GEOMETRIC REALIZATION OF CLUSTER ALGEBRAS OF FINITE TYPE, II

BASED ON THE TALK BY ANDREI ZELEVINSKY

The results presented during the talk were obtained joint with Shih Wei Yang.

Assumption.

Throughout the talk Q is a fixed acyclic Dynkin quiver with n vertices.

Reminder.

Each cluster variable z is uniquely detemined by its g-vector $\mathbf{g}_z$ and polynomial F_z.

Notation.

By G we denote the simply connected semisimple algebraic group over $\mathbb{C}$ associated to Q. Let H denote a fixed maximal torus of G, $W := \mathrm{Norm}_G(H)/H$, and $P = \mathrm{Hom}(H, \mathbb{C}^*)$. Let c be the Coxeter element corresponding to the orientation of Q, i.e. if $c = s_1 \cdots s_n$ for simple reflections $s_{x_1}, \ldots, s_{x_n}$, then there is no arrow from x_i to x_j in Q for $i < j$. We denote by $\omega_1, \ldots, \omega_n$ the fundamental weights in P. We have the isomorphism $\mathbb{Z}^n \to P$ which sends $\mathbf{g}$ to $g_1\omega_1 + \cdots + g_n\omega_n$. We denote by $\Pi(c)$ the subset of P corresponding to the set of g-vector of cluster variables. If $\gamma, \delta \in P$, then we write $\gamma \geq \delta$ if $\gamma - \delta$ is a sum of simple roots. If w_0 is the longest element of W, then $w_i^* = -w_0 w_i$ for $i \in [1, n]$.

For each simple root we the SL_2-embedding $\varphi_i : \mathrm{SL}_2 \to G$. Let

$$x_i(t) = \varphi_i\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) \qquad \text{and} \qquad \overline{x}_i(t) = \varphi_i\left(\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\right).$$

Then almost every $x \in G$ has the Gauss decomposition $x = [x]_- \cdot [x]_0 \cdot [x]_+$, where $[x]_-$ belongs to th subgroup generated by $\overline{x}_i(t)$, $[x]_0 \in H$, and $[x]_+$ belongs to the subgroup generated by $x_i(t)$. The maps $\Delta_{\omega_i,\omega_i}(x) = ([x]_0)^{\omega_i}$ extends to a regular function on G. More generally, if $\gamma = u\omega_i$ and $\delta = v\omega_i$, then $\Delta_{\gamma,\delta}(x) = \Delta_{\omega_i,\omega_i}(\overline{u}^{-1} x \overline{v})$, where $\overline{u}$ and $\overline{v}$ are obtained by extending the definition $\overline{s}_i = \varphi_i\left(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\right)$ using the reduced presentations.

Proposition.

For each $i \in [1, n]$ there exists $h(i, c) \geq 1$ such that

$$w_i > cw_i > \cdots > c^{h(i,c)} w_i = -w_i^*.$$

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Theorem.

$\Pi(c) = \{c^m w_i \mid i \in [1, n],\ m \in [0, h(i, c)]\}$ and

$$F_\gamma(t_1, \dots, t_n) = \Delta_{\gamma,\gamma}(\overline{x}_1(1) \cdots \overline{x}_n(1) x_n(t_n) \cdots x_1(t_1)).$$

Example.

If Q is an equioriented quiver of type $\mathbb{A}_n$, then $G = \mathrm{SL}_{n+1}$, H consists of the diagonal matrices, $W = S_{n+1}$, $\omega_i := [1, i]$ and the corresponding map is given by the appropriate left-upper minor, $i \in [1, n]$, and $\Pi(c) = \{[i, j] \mid (i, j) \neq (1, n + 1)\}$. If $\gamma c^m \omega_k$, then $F_\gamma = 1 + t_m + \cdots + t_m \cdots t_{m+k+1}$.