PERVERSE EQUIVALENCES, MUTATIONS AND APPLICATIONS

BASED ON THE TALK BY SEFI LADKANI

Throughout the talk K is an algebraically closed field.

1. Introduction

The following theorem serves as a motivation for the presentation.

Theorem 1.1. *Let s be a sink in a quiver Q without oriented cycles. If $\sigma_s Q$ is the BGP-reflection of Q at s, then KQ and $K(\sigma_s Q)$ are derived equivalent.*

We recall also the following structure theorem for the derived equivalence.

Theorem 1.2 (Rickard)**.** *Let R and S be algebras. Then R and S are derived equivalent if and only there exists a tilting R-complex T such that $S \simeq \operatorname{End}_{\mathcal{D}^b(R)}(T)$.*

2. Mutations

2.1. **Quiver mutations (Fomin/Zelevinsky).** Given a quiver Q we denote by B_Q the $Q_0 \times Q_0$-matrix defined by

$$\begin{aligned}(B_Q)_{i,j} := \#\{&\alpha \in Q_1 : s\alpha = j \text{ and } t\alpha = i\} \\ &- \#\{\alpha \in Q_1 : s\alpha = i \text{ and } t\alpha = j\} \qquad (i, j \in Q_0).\end{aligned}$$

If Q is a quiver without loops and 2-cycles, then B_Q determines Q uniquely up to an isomorphism fixing vertices. For a quiver Q without loops and $k \in Q_0$ we define the $Q_0 \times Q_0$-matrices r_k^- and r_k^+ by

$$(r_k^-)_{i,j} := \begin{cases} \delta_{i,j} & i \neq k, \\ \#\{\alpha \in Q_1 : s\alpha = j \text{ and } t\alpha = i\} - \delta_{i,j} & i = k, \end{cases}$$

and

$$(r_k^+)_{i,j} := \begin{cases} \delta_{i,j} & i \neq k, \\ \#\{\alpha \in Q_1 : s\alpha = i \text{ and } t\alpha = j\} - \delta_{i,j} & i = k, \end{cases}$$

$(i, j \in Q_0)$. Finally, for a quiver Q without loops and 2-cycles, and $k \in Q_0$, we denote by $\mu_k(Q)$ the quiver mutation of Q at k.

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Lemma 2.1 (Berenstein/Fomin/Zelevinsky, Geiss/Leclerc/Schröer)**.** *If Q is a quiver without loops and 2-cycles, then*

$$B_{\mu_k(Q)} = (r_k^+)^{\mathrm{T}} \cdot B_Q \cdot r_k^+ \qquad \textit{and} \qquad B_{\mu_k(Q)} = (r_k^-)^{\mathrm{T}} \cdot B_Q \cdot r_k^-$$

for each $k \in Q_0$.

2.2. **Algebra mutations.** Throughout this subsection A is the path algebra of a bound quiver (Q, I) and $k \in Q_0$.

By L_k we denote the cone of the canonical map

$$P_k \to \bigoplus_{\substack{\alpha \in Q_1 \\ t\alpha = k}} P_{s\alpha},$$

and we put

$$T_k^- := L_k \oplus \bigoplus_{\substack{i \in Q_0 \\ i \neq k}} P_i.$$

We define T_k^+ dually. Both T_k^- and T_k^+ are perfect generators of $\mathcal{D}^b(A)$. Moreover, they are both silting, i.e.

$$\operatorname{Hom}_{\mathcal{D}^b(A)}(T_k^-, T_k^-[r]) = 0 = \operatorname{Hom}_{\mathcal{D}^b(A)}(T_k^+, T_k^+[r])$$

for each $r \in \mathbb{N}_+$. Next, T_k^- is tilting if and only if

$$\operatorname{Hom}_{\mathcal{D}^b(A)}(P_i, L_k[-1]) = 0$$

for each $i \in Q_0$, $i \neq k$, which leads to a combinatorial condition. If T_k^- is tilting, then we put

$$\mu_k^-(A) := \operatorname{End}_{\mathcal{D}^b(A)}(T_k^-).$$

Similarly, if T_k^+ is tilting, then we put

$$\mu_k^+(A) := \operatorname{End}_{\mathcal{D}^b(A)}(T_k^+).$$

We call $\mu_k^-(A)$ and $\mu_k^+(A)$ the mutations of A at k. For example, if A is the path algebra of the quiver

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}\ ,$$

then $\mu_1^-(A)$ and $\mu_3^+(A)$ are not defined, $\mu_1^+(A)$ and $\mu_3^-(A)$ are the path algebras of the quivers

$$\underset{1}{\bullet} \longleftarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \qquad \text{and} \qquad \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet}\ ,$$

respectively, while $\mu_2^-(A)$ and $\mu_2^+(A)$ are the path algebras of the bound quivers

$$\underset{2}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{3}{\bullet} \qquad \text{and} \qquad \underset{1}{\bullet} \longrightarrow \underset{3}{\bullet} \longrightarrow \underset{2}{\bullet}\ ,$$

respectively. If T_k^- is tilting, then $\mathbb{R}\operatorname{Hom}_A(T_k^-, -)$ induces a perverse Morita equivalence in a sense of Chuang and Rouquier with filtration

$\varnothing \subset \{k\} \subset Q_0$ and perversity $(0, -1)$. Finally, let T_k^{BB} be the Brenner–Butler module, i.e.

$$T_k^{BB} := \tau_A^- S_k \oplus \bigoplus_{\substack{i \in Q_0 \\ i \neq k}} P_i.$$

If T_k^{BB} is tilting, then $T_k^{BB} \simeq T_k^-$. On other hand, if T_k^- is tilting and isomorphic to a module, then $T_k^- \simeq T_k^{BB}$.

Recall that by C_A we denote the Cartan matrix of A, i.e.

$$(C_A)_{i,j} := \dim_K \operatorname{Hom}_A(P_i, P_j) \qquad (i, j \in Q_0).$$

Moreover, if $\operatorname{gl.dim} A < \infty$, then by c_A we denote the matrix of the Euler form of A, i.e. $c_A := C_A^{-\mathrm{T}}$.

Lemma 2.2. *Assume T_k^- is tilting. Then*

$$C_{\mu_k^-(A)} = r_k^- \cdot C_A \cdot (r_k^-)^{\mathrm{T}}.$$

In particular, if $\operatorname{gl.dim} A < \infty$*, then*

$$c_{\mu_k^-(A)} = (r_k^-)^{\mathrm{T}} \cdot c_A \cdot r_k^-.$$

3. Application

Throughout this section $\mathcal{C}$ is a K-linear, Hom-finite, 2-Calabi-Yau triangulated category. By $\operatorname{ind} \mathcal{C}$ we denote a chosen set of representatives of the isomorphism classes of the indecomposable objects in $\mathcal{C}$.

By a cluster tilting sequence in $\mathcal{C}$ we mean a sequence

$$\mathcal{U} = (U_1, \ldots, U_n)$$

of objects from $\operatorname{ind} \mathcal{C}$ such that $U := \bigoplus_{i \in [1,n]} U_i$ is maximal rigid, i.e. $\operatorname{Hom}_{\mathcal{C}}(U, U[1]) = 0$ and if $\operatorname{Hom}_{\mathcal{C}}(U, X[1]) = 0$ for $X \in \operatorname{ind} \mathcal{C}$, then $X = U_i$ for some $i \in [1, n]$. If $\mathcal{U} = (U_1, \ldots, U_n)$ is a cluster tilting sequence in $\mathcal{C}$ and $k \in [1, n]$, then there exists unique $X \in \operatorname{ind} \mathcal{C}$ such that $X \not\simeq U_k$ and the sequence $\mathcal{U}' = (U_1', \ldots, U_n')$ defined by

$$U_i' := \begin{cases} U_i & i \neq k, \\ X & i = k, \end{cases} \qquad (i \in [1, n])$$

is cluster tilting (by results of Buan/Marsch/Reineke/Reiten/Todorov and Iyama/Yoshino). In the above situation, we put $\mu_k(\mathcal{U}) := \mathcal{U}'$ and call $\mathcal{U}'$ the cluster tilting mutation of $\mathcal{U}$ at k. For a cluster tilting sequence $\mathcal{U} = (U_1, \ldots, U_n)$ we put

$$\operatorname{End}_{\mathcal{C}}(\mathcal{U}) := \operatorname{End}_{\mathcal{C}}\Big(\bigoplus_{i \in [1,n]} U_i\Big).$$

Theorem 3.1. *Let $\mathcal{U} = (U_1, \dots, U_n)$ be a cluster tilting sequence, $k \in [1, n]$, and*

$$\Lambda := \operatorname{End}_{\mathcal{C}}(\mathcal{U}) \qquad \textit{and} \qquad \Lambda' := \operatorname{End}_{\mathcal{C}}(\mu_k(\mathcal{U})).$$

If $\mu_k^{BB}(\Lambda)$ and $\mu_k^{BB}(\Lambda'^{\mathrm{op}})$ ar defined, then $\Lambda' = \mu_k^{BB}(\Lambda)$.