1. Motivation and Background

The purpose of this talk is to provide a survey on the representation theory of Lie algebras over fields of positive characteristic. By now, this theory is about 70 years old, its origin dating back to Witt’s definition of the first non-classical simple Lie algebra. Since then, this field has seen the introduction of a variety of methods and the statements of challenging conjectures, each furnishing a new impetus for further investigations.

Given any associative algebra $\Lambda$ over an algebraically closed field $k$, one basic problem is the classification of the irreducible $\Lambda$-modules. This is usually not possible, so one looks instead for good parametrizations or at least for information concerning the dimensions of these modules. The center $C(\Lambda)$ acts on finite-dimensional irreducible modules via algebra homomorphisms $\chi : C(\Lambda) \to k$, the so-called central characters. This leads to a first subdivision of the class of finite-dimensional irreducible modules.

In this talk we shall take a closer look at the following situation. We consider a finite-dimensional Lie algebra $g$ over $k$, as well as its universal enveloping algebra $U(g) \supset g$. The universal property of the latter ensures that every representation $g \to \text{gl}(V)$ of our Lie algebra extends in a unique fashion to a representation $U(g) \to \text{End}_k(V)$. While the resulting identification of the representation theories of $g$ and $U(g)$ makes no particular reference to the base field, most of the classical results heavily depend on the characteristic of $k$. To better appreciate the peculiarities of Lie algebras of positive characteristic, let me begin by recalling a few facts concerning their classical precursors for complex numbers. This structure theory is somewhat older, of course, its pioneers being Lie, Engel, Killing and E. Cartan.

In the sequel all Lie algebras are assumed to be finite-dimensional.

**Theorem 1** (Lie’s Theorem). If $\text{char}(k) = 0$ and $g$ is solvable, then every finite-dimensional irreducible $g$-module is one-dimensional.

This theorem does in fact have a characteristic free version, the so-called Lie Kolchin Theorem. It implies the validity of a weak analogue of Lie’s Theorem in case $g = \text{Lie}(G)$ is the Lie algebra of a smooth algebraic group. However, the general failure of this result for arbitrary Lie algebras of positive characteristic constitutes one of the obstacles the classification of simple Lie algebras of positive characteristic has to deal with.
Example. Let $g := kx \oplus ky \oplus kz$, $[x, y] = z$; $[x, z] = [y, z] = 0$ be the three-dimensional Heisenberg algebra. If char($k$) = $p > 0$, then the truncated polynomial ring $V := k[Y]/(Y^p)$ is an irreducible $g$-module, if one lets $x$, $y$ and $z$ act via differentiation, left multiplication by $Y$, and the identity, respectively.

So let us assume that char($k$) = 0. By the theorem of Levi-Malcev, our Lie algebra $g$ can be written as a semidirect product $g = s \oplus \text{rad}(g)$. Here $\text{rad}(g)$ denotes the solvable radical of $g$, that is, the largest solvable ideal of $g$. Those Lie algebras, where $\text{rad}(g)$ coincides with the center $C(g)$ are the so-called, reductive Lie algebras (for instance, the Lie algebra $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus kI_n$ is of that type), those, for which $\text{rad}(g) = (0)$ are semisimple. Semisimple Lie algebras have a particularly nice representation theory:

**Theorem 2** (Weyl’s Theorem). If $g$ is semisimple, then every finite-dimensional $g$-module is completely reducible.

So where does the algebra $U(g)$ and its central characters enter? A semisimple Lie algebra $g$ affords a triangular decomposition, that is,

$$g = n^- \oplus \mathfrak{h} \oplus n^+,$$

where the three constituents are subalgebras such that $[\mathfrak{h}, \mathfrak{h}] = (0)$ (Cartan subalgebra), $[\mathfrak{h}, n^\pm] \subset n^\pm$. Instead of studying all $g$-modules one considers a full subcategory, the so-called Bernstein-Gel’fand-Gel’fand category $\mathcal{O}$. This category has nice closure properties and it comprises the category of finite-dimensional $g$-modules. In particular, each simple object of $\mathcal{O}$ is an irreducible $g$-module which is uniquely determined by a linear form $\lambda \in \mathfrak{h}^*$. The irreducible module $L(\lambda)$ is finite-dimensional if and only if $\lambda$ takes non-negative integer values on a certain basis of $\mathfrak{h}$. The dimension of $L(\lambda)$ can then be computed from Weyl’s character formula.

The so-called Killing-form of $g$ gives rise to a positive definite symmetric form on $\mathfrak{h}$. There is a finite reflection group $W \subset O(\mathfrak{h})$, called the Weyl group that provides a lot of information on the representation theory. As it turns out, each irreducible module $L(\lambda)$ defines a central character $\chi_\lambda : C(U(g)) \rightarrow k$, which obeys the following linkage principle:

**Theorem 3** (Harish-Chandra). Let $g$ be semisimple. Then the following statements hold:

1. We have $C(U(g)) \cong U(\mathfrak{h})^W$, so that $C(U(g))$ is a polynomial ring.
2. $\chi_\lambda = \chi_\mu \iff \mu \in W.\lambda$.

Since the linear forms $\lambda$ corresponding to finite-dimensional irreducible modules constitute a fundamental domain for the action of the Weyl group on $\mathfrak{h}^*$, there is a one-to-one correspondence between finite-dimensional irreducible $g$-modules and central characters. In particular, there are no non-trivial extensions between such modules. Thus, we retrieve essentially Weyl’s Theorem from the above linkage principle.
2. Irreducible Representations of Modular Lie Algebras

The Theorem of Harish-Chandra shows the importance of the center of the universal enveloping algebra. One of the main differences between the classical case and that of modular Lie algebras resides in the drastic change of the size of the center of $U(g)$. We are going to delineate some of the representation-theoretic manifestations in this section. Hence we assume from now on that $k$ is an algebraically closed field of characteristic $p > 0$.

We begin with a result, that combines early work by Curtis, Jacobson, and Zassenhaus.

Theorem 4. Let $g$ be an $n$-dimensional Lie algebra over $k$.

1. There exists a subalgebra $O(g) \subset C(U(g))$ such that
   - $O(g) \cong k[X_1, \ldots, X_n]$, and
   - $U(g)$ is a free $O(g)$-module of finite rank.

2. The dimensions of the irreducible $U(g)$-modules are bounded, with the maximum dimension $d(g)$ being a power of $p$.

The foregoing result has a number of interesting consequences, all of which markedly contrast with the classical results mentioned in Section 1. Our next result shows that an analogue of Weyl’s Theorem only holds for $g = (0)$.

Theorem 5. Let $M \neq (0)$ be a finite-dimensional $g$-module. Then there exists a finite-dimensional $g$-module $N$ such that

$$\text{Ext}^n_{U(g)}(M, N) \neq (0) \neq \text{Ext}^n_{U(g)}(N, M)$$

for $0 \leq n \leq \dim_k g$.

Since the universal enveloping algebra has global dimension $\dim_k g$, the theorem entails that all extension groups that can possibly be non-trivial do in fact have this property. For semisimple Lie algebras of characteristic zero, Whiteheads classical results, which can be used to prove Weyl’s Theorem, assert that $\text{Ext}^n_{U(g)}(M, N) = (0)$ $1 \leq n \leq 2$ for finite dimensional $g$-modules $M$ and $N$.

In the early 1940’s Jacobson studied purely inseparable field extensions of exponent 1. In this context the class of restricted Lie algebras arose for the first time.

Recall that any associative $k$-algebra $\Lambda$ gives rise to a Lie algebra $\Lambda^-$ via the commutator product

$$[a, b] := ab - ba \quad \forall \ a, b \in \Lambda.$$ 

A Lie subalgebra $g \subset \Lambda^-$ satisfying $a^p \in g$ for every $a \in g$ is referred to as a restricted Lie algebra. These algebras can, of course, also be defined axiomatically: then a restricted Lie algebra is a pair $(g, [p])$ consisting of a Lie algebra $g$ and a map $[p] : g \rightarrow g$ satisfying the formal properties of a $p$-power operator.

Restricted Lie algebras are more tractable than general modular Lie algebras for a number of technical reasons, one being the availability of a Jordan-Chevalley decomposition for elements of $g$. Let me draw your attention to an important feature of the representation theory. By Theorem 4 every irreducible $g$-module $V$ gives rise to a central character $\chi_V : C(U(g)) \rightarrow k$. If $(g, [p])$ is a restricted Lie algebra, then

$$x^p - x^{[p]} \in C(U(g)) \quad \text{for every} \ x \in g.$$
Moreover, we can choose $O(g) = \text{alg}_k(\{x^p - x^{|p|}; \ x \in g\})$. In 1971 Kac and Weisfeiler observed that $\chi_V|_{O(g)}$ determines a linear form $\chi \in g^*$ via

$$\chi_V(x^p - x^{|p|}) = \chi(x)^p \ \forall \ x \in g.$$  

Accordingly, we can associate to every irreducible $g$-module $V$ a linear form $\chi \in g^*$, its so-called $p$-character, such that

$$x^p \cdot v - x^{|p|} \cdot v - \chi(x)^p v = 0$$

for every $x \in g$ and $v \in V$. This observation provides a first subdivision of the set of irreducible representations. It also leads us to one of the main conjectures in the field.

Given a linear form $\chi \in g^*$, we consider the $p$-subalgebra

$$z_g(\chi) := \{x \in g; \chi([x,y]) = 0 \ \forall \ y \in g\}.$$  

Since the bilinear form $(x,y) \mapsto \chi([x,y])$ is alternating, $z_g(\chi)$ has even codimension in $g$. We put

$$r(g) := \min_{\chi \in g^*} \dim_k z_g(\chi).$$

Conjecture (Kac-Weisfeiler). Let $(g, [p])$ be a restricted Lie algebra of dimension $n$. Then

$$d(g) = p^{\frac{1}{2}(n-r(g))}.$$  

3. Reduced Enveloping Algebras

According to the observation by Kac-Weisfeiler, every irreducible module $V$ of a restricted Lie algebra $(g, [p])$ defines a linear map such that the ideal $I_\chi < U(g)$ that is generated by $\{x^p - x^{|p|} - \chi(x)^p1; \ x \in g\}$ is contained in the annihilator of the irreducible $U(g)$-module $V$. Accordingly, $V$ is a module for the algebra

$$U_\chi(g) := U(g)/I_\chi.$$  

The algebra $U_\chi(g)$ is the reduced enveloping algebra defined by $\chi$. The family $(U_\chi(g))_{\chi \in g^*}$ plays an important role in the treatment of the Kac-Weisfeiler conjecture as well as in other problems. Via reduced enveloping algebras the methods from the representation theory of finite-dimensional associative algebras can be brought to bear:

- By the modular version of the Theorem of Poincaré-Birkhoff-Witt, each $U_\chi(g)$ has dimension $p^{\dim_k g}$.
- $U_\chi(g)$ is a Frobenius algebra, which is not necessarily symmetric.
- $U_0(g)$ is a Hopf algebra (whose structure is inherited from that of $U(g)$).
- Every automorphism $\varphi : g \rightarrow g$ of restricted Lie algebras induces an isomorphism $U_\chi(g) \xrightarrow{\cong} U_{\chi \circ \varphi^{-1}}(g)$.

The last observation is usually applied for Lie algebras of algebraic groups. Let $G$ be an algebraic group, $g := \text{Lie}(G)$ its Lie algebra. Then $g$ naturally has the structure of a restricted Lie algebra, and $G$ acts on $g$ via the adjoint representation. Let $G \times g^* \rightarrow g^*; (g, \chi) \mapsto g \cdot \chi$ be the co-adjoint action, that is,

$$g \cdot \chi = \chi \circ \text{Ad}(g^{-1}),$$

then we have $U_\chi(g) \cong U_{g \cdot \chi}(g)$ for every $g \in G$ and $\chi \in g^*$. 
Let us return to the Kac-Weisfeiler conjecture. When would we expect to get irreducible modules of maximal dimension? Since all reduced enveloping algebras have the same dimension, the semisimple ones are good candidates. Thus, given a restricted Lie algebra $\mathfrak{g}$ of dimension $n$, we study the morphism

$$\mathfrak{g}^* \longrightarrow \text{Alg}_k(p^n) ; \quad \chi \mapsto U_{\chi}(\mathfrak{g})$$

of varieties. In fact, one does this simultaneously for certain deformations of $\mathfrak{g}$, which degenerate to the case, where $\mathfrak{g}$ is an abelian Lie algebra with trivial $p$-mapping.

We have noted earlier that only $\mathfrak{g} = (0)$ satisfies Weyl’s Theorem. This raises the question, when $U_{\chi}(\mathfrak{g})$ is semisimple. Given a restricted Lie algebra $(\mathfrak{g}, [\mathfrak{p}])$, we define its nullcone via

$$\mathcal{V}_{\mathfrak{g}} := \{ x \in \mathfrak{g} ; \quad \mathfrak{z}_{\mathfrak{g}}(\chi) = 0 \}.$$ 

Thus, $\mathcal{V}_{\mathfrak{g}}$ is a closed, conical subset of $\mathfrak{g}$ and we say that $\mathfrak{g}$ is a torus if $\mathcal{V}_{\mathfrak{g}} = \{0\}$. In the early 1950’s Hochschild proved that $\mathfrak{g}$ is a torus if and only if $U_0(\mathfrak{g})$ is semisimple. Tori also figure in the following result, which confirms the Kac-Weisfeiler conjecture in an important case.

**Theorem 6** (Premet-Skryabin). Let $(\mathfrak{g}, [\mathfrak{p}])$ be an $n$-dimensional restricted Lie algebra, and suppose there exists $\chi_0 \in \mathfrak{g}^*$ such that $\mathfrak{z}_{\mathfrak{g}}(\chi_0)$ is a torus. Then

$$\mathcal{O} := \{ \chi \in \mathfrak{g}^* ; \quad U_{\chi}(\mathfrak{g}) \text{ is semisimple} \}$$

is a non-empty open subset of $\mathfrak{g}^*$. For each $\chi \in \mathcal{O}$ the algebra $U_{\chi}(\mathfrak{g})$ has $p r(\mathfrak{g})$ pairwise non-isomorphic irreducible modules, each of dimension $p^{\frac{1}{2}(n-r(\mathfrak{g}))}$.

Let me indicate some general arguments which ensure that the foregoing result actually holds for Lie algebras of reductive groups. Suppose that $\mathfrak{g}$ affords a non-degenerate bilinear form $(, ) : \mathfrak{g} \times \mathfrak{g} \longrightarrow k$ satisfying

$$(\left[ x, y \right], z) = (x, \left[ y, z \right]) \quad \forall \quad x, y, z \in \mathfrak{g}.$$ 

For instance, the Killing-Form has this invariance property. Given $\chi \in \mathfrak{g}^*$, there exists an element $a \in \mathfrak{g}$ such that

$$\chi(x) = (a, x) \quad \forall \quad x \in \mathfrak{g}.$$ 

Since the form is invariant, we obtain $\mathfrak{z}_{\mathfrak{g}}(\chi) = C_{\mathfrak{g}}(a) := \{ x \in \mathfrak{g} ; \quad [a, x] = 0 \}$. Now suppose that $\mathfrak{g}$ contains a self-centralizing maximal torus $\mathfrak{t}$ (such a torus does not necessarily exist). Then there exists $t_0 \in \mathfrak{t}$ such that

$$\mathfrak{t} = C_{\mathfrak{g}}(t_0).$$ 

Defining $\chi(x) := (t_0, x)$ for every $x \in \mathfrak{g}$, we obtain a linear form with $\dim_k \mathfrak{z}_{\mathfrak{g}}(\chi) = r(\mathfrak{g}) = \dim_k \mathfrak{t}$. 
4. Representation Theory of $U_\chi(\mathfrak{g})$

Given a restricted Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, we have seen in the last section that the classification problem of irreducible $\mathfrak{g}$-modules motivates the study of an algebraic family $(U_\chi(\mathfrak{g}))_{\chi \in \mathfrak{g}^*}$ of finite-dimensional Frobenius algebras. These algebras are usually not symmetric, a Nakayama automorphism $\mu : U_\chi(\mathfrak{g}) \longrightarrow U_\chi(\mathfrak{g})$ being defined via

$$\mu(x) = x - \text{tr(ad} x) \quad \forall x \in \mathfrak{g}.$$ Nevertheless, these algebras still share a lot of features with group algebras of finite groups. For instance, if $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, then $U_\chi(\mathfrak{h})$ is a subalgebra of $U_\chi(\mathfrak{g})$, with the latter being a free module over the former. As a result, the induction functor

$$\text{mod}
\begin{array}{c}
U_\chi(\mathfrak{h}) \\
\longrightarrow
\end{array}
\text{mod}
\begin{array}{c}
U_\chi(\mathfrak{g}) \\
M \mapsto U_\chi(\mathfrak{g}) \otimes U_\chi(\mathfrak{h}) M
\end{array}$$

is exact.

The affinity with the modular representation theory of finite groups has led to a number of paradigms such as

- the "generic point" $U_0(\mathfrak{g})$ being the most complicated among the algebras $U_\chi(\mathfrak{g})$,
- the principal block of the Hopf algebra $U_0(\mathfrak{g})$ being the most complicated block.

There is no precise meaning to "most complicated", yet one certainly would expect it to be reflected in the notion of representation type. By way of illustration, let me recall the relevant methods from the modular representation theory of finite groups.

If $G$ is a finite group, then every block $\mathcal{B} \subset k[G]$ gives rise to a $p$-subgroup $D \subset G$ that measures the complexity of $\mathcal{B}$. More precisely, setting $H := DC_G(D)$, there is a block $\mathcal{B}_H \subset k[H]$ such that $\mathcal{B}$ is a Brauer correspondent of $\mathcal{B}_H$. In particular, if $\mathcal{B}$ is tame or representation-finite, then $\mathcal{B}_H$ also enjoys this property. Moreover, since $D$ is a normal $p$-subgroup of $H$, one can show that $\mathcal{B}_H \cong \text{Mat}_n(k[D])$. Thus, we are reduced to the case of a $p$-group. If $\mathcal{B}$ is the principal block of $k[G]$, then $D$ is a Sylow-$p$-subgroup of $G$, implying that this block has the most complicated structure.

In default of analogous descent techniques for reduced enveloping algebras, one resorts to geometric methods related to cohomological support varieties and rank varieties.

**Definition.** Let $M$ be a $U_\chi(\mathfrak{g})$-module. Then

$$\mathcal{V}_\mathfrak{g}(M) := \{ x \in \mathcal{V}_\mathfrak{g} ; M|_{U_\chi_{\mathfrak{h}_x}(k)} \text{ is not projective} \} \cup \{0\}$$

is called the rank variety of $M$.

In the context of restricted Lie algebras, these varieties were introduced by Friedlander and Parshall in the mid 1980’s. The motivation for their work was Carlson’s realization of support varieties of finite groups via $p$-elementary abelian subgroups. Since then a lot of work in modular representation theory has focused on the computation and interpretation of rank varieties. Let me summarize some of it in the following result:

**Theorem 7.** Let $\chi \in \mathfrak{g}^*$ be a linear form, $\mathcal{B} \subset U_\chi(\mathfrak{g})$ a block, $M$ a $\mathcal{B}$-module.

1. The dimension $\dim \mathcal{V}_\mathfrak{g}(M)$ coincides with the complexity $c_{U_\chi(\mathfrak{g})}(M)$.
2. If $M$ is irreducible, and $\dim \mathcal{V}_\mathfrak{g}(M) \leq 1$, then $\mathcal{B}$ is a Nakayama algebra with 1 or $p$ irreducible modules.
If $B$ is tame, then $\dim V_g(M) \leq 2$.

Rank varieties also arise in the Auslander-Reiten theory of the Frobenius algebra $U_\chi(g)$. Let $\Gamma_s(g, \chi)$ be its stable Auslander-Reiten quiver. Given a connected component $\Theta \subset \Gamma_s(g, \chi)$, we have

$$V_g(M) = V_g(N) \quad \forall \ M, N \in \Theta.$$ 

Thus, we can speak of the rank variety of $V_g(\Theta)$ of $\Theta$.

Let me provide an example for the utility of this observation. Suppose that $\Theta$ is a component of Euclidean tree class. Then $\Theta$ is attached to a principal indecomposable module, so $\Omega^{-1}(\Theta) \cong \Theta$ contains an irreducible vertex $S$. By general principles, $S$ is isomorphic to its twist by any element of the connected component $\text{Aut}(U_\chi(g))^0$ of the automorphism group of $U_\chi(g)$. This is particularly useful if $g = \text{Lie}(G)$ is the Lie algebra of a smooth algebraic group $G$, as it implies that $V_g(\Theta)$ is invariant under the connected component of the stabilizer $G_\chi$ of $\chi$ in $G$.

We have the following analogue of Webb’s classification of AR-components for finite groups.

**Theorem 8.** Let $\Theta \subset \Gamma_s(g, \chi)$ be a connected component.

1. If $\dim V_g(\Theta) = 1$, then $\Theta \cong \mathbb{Z}[A_n]/(\tau^i)$ for $n \in \mathbb{N} \cup \{\infty\}$ and $i \in \{1, p\}$.
2. If $\dim V_g(\Theta) = 2$, then $\Theta$ is either of Euclidean or infinite Dynkin type.
3. If $\dim V_g(\Theta) \geq 3$, then $\Theta \cong \mathbb{Z}[A_\infty]$.

It is not clear, whether all the components in (2) actually occur. These questions along with that concerning the description of enveloping algebras of tame representation type were the central theme of joint work with Andrzej Skowroński. We established

- the existence of components of type $\mathbb{Z}[A_\infty]$,
- an example, where $U_0(g)$ is tame, but $U_\chi(g)$ is wild, and
- an example of $U_0(g)$ having a tame principal block, with other blocks being wild.

Hence one generally cannot expect to have a defect theory and a correspondence in the sense of Brauer.

5. **Reductive Lie algebras**

We have seen before that the representation theory of a Lie algebra is highly sensitive to the characteristic of the underlying base field. But even within the universe of modular Lie algebras there are major differences, depending on the fact whether our Lie algebra is of “classical” type or not. Initially, Lie algebras did not have a life of their own, but arose as tangent spaces of Lie groups. Analogously, every restricted Lie algebra $g$ is of the form $g = \text{Lie}(G)$, where $G$ is a an algebraic group scheme. In the classical situation this group scheme is smooth or reduced and one may identify the representations of $G$ with the rational representations of the group $G := G(k)$ of rational points. It turns out that representations of such Lie algebras are much better behaved than those of arbitrary ones.
Examples. (1) Suppose that $G$ is solvable. By the Lie-Kolchin theorem every irreducible $U_0(g)$-module is one-dimensional.

(2) If $g = \text{Lie}(G)$, and the principal block of $U_0(g)$ is tame, then $U_0(g) \cong U_0(\mathfrak{sl}(2))^n$ for some $n \geq 1$, and $U_0(g)$ is domestic. In general, tame restricted Lie algebras may not even have polynomial growth.

It turns out that certain features from the classical characteristic zero theory transfer to the modular case if one considers reductive Lie algebras, that is, Lie algebras of smooth reductive groups. If $G$ is such a group with maximal torus $T \subset G$, then the irreducible $G$-modules are of the form $L(\lambda)$, where $\lambda \in X(T)^+$ belongs to a subset of the character group of $T$.

Every $G$-module naturally has the structure of an $U_0(g)$-module, but $L(\lambda)$ may no longer be an irreducible $U_0(g)$-module. Let us call the characters $\lambda$ for which irreducibility continues to hold, the restricted weights. Given an element $\lambda \in X(T)^+$, we write

$$\lambda = \sum_{i=0}^r p^i \lambda_i$$

with each $\lambda_i$ being restricted. Steinberg’s tensor product theorem provides an isomorphism

$$L(\lambda) \cong L(\lambda_0) \otimes_k L(\lambda_1)^{[1]} \otimes_k L(\lambda_2)^{[2]} \otimes_k \cdots \otimes_k L(\lambda_r)^{[r]},$$

where the superscripts indicate the twist of a module by the corresponding power of the Frobenius endomorphism of $G$. The point is, that if you know the irreducible $U_0(g)$-modules, then you have good control over the irreducible $G$-modules, whose dimensions can no longer be computed from Weyl’s character formula. This is the background for the interest in representations of $U_\chi(g)$ in this context.

Thus, let $G$ be a reductive group with Lie algebra $g$. Just as in the classical case, the restricted Lie algebra $g$ affords a triangular decomposition

$$g = n^- \oplus h \oplus n^+$$

of $p$-subalgebras. Recall that $G$ acts on $g$ and $g^*$ via the adjoint action and co-adjoint action, respectively. If $g$ has a non-degenerate Killing form, then these two actions are equivalent, and one can use this fact to understand the following observation. For every $\chi \in g^*$ there exists $g \in G$ such that

$$(g, \chi)(n^+) = 0.$$

Since $U_\chi(g) \cong U_{g,\chi}(g)$ we may therefore always assume that $\chi|_{n^+} = 0$. This has the following important consequence: Letting $b := h \oplus n^+$ be the Borel subalgebra of $g$, we now easily have an analogue of the Lie-Kolchin theorem. Every irreducible $U_{\chi|b}(b)$-module is one-dimensional.

Now let $S$ be an arbitrary irreducible $U_\chi(g)$-module. Then $S$ contains a one-dimensional $U_{\chi|b}(b)$-module $k_\lambda$, and there results a surjection

$$U_\chi(g) \otimes_{U_{\chi|b}(b)} k_\lambda \rightarrow S.$$  

The induced module is called a Baby Verma Module and is denoted $Z_{\chi}(\lambda)$. These modules play an important role in the representation theory of reductive Lie algebras.

Thanks to a result by Friedlander and Parshall, passage to a suitable reductive $p$-subalgebra $g' \subset g$ yields a Morita equivalent algebra $U_{\chi|b'}(g')$, whose linear form is nilpotent, that is, it
satisfies $\chi(b) = 0$. For such forms the case best understood so far is the one where $\chi$ has \textit{standard Levi form}. By definition, such a nilpotent form satisfies

$$\chi(g - \alpha) \neq (0) \Rightarrow \text{the root } \alpha \text{ is simple.}$$

In this situation one has:

- the irreducible $U_\chi(g)$-modules are of the form $L_\chi(\lambda)$ with $\lambda \in \mathfrak{h}^*$ a certain linear form,
- a linkage principle similar to that of Section 1, yet involving the affine Weyl group of the group $G$,
- a conjecture concerning the multiplicities $[Z_\chi(\lambda) : L_\chi(\mu)]$, generally referred to as \textit{Lusztig's Hope}.

\textbf{References}


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