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DISJOINTNESS OF INTERVAL EXCHANGE TRANSFORMATIONS FROM SYSTEMS OF PROBABILISTIC ORIGIN

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Abstract. It is proved that almost every interval exchange transformation given by the symmetric permutation

$$\begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ m & m-1 & \dots & 2 & 1 \end{pmatrix},$$

where $m \geq 5$ is an odd number, is disjoint from ELF systems. Some disjointness properties of special flows built over interval exchange transformations and under piecewise constant roof function are investigated as well.

1. **Introduction.** The notion of ELF systems was introduced in paper [8] to express the fact that a given system is of probabilistic origin. An automorphism $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is said to has the ELF property, if it is ergodic and the weak closure of the set of all its iterations $\{T^n : n \in \mathbb{N}\}$, considered as Markov operators in $L^2(X, \mathcal{B}, \mu)$, consists of indecomposable Markov operators. The following standard classes of systems of probabilistic origin enjoy the ELF property: mixing systems, ergodic Gaussian systems (see [13] and [8]), Poisson suspensions, dynamical systems coming from stationary symmetric α–stable processes (see [6]) and infinitely divisible processes (see [18]).

This work continues the research programme begun in [8]. The purpose of that project is to study deterministic systems that are disjoint from systems of probabilistic origin. This is closely related to some problems of smooth realization for systems of probabilistic origin (see [9]). Recall that two measure-preserving automorphisms of standard probability spaces are disjoint if they have only one joining equal to the product measure (see [10]). According to the programme, in [9], it was

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proved that smooth ergodic flows of compact orientable smooth surfaces having only non–degenerate saddles as isolated critical points (and having a "good" transversal) are disjoint from ELF flows. A substantial contribution to the project was provided in [6] where some classes of automorphisms disjoint from ELF automorphisms were found. The proofs of disjointness in [8] and [9] were based mainly on the following two results.

Proposition 1.1 ([9]). *Suppose that $\mathcal{T} = \{T_t\}_{t\in\mathbb{R}}$ is an ergodic flow on $(X, \mathcal{B}, \mu)$ such that there exist a sequence $\{t_n\} \subset \mathbb{R}$, $t_n \to +\infty$, $0 < \alpha \le 1$, $J \in \mathcal{J}(\mathcal{T})$ and a probability Borel measure on $\mathbb{R}$ for which*

$$T_{t_n} \to \alpha \int_{\mathbb{R}} T_s dP(s) + (1-\alpha)J \text{ weakly in } \mathcal{L}(L^2(X, \mathcal{B}, \mu)).$$

If P is not a Dirac measure then $\mathcal{T}$ is disjoint from all weakly mixing ELF flows.

Proposition 1.2 ([6]). *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a weakly mixing ergodic automorphism. Assume that there exist an increasing sequence $\{t_n\}$ of natural numbers, $0 < \alpha \le 1$, $J \in \mathcal{J}(T)$ and a probability measure P on $\mathbb{Z}$ such that*

$$T^{t_n} \to \alpha \sum_{k\in\mathbb{Z}} P(\{k\})T^k + (1-\alpha)J \text{ weakly in } \mathcal{L}(L^2(X, \mathcal{B}, \mu)).$$

If P is not Dirac then T is disjoint from all ELF automorphisms.

It was shown in [6] that weakly mixing simple but not mixing automorphisms and almost all interval exchange transformations for some special irreducible permutations are disjoint from ELF automorphisms. More precisely, if π is an irreducible permutation of m elements such that $\pi(i) + 1 \ne \pi(i + 1)$ for $1 \le i \le m - 1$ and

$$\pi(\pi^{-1}(1) - 1) = \pi(1) - 1 \quad \text{or} \quad \pi(\pi^{-1}(m) + 1) = \pi(m) + 1 \tag{1}$$

then almost every corresponding interval exchange transformation is disjoint from all ELF systems. This class of permutation is in a sense marginal because the corresponding IETs can be "reduced" to exchanges of $m-1$ intervals. In this paper we will prove the disjointness of almost every IET from ELF automorphisms (see Theorem 6.3) for more interesting set of permutations lying in the Rauzy class generated by the symmetric permutation

$$\tau_m^{sym} = \begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ m & m-1 & \dots & 2 & 1 \end{pmatrix},$$

where $m \ge 5$ is odd. As τ_3^{sym} fulfills (1), the disjointness for $m = 3$ was proved in [6].

The idea of the proof consists in finding some IETs which have segments of orbits with special code properties. For $m = 5$ we deal with IETs of periodic type. Since such IETs are isomorphic to substitutional dynamical systems, we can use an approach developed in [2] for substitutions for which one is an eigenvalue of the associated matrix. It allows us to find the required segments of orbits. In order to find appropriate IETs of periodic type we use a method of searching some paths in Rauzy graphs developed in [14] and [19]. Moreover, we introduce a procedure which helps us to reduce the problem of finding appropriate IETs to a smaller number of intervals. Since the existence of the required segments of orbits is an open condition, we can use standard technics introduced by Veech in [23] to prove the existence of such segments for almost all IETs in the Rauzy class of the permutation τ_m^{sym}. Finally, this allows us to apply Proposition 1.2.

As a byproduct, we obtain (using Maple) concrete examples of IETs of periodic type disjoint from ELF systems.

Using Proposition 1.1, for some irreducible permutations we show that in the class of special flows built over corresponding IETs and under roof functions constant over the exchanged intervals almost every such flow is disjoint from weakly mixing ELF flows (see Corollary 4.12).

2. **Preliminaries.**

2.1. **Joinings.** For background information on the theory of joinings we refer the reader to [11] and [5].

Let T and S be ergodic automorphisms of standard probability Borel spaces $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ respectively. By a *joining* between T and S we mean any $T\times S$-invariant probability measure ρ on $(X\times Y,\mathcal{B}\otimes\mathcal{C})$ whose projections on X and Y are equal to μ and ν respectively. If we consider flows $\mathcal{T}=\{T_t\}_{t\in\mathbb{R}}$ and $\mathcal{S}=\{S_t\}_{t\in\mathbb{R}}$, by a joining between $\mathcal{T}$ and $\mathcal{S}$ we mean any probability $\{T_t\times S_t\}_{t\in\mathbb{R}}$-invariant measure on $(X\times Y,\mathcal{B}\otimes\mathcal{C})$ with the same property. The set of joinings between automorphisms T and S is denoted by $J(T,S)$. If the automorphism $T\times S$ on $(X\times Y,\mathcal{B}\otimes\mathcal{C},\rho)$ is ergodic then the joining ρ is called ergodic. The set of ergodic joinings is denoted by $J^e(T,S)$.

Any automorphism $R:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ determines a unitary operator on $L^2(X,\mathcal{B},\mu)$, still denoted by R, by the formula $R(f)=f\circ R$. An operator $\Phi:L^2(X,\mathcal{B},\mu)\to L^2(Y,\mathcal{C},\nu)$ is called a *Markov operator* if

$$\Phi(1)=\Phi^*(1)=1 \text{ and } \Phi(f)\geq 0 \text{ for all } f\geq 0.$$

Denote by $\mathcal{J}(S,T)$ the set of all Markov operators from $L^2(X,\mathcal{B},\mu)$ to $L^2(Y,\mathcal{C},\nu)$ such that $T\circ\Phi=\Phi\circ S$. For $\rho\in J(S,T)$, define an operator $\Phi_\rho:L^2(X,\mathcal{B},\mu)\to L^2(Y,\mathcal{C},\nu)$ by the formula

$$\langle f,g\rangle_{L^2(X\times Y,\rho)}=\langle\Phi_\rho(f),g\rangle_{L^2(Y,\nu)} \tag{2}$$

for each $f\in L^2(X,\mathcal{B},\mu)$ and $g\in L^2(Y,\mathcal{C},\nu)$. Then $\phi_\rho\in\mathcal{J}(T,S)$ and $J(T,S)\ni\rho\mapsto\phi_\rho\in\mathcal{J}(T,S)$ establishes a 1-1 affine correspondence between the sets $\mathcal{J}(T,S)$ and $J(T,S)$. The set of intertwining Markov operators $\mathcal{J}(S,T)$ is compact in the weak operator topology, thus the set $J(S,T)$ is also compact (on $J(T,S)$ we transport the topology of $\mathcal{J}(T,S)$). Markov operators corresponding to ergodic joinings are called *indecomposable*.

Following [10], automorphisms S and T are called *disjoint* if $J(S,T)=\{\mu\otimes\nu\}$. Note that $\Phi_{\mu\otimes\nu}f=\int_X f\,d\mu$.

If $S=T$ we will write $J(T)$, $J^e(T)$, $\mathcal{J}(T)$ and $\mathcal{J}^e(T)$ instead of $J(T,T)$, $J^e(T,T)$, $\mathcal{J}(T,T)$ and $\mathcal{J}^e(T,T)$ respectively, and the elements of $J(T)$ are called *self-joinings* of T.

By $C(T)$ we denote the centralizer of T, that is the set of all automorphisms of $(X,\mathcal{B},\mu)$ commuting with T. If $R\in C(T)$ then the formula $\mu_R(A\times B)=\mu(A\cap R^{-1}B)$ determines an ergodic self-joining of T supported on the graph of R. Moreover, $\Phi_{\mu_R}=R$.

Definition 2.1. Let T be an ergodic automorphism of a standard probability space $(X,\mathcal{B},\mu)$. Following [8], we say that T is an *ELF automorphism*, if $\overline{\{T^n:n\in\mathbb{Z}\}}\subset\mathcal{J}^e(T)$ (the closure is taken in the weak operator topology). An ergodic flow $\mathcal{T}=\{T_t\}_{t\in\mathbb{R}}$ on $(X,\mathcal{B},\mu)$ is said to be an *ELF flow* if $\overline{\{T_t:t\in\mathbb{R}\}}\subset\mathcal{J}^e(\mathcal{T})$.

Lemma 2.2. *Let $\mathcal{T}=\{T_t\}_{t\in\mathbb{R}}$ be a measure-preserving flow on a standard probability space $(X,\mathcal{B},\mu)$ and let $\Phi\in\mathcal{J}(\mathcal{T})$. Suppose that $\{t_n\}$ is a real sequence such that $t_n\to+\infty$ and the sequence of operators $\{T_{t_n}\}$ converges in the weak operator topology. For every $0<\alpha\le 1$ the following two statements are equivalent:*

- *(i) There exists $\Phi'\in\mathcal{J}(\mathcal{T})$ such that $T_{t_n}\to\alpha\Phi+(1-\alpha)\Phi'$ weakly.*
- *(ii) For all $A,B\in\mathcal{B}$ we have $\lim_{n\to\infty}\langle T_{t_n}\chi_A,\chi_B\rangle\ge\alpha\langle\Phi\chi_A,\chi_B\rangle$.*

Proof. Suppose that $T_{t_n}\to\alpha\Phi+(1-\alpha)\Phi'$ weakly for some $\Phi'\in\mathcal{J}(\mathcal{T})$. Then

$$\langle T_{t_n}\chi_A,\chi_B\rangle\to\alpha\langle\Phi\chi_A,\chi_B\rangle+(1-\alpha)\langle\Phi'\chi_A,\chi_B\rangle$$

for all $A,B\in\mathcal{B}$. Since $\Phi'\chi_A\ge 0$, we have $\langle\Phi'\chi_A,\chi_B\rangle\ge 0$, which implies (ii).

The converse follows directly from Lemma 3 in [9]. □

Remark 2.3. Note that if $\rho,\rho_0\in J(\mathcal{T})$ are self-joinings such that $T_{t_n}\to\Phi_\rho$ weakly and $\Phi_{\rho_0}=\Phi$ then condition (ii) in Lemma 2.2 is equivalent to $\rho\ge\alpha\rho_0$.

Lemma 2.4. *Let $\mathcal{T}=\{T_t\}_{t\in\mathbb{R}}$ be an ergodic flow on a standard probability space $(X,\mathcal{B},\mu)$. Suppose that there exist a real sequence $\{t_n\}$ with $t_n\to+\infty$, $\theta\neq 0$, $0<\alpha_1,\alpha_2\le 1$ and $\Phi_1,\Phi_2\in\mathcal{J}(\mathcal{T})$ such that*

$$T_{t_n}\to\alpha_1 Id+(1-\alpha_1)\Phi_1 \text{ and } T_{t_n}\to\alpha_2T_\theta+(1-\alpha_2)\Phi_2$$

weakly in $\mathcal{L}(L^2(X,\mathcal{B},\mu))$. Then $\alpha_1+\alpha_2\le 1$ and there exists $\Phi\in\mathcal{J}(\mathcal{T})$ such that

$$T_{t_n}\to\alpha_1 Id+\alpha_2T_\theta+(1-\alpha_1-\alpha_2)\Phi \text{ weakly in } \mathcal{L}(L^2(X,\mathcal{B},\mu)).$$

Proof. Let $\rho\in J(\mathcal{T})$ be a self-joining such that $T_{t_n}\to\Phi_\rho$ weakly. By Lemma 2.2, $\rho\ge\alpha_1\mu_{Id}$ and $\rho\ge\alpha_2\mu_{T_\theta}$. By the ergodicity of $\mathcal{T}$, $\mu(\{x\in X: T_\theta x=x\})=0$, and hence the measures μ_{Id} and μ_{T_θ} are orthogonal. It follows that

$$\rho\ge\alpha_1\mu_{Id}+\alpha_2\mu_{T_\theta}=(\alpha_1+\alpha_2)\frac{\alpha_1\mu_{Id}+\alpha_2\mu_{T_\theta}}{\alpha_1+\alpha_2}.$$

Since ρ is probabilistic, we obtain $\alpha_1+\alpha_2\le 1$. Now we apply Lemma 2.2 to the operator associated with the self-joining $\frac{\alpha_1\mu_{Id}+\alpha_2\mu_{T_\theta}}{\alpha_1+\alpha_2}\in J(\mathcal{T})$. This completes the proof. □

2.2. **Special flows.** Assume that T is an ergodic automorphism of a standard probability space $(X,\mathcal{B},\mu)$ and let $f\in L^1(X,\mathcal{B},\mu)$ be a positive function. Denote by λ the Lebesgue measure on $\mathbb{R}$. Let $X^f=\{(x,r)\in X\times\mathbb{R};\ 0\le r<f(x)\}$ and let $\mathcal{B}^f$ and μ^f be the restrictions of $\mathcal{B}\otimes\mathcal{B}(\mathbb{R})$ and $\mu\otimes\lambda$ to X^f. Denote by $T^f=\{(T^f)_t\}_{t\in\mathbb{R}}:(X^f,\mathcal{B}^f,\mu^f)\to(X^f,\mathcal{B}^f,\mu^f)$ the *special flow* built over T and under f. The special flow moves each point in X^f vertically at unit speed and points $(x,f(x))$ and $(Tx,0)$ are identified. For any $n\in\mathbb{Z}$ let

$$f^{(n)}(x)=\begin{cases} f(x)+\ldots+f(T^{n-1}x) & \text{if} \quad n>0,\\ 0 & \text{if} \quad n=0,\\ -(f(T^nx)+\ldots+f(T^{-1}x)) & \text{if} \quad n<0.\end{cases}$$

Consider the skew product $S_{-f}:(X\times\mathbb{R},\mathcal{B}\otimes\mathcal{B}(\mathbb{R}),\mu\otimes\lambda)\to(X\times\mathbb{R},\mathcal{B}\otimes\mathcal{B}(\mathbb{R}),\mu\otimes\lambda)$ given by $S_{-f}(x,r)=(Tx,-f(x)+r)$. Thus $(S_{-f})^k(x,r)=(T^kx,-f^{(k)}(x)+r)$ for each $k\in\mathbb{Z}$. Moreover,

$$T^f_t(x,r)=(S_{-f})^k(x,r+t)=(T^kx,r+t-f^{(k)}(x)),$$

where $k\in\mathbb{Z}$ is given by $f^{(k)}(x)\le r+t<f^{(k+1)}(x)$

If $f\equiv 1$, the special flow T^f is called a *suspension flow*. For more information on special flows we refer the reader to [4].

To apply Propositions 1.1 and 1.2 we will need the following result from [9].

Proposition 2.5. *Let (X,d) be a compact metric space. Let $\mathcal{B}$ stand for the $\sigma-$algebra of Borel subsets of X and let μ be a probability Borel measure on X. Suppose that $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is an ergodic measure–preserving automorphism and there exist an increasing sequence of natural numbers $\{q_n\}$ and a sequence of Borel sets $\{C_n\}$ such that*

$$\mu(C_n)\to\alpha>0,\ \ \mu(C_n\triangle T^{-1}C_n)\to 0\quad\textit{and}\ \sup_{x\in C_n} d(x,T^{q_n}x)\to 0.$$

Assume that $f\in L^2(X,\mu)$ a positive function bounded away from zero and $\{a_n\}$ is a sequence of real numbers such that the sequence $\left\{\int_{C_n}|f^{(q_n)}(x)-a_n|^2d\mu(x)\right\}$ is bounded. Suppose that

$$\frac{1}{\mu(C_n)}\left((f^{(q_n)}-a_n)|_{C_n}\right)_*(\mu|_{C_n})\to P$$

weakly in the set of probability Borel measures on $\mathbb{R}$. Then $\{(T^f)_{a_n}\}$ converges weakly to the operator

$$\alpha\int_{\mathbb{R}}(T^f)_{-t}\,dP(t)+(1-\alpha)J,$$

where $J\in\mathcal{J}(T^f)$.

2.3. **Substitutions.** Let us consider a finite alphabet $A=\{0,\dots,r-1\}$ and let A^* stand for the set of nonempty finite words over A. Each map $\sigma:A\to A^*$ is called a *substitution*. The map σ can be extended to $\sigma:A^*\to A^*$, $\sigma:A^{\mathbb{N}}\to A^{\mathbb{N}}$ and $\sigma:A^{\mathbb{Z}}\to A^{\mathbb{Z}}$ by taking concatenations. The *associated matrix* with the substitution σ is the matrix $M=[m_{ij}]_{0\le i,j\le r-1}$, where $m_{i,j}$ is the number of occurrences of the symbol i in the word $\sigma(j)$. A substitution σ is called *primitive*, if there exists $n\in\mathbb{N}$ such that all entries of M^n are strictly positive. Perron-Frobenius theorem states that for each primitive matrix, there exists an eigenavalue $\theta>0$, which is greater than the absolute value of any other eigenvalue of M. Moreover, there exist left and right eigenvectors associated with θ with positive entries.

The space $A^{\mathbb{Z}}$ is endowed with the metric

$$d(x,y)=\frac{1}{1+\inf\{|k|;x_k\neq y_k\}}$$

for each $x,y\in A^{\mathbb{Z}}$. For any primitive substitution σ there is at least one sequence $u\in A^{\mathbb{Z}}$ such that $u=\sigma^k(u)$ for some $k\ge 1$. If the sequence u is not periodic, the substitution σ is called *aperiodic*. Let $S:A^{\mathbb{Z}}\to A^{\mathbb{Z}}$ stand for the left shift defined by $(Sx)_n=x_{n+1}$. Let $X_\sigma=\overline{\{S^nu:\ n\in\mathbb{Z}\}}$. Denote by $\mathcal{L}_\sigma\subset A^*$ the language which contains of all finite words which occur in the sequence u. Therefore $\{x_n\}_{n\in\mathbb{Z}}\in X_\sigma$ if and only if $x_n\dots x_{n+k}\in\mathcal{L}_\sigma$ for all $n\in\mathbb{Z}$ and $k\ge 0$. Since X_σ is S-invariant, we can consider the restriction of S to X_σ. This topological dynamical system will be denoted by $S_\sigma:X_\sigma\to X_\sigma$ and is called a *substitution dynamical system*. If σ is primitive and aperiodic then the dynamical system (X_σ,S_σ) is minimal and uniquely ergodic (see [16]). Denote by μ_σ the unique S_σ-invariant probability measure.

For any $\bar{w}=w_0\dots w_{m-1}\in A^*$, the vector $l(\bar{w})=(l_0(\bar{w}),\dots,l_{r-1}(\bar{w}))$ with $l_i(\bar{w})=\#\{j:w_j=i,0\le j\le m-1\}$ for any $0\le i\le r-1$, is called the *population vector* of the word $\bar{w}$. Following [3], a finite word $\bar{w}=w_0\dots w_{k-1}\in\mathcal{L}_\sigma$ is called a *recurrence word* of σ if $\bar{w}w_0=w_0\dots w_{k-1}w_0\in\mathcal{L}_\sigma$.

3. Interval exchange transformations.

3.1. Introduction. Let $m \geq 2$ be a fixed natural number and let

$$\mathbb{R}^m_+ = \{\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m \setminus \{0\} : \ \lambda_i \geq 0,\ 1 \leq i \leq m\}.$$

Denote by S_m the set of all permutations of $\{1, \dots, m\}$. A permutation π is called *irreducible*, if $\pi\{1, \dots, k\} \neq \{1, \dots, k\}$ for any $1 \leq k \leq m-1$. The set of all irreducible permutations is denoted by S^0_m. Given $(\lambda, \pi) \in \mathbb{R}^m_+ \times S^0_m$ let $T = T_{\lambda,\pi} : [0, |\lambda|) \to [0, |\lambda|)$ ($|\lambda| = \sum_{i=1}^m \lambda_i$) stand for the *interval exchange transformation (IET)* of m intervals $\Delta_j = \left[\sum_{i=1}^{j-1} \lambda_i, \sum_{i=1}^{j} \lambda_i\right)$, $j = 1, \dots, m$, which are rearranged according to the permutation π. Denote by μ the restriction of the Lebesgue measure on $\mathbb{R}$ to the interval $[0, |\lambda|)$. Then $T : ([0, |\lambda|), \mu) \to ([0, |\lambda|), \mu)$ is a measure–preserving automorphism. Note that T^{-1} is also an IET of intervals $T\Delta_{\pi^{-1}(1)}, \dots, T\Delta_{\pi^{-1}(m)}$.

Denote by $\beta_0, \dots, \beta_{m-1}$ the left endpoints of the intervals $\Delta_1, \dots, \Delta_m$. Let $Orb(x) = \{T^n x : \ n \in \mathbb{Z}\}$ stand for the orbit of point x. Following [12], we say that T satisfies the *infinite distinct orbit condition (IDOC)*, if every orbit of β_s, $1 \leq s \leq m-1$, is infinite and $Orb(\beta_s) \cap Orb(\beta_t) = \emptyset$ for all $1 \leq s \neq t \leq m-1$. Recall that (see [12]) if $\pi \in S^0_m$ then $T_{\lambda,\pi}$ satisfies the IDOC for a.e. $\lambda \in \mathbb{R}^m_+$. Moreover, every IET satisfying the IDOC is minimal, i.e. every orbit is dense (see [12]).

For any subinterval $Z \subset [0, |\lambda|)$ we can define the induced IET $T_Z : Z \to Z$ by the formula $T_Z(x) = T^{k(x)}(x)$ for any $x \in Z$, where $k(x) \in \mathbb{N}$ is the first positive return time of x to Z. The induced map T_Z is an exchange of at most $m+2$ intervals (see e.g. [4]).

Let $d(\lambda, \lambda') = \sum_{i=1}^m |\lambda_i - \lambda'_i|$ for all $\lambda, \lambda' \in \mathbb{R}^m_+$. We will consider on $\mathbb{R}^m_+ \times S^0_m$ the metric

$$\bar{d}((\lambda, \pi), (\lambda', \pi')) = d(\lambda, \lambda') + \delta_{\pi,\pi'}.$$

3.2. Rauzy induction. Suppose that $T = T_{\lambda,\pi} : [0, |\lambda|) \to [0, |\lambda|)$ is an IET given by $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m_+$ and permutation $\pi \in S^0_m$. Assume that $T_{\lambda,\pi}$ satisfies the IDOC. By *Rauzy induction* of IETs we mean a special kind of induction, where $Z := [0, |\lambda| - \min\{\lambda_m, \lambda_{\pi^{-1}(m)}\})$. By IDOC, $\lambda_m \neq \lambda_{\pi^{-1}(m)}$. Then T_Z is again an exchange of m intervals and we can write $T_Z = T_{\lambda',\pi'}$, where $(\lambda', \pi') \in \mathbb{R}^m_+ \times S^0_m$. Let us consider two maps $a, b : S^0_m \to S^0_m$

$$a\pi(i) = \begin{cases} \pi(i), & i \leq \pi^{-1}(m) \\ \pi(m), & i = \pi^{-1}(m) + 1, \\ \pi(i-1), & i > \pi^{-1}(m) + 1 \end{cases} \quad b\pi(i) = \begin{cases} \pi(i), & \pi(i) \leq \pi(m) \\ \pi(i) + 1, & \pi(m) < \pi(i) < m \\ \pi(m) + 1, & \pi(i) = m. \end{cases}$$

and two $m \times m$ matrices

$$A(a, \pi) = \begin{array}{cl} \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow \pi^{-1}(m) \\ \\ \\ \\ \\ \end{array} \end{array}$$

$$A(b,\pi)=\begin{array}{c}\begin{pmatrix}1 & & \dots & & 0\\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0\,1\,0 & \dots & 1\end{pmatrix}\\ \uparrow \\ \pi^{-1}(m)\end{array}.$$

The induced IET is then given by $(\lambda',\pi')=(A(c,\pi)^{-1}\lambda, c(\pi))$, where

$$c=c(\lambda,\pi)=\begin{cases} a & \text{if} \quad \lambda_m<\lambda_{\pi^{-1}(m)}\\ b & \text{if} \quad \lambda_m>\lambda_{\pi^{-1}(m)}.\end{cases}$$

By IDOC, we can define Rauzy induction for the IET given by (λ',π'). In fact, this procedure can be continued for any finite number of steps. Let $\Lambda_m=\{\lambda\in\mathbb{R}_+^m : |\lambda|=1\}$. Define maps

$$J:\mathbb{R}_+^m\times S_m^0\to\mathbb{R}_+^m\times S_m^0,\ J(\lambda,\pi)=(A^{-1}(c,\pi)\lambda, c\pi),\ c=c(\lambda,\pi)$$

and

$$P:\Lambda_m\times S_m^0\to\Lambda_m\times S_m^0,\ P(\lambda,\pi)=\left(\frac{A^{-1}(c,\pi)\lambda}{|A^{-1}(c,\pi)\lambda|}, c\pi\right),\ c=c(\lambda,\pi).$$

If $(\lambda',\pi'):=J^n(\lambda,\pi)$ then

$$\pi'=\pi_n,\ \lambda'=A(c_n,\pi_{n-1})^{-1}A(c_{n-1},\pi_{n-2})^{-1}\dots A(c_2,\pi_1)^{-1}A(c_1,\pi)^{-1}\lambda,$$

where

$$c_k=c_k(\lambda,\pi)=c(J^{k-1}(\lambda,\pi)),\ \pi_k=c_k\circ\ldots\circ c_1\circ\pi.$$

Hence

$$\lambda=A^{(n)}(\lambda,\pi)\cdot\lambda',\ \text{where } A^{(n)}(\lambda,\pi)=A(c_1,\pi)\dots A(c_n,\pi_{n-1}).$$

By *Rauzy graph* we mean a directed graph whose vertices are irreducible permutations and edges connect permutations obtained one from the other by applying maps a or b and are labeled according to the type, a or b respectively. Any connected component of the Rauzy graph is called a *Rauzy class*. Denote by $\mathcal{R}(\pi)\subset S_m^0$ the Rauzy class containing the permutation $\pi\in S_m^0$. See [17], [22], [23] for more details.

Theorem 3.1 ([22]). *Let $\mathcal{R}\subset S_m^0$ be a fixed Rauzy class. There exists on $\Lambda_m\times\mathcal{R}$ a smooth positive σ-finite P-invariant measure κ, which is ergodic and conservative with respect to P and which is equivalent to the Lebesgue measure.*

Theorem 3.2 ([15], [22]). *If $\pi\in S_m^0$ then for a.e. $\lambda\in\mathbb{R}_+^m$ the IET $T_{\lambda,\pi}$ is uniquely ergodic.*

Notation. Let $T_{\lambda,\pi}:[0,|\lambda|)\to[0,|\lambda|)$ be an IET. Each point $x\in[0,|\lambda|)$ is coded by a sequence $\{w_n\}_{n\in\mathbb{Z}}\in\{1,\dots,m\}^{\mathbb{Z}}$ so that $T^nx\in\Delta_{w_n}$ for $n\in\mathbb{Z}$. Denote by $\mathcal{L}(T_{\lambda,\pi})\subset\{1,\dots,m\}^*$ the language determined by all such sequences, i.e. $\mathcal{L}(T_{\lambda,\pi})$ is the set of all finite words (over the alphabet $\{1,\dots,m\}$) which occur in such sequences. Therefore $\bar{w}=w_0\dots w_{k-1}\in\mathcal{L}(T_{\lambda,\pi})$ if and only if the set $\bigcap_{j=0}^{k-1}T_{\lambda,\pi}^{-j}\Delta_{w_j}$ is not empty. A word $\bar{w}=w_0\dots w_{k-1}\in\mathcal{L}(T_{\lambda,\pi})$ is called *recurrence* if $w_0\dots w_{k-1}w_0\in\mathcal{L}(T_{\lambda,\pi})$. As in Subsection 2.3 for every $\bar{w}=w_0\dots w_{k-1}\in\mathcal{L}(T_{\lambda,\pi})$ denote by $l(\bar{w})=(l_1(\bar{w}),\dots,l_m(\bar{w}))^T$ the *population vector*, i.e. $l_i(\bar{w})=\#\{j:w_j=i, 0\le j\le k-1\}$ for $1\le i\le m$.

Remark 3.3. Suppose that $T = T_{\lambda,\pi} : [0, |\lambda|) \to [0, |\lambda|)$ fulfills the IDOC. Consider $T_{(n)} = T_{\lambda^{(n)},\pi^{(n)}}$, where $(\lambda^{(n)}, \pi^{(n)}) = J^n(\lambda, \pi)$. Let $\Delta_1^{(n)}, \ldots, \Delta_m^{(n)}$ stand for the intervals exchanged by $T_{(n)}$. Then the action of the initial IET T can be seen in terms of Rohlin towers over $\Delta_1^{(n)}, \ldots, \Delta_m^{(n)} \subset [0, |\lambda|)$. Let

$$h^{(n)} = (h_1^{(n)}, \ldots, h_m^{(n)}) = (1, \ldots, 1)A^{(n)}(\lambda, \pi).$$

Then $h_i^{(n)}$ is the first return time for the action of T on $\Delta_i^{(n)}$ to $[0, |\lambda^{(n)}|)$ for $1 \le i \le m$ and

$$A^{(n)}(\lambda, \pi)_{ij} = \#\{0 \le k < h_i^{(n)} : T^k \Delta_j^{(n)} \subset \Delta_i\}.$$

Moreover, $\Xi_i = \{T^j \Delta_i^{(n)} : 0 \le j < h_j^{(n)}\}$ is a Rohlin tower of intervals and the towers $\Xi_1, \ldots, \Xi_m$ are pairwise disjoint and fill the whole interval $[0, |\lambda|)$. It follows that

$$\sum_{i=1}^{m} h_j^{(n)} \lambda_j^{(n)} = |\lambda|. \tag{3}$$

Remark 3.4. For any $\bar{w} = w_0 \ldots w_{K-1} \in \mathcal{L}(T_{(n)})$ let $q_{\bar{w}}^{(n)} = \sum_{i=0}^{K-1} h_{w_i}^{(n)}$. If $x \in [0, |\lambda^{(n)}|)$ and $T_{(n)}^i x \in \Delta_{w_i}^{(n)}$ for $0 \le i < K$ then $T_{(n)}^K x = T^{q_{\bar{w}}^{(n)}} x$. Moreover,

$$q_{\bar{w}}^{(n)} = \sum_{i=0}^{K-1} h_{w_i}^{(n)} = h^{(n)} l(w_0 \ldots w_{K-1}) = |A^{(n)}(\lambda, \pi) l(w_0 \ldots w_{K-1})|, \tag{4}$$

where $l(w_0 \ldots w_{K-1})$ is the population vector of the word $w_0 \ldots w_{K-1}$. Next consider the word $\bar{w}' = w_0' \ldots w'_{q_{\bar{w}}^{(n)}-1} \in \mathcal{L}(T)$ such $T^j x \in \Delta_{w_j'}$ for $0 \le j < q_{\bar{w}}^{(n)}$. Thus

$$l(\bar{w}') = A^{(n)}(\lambda, \pi) l(\bar{w}). \tag{5}$$

Definition 3.5 (see [19])**.** An IET $T_{\lambda,\pi}$ is called of *periodic type*, if the following holds:

(i) There exists $k > 1$ such that $P^{n+k}(\lambda, \pi) = P^n(\lambda, \pi)$ for all $n \in \mathbb{N}$;
(ii) The matrix $A^{(k)}(\lambda, \pi)$ has strictly positive entries.

The matrix $(A^{(k)}(\lambda, \pi))^T$ is also known as a *period matrix*. It was shown in [14] how to produce matrices with strictly positive entries by walking on the Rauzy graph. Furthermore, in [19] (see Lemma 6) the authors presented a simple method for searching IETs of periodic type using closed paths in the Rauzy graph.

Remark 3.6. Each IET of periodic type satisfies the IDOC (see [19]).

3.3. **Permutation η_π and subspace $H(\pi)$.** Let $\pi \in S_m^0$. Following [22], define the permutation η_π on $\{0, \ldots, m\}$ as follows:

$$\eta_\pi(i) = \begin{cases} \pi^{-1}(1) - 1 & \text{if } i = 0 \\ m & \text{if } i = \pi^{-1}(m) \\ \pi^{-1}(\pi(i) + 1) - 1 & \text{otherwise.} \end{cases}$$

The set $\{0, \ldots, m\}$ is partitioned by η_π into *cyclic subsets*, i.e. into the orbits of η_π. Denote by $\Sigma(\pi)$ the set of all cyclic sets of the permutation η_π. To each $S \in \Sigma(\pi)$ we associate a vector $b(S) \in \mathbb{Z}^m$ given by

$$b(S)_i = \chi_S(i - 1) - \chi_S(i), \; 1 \le i \le m,$$

where χ_S is the characteristic function of S. By $|b(S)|$ denote the sum of entries of the vector $b(S)$.

Proposition 3.7 ([23], Proposition 5.11)**.** *For each* $S \in \Sigma(\pi)$*,*

$$|b(S)| = \begin{cases} 1 & \text{if } 0 \in S,\ m \notin S, \\ -1 & \text{if } 0 \notin S,\ m \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.8 ([23], Lemma 5.6)**.** *Let* $\pi \in S_m^0$ *and* $c = a$ *or* b*. There is a bijection* $\Sigma(\pi) \ni S \mapsto cS \in \Sigma(c\pi)$ *such that*

$$b(S) = A(c,\pi)b(cS) \text{ for every } S \in \Sigma(\pi).$$

Let us recall the definition of the alternating $m \times m$ matrix L^π:

$$L^\pi_{ij} = \begin{cases} 1 & \text{if } i < j \text{ and } \pi(i) > \pi(j), \\ -1 & \text{if } i > j \text{ and } \pi(i) < \pi(j), \\ 0 & \text{otherwise.} \end{cases}$$

Define $H(\pi) = L^\pi(\mathbb{R}^m)$. Since L^π is anti-symmetric, $H(\pi) = (\ker L^\pi)^\perp$.

Proposition 3.9 ([23], Proposition 5.2)**.** *A vector* $h \in H(\pi)$ *if and only if* $h \cdot b(S) = 0$ *for every* $S \in \Sigma(\pi)$*.*

Remark 3.10. Let $\widetilde{S}_m^0$ stand for the subset of irreducible permutations π of m elements such that if $S \in \Sigma(\pi)$, then $|b(S)| = \pm 1$. The condition $\pi \in \widetilde{S}_m^0$ is quite restrictive. In view of Proposition 3.7, this implies $\#\Sigma(\pi) = 2$, hence if $\Sigma(\pi) = \{S_0, S_1\}$ with $0 \in S_0$ then $m \in S_1$ and $b(S_1) = -b(S_0)$. Note that if m is odd then $\tau_m^{sym} \in \widetilde{S}_m^0$ and $b(S_0) = (1, -1, 1, \ldots, -1, 1)$.

Let $E = [E_{ij}]_{1 \le i,j \le m}$ be a matrix with strictly positive entries. Following [21] set

$$\nu(E) = \max_{1 \le i,j,k \le m} \frac{E_{ij}}{E_{ik}} \text{ and } e_j = \sum_{i=1}^m E_{ij}.$$

Then

$$e_j \le \nu(E)e_k,\ 1 \le j,k \le m \text{ and } \nu(FE) \le \nu(E), \tag{6}$$

for any nonnegative nonsingular matrix $F = [F_{ij}]_{1 \le i,j \le m}$.

3.4. **IETs and substitutions.** For any IET $T := T_{\lambda,\pi} : [0, |\lambda|) \to [0, |\lambda|)$ of m intervals $\Delta_1, \ldots, \Delta_m$ and $n \in \mathbb{N}$ we can define a substitution on m symbols (see [7]). We shortly describe this procedure. Let $(\lambda', \pi') = J^n(\lambda, \pi)$. Let $\Delta'_1, \ldots, \Delta'_m$ be the exchanged intervals for $T_{\lambda',\pi'}$. Let $k : [0, |\lambda'|) \to \mathbb{N}$ stand for the first return time map to the interval $[0, |\lambda'|)$. Let $A = \{1, \ldots, m\}$. We define a map $cod : [0, |\lambda'|) \to A^*$ in the following way: $cod(x) = i_1 \ldots i_{k(x)}$ if $T^j x \in \Delta_{i_j}$ for $0 \le j < k(x)$. Since the map $cod : [0, |\lambda'|) \to A^*$ is constant on every interval $\Delta'_1, \ldots, \Delta'_m$, we can define a substitution $\sigma : A \to A^*$ so that $\sigma(i) = cod(x)$ for each $x \in \Delta'_i$ and $i = 1, \ldots, m$.

Remark 3.11. Suppose that $T_{\lambda,\pi}$ is an IET of periodic type for which $A^{(n)}(\lambda,\pi)^T$ is its periodic matrix. Let us consider the corresponding substitution $\sigma : A \to A^*$. Then $T_{\lambda,\pi}$ is measure-theoretically isomorphic to the substitution system S_σ. The isomorphism is established by the map $\phi : [0, |\lambda|) \to A^{\mathbb{Z}}$,

$$(\phi(x))_i = j, \text{ if } T^i x \in \Delta_j,\ 1 \le j \le m, \text{ for all } i \in \mathbb{Z}.$$

It follows that $\mathcal{L}(T_{\lambda,\pi}) = \mathcal{L}_\sigma$. Moreover, $A^{(n)}(\lambda,\pi)$ is the substitution matrix of σ.

4. **Disjointness from ELF system.**

4.1. **Perturbation of IET.** In this section we state three lemmas which, roughly speaking, say that any type of finite orbit combinatorics of any IET satisfying the IDOC is preserved in the passage to its slight perturbation. The proof of the lemmas is rather straightforward and we leave it to the reader.

Let $T := T_{\lambda,\pi} : [0,|\lambda|) \to [0,|\lambda|)$ be an IET of m intervals $\Delta_1,\ldots,\Delta_m$ of lengths $\lambda_1,\ldots,\lambda_m \geq 0$ and given by an irreducible permutation π. Recall that

$$Tx = x + \sum_{\pi(j)<\pi(i)} \lambda_{\pi(j)} - \sum_{j<i} \lambda_j \text{ if } x \in \Delta_i,$$

$$T^{-1}x = x - \sum_{\pi(j)<\pi(i)} \lambda_{\pi(j)} + \sum_{j<i} \lambda_j \text{ if } x \in T\Delta_i,$$

and $\beta_0 = 0$, $\beta_s = \sum_{j=1}^{s} \lambda_j$ for $1 \leq s \leq m-1$. Assume that T satisfies the *weak IDOC*, i.e. every orbit of β_s, $1 \leq s \leq m-1$ is infinite and $Orb(\beta_s) \cap Orb(\beta_t) \neq \emptyset$ for $1 \leq s,t \leq m-1$ implies $\beta_s = \beta_t$.

Fix $K \geq 1$. Let

$$0 < \delta < \min\{|T^{-j_1}\beta_{t_1} - T^{-j_2}\beta_{t_2}| : |j_1|,|j_2| \leq K,\ 1 \leq t_1,t_2 \leq m-1, \beta_{t_1} \neq \beta_{t_2}\}.$$

Fix an $\varepsilon = \varepsilon_{T,K} > 0$ such that

$$m(2K+4)\varepsilon < \frac{1}{10}\delta. \tag{7}$$

Let

$$\mathcal{K} := \{(\lambda^\varepsilon,\pi) \in \mathbb{R}_+^m \times \{\pi\} :\ |\lambda^\varepsilon| = |\lambda|,\ |\lambda_i - \lambda_i^\varepsilon| < \varepsilon,\ 1 \leq i \leq m\}.$$

Suppose that $T_\varepsilon := T_{\lambda^\varepsilon,\pi} : [0,|\lambda|) \to [0,|\lambda|)$ is an IET such that $(\lambda^\varepsilon,\pi) \in \mathcal{K}$. Let $\Delta_1^\varepsilon,\ldots,\Delta_m^\varepsilon$ be the intervals exchanged by T_ε and let $\beta_0^\varepsilon = 0$, $\beta_i^\varepsilon = \sum_{j\leq i} \lambda_j^\varepsilon$, $i = 1,\ldots,m-1$. Note that

$$|\beta_i - \beta_i^\varepsilon| < m\varepsilon \text{ and } |T\beta_i - T_\varepsilon\beta_i^\varepsilon| < m\varepsilon \text{ for } 0 \leq i \leq m-1.$$

Indeed,

$$|\beta_t - \beta_t^\varepsilon| = |\sum_{j\leq t} \lambda_j - \sum_{j\leq t} \lambda_j^\varepsilon| \leq \sum_{j\leq t} |\lambda_j - \lambda_j^\varepsilon| < m\varepsilon,$$

$$|T\beta_t - T_\varepsilon\beta_t^\varepsilon| = |\sum_{\pi(j)<\pi(t+1)} \lambda_j - \sum_{\pi(j)<\pi(t+1)} \lambda_j^\varepsilon| \leq \sum_{\pi(j)<\pi(t+1)} |\lambda_j - \lambda_j^\varepsilon| < m\varepsilon.$$

Lemma 4.1. *For every $0 \leq s \leq K+1$ and $0 \leq t \leq m-1$*

$$|T^{-s+1}\beta_t - T_\varepsilon^{-s+1}\beta_t^\varepsilon| < m(2s+1)\varepsilon \text{ and } T^{-s}\beta_t \in \Delta_i \text{ implies } T_\varepsilon^{-s}\beta_t^\varepsilon \in \Delta_i^\varepsilon.$$

Moreover,

$$|T^{s}\beta_t - T_\varepsilon^{s}\beta_t^\varepsilon| < m(2s+1)\varepsilon \text{ and } T^{s}\beta_t \in \Delta_i \text{ implies } T_\varepsilon^{s}\beta_t^\varepsilon \in \Delta_i^\varepsilon$$

for all $0 \leq s \leq K$ and $0 \leq t \leq m-1$. □

Lemma 4.2. *For all $0 \leq s_1,s_2 \leq K$ and $0 \leq t_1,t_2 \leq m-1$,*

$$T^{-s_1}\beta_{t_1} < T^{-s_2}\beta_{t_2} \iff T_\varepsilon^{-s_1}\beta_{t_1}^\varepsilon < T_\varepsilon^{-s_2}\beta_{t_2}^\varepsilon.$$

□

Take $\bar{w} = w_0 \dots w_K \in \{1, \dots, m\}^{K+1}$ such that the set $I_{\bar{w}} = \bigcap_{i=0}^{K} T^{-i}\Delta_{w_i}$ is not empty, i.e. $\bar{w} \in \mathcal{L}(T)$. The set $I_{\bar{w}}$ is an interval of the form $[T^{-k_1}\beta_{t_1}, T^{-k_2}\beta_{t_2})$, where $0 \le k_1, k_2 \le K$ and $0 \le t_1, t_2 \le m-1$. It follows that $|\bigcap_{i=0}^{K} T^{-i}\Delta_{w_i}| > \delta$. Let

$$I_{\bar{w}}^{\varepsilon} = [T_{\varepsilon}^{-k_1}\beta_{t_1}^{\varepsilon}, T_{\varepsilon}^{-k_2}\beta_{t_2}^{\varepsilon}).$$

In view of (7), the following result is a simple consequence of Lemmas 4.1 and 4.2.

Lemma 4.3. *For each $\bar{w} = w_0 \dots w_K \in \{1, \dots, m\}^{K+1}$ if $I_{\bar{w}} \neq \emptyset$ and $(\lambda^{\varepsilon}, \pi) \in \mathcal{K}$ then*

$$|I_{\bar{w}}^{\varepsilon}| \ge \frac{4}{5}|I_{\bar{w}}| \text{ and } I_{\bar{w}}^{\varepsilon} \subset \bigcap_{i=0}^{K} T_{\varepsilon}^{-i}\Delta_{w_i}^{\varepsilon}.$$

□

4.2. **Disjointness theorems.** Fix $\tau \in S_m^0$. Let $T_{\lambda,\tau} : [0,1) \to [0,1)$ be an IET with the IDOC, which has two recurrence words $\bar{w}_1 \in \mathcal{L}(T_{\lambda,\tau})$ and $\bar{w}_2 \in \mathcal{L}(T_{\lambda,\tau})$ with lengths K_1, K_2 respectively and such that

$$l(\bar{w}_1) - l(\bar{w}_2) = b(S) \text{ for some } S \in \Sigma(\tau). \tag{8}$$

Let $K = \max\{K_1, K_2\} + 1$ and choose $\varepsilon = \varepsilon_{T_{\lambda,\tau},K} > 0$ such that (7) holds. Let

$$\mathcal{K} = \{(\lambda^{\varepsilon}, \tau) : \ |\lambda^{\varepsilon}| = |\lambda| = 1, \ |\lambda_i - \lambda_i^{\varepsilon}| < \varepsilon, \ 1 \le i \le m\}.$$

Choose $\bar{w}_1^{ext} = w_0^1 \dots w_K^1 \in \mathcal{L}(T_{\lambda,\tau})$ and $\bar{w}_2^{ext} = w_0^2 \dots w_K^2 \in \mathcal{L}(T_{\lambda,\tau})$ which are extensions of $\bar{w}_1, \bar{w}_2$ such that $w_{K_1}^1 = w_0^1$ and $w_{K_2}^2 = w_0^2$. Let $\theta_r = |\bigcap_{i=0}^{K} T_{\lambda,\tau}^{-i}\Delta_{w_i^r}^{\lambda,\tau}|$ for $r = 1, 2$.

Using some standard Veech's arguments (see [23]), there exist $k > 1$ and maps $c_1, \dots, c_k$ ($c_i = a$ or b for $i = 1, \dots, k$) such that if $\tau_i = c_i \circ \dots \circ c_1 \circ \tau$ ($i = 1, \dots, k$) then

(i) $\tau_k = \tau$,
(ii) the matrix $B = A(c_1, \tau)A(c_2, \tau_1) \dots A(c_k, \tau_{k-1})$ has strictly positive entries.
(iii) $c_i = c_i(\rho, \tau)$ for any $1 \le i \le k$.

Let $\mathcal{M} = \{(B\lambda^{\varepsilon}/|B\lambda^{\varepsilon}|, \tau) : \ (\lambda^{\varepsilon}, \tau) \in \mathcal{K}\}$. Since $\mathcal{K}$ is open, the set $\mathcal{M}$ is also open in $(\Lambda_m, \bar{d})$, hence $\mathcal{M} \subset \Lambda_m \times \mathcal{R}(\tau)$ has positive measure κ, where κ is the invariant measure from Theorem 3.1.

Notation. Denote by $\mathcal{A}$ the set of all $(\rho, \pi) \in \Lambda_m \times \mathcal{R}(\tau)$ such that the IET $T_{\rho,\pi}$ satisfies the IDOC and there exists an increasing sequence $\{k_n\}_{n\in\mathbb{N}}$ of natural numbers such that $P^{k_n}(\rho, \pi) \in \mathcal{M}$.

Remark 4.4. By the ergodicity and conservativity of $P : (\Lambda_m \times \mathcal{R}(\tau), \kappa) \to (\Lambda_m \times \mathcal{R}(\tau), \kappa)$ (see Theorem 3.1), $\kappa((\Lambda_m \times \mathcal{R}(\tau)) \setminus \mathcal{A}) = 0$.

Take $(\rho, \pi) \in \mathcal{A}$ and let $T = T_{\rho,\pi} : [0,1) \to [0,1)$. Let $\Delta_1, \dots, \Delta_m$ stand for the intervals exchanged by T. Let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence such that $P^{k_n}(\rho, \pi) \in \mathcal{M}$. By the definition of $\mathcal{M}$, $P^{k_n+k}(\rho, \pi) \in \mathcal{K}$ and $A^{(k_n+k)}(\rho, \pi) = A^{(k_n)}(\rho, \pi)B$.

Let $(\rho^{(k_n)}, \tau) := J^{k_n+k}(\rho, \pi)$ and let $\Delta_1^{(k_n)}, \dots, \Delta_m^{(k_n)}$ stand for the intervals exchanged by $T_{\rho^{(k_n)},\tau}$. Then $(\rho^{(k_n)}/|\rho^{(k_n)}|, \tau) := P^{k_n+k}(\rho, \pi) \in \mathcal{K}$. Let us apply Lemma 4.3 to $T_{\rho^{(k_n)}/|\rho^{(k_n)}|,\tau}$ and the words $\bar{w}_1^{ext}$, $\bar{w}_2^{ext}$. After rescaling we obtain two intervals

$$I_1^{(k_n)} \subset \bigcap_{i=0}^{K_1} T_{\rho^{(k_n)},\tau}^{-i}\Delta_{w_i^1}^{(k_n)} \text{ and } I_2^{(k_n)} \subset \bigcap_{i=0}^{K_2} T_{\rho^{(k_n)},\tau}^{-i}\Delta_{w_i^2}^{(k_n)}$$

such that $|I_1^{(k_n)}| \geq \frac{4}{5}\theta_1|\rho^{(k_n)}|$ and $|I_2^{(k_n)}| \geq \frac{4}{5}\theta_2|\rho^{(k_n)}|$.

By Remark 3.3, $h_j^{(k_n)} = \sum_{i=1}^m A_{ij}^{(k_n+k)}$ is the first return time of the interval $\Delta_j^{(k_n)}$ to $[0,|\rho^{(k_n)}|)$ for the action of T. Let

$$C_r^{(k_n)} := \bigcup_{i=0}^{h_{w_0^r}^{(k_n)}-1} T^i I_r^{(k_n)} \text{ for } r = 1,2.$$

Lemma 4.5. *There exists $\alpha > 0$ such that $\mu(C_r^{(k_n)}) \geq \alpha > 0$ for any $n \in \mathbb{N}$ and for $r = 1,2$.*

Proof. In view of (3), $1 = \sum_{j=1}^m h_j^{(k_n)}\rho_j^{(k_n)}$. Moreover, from (6),

$$h_j^{(k_n)} \leq \nu(A^{(k_n+k)})h_l^{(k_n)} = \nu\left(A^{(k_n)}(\rho,\pi)B\right)h_l^{(k_n)} \leq \nu(B)h_l^{(k_n)}$$

for all $1 \leq j, l \leq m$. It follows that

$$1 \leq \sum_{j=1}^m \nu(B)h_l^{(k_n)}\rho_j^{(k_n)} = \nu(B)h_l^{(k_n)}|\rho^{(k_n)}|$$

for any $1 \leq l \leq m$. Since $|I_r^{(k_n)}| \geq \frac{4}{5}\theta_r|\rho^{(k_n)}|$ for $r = 1,2$, we obtain

$$1 \leq \nu(B)h_{w_0^r}^{(k_n)}\frac{|I_r^{(k_n)}|}{\frac{4}{5}\theta_r}.$$

Hence

$$0 < \alpha := \frac{4\min(\theta_1,\theta_2)}{5\nu(B)} \leq h_{w_0^r}^{(k_n)}|I_r^{(k_n)}| = \mu\left(\bigcup_{i=0}^{h_{w_0^r}^{(k_n)}-1} T^i I_r^{(k_n)}\right) = \mu(C_r^{(k_n)})$$

for $r = 1,2$. □

Let

$$q_r^{(k_n)} = |A^{(k_n+k)}(\rho,\pi)l(\bar{w}_r)| = \sum_{j=0}^{K_r-1} h_{w_j^r}^{(k_n)} \text{ for } r = 1,2.$$

Since $I_r^{(k_n)} \subset \bigcap_{i=0}^{K_r} T_{\rho^{(k_n)},\tau}^{-i}\Delta_{w_i^r}^{(k_n)}$, by Remark 3.4,

$$T_{\rho^{(k_n)},\tau}^{K_r}x = T^{q_r^{(k_n)}}x \text{ for every } x \in I_r^{(k_n)}. \tag{9}$$

Lemma 4.6. *For $r = 1,2$ we have*

(i) $\mu(C_r^{(k_n)} \triangle T^{-1}C_r^{(k_n)}) \to 0$ *as* $n \to \infty$,

(ii) $\sup_{x\in C_r^{(k_n)}}|x - T^{q_r^{(k_n)}}x| \to 0$ *as* $n \to \infty$.

Proof. (i). Since $\{T^i I_r^{(k_n)} : 0 \leq i < h_{w_0^r}^{(k_n)}\}$ is a Rohlin tower,

$$TC_r^{(k_n)} \triangle C_r^{(k_n)} \subset I_r^{(k_n)} \cup T^{h_{w_0^r}^{(k_n)}} I_r^{(k_n)}.$$

It follows that

$$\mu(T^{-1}C_r^{(k_n)} \triangle C_r^{(k_n)}) \leq 2\mu(I_r^{(k_n)}) \leq 2|\rho^{(k_n)}| \to 0 \text{ as } n \to \infty.$$

(ii). For each $x \in C_r^{(k_n)}$ there exist $y \in I_r^{(k_n)}$ and $0 \le i < h_{w_0^r}^{(k_n)}$ such that $x = T^i y$. In view of (9) we have $T^{q_r^{(k_n)}} y = T^{K_r}_{\rho^{(k_n)},\tau} y$, so

$$T^{q_r^{(k_n)}} x = T^{q_r^{(k_n)}} T^i y = T^i T^{q_r^{(k_n)}} y = T^i T^{K_r}_{\rho^{(k_n)},\tau} y.$$

Since $I_r^{(k_n)} \subset \bigcap_{i=0}^{K_r} T^{-i}_{\rho^{(k_n)},\tau} \Delta^{(k_n)}_{w_i^r}$, we have $y \in \Delta^{(k_n)}_{w_0^r}$ and $T^{K_r}_{\rho^{(k_n)},\tau} y \in \Delta^{(k_n)}_{w^r_{K_r}} = \Delta^{(k_n)}_{w_0^r}$. It follows that $x = T^i y \in T^i \Delta^{(k_n)}_{w_0^r}$ and $T^{q_r^{(k_n)}} x = T^i T^{K_r}_{\rho^{(k_n)},\tau} y \in T^i \Delta^{(k_n)}_{w_0^r}$. Since $T^i \Delta^{(k_n)}_{w_0^r}$ is an interval of length $|\Delta^{(k_n)}_{w_0^r}|$, we obtain

$$|x - T^{q_r^{(k_n)}} x| < |\Delta^{(k_n)}_{w_0^r}|.$$

Consequently, in view of the IDOC,

$$\sup_{x \in C_r^{(k_n)}} |x - T^{q_r^{(k_n)}} x| < |\Delta^{(k_n)}_{w_0^r}| \to 0 \text{ as } n \to \infty.$$

$\square$

Let $b_\tau := b(S)$ stand for the vector associated with a cyclic set $S \subset \Sigma(\tau)$ so that (8) holds. Let

$$b_\pi^{(n)} = A^{(k_n+k)} b_\tau.$$

By Lemma 3.8, $b_\pi^{(n)} = b(S')$ for some $S' \in \Sigma(\pi)$.

Remark 4.7. Suppose that $x_r \in I_r^{(k_n)} \subset \bigcap_{i=0}^{K_r} T^{-i}_{\rho^{(k_n)},\tau} \Delta^{(k_n)}_{w_i^r}$ for $r = 1, 2$. Let us consider the word $\bar{w}'_r \in \{1, \ldots, m\}^{q_r^{(k_n)}}$ determined by $T^i_{\rho,\pi} x_r \in \Delta_{w_i'^r}$ for $0 \le i < q_r^{(k_n)}$ and $r = 1, 2$. From (5) we have $l(\bar{w}'_r) = A^{(k_n+k)} l(\bar{w}_r)$ for $r = 1, 2$. It follows that

$$l(\bar{w}'_1) - l(\bar{w}'_2) = A^{(k_n+k)}(l(\bar{w}_1) - l(\bar{w}_2)) = A^{(k_n+k)} b_\tau = b_\pi^{(n)} = b(S') \tag{10}$$

for some $S' \in \Sigma(\pi)$. Choose n such that $[0, |\rho^{(k_n)}|) \subset \Delta_1$. Since $T^{q_r^{(k_n)}}_{\rho,\pi} x_r = T^{K_r}_{\rho^{(k_n)},\tau} x_r \in \Delta^{(k_n)}_{w_0^r}$, $x_r \in \Delta^{(k_n)}_{w_0^r}$ and $\Delta^{(k_n)}_{w_0^r} \subset \Delta_1$, we conclude that $\bar{w}'_r$ is a recurrence word of $T_{\rho,\pi}$ for $r = 1, 2$.

This gives the following conclusion.

Lemma 4.8. *Suppose that $(\lambda, \tau) \in \Lambda_m \times S_m^0$ is a pair such that the IET $T_{\lambda,\tau}$ satisfies the IDOC and $T_{\lambda,\tau}$ has recurrence words $\bar{w}_1,\ \bar{w}_2 \in \mathcal{L}(T_{\lambda,\tau})$ such that $l(\bar{w}_1) - l(\bar{w}_2) = b(S)$ for some $S \in \Sigma(\tau)$. If $\pi \in \mathcal{R}(\tau)$ then for almost every $\rho \in \Lambda_m$ the IET $T_{\rho,\pi}$ has recurrence words $\bar{w}'_1,\ \bar{w}'_2 \in \mathcal{L}(T_{\rho,\pi})$ such that $l(\bar{w}'_1) - l(\bar{w}'_2) = b(S')$ for some $S' \in \Sigma(\pi)$.* $\square$

For any $h = (h_1, \ldots, h_m) \in \mathbb{R}^m_+$ denote by $f_h : [0, 1) \to \mathbb{N}$ the step function $f_h = \sum_{i=1}^m h_i \chi_{\Delta_i^\rho}$.

Lemma 4.9. *For every $x \in C_r^{(k_n)}$, $r = 1, 2$ we have*

$$f_h^{(q_r^{(k_n)})}(x) = h \cdot A^{(k_n+k)} \cdot l(\bar{w}_r).$$

Proof. First suppose that $x \in I_r^{(k_n)} \subset \bigcap_{i=0}^{K_r} T^{-i}_{\rho^{(k_n)},\tau} \Delta^{(k_n)}_{w_i^r}$. Since $T^i_{\rho,\pi} x \in \Delta_{w_i'^r}$ for $0 \le i < q_r^{(k_n)}$, by the definition of f_h,

$$f_h^{(q_r^{(k_n)})}(x) = \sum_{i=0}^{q_r^{(k_n)}-1} h_{w_i'^r} = h \cdot l(\bar{w}'_r).$$

In view of Remark 4.7,
$$f_h^{(q_r^{(k_n)})}(x) = h \cdot l(\bar{w}_r') = h \cdot A^{(k_n+k)} \cdot l(\bar{w}_r).$$

If $x \in C_r^{(k_n)}$ then there exist $y \in I_r^{(k_n)}$ and $0 \le i < h_{w_0^r}^{(k_n)}$ such that $x = T^i y$. In view of the proof of Lemma 4.6, $x,\ T^{q_r^{(k_n)}} x \in T^i \Delta_{w_0^r}^{(k_n)}$, and hence $x,\ T^{q_r^{(k_n)}} x \in \Delta_j$ for some $1 \le j \le m$. Thus $f_h(T^{q_r^{(k_n)}} x) = f_h(x)$ for every $x \in C_r^{(k_n)}$. It follows that
$$\begin{aligned} f_h^{(q_r^{(k_n)})}(x) &= f_h^{(q_r^{(k_n)})}(T^i y) = f_h^{(q_r^{(k_n)})}(y) + f_h^{(i)}(T^{(q_r^{(k_n)})} y) - f_h^{(i)}(y) \\ &= f_h^{(q_r^{(k_n)})}(y) + \sum_{l=0}^{i-1} (f_h(T^{(q_r^{(k_n)})} T^l y) - f_h(T^l y)) \\ &= f_h^{(q_r^{(k_n)})}(y) = h \cdot A^{(k_n+k)} \cdot l(\bar{w}_r). \end{aligned}$$

□

Setting $a_r^{(k_n)} := h \cdot A^{(k_n+k)} \cdot l(\bar{w}_r)$ we obtain
$$\frac{1}{\mu(C_r^{(k_n)})} \left(\left(f_h^{(q_r^{(k_n)})} - a_r^{(k_n)} \right) \Big|_{C_r^{(k_n)}} \right)_* \left(\mu|_{C_r^{(k_n)}} \right) = \delta_0 \text{ for } r = 1, 2.$$

Now we apply Proposition 2.5 together with Lemmas 4.5 and 4.6 to obtain the following result.

Theorem 4.10. *For $r = 1, 2$ there exist $\alpha_r > 0$ and $\Phi_r \in \mathcal{J}(T^{f_h})$ such that*
$$(T^{f_h})_{a_r^{(k_n)}} \to \alpha_r Id + (1 - \alpha_r)\Phi_r$$
in weak operator topology as $n \to \infty$. □

Theorem 4.11. *Assume that T is ergodic, $h \notin H(\pi)$ and $\theta_h = h \cdot b_\pi^{(n)} \ne 0$. Then there exist a sequence $\{a_n\}$, $a_n \to +\infty$, positive numbers α_1, α_2 with $\alpha_1 + \alpha_2 \le 1$, and $\Phi \in \mathcal{J}(T^{f_h})$ such that*
$$(T^{f_h})_{a_n} \to \alpha_1 Id + \alpha_2 (T^{f_h})_{\theta_h} + (1 - \alpha_1 - \alpha_2)\Phi$$
in weak operator topology as $n \to \infty$. Hence the special flow T^{f_h} is disjoint from all weakly mixing ELF flows.

Proof. By the definitions of $b_\pi^{(n)}$, θ_h and (10),
$$\begin{aligned} a_1^{(k_n)} - a_2^{(k_n)} &= h \cdot A^{(k_n+k)} \cdot l(\bar{w}_1) - h \cdot A^{(k_n+k)} \cdot l(\bar{w}_2) \\ &= h \cdot A^{(k_n+k)} \cdot (l(\bar{w}_1) - l(\bar{w}_2)) \\ &= h \cdot A^{(k_n+k)} \cdot b_\tau = h \cdot b_\pi^{(n)} = \theta_h \ne 0. \end{aligned}$$
Set $a_n := a_1^{(k_n)}$. By Theorem 4.10, it follows that
$$(T^{f_h})_{a_n} \to \alpha_1 Id + (1 - \alpha_1)\Phi_1$$
and
$$(T^{f_h})_{a_n} \to \alpha_2 (T^{f_h})_{\theta_h} + (1 - \alpha_2)(T^{f_h})_{\theta_h} \circ \Phi_2.$$
Since T^{f_h} is ergodic, by Lemma 2.4, $\alpha_1 + \alpha_2 \le 1$ and there exists $\Phi \in \mathcal{J}(T^{f_h})$ for which
$$(T^{f_h})_{a_n} \to \alpha_1 Id + \alpha_2 (T^{f_h})_{\theta_h} + (1 - \alpha_1 - \alpha_2)\Phi.$$
Therefore
$$(T^{f_h})_{a_n} \to (\alpha_1 + \alpha_2) \int_{\mathbb{R}} (T^f)_t \, dP(t) + (1 - \alpha_1 - \alpha_2)\Phi,$$

where $P = \frac{\alpha_1}{\alpha_1+\alpha_2}\delta_0 + \frac{\alpha_2}{\alpha_1+\alpha_2}\delta_{\theta_h}$. Now it suffices to apply Proposition 1.1 to complete the proof. $\square$

Corollary 4.12. *If $\pi \in S_m^0$ and $\#\Sigma(\pi) \neq 1$ then for almost every $\lambda \in \mathbb{R}_+^m$ and $h \in \mathbb{R}_+^m$ such that $h \cdot b(S) \neq 0$ for all $S \in \Sigma(\pi)$ the special flow $T_{\lambda,\pi}^{f_h}$ is disjoint from weakly mixing ELF flows. In particular, for a.e. $\lambda \in \mathbb{R}_+^m$ and a.e. $h \in \mathbb{R}_+^m$ the special flow $T_{\lambda,\pi}^{f_h}$ is disjoint from weakly mixing ELF flows.* $\square$

Theorem 4.13. *If $(\rho, \pi) \in \mathcal{A} \cap (\Lambda_m \times \widetilde{S}_m^0)$ then the IET $T = T_{\rho,\pi} : [0,1) \to [0,1)$ is weakly mixing and disjoint from all ELF automorphisms.*

Proof. Let $h = (1,\dots,1)$, hence $f_h \equiv 1$. Since $\pi \in \widetilde{S}_m^0$, it follows that $\theta_1 = (1,\dots,1) \cdot b(S) = |b(S)| = \pm 1$ and hence $h \notin H(\pi)$. Without loss of generality we can assume that $\theta_1 = 1$. By Theorem 4.11, there exists a sequence $\{a_n\}$ of natural numbers ($a_n = h \cdot A^{(k_n+k)} \cdot l(\bar{w}_1) \in \mathbb{N}$), positive numbers α_1, α_2 with $\alpha_1 + \alpha_2 \leq 1$, and $\Phi \in \mathcal{J}(T^{f_1})$ such that

$$(T_1^{f_1})^{a_n} \to \alpha_1 Id_{[0,1)\times[0,1)} + \alpha_2 T_1^{f_1} + (1 - \alpha_1 - \alpha_2)\Phi.$$

Since $T_1^{f_1} = (T^1)_1 = T \otimes Id$ ($Id := Id_{[0,1)}$), we have

$$(T^{a_n} \otimes Id) \to \alpha_1 (Id \otimes Id) + \alpha_2 (T \otimes Id) + (1 - \alpha_1 - \alpha_2)\Phi, \tag{11}$$

and hence

$$(T^{a_n} - \alpha_1 Id - \alpha_2 T) \otimes Id \to (1 - \alpha_1 - \alpha_2)\Phi. \tag{12}$$

Suppose that $\alpha_1 + \alpha_2 < 1$. If $\alpha_1 + \alpha_2 = 1$ then the last member of (11) vanishes and the proof becomes even easier.

Consider the subspace

$$H = L^2([0,1)\times[0,1), \mathcal{B}_{[0,1)} \otimes \{\emptyset, [0,1)\}, \mu\otimes\mu) \subset L^2([0,1)\times[0,1), \mathcal{B}_{[0,1)} \otimes \mathcal{B}_{[0,1)}, \mu\otimes\mu).$$

The space H can be identified with $L^2([0,1), \mathcal{B}_{[0,1)}, \mu)$ via the map

$$L^2([0,1), \mathcal{B}_{[0,1)}, \mu) \ni f \mapsto \overline{f} \in H, \quad \overline{f}(x,y) = f(x).$$

Since H is closed and $(T^{a_n} - \alpha_1 Id - \alpha_2 T) \otimes Id$-invariant, by (12), it follows that H is also Φ-invariant. Denote by $\Phi' : L^2([0,1), \mathcal{B}, \mu) \to L^2([0,1), \mathcal{B}, \mu)$ the restriction of Φ to H, more precisely, Φ' is determines by $\overline{\Phi'(f)} = \Phi(\overline{f})$ for $f \in L^2([0,1), \mathcal{B}, \mu)$. In view of (11), it follows that

$$T^{a_n} \to \alpha_1 Id + \alpha_2 T + (1 - \alpha_1 - \alpha_2)\Phi'. \tag{13}$$

Since Φ is a Markov operator, $\overline{\Phi'(f)} = \Phi(\overline{f}) \geq 0$ for every $f \geq 0$. Moreover, $\overline{\Phi'(1)} = \Phi(\overline{1}) = \overline{1}$ and

$$\int_{[0,1)} \Phi'(f)\, d\mu = \int_{[0,1)\times[0,1)} \Phi(\overline{f})\, d\mu\otimes\mu = \int_{[0,1)\times[0,1)} \overline{f}\, d\mu\otimes\mu = \int_{[0,1)} f\, d\mu,$$

and hence Φ' is a Markov operator. As $\Phi \in \mathcal{J}(T^{f_1})$, the operators Φ and $T \otimes Id$ commute. It follows that

$$\overline{\Phi' \circ T(f)} = \Phi \circ (T \otimes Id)(\overline{f}) = (T \otimes Id) \circ \Phi(\overline{f}) = \overline{T \circ \Phi'(f)}.$$

Therefore $\Phi' \in \mathcal{J}(T)$.

Finally we will show that T is weakly mixing. Suppose that $fT = e^{ia} f$ for some $f \in L^2([0,1))$ with $\|f\| = 1$ and $a \in \mathbb{R}$. From (13),

$$\begin{aligned} 1 &= \|f\|^2 = |\langle f, fT^{a_n}\rangle| = |\alpha_1 \langle f, f\rangle + \alpha_2 \langle f, fT\rangle + (1 - \alpha_1 - \alpha_2)\langle f, \Phi' f\rangle| \\ &\leq |\alpha_1 + \alpha_2 e^{ia}| + (1 - \alpha_1 - \alpha_2)|\langle f, \Phi' f\rangle| \leq |\alpha_1 + \alpha_2 e^{ia}| + (1 - \alpha_1 - \alpha_2). \end{aligned}$$

It follows that
$$1 \leq \left| \frac{\alpha_1}{\alpha_1+\alpha_2} + \frac{\alpha_2}{\alpha_1+\alpha_2} e^{ia} \right|,$$
thus $e^{ia}=1$, which proves the weak mixing of T. Now we can apply Proposition 1.2 to complete the proof. □

In view of Remark 4.4 we obtain the following conclusion.

Theorem 4.14. *Suppose that $(\lambda,\tau) \in \Lambda_m \times \widetilde{S}^0_m$ is a pair such that the IET $T_{\lambda,\tau}$ satisfies the IDOC and $T_{\lambda,\tau}$ has recurrence words $\bar{w}_1,\ \bar{w}_2 \in \mathcal{L}(T_{\lambda,\tau})$ such that $l(\bar{w}_1) - l(\bar{w}_2) = b(S)$ for some $S \in \Sigma(\pi)$. If $\pi \in \mathcal{R}(\tau) \cap \widetilde{S}^0_m$ then for almost every $\rho \in \Lambda_m$ the IET $T_{\rho,\pi}$ is disjoint from ELF automorphisms.* □

5. **An example of 5-IET.** In this subsection we give an explicit example of 5-IET of periodic type which fulfills the hypothesis of Theorem 4.14. This example gives also an example of IET which is disjoint from ELF automorphisms. To find IETs of periodic type we use a method introduced in [19]. This method is based on searching for some closed paths in Rauzy graphs.

Let $c = c_1 \ldots c_n$ be a word over the alphabet $\{a,b\}$. Denote by $c(\tau)$ the path of length n in the Rauzy class $\mathcal{R}(\tau)$ starting from the permutation τ then we apply consecutively operations $c_1,\ldots,c_n$. Suppose that $c(\tau)$ is a closed path, i.e. $\tau = c_n \circ \ldots \circ c_1(\tau)$, and
$$A(c(\tau)) = A(c_1,\tau)A(c_2,c_1(\tau))\ldots A(c_n, c_{n-1}\circ\ldots\circ c_1(\tau))$$
is a primitive matrix. A method of searching for such paths was described in [14]. Let $\theta > 1$ stand for the Perron-Frobenius eigenvalue of $A(c(\tau))$ and let $\lambda \in \mathbb{R}^m_+$ be a right Perron-Frobenius eigenvector. Since
$$\theta\lambda = A(c_1,\tau)A(c_2,c_1(\tau))\ldots A(c_n, c_{n-1}\circ\ldots\circ c_1(\tau))\lambda,$$
we conclude that $c(J^{k-1}(\lambda,\tau)) = c_k$ and
$$J^k(\lambda,\tau) = (A(c_k, c_{k-1}\circ\ldots\circ c_1(\tau))^{-1}\ldots A(c_2,c_1(\tau))^{-1}A(c_1,\tau)^{-1}\lambda, c_k\circ\ldots\circ c_1(\tau))$$
for $1 \leq k \leq n$. It follows that $J^n(\lambda,\tau) = (\theta^{-1}\lambda,\tau)$, and hence $P^n(\lambda,\tau) = (\lambda,\tau)$, which shows that $T_{\lambda,\tau}$ has periodic type.

Let us consider the permutation τ_5^{sym}. In this case
$$\eta_{\tau_5^{sym}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 0 & 1 & 2 & 3 \end{pmatrix}$$
and $\eta_{\tau_5^{sym}}$ has two cyclic sets $S_0 = \{0,2,4\}$ and $S_1 = \{1,2,5\}$ for which $b(S_0) = (1,-1,1,-1,1)$ and $b(S_1) = (-1,1,-1,1,-1)$. Therefore $\tau_5^{sym} \in \widetilde{S}^0_5$.

Next consider the closed path $c(\tau_5^{sym})$, where $c = bbaababaaaba$, in the Rauzy class $\mathcal{R}(\tau_5^{sym})$ (see Figure 1). Here
$$A = A(c(\tau_5^{sym})) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 2 & 3 & 2 & 2 & 2 \end{pmatrix}, \quad B = A^2 = \begin{pmatrix} 4 & 6 & 5 & 8 & 6 \\ 3 & 5 & 1 & 1 & 1 \\ 4 & 6 & 8 & 16 & 11 \\ 2 & 3 & 2 & 6 & 4 \\ 9 & 14 & 10 & 16 & 12 \end{pmatrix},$$

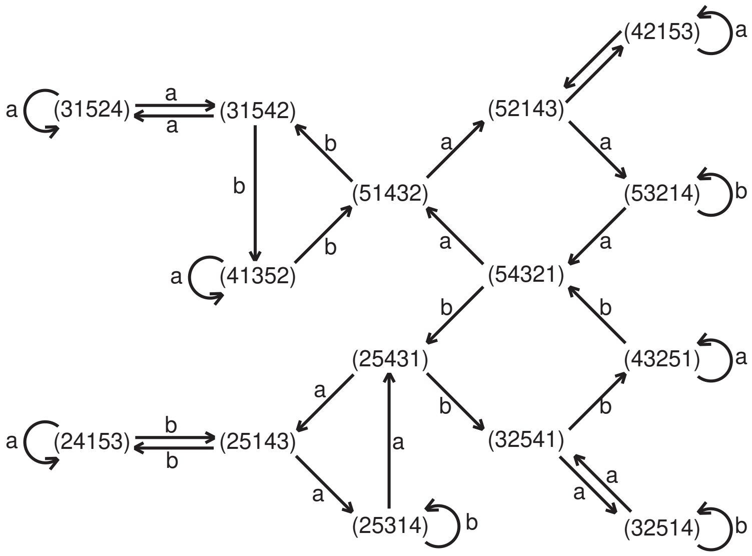

FIGURE 1. Rauzy class $\mathcal{R}(\tau_5^{sym})$

$\theta = 2+\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{15+8\sqrt{3}} \approx 5.55$ is the Perron-Frobenius eigenvalue of A and

$$\begin{aligned}\lambda \;=\; &\Big(\sqrt{3}, \frac{3}{2}-\sqrt{3}+\sqrt{15+8\sqrt{3}}-\frac{1}{2}\sqrt{3}\sqrt{15+8\sqrt{3}},\\ &-1+\frac{3}{2}\sqrt{3}-\frac{3}{2}\sqrt{15+8\sqrt{3}}+\sqrt{3}\sqrt{15+8\sqrt{3}}, 1, \frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{15+8\sqrt{3}}\Big)\end{aligned}$$

is a right Perron-Frobenius eigenvector. Therefore the IET $T := T_{\lambda,\tau_5^{sym}} : [0,|\lambda|) \to [0,|\lambda|)$ has periodic type, hence is minimal. Moreover, $T_{\lambda,\tau_5^{sym}}$ is weakly mixing. This is a consequence of Corollary 1 in [19] because the characteristic polynomial of A is equal to $p(x) = (x-1)(x^4-8x^3+15x^2-8x+1)$ and the Galois groups of p has five elements.

In view of Remark 3.11, every IET of periodic type is isomorphic to a substitution dynamical system $S_\sigma : X_\sigma \to X_\sigma$. For the IET $T_{\lambda,\tau_5^{sym}}$ the substitution σ is defined over the alphabet $\{1,\dots,5\}$ and

$$\sigma(1) = 1525,\ \sigma(2) = 152525,\ \sigma(3) = 15335,\ \sigma(4) = 15343435,\ \sigma(5) = 153435.$$

Since every word $\sigma(i)$ for $1 \le i \le 5$ starts with the symbol 1, if $\bar{w} \in \mathcal{L}_\sigma$ then $\sigma(\bar{w}) \in \mathcal{L}_\sigma$ is a recurrence word for σ. As $\mathcal{L}_\sigma = \mathcal{L}(T_{\lambda,\tau_5^{sym}})$ it follows that $\sigma(\bar{w})$ is a recurrence word for the IET T.

Let $u \in \{1,\dots,5\}^{\mathbb{N}}$ stand for the unique fixed sequence for $\sigma : \{1,\dots,5\}^{\mathbb{N}} \to \{1,\dots,5\}^{\mathbb{N}}$ starting from the symbol 1, i.e.

$$u = 15\mathbf{251534}351525251534351525153435\mathbf{1533351}534343515335153435152\dots$$

Note that $251534,\ 5153351 \in \mathcal{L}_\sigma$ and $l(251534) - l(5153351) = (-1,1,-1,1,-1)$. Let $\bar{w}_1 = \sigma(251534)$ and $\bar{w}_2 = \sigma(5153351)$. Then $\bar{w}_1,\ \bar{w}_2$ are recurrence words for T and

$$\begin{aligned} l(\bar{w}_1) - l(\bar{w}_2) &= Al(251534) - Al(5153351) = A(-1,1,-1,1,-1)\\ &= (-1,1,-1,1,-1) = b(S_1).\end{aligned}$$

The closed path $c(\tau_5^{sym})$, the substitution σ and the recurrence words $\bar{w}_1$, $\bar{w}_2$ were found with the help of Maple.

Now can we apply Theorems 4.13 and 4.14 to have the following result.

Theorem 5.1. *If $\pi \in \mathcal{R}(\tau_5^{sym}) \cap \widetilde{S}_5^0$ then for almost every $\rho \in \Lambda_5$ the IET $T_{\rho,\pi}$ is disjoint from ELF automorphisms. Moreover, the periodic type IET $T_{\lambda,\tau_5^{sym}}$ is also disjoint from ELF automorphisms.*

Proof. The first part of the theorem is a simple consequence of Theorem 4.14.

To prove the second part let us consider the set

$$\mathcal{M} = \left\{ \left(\frac{B\lambda^\varepsilon}{|B\lambda^\varepsilon|}, \tau_5^{sym} \right) : \ (\lambda^\varepsilon, \tau_5^{sym}) \in \mathcal{K} \right\}$$

with $B = A^2 = A(cc(\tau_5^{sym})) = A^{(24)}(\lambda, \tau_5^{sym})$. (See the beginning of Subsection 4.1 for the definition of $\mathcal{K}$). Since $B\lambda = \theta\lambda$ and $P^{24}(\lambda, \tau_5^{sym}) = (\lambda, \tau_5^{sym})$ we have $P^{24k}(\lambda, \tau_5^{sym}) \in \mathcal{M}$ for every natural k. Consequently, $(\lambda, \tau_5^{sym}) \in \mathcal{A}$, which implies, by Theorem 4.13, the disjointness of $T_{\lambda,\tau_5^{sym}}$ from ELF automorphisms. □

6. **Procedure of reduction.** In this section we describe a procedure which helps us to reduce the problem of searching recurrence words satisfying (8) to a smaller number of intervals.

Assume that $m \geq 5$ is odd. Let

$$\tau_m^{sym} = \begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ m & m-1 & \dots & 2 & 1 \end{pmatrix}$$

and

$$\tau_m = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & m-3 & m-2 & m-1 & m \\ m-1 & 1 & m-2 & m-3 & \dots & 4 & 3 & m & 2 \end{pmatrix}.$$

Remark 6.1. The cyclic sets of $\eta_{\tau_m^{sym}}$ are of the form

$$S_0^m = \{0, 2, 4, \dots, m-3, m-1\}, \ S_1^m = \{1, 3, \dots, m-2, m\}$$

with associated vectors

$$b(S_0^m) = (1, -1, 1, \dots, 1, -1, 1), \ b(S_1^m) = (-1, 1, -1, \dots, -1, 1, -1).$$

The cyclic sets of η_{τ_m} are

$$Q_0^m = \{0, 1, 3, \dots, m-4, m-2\}, \ Q_1^m = \{2, 4, \dots, m-3, m-1, m\}$$

with

$$b(Q_0^m) = (0, 1, -1, 1, \dots, 1, -1, 1, 0), \ b(Q_1^m) = (0, -1, 1, -1, \dots, -1, 1, -1, 0). \quad (14)$$

Hence $\tau_m, \tau_m^{sym} \in \widetilde{S}_m^0$.

Lemma 6.2. *For any odd $m \geq 5$ we have $\tau_m \in \mathcal{R}(\tau_m^{sym})$.*

Proof. First note that

$$a\tau_m^{sym}(i) = \begin{cases} m & \text{if} \quad i = 1, \\ 1 & \text{if} \quad i = 2, \\ m - i + 2 & \text{if} \quad 3 \leq i \leq m. \end{cases}$$

Set $b^k = \overbrace{b \circ \ldots \circ b}^{k}$ for $k \geq 1$ and $b^0 = Id$. Thus for $1 \leq k \leq m-3$,

$$b^k a\tau_m^{sym}(i) = \begin{cases} 1 & \text{if } i = 2, \\ 2 & \text{if } i = m, \\ 3 & \text{if } i = (b^{k-1}a\tau_m^{sym})^{-1}(m), \\ b^{k-1}a\tau_m^{sym}(i) + 1 & \text{otherwise.} \end{cases}$$

It follows that $b^{m-3}a\tau_m^{sym} = \tau_m$. □

Theorem 6.3. *For every odd $m \geq 5$ there exists an IET $T_{\lambda,\tau_m^{sym}}$ fulfilling the IDOC which has two recurrence words $\bar{w}_1$, $\bar{w}_2$ such that $l(\bar{w}_1) - l(\bar{w}_2) = b(S)$ for some $S \in \Sigma(\tau_m^{sym})$.*

If $\pi \in \widetilde{S}_m^0 \cap \mathcal{R}(\tau_m^{sym})$ then for almost all $\rho \in \Lambda_m$, $T_{\rho,\pi}$ is disjoint from ELF automorphisms.

Proof. The proof of the first part is by induction on m. In the base step at $m = 5$ we use the example from Section 5.

In the inductive step suppose that $m \geq 7$ and there exists an IET $T_{\lambda,\tau_{m-2}^{sym}}$ ($|\lambda| = 1$) with the IDOC which has two recurrence words $\bar{w}_1$, $\bar{w}_2 \in \{1, \ldots, m-2\}^*$ of length K_1, K_2 respectively, such that $l(\bar{w}_1) - l(\bar{w}_2) = b(S_0^{m-2})$, where $b(S_0^{m-2}) = (1, -1, 1, \ldots, -1, 1)$. Set

$$\widetilde{\lambda} = (0, \lambda_1, \ldots, \lambda_{m-2}, 0) \in \mathbb{R}_+^m$$

and

$$\widetilde{w}_r = w_1^r + 1 \ldots w_{K_r}^r + 1 \in \{1, \ldots, m\}^{K_r} \text{ for } r = 1, 2.$$

Since $T_{\widetilde{\lambda},\tau_m} \equiv T_{\lambda,\tau_{m-2}^{sym}}$, the m-IET $T_{\widetilde{\lambda},\tau_m}$ satisfies the weak IDOC. If $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_m$ are intervals exchanged by $T_{\widetilde{\lambda},\tau_m}$ then $T^k_{\widetilde{\lambda},\tau_m} x \in \widetilde{\Delta}_{j+1}$ if and only if $T^k_{\lambda,\tau_{m-2}^{sym}} x \in \Delta_j$. It follows that $\widetilde{w}_1$, $\widetilde{w}_2$ are recurrence words for $T_{\widetilde{\lambda},\tau_m}$. Moreover,

$$l(\widetilde{w}_r) = (0, l(\bar{w}_r), 0) \text{ for } r = 1, 2,$$

and hence

$$l(\widetilde{w}_1) - l(\widetilde{w}_2) = (0, l(\bar{w}_1) - l(\bar{w}_2), 0) = (0, b(S_0^{m-2}), 0).$$

In view of (14), we have

$$l(\widetilde{w}_1) - l(\widetilde{w}_2) = (0, b(S_0^{m-2}), 0) = b(Q_0^m).$$

Let $K = \max(K_1, K_2) + 1$ and take $\varepsilon = \varepsilon_{T_{\widetilde{\lambda},\tau_m},K}$ (see (7) in Subsection 4.1). Since

$$\mathcal{K} = \{(\lambda', \tau_m) \in \Lambda_m \times \{\tau_m\} : \ |\lambda'| = 1, \ d(\lambda', \widetilde{\lambda}) < \varepsilon\}$$

has positive Lebesgue measure on $\Lambda_m \times \{\tau_m\}$, we can find $(\lambda', \tau_m) \in \mathcal{K}$ such that T_{λ',τ_m} satisfies the IDOC. By Lemma 4.3, $\widetilde{w}_1$, $\widetilde{w}_2$ are recurrence words for T_{λ',τ_m}. In view of Lemma 4.8, there exists $(\widehat{\lambda}, \tau_m^{sym}) \in \Lambda_m \times \{\tau_m^{sym}\}$ such that the IET $T_{\widehat{\lambda},\tau_m}$ satisfies the IDOC and it has two recurrent words $\widehat{w}_1$ and $\widehat{w}_2$ such that $l(\widehat{w}_1) - l(\widehat{w}_2) = b(S)$ for some $S \in \Sigma(\tau_m^{sym})$.

Now we apply Theorem 4.14 to complete the proof of the second part. □

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