Mildly mixing flows on surfaces

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AMS-PTM Meeting

Frączek, Lemańczyk, *On mild mixing of special flows over irrational rotations under piecewise smooth functions*, Ergod. Th. Dynam. Sys.

Frączek, Lemańczyk, *Smooth singular flows in dimension 2 with the minimal self-joining property*, preprint

Frączek, Lemańczyk, Lesigne, *Mild mixing property for special flows under piecewise constant functions*, to appear in DCDS

Mild mixing

Definition. [Furstenberg, Weiss, 1978] A finite measure–preserving transformation $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is mildly mixing if its Cartesian product with an arbitrary ergodic conservative measure–preserving transformation $S : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ remains ergodic (ν can be finite or infinite).

Proposition. [Furstenberg, Weiss, 1978] *A finite measure–preserving transformation $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is mildly mixing iff T has no nontrivial rigid factors, i.e.*

$$\liminf_{n\to\infty} \mu(T^{-n}A \triangle A) > 0 \text{ for any } A \in \mathcal{B}, 0 < \mu(A) < 1.$$

strong mixing $\Longrightarrow$ mild mixing $\Longrightarrow$ weak mixing

Schmidt, Walters (1982) Characterization of the mild mixing property for measure–preserving actions of locally compact (second countable) but non–compact Abelian groups.

Measure–preserving systems which are mildly mixing but not mixing

- Chacon transformation - the mild mixing property follows form the minimal self–joining property.
- Abelian group actions - Schmidt (1984) Gaussian processes (spectral approach)
- None of the known examples of mild mixing but not mixing systems was proved to be smooth.
- Interval exchange transformations with minimal self–joining prop.

Mildly mixing flows

One of the main tools which we use to prove the mild mixing property is so called Ratner property, but in actual fact we use a modification of this property, which we will call R–property.

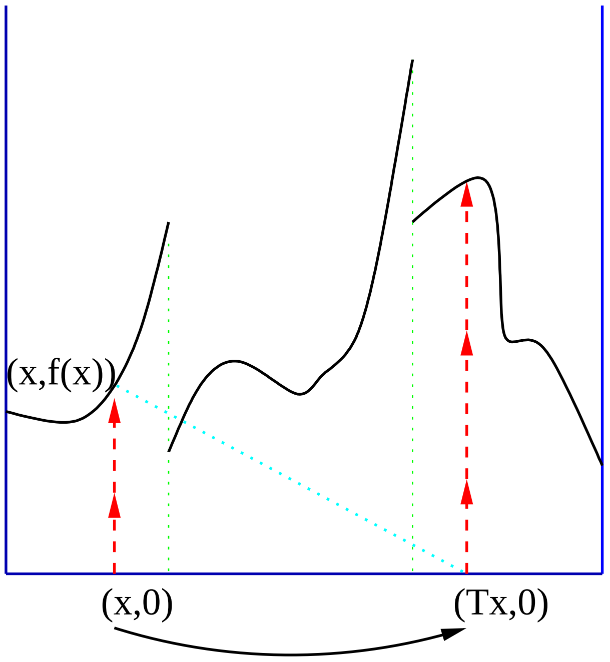
Proposition. [Ratner, 1983] *If a measure–preserving flow* $(S_t)_{t\in\mathbb{R}}$ *satisfies the R–property then* $(S_t)_{t\in\mathbb{R}}$ *is a finite extension of each its nontrivial factor.*

$(S_t)_{t\in\mathbb{R}}$ is partially rigid if there exists $t_n \to +\infty$ and $0 < u \leqslant 1$ such that $\liminf_{n\to\infty} \mu(S_{-t_n}A \cap A) \geqslant u\mu(A)$ for any $A \in \mathcal{B}$

Proposition. [Fr-Le] If $(S_t)_{t\in\mathbb{R}}$ satisfies the R–property and is not partially rigid then it is mildly mixing.

Special flows over irrational rotations

Let $T:\mathbb{T}\to\mathbb{T}$, $Tx=x+\alpha$ be an irrational rotation and let $f:\mathbb{T}\to\mathbb{R}$ be a positive piecewise absolutely continuous function. Consider the special flow $T^f=(T^f_t)_{t\in\mathbb{R}}$ built over T and under f. T^f acts on $\mathbb{T}^f=\{(x,s):x\in\mathbb{T},\ 0\leqslant s<f(x)\}$.



Proposition. [von Neuman, 30's] *If the sum of jumps of f (denoted by $S(f)$) is not zero then T^f is weakly mixing.*

T^f is not mixing (Kochergin for roof functions with bounded variation).

Theorem. [Fr-Le] *If* $S(f) \neq 0$ then T^f is not partially rigid. If additionally α has bounded partial quotients in its continued fraction expansion ($\left|\alpha - \frac{p}{q}\right| \leqslant \frac{C}{q^2}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$) then T^f satisfies the R–property, and finally T^f is mildly mixing.

The case $S(f) = 0$: Suppose that α has bounded partial quotients and $f' \in L^2$. Then we can assume that f is piecewise constant. Let $\{\xi_1, \ldots, \xi_p\}$ stand for the set of discontinuities of f and

$$d_i = \lim_{x \to \xi_i^-} f(x) - \lim_{x \to \xi_i^+} f(x) \text{ for } i = 1, \ldots, p.$$

Consider two additional properties that f may satisfies:

(P1) whenever a nontrivial sum $\sum_{i=1}^{p} n_i d_i$ with $n_i \in \mathbb{Z}$, $i = 1, \ldots, p$ equals zero, there exist $1 \leqslant i < i' \leqslant p$ such that $n_i \neq 0, n_{i'} \neq 0$ and $\xi_i - \xi_{i'} \in (\mathbb{Q} + \mathbb{Q}\alpha) \setminus (\mathbb{Z} + \mathbb{Z}\alpha)$;

(P2) $\int f(x)\, dx \notin \sum_{i=1}^{p} (\xi_i + \mathbb{Q} + \mathbb{Q}\alpha) d_i$.

Example. If $f = a + b\chi_{[0,\xi)}$, where $a, b > 0$ and $a/b \notin \mathbb{Q} + \mathbb{Q}\alpha$ and $\xi \in (\mathbb{Q} + \mathbb{Q}\alpha) \setminus (\mathbb{Z} + \mathbb{Z}\alpha)$, then f satisfies (P1) and (P2).

Theorem. [Fr-Le-Le] *If a piecewise constant function f satisfies (P1) and (P2) then T^f is weakly mixing and satisfies the R–property. T^f is partially rigid, nevertheless T^f is mildly mixing.*

Velocity changes for quasi-periodic Hamiltonian flows on 2-torus

Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a quasi-periodic C^∞ Hamiltonian, i.e.

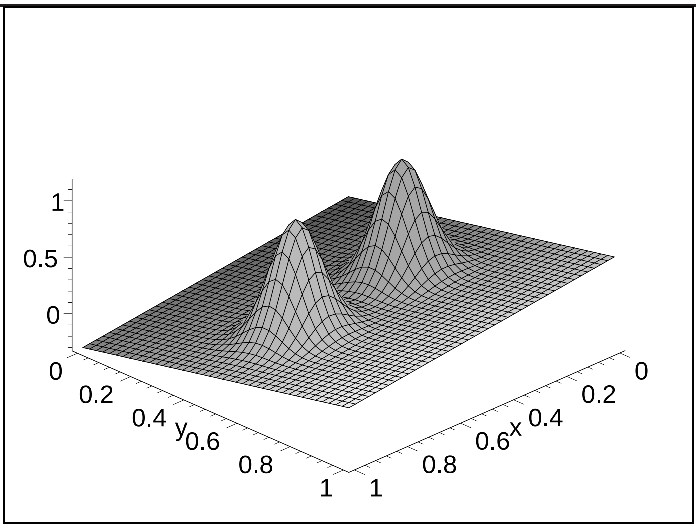
$$H(x+m, y+n) = H(x,y) + m\alpha_1 + n\alpha_2$$

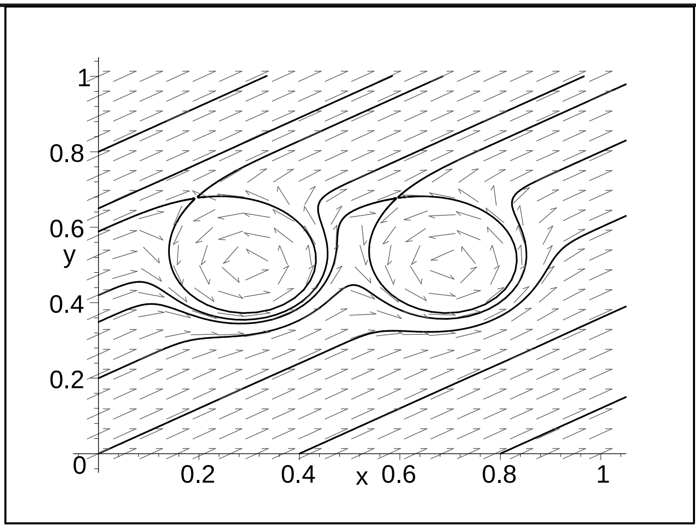
and $\alpha = \alpha_1/\alpha_2$ is irrational and has bounded partial quotients.

Suppose that H has critical points and H is in the general position (every critical point is non-degenerated and all critical values are distinct).

H detemind the (quasi–periodic) Hamiltonian flow $(h_t)_{t\in\mathbb{R}}$ on $\mathbb{T}^2$ associated with the Hamiltonian equation:

$$\frac{dx}{dt} = \frac{\partial}{\partial y}H(x,y),\ \frac{dy}{dt} = -\frac{\partial}{\partial x}H(x,y)$$





Next let us change the velocity in (h_t). Let $g : \mathbb{T}^2 \longrightarrow \mathbb{R}^+$ be a non-negative C^∞ function which is positive except the set Crit$\subset \mathbb{T}^2$ of all critical points of H. Let us consider $(\varphi_t)_{t\in\mathbb{R}}$ on $\mathbb{T}^2\backslash$Crit associated with

$$X(x,y) = \left(\frac{\partial}{\partial y}H(x,y), -\frac{\partial}{\partial x}H(x,y)\right)/g(x,y)$$

Orbits of $(\varphi_t)_{t\in\mathbb{R}}$ and $(h_t)_{t\in\mathbb{R}}$ are the same on $\mathbb{T}^2\setminus\mathrm{Crit}$ and the phase space of (φ_t) decomposes into an invariant set filled with periodic orbits and separatrices and one ergodic component EC of positive Lebesgue measure (Arnold (1991)).

Moreover, (φ_t) on EC is isomorphic to the special flow built over the circle rotation $Tx = x + \alpha$ and under a piecewise absolutely continuous roof function $f : \mathbb{T} \to \mathbb{R}$ such that $f' \in L^2$.

$$S(f) = \int_{\boldsymbol{EC}} d\omega, \text{ where } \omega(Y) = \frac{\langle X, Y\rangle}{\langle X, X\rangle}$$

Theorem. [Fr-Le] *If* α has bounded partial quotients and $\int_{\boldsymbol{EC}} d\omega \neq 0$ then (φ_t) is mildly mixing but not mixing.

Let $\{x_1, \ldots, x_p\} = \partial EC \cap \mathrm{Crit}$. Let B_i stand for the connected component of ∂EC which contains x_i and put

$$d_i = -\int_{\boldsymbol{B_i}} \omega_X,\ \xi_i = \int_{\boldsymbol{x_1}}^{\boldsymbol{x_i}} g(x,y) i_X(dx \wedge dy) \text{ for } i = 1, \ldots, p.$$

Theorem. [Fr-Le-Le] *Suppose that α is an irrational number with bounded partial quotients and $\int_{\boldsymbol{EC}} d\omega_X = 0$. If X satisfies the following properties:*

(P'1) whenever a nontrivial sum $\sum_{i=1}^{p} n_i d_i$ with $n_i \in \mathbb{Z}$, $i = 1, \ldots, p$ equals to zero, there exist $1 \leqslant i < i' \leqslant p$ such that n_i and $n_{i'}$ are nonzero and $\xi_i - \xi_{i'} \in (\mathbb{Q}\alpha_1 + \mathbb{Q}\alpha_2) \setminus (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2)$;

(P'2) $\int_{\boldsymbol{EC}} g(x,y)\, dx\, dy \notin \sum_{i=1}^{p} (\xi_i + \mathbb{Q}\alpha_1 + \mathbb{Q}\alpha_2) d_i$,

then the flow (φ_t) is mildly mixing but not mixing.