x_n + g_n) \quad \text{for all } (x, g) \in X \times G. \tag{3.5}$$

By (3.5) it follows directly that the cocycle identity (3.1) is satisfied and moreover for $x \in F_n$ and $g \in G_n$ we have

$$\phi_g(x) = \prod_{k=0}^{\infty}\varepsilon_k(g_k) = \prod_{k=0}^{n-1}\varepsilon_k(g_k),$$

so $\phi_g$ is constant on $F_n$ for $g \in G_n$ ($\phi_g(x) = \phi_g(0)$) and in fact since

$$\varphi(T_hx, g)\varphi(x, g) = \varphi(x, g + h),$$

we also have $\varphi_g$ is constant on each level $T_hF_n, h \in G_n$, whenever $g \in G_n$: $\phi_g|_{T_hF_n} = \phi_{g+h}(0)\phi_g(0)$. [In fact, as shown in [14], each Morse cocycle is defined by a $\mathbb{Z}_2$-valued sequence $(\varepsilon_n(j))_{0\le j\le p_n}, n \ge 0)$.] The following has been observed in [14]:

For each Morse cocycle $\varphi$, the representation $V$ on $L_1$ has simple spectrum.

Now by the purity law of Helson [17], it follows that once the spectral measure of $\mathbb{1}_X$ (for $V$) is continuous, the spectrum on $L_1$ is continuous and therefore the Koopman representation $U_{T_\varphi}$ has simple spectrum whenever $\varphi$ is a Morse cocycle.

Set $\varepsilon_n(k) = \left(\frac{k}{p_n}\right)$ for $0 < k < p_n$ and $\varepsilon_n(0) = 1$, where $\left(\frac{k}{p_n}\right)$ is the Legendre symbol, that is, it is equal to 1 if $k \neq 0$ is a square modulo $p_n$, $-1$ if not and $\left(\frac{0}{p_n}\right) = 0$.

**Theorem 3.2.** [14] The spectral measure $\sigma_0 = \sigma_{V,\mathbb{1}_X}$ of the function $\mathbb{1}_X$ is the product measure $\bigotimes_{n=0}^{\infty} |P_n|^2 m_{\mathbb{Z}/p_n\mathbb{Z}}$ where $P_n$ is defined on $\mathbb{Z}/p_n\mathbb{Z}$ by

$$P_n(x) = \frac{1}{\sqrt{p_n}}\left(1 + \sum_{k=1}^{p_n-1}\left(\frac{k}{p_n}\right)e^{-2i\pi\frac{kx}{p_n}}\right).$$

Moreover, $\sigma_0$ is equivalent to the Haar measure of the dual group $\widehat{G}$ with

$$\left(\bigotimes_{n=0}^{\infty} |P_n|^2 m_{\mathbb{Z}/p_n\mathbb{Z}}\right)\widehat{\ }(g) = \widehat{\sigma}_0(g) \ \text{ for each } g \in G. \tag{3.6}$$

It follows that the sequence $\left(\bigotimes_{n=0}^{N} |P_n|^2 m_{\mathbb{Z}/p_n\mathbb{Z}}\right)_{N\ge1}$ of measures converges weakly to $\sigma_0$ and since $\sigma_0 \ll m_X$ we obtain the following.

**Corollary 3.3.** *The sequence* $\left(\bigotimes_{n=0}^{N} |P_n|^2\right)_{N\ge1}$ *of polynomials on* $X$ *converges weakly in* $L^1(X, m_X)$ *to* $\frac{d\sigma_0}{dm_X}$.

Notice that the sequence $(|P_n|^2)_{n\ge0}$ meant as polynomials on $X$ ($P_n(x) = P_n(x_n)$ for $x \in X$) is *ultra flat*, that is

$$\frac{\|P_n|\|_{L^2}}{\|P_n\|_{L^\infty}} \to 1;$$

indeed, $P_n(0) = \frac{1}{\sqrt{p_n}}$ and for $x \neq 0$ by the Gauss formula [3,18]

$$\begin{aligned} P_n(x) &= \frac{1}{\sqrt{p_n}}\left(1 + \sum_{k=1}^{p_n-1}\left(\frac{k}{p_n}\right)e^{-2i\pi\frac{kx}{p_n}}\right) \\ &= \frac{1}{\sqrt{p_n}}\left(1 + \sum_{k=0}^{p_n-1} e^{-2i\pi k^2 x/p_n} - \sum_{k=0}^{p_n-1} e^{-2i\pi kx/p_n}\right) \\ &= \frac{1}{\sqrt{p_n}}\left(1 + \delta_{p_n}\sqrt{p_n}\left(\frac{x}{p_n}\right)\right) \end{aligned}$$

with $\delta_{p_n} = 1$ if $p_n \equiv 1 \bmod 4$ or $\delta_{p_n} = i$ if $p_n \equiv 3 \bmod 4$; whence

$$1 - \frac{1}{\sqrt{p_n}} \leq |P_n(x)| \leq 1 + \frac{1}{\sqrt{p_n}}. \tag{3.7}$$

Recall also the following elementary fact.

**Lemma 3.4.** Let $a \in (0, 1)$. Then we have

$$\prod_{n=1}^{\infty}(1 + a^n) \leq \exp\left(\frac{a}{1-a}\right).$$

In view of (3.7) and Corollary 3.3

$$\frac{d\sigma_0}{dm_X} \leq \prod_{n=0}^{\infty}\left(1 + \frac{1}{\sqrt{p_n}}\right)^2. \tag{3.8}$$

Let us choose the sequence $(p_n)_{n\geq 0}$ so that $\sqrt{p_n} \geq 5^{n+1}$ for any $n \geq 0$. By Lemma 3.4 we have

$$\prod_{n=0}^{\infty}\left(1 + \frac{1}{\sqrt{p_n}}\right) \leq e^{1/4}.$$

Hence by (3.8), $\frac{d\sigma_0}{dm_X} \leq e^{1/2} < 2$ and the proof of Theorem 1.1 is complete.

**Remark 3.5.** The action of the group $G = \bigoplus_{n=0}^{\infty} \mathbb{Z}/p_n\mathbb{Z}$ on its dual by translations (as above) is a particular case of so called $(C, F)$-actions of the group $G$ (see [7]), all such actions have the funny rank one property. It would be interesting to see whether it is possible to carry out Guenais' construction of "good" Morse cocycles over a weakly mixing $(C, F)$-action of the group $G$ to obtain a system which is not AT; as Morse cocycles over $(C, F)$-actions yield systems which have simple spectrum on $L_1$ we have additionally to know that the spectral types on $L_0$ and $L_1$ are mutually singular. Such a construction would be an analog of Ageev's constructions [2] for $\mathbb{Z}$-actions. It would be even more interesting if we could carry out the above for the mixing $(C, F)$-actions of the group $G$ constructed by Danilenko in [6].

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## References

[1] E. H. el Abdalaoui and M. Lemańczyk, Approximate transitivity property and Lebesgue spectrum, Monatsh. Math. **161** (2010), 121–144.

[2] O. N. Ageev, Dynamical systems with an even-multiplicity Lebesgue component in the spectrum, Math. USSR Sbornik **64** (1989), 305–316.

[3] B. C. Berndt, R. J. Evans, and S. W. Kenneth, Gauss and Jacobi Sums, Wiley and Sons, 1998.

[4] M. Blümlinger, Rajchman measures on compact groups, Math. Ann. **284** (1989), 55–62.

[5] A. Connes and G. J. Woods, Approximately transitive flows and ITPFI factors, Ergodic Theory Dynam. Systems **5** (1985), 203–236.

[6] A. Danilenko, Mixing rank-one actions for infinite sums of finite groups, Israel J. Math. **156** (2006), 341–358.

[7] A. Danilenko, $(C, F)$-actions in ergodic theory. Geometry and dynamics of groups and spaces, 325–351, Progr. Math. **265**, Birkhäuser, Basel, 2008.

[8] M.-C. David, Sur quelques problèmes de théorie ergodique non commutative, PhD thesis, 1979.

[9] A. Dooley and A. Quas, Approximate transitivity for zero-entropy systems, Ergodic Theory Dynam. Systems **25** (2005), 443–453.

[10] S. Ferenczi, Systèmes de rang un gauche [Funny rank-one systems], Ann. Inst. H. Poincaré Probab. Statist. **21** (1985), 177–186.

[11] S. Ferenczi, Tiling and local rank properties of the Morse sequence, Theoret. Comput. Sci. **129** (1994), 369–383.

[12] T. Giordano and D. Handelman, Matrix-valued random walks and variations on AT property, Münster J. Math. **1** (2008), 15–72.

[13] V. Ya. Golodets, Approximately Transitive Actions of Abelian Groups and Spectrum, `http://ftp.esi.ac.at/pub/Preprints/esi108.ps`.

[14] M. Guenais, Morse cocycles and simple Lebesgue spectrum, Ergodic Theory Dynam. Systems **19** (1999), 437–446.

[15] J. M. Hawkins, Properties of ergodic flows associated to product odometers, Pacific J. Math. **141** (1990), 287–294.

[16] J. M. Hawkins and E. A. Robinson, Jr., Approximately transitive (2) flows and transformations have simple spectrum, Dynamical systems (College Park, MD, 1986–87), 261–280, Lecture Notes in Math. **1342**, Springer-Verlag, Berlin, 1988.

[17] H. Helson, Cocycles on the circle, J. Operator Theory **16** (1986), 189–199.

[18] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory, Springer-Verlag, 1990.

[19] A. del Junco, A simple map with no prime factors, Israel J. Math. **104** (1998), 301–320.

[20] A. Katok and J.-P Thouvenot, Spectral Properties and Combinatorial Constructions in Ergodic Theory, in: Handbook of dynamical systems. Vol. 1B, 649–743, Elsevier B. V., Amsterdam, 2006.

[21] M. Lemańczyk, Spectral Theory of Dynamical Systems, Encyclopedia of Complexity and System Science, Springer-Verlag (2009), 8554–8575.

[22] A. M. Sokhet, Les actions approximativement transitives dans la théorie ergodique, PhD thesis, Paris 1997.

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