GRADED CENTERS OF SOME TRIANGULATED CATEGORIES

BASED ON THE TALK BY YU YE

Assumption.
Throughout the talk we consider categories over a fixed field k.

Definition.
For a category $\mathscr{A}$ we define the center $Z(\mathscr{A})$ of $\mathscr{A}$ by

$$Z(\mathscr{A}) := \{\eta : \mathrm{Id}_{\mathscr{A}} \to \mathrm{Id}_{\mathscr{A}}\}.$$

Remark.
If $\mathscr{A}$ is a category, then $Z(\mathscr{A})$ is a commutative ring.

Example.
If A is a k-algebra, then $Z(\operatorname{Mod} A) = Z(\operatorname{mod} A) = Z(A)$.

Example.
If A is a k-algebra, then we have an inclusion $Z(A) \hookrightarrow Z(\mathscr{D}^b(\operatorname{mod} A))$, which is an isomorphism provided A is hereditary.

Example.
If $A = k[X]/(X^n)$, then $Z(\underline{\operatorname{mod}}\, A) = k[X]/(X^{\lfloor \frac{n}{2} \rfloor})$.

Definition.
For a triangulated category $\mathscr{T}$ with the suspension functor Σ we define the graded center $Z^*(\mathscr{T}) = Z^*(\mathscr{T}, \Sigma)$ by

$$Z^n(\mathscr{T}) := \{\eta : \mathrm{Id}_{\mathscr{T}} \to \Sigma^n \mid \eta\Sigma = (-1)^n \Sigma\eta\}$$

for $n \in \mathbb{Z}$.

Remark.
If $\mathscr{T}$ is a category, then $Z^*(\mathscr{T})$ is a graded commutative algebra.

Remark.
The above definition makes sense for an arbitrary category endowed with the shift functor.

Remark.
If $\mathscr{T}$ is a triangulated category, then $Z^0(\mathscr{T}) \subseteq Z(\mathscr{T})$.

Remark.
In general, the graded center of a triangulated category is not a set.

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Remark.

If M is an object of a triangulated category $\mathscr{T}$, then we have the evaluation map $\varphi_M : Z^*(\mathscr{T}) \to \operatorname{Ext}^*_{\mathscr{T}}(M,M) := \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathscr{T}}(M, \Sigma^n M)$. In particular, if A is an algebra, then we have a ring homomorphism $\operatorname{HH}^*(A) \to \operatorname{Ext}^*_{\mathscr{T}}(M,M)$.

Remark.

If $\mathscr{S}$ is a triangulated subcategory of a triangulated category $\mathscr{T}$, then we have the induced maps $i^* : Z(\mathscr{T}) \to Z(\mathscr{S})$ and $\pi_* : Z(\mathscr{T}) \to Z(\mathscr{T}/\mathscr{S})$. It is an open problem if (i^*, π_*) is injective.

Notation.

For a full subcategory $\mathscr{C}$ of an abelian category we denote by $\mathscr{K}(\mathscr{C})$ the full subcategory of the homotopy category of complexes over $\mathscr{C}$ formed by all X such that $H_n(X) = 0$ for all but a finite number of $n \in \mathbb{Z}$. By $\mathscr{K}^-(\mathscr{C})$ we denote the full subcategory of $\mathscr{K}(\mathscr{C})$ formed by all X such that $X_n = 0$ for all but a finite number of $n \in \mathbb{N}$. We define $\mathscr{K}^+(\mathscr{C})$ dually and we put $\mathscr{K}^b(\mathscr{C}) := \mathscr{K}^-(\mathscr{C}) \cap \mathscr{K}^+(\mathscr{C})$.

Notation.

For $n \in \mathbb{Z}$ and a full subcategory $\mathscr{C}$ of an abelian category we denote by ι^n the map which assigns to $X \in \mathscr{K}(\mathscr{C})$ the truncated complex $\iota_n X \in \mathscr{K}^-(\mathscr{C})$. Moreover, for $X \in \mathscr{K}(\mathscr{C})$ we also denote by i_n^X the corresponding inclusion map $\iota_n X \to X$.

Lemma.

Let $f : X \to Y$ for $X, Y \in \mathscr{K}(\operatorname{proj} \mathscr{A})$ and an abelian category $\mathscr{A}$. If $n \in \mathbb{Z}$ is such that $H_m(Y) = 0$ for all $m \in [n+1, \infty)$, then f is null-homotopic if and only if $f \circ i_n^X$ is null-homotopic.

Proof.

Let $m \in [n+1, \infty)$ and assume there exist $s_i : X_i \to Y_{i+1}$, $i \in (-\infty, m-1]$, such that $f_i = s_{i-1} d_i^X + d_{i+1}^Y s_i$ for each $i \in (-\infty, m-1]$. Observe that $d_m^Y (f_m - s_{m-1} d_m^X) = 0$, hence $\operatorname{Im}(f_m - s_{m-1} d_m^X) \subseteq \operatorname{Ker} d_m^Y = \operatorname{Im} d_{m+1}^Y$. Since $X_m \in \operatorname{proj} \mathscr{A}$, there exists $s_m : X_m \to Y_{m+1}$ such that $f_m - s_{m-1} d_m^X = d_{m+1}^Y s_m$, hence the claim follows by obvious induction.

Proposition.

If $\eta \in Z^t(\mathscr{K}^b(\operatorname{proj} \mathscr{A}))$ for an abelian category $\mathscr{A}$ and $t \in \mathbb{Z}$, then there exists unique $\theta \in Z^t(\mathscr{K}^+(\operatorname{proj} \mathscr{A}))$ such that $\eta = i^* \theta$.

Theorem (Krause/Ye).

If $\mathscr{A}$ is an abelian category, then $Z^*(\mathscr{K}^b(\operatorname{proj} \mathscr{A})) \to Z^*(\mathscr{K}^+(\operatorname{proj} \mathscr{A}))$ is an isomorphism.

Lemma.

If A is a hereditary algebra, then

$$Z^*(\mathscr{D}^b(\operatorname{mod} A)) = Z^0(\mathscr{D}^b(\operatorname{mod} A)) \oplus Z^1(\mathscr{D}^b(\operatorname{mod} A))$$

and

$$Z^0(\mathscr{D}^b(\operatorname{mod} A)) = Z(A).$$

Proposition.

If $\mathscr{H}_1$ and $\mathscr{H}_2$ are additive subcategories of a hereditary category $\mathscr{H}$ such that $\mathscr{H} = \mathscr{H}_1 \vee \mathscr{H}_2$ and $\operatorname{Hom}_{\mathscr{H}}(\mathscr{H}_2, \mathscr{H}_1) = 0 = \operatorname{Ext}^1_{\mathscr{H}}(\mathscr{H}_1, \mathscr{H}_2)$, then

$$Z^1(\mathscr{D}^b(\mathscr{H})) \simeq Z^1(\mathscr{D}^b(\Sigma^* \mathscr{H}_1)) \times Z^1(\mathscr{D}^b(\Sigma^* \mathscr{H}_2)).$$

Proposition.

If Q is a quiver of Dynkin type, then $Z^*(\mathscr{D}^b(\operatorname{mod} kQ)) \simeq k$.

Proposition.

If Q is a quiver of Euclidean type, then

$$Z^*(\mathscr{D}^b(\operatorname{mod} kQ)) \simeq k \ltimes \Big(\prod_{\mathscr{T}_1 \times \mathbb{Z}} k \Big),$$

where $\mathscr{T}_1$ indexes the homogeneous tubes.