Asymptotic Independent Representations for Sums and Order Statistics of Stationary Sequences

Adam Jakubowski

Preface

This is a report on the author's research initiated during the stay in Göttingen in 1987/88 and continued over the past few years. The question posed at the very beginning was simple: to identify the area where we can still apply the classical limit theory for sums and maxima of *independent* random variables. Existence of asymptotic independent representations became a natural frame for the considerations. Obtained this way theory is a unification of several trends in limit theory for sums and helps in understanding the asymptotic structure of order statistics.

This is not a monograph on limit theory for sums and order statistics of dependent stationary random variables. The choice of presented results and examples depends heavily on the author's taste and interest. In particular, we concentrate on results which we can obtain by means developed in the paper, and on examples demonstrating how weak are the assumptions usually made in the paper.

The author wishes to express his gratitude to the Alexander von Humboldt Foundation for supporting his stay in Göttingen and for active help every year since that time. Some research presented in the paper was supported by grants CPBP 01.01(2/30) and CPBP 01.02(II.8/15).

Toruń, April 1991 Adam Jakubowski

Contents

Introduction and Announcement of Results

Asymptotic Independent Representations for Sums

"From independence to dependence": the head of Chapter IX in M. Loève's book ([Loè78]) is the shortest program for any attempt to build a limit theory extending the classical one for independent random variables.

Considering sums of random variables, Loève himself suggested "comparison of summands"(cf, p.41), which, in the simplest case, led to suppressing the dependence between summands. Loève's conditions were, however, hardly applicable.

Much more fruitful is another approach, known as "Bernstein's method" (see [IbLi71], [Ios77]). Here the main idea is to divide the sum into almost independent *segments*. It is possible, if summands possess "mixing" properties, describable in various ways. We refer to [Bra86] and [Pel86] for the nearly up-to-dated survey on the present stage of the theory.

Some results obtained on the base of Bernstein's method can be "visualized" in the form of an almost sure invariance principle (ASIP): the original (dependent) sequence can be redefined (without changing its law) onto another probability space, on which an accompanied independent sequence exists, with sums of both sequences being close in a strong sense (see [Phi86] for the survey). ASIP is a very powerful tool: as a rule it implies functional convergence of the corresponding partial-sum process. On the other hand, it is easy to find examples of 1-dependent sequences with partial sums weakly convergent, when properly normalized, to a p-stable distribution, but *not* convergent in the functional manner (see 5.8, 5.19 below, also [Sze89]). It follows that looking for the general theory, *we cannot expect results like ASIP* (or even invariance principle in probability). Therefore we suggest restricting the attention to the weakest approximation, which is still of interest.

Let $\{X_j\}_{j\in \mathbb{N}}$ be a strictly stationary sequence of random variables with partial sums $S_0 = 0$, $S_n = \sum_{j=1}^{n} X_j$, $n \in \mathbb{N}$. We will say that $\{X_j\}$ admits (or possesses) *asymptotic independent representation (a.i.r.) for partial sums*, if there exist independent, identically distributed random variables $\{\widetilde{X}_j\}_{j\in \mathbb{N}}$ with partial sums $\widetilde{S}_n$, such that

$$\sup_{x\in \mathbb{R}^1} |P(S_n \leq x) - P(\widetilde{S}_n \leq x)| \longrightarrow 0 \quad \text{as } n \to +\infty.$$

Trivial examples show that a.i.r. is not unique . However, *if exists, it determines the class of possible limit laws and the way of normalization and centering in limit theorems* for S_n's. In particular, we will study only *regular* a.i.r., i.e. sequences $\{\widetilde{X}_j\}$ such that $\widetilde{S}_n/B_n \longrightarrow_{\mathcal{D}} \mu$ for some normalizing constants $B_n \to +\infty$ and some *non-degenerate* law μ.

Suppose $\{\widetilde{X}_j\}$ is a regular a.i.r. for $\{X_j\}$. By the very definition also $S_n/B_n \longrightarrow_{\mathcal{D}} \mu$, and by the theory for independent identically distributed summands, B_n must be a $1/p$-regularly varying sequence and μ — a strictly p-stable distribution, for some p, $0 < p \leq 2$. Further, if $p = 2$, then B_n^2/n must be equivalent to a non-decreasing positive sequence. Our first observation (Theorem 1.1) states the converse: a regular asymptotic representation for sums exists if, and only if, $S_n/B_n \longrightarrow_{\mathcal{D}} \mu$ for some $\{B_n\}$ and μ specified above.

Notice the exceptionality of $p = 2$. It is interesting, that we can get rid of restrictions on B_n^2/n, if we admit approximation by stationary and independent in rows arrays $\{\widetilde{X}_{n,j} : j, n \in I\!N\}$: for every $\theta > 1$

$$\max_{n < m \leq \theta \cdot n} \sup_{x \in I\!R^1} |P(S_m \leq x) - P(\sum_{j=1}^{m} \widetilde{X}_{n,j} \leq x)| \longrightarrow 0 \quad \text{as } n \to +\infty.$$

Suppose, as before, that $S_n \longrightarrow_{\mathcal{D}} \mu$, where $B_n \to +\infty$ and μ is non-degenerated. Then Theorem 1.4 says, that the above asymptotic independent representation (a.i.r. "in the array setting") exists if, and only if, for some p, $0 < p \leq 2$, B_n is $1/p$-regularly varying and μ is strictly p-stable.

Observe, that approximation by arrays does not extend the class of possible limit laws (to all infinitely divisible distributions, for example) and does not bring any new phenomena in the case $0 < p < 2$. On the other hand it exhibits an interesting structure of the Central Limit Theorem for stationary sequences.

Theorem 1.4 restricts our attention to **p-stable limit theorems**, i.e. results on weak convergence of S_n/B_n to strictly p-stable laws, when B_n is a $1/p$-regularly varying sequence.

We aim at finding necessary and sufficient conditions for p-stable limit theorems.

All results in Chapters 1-4 without stated references are taken from [Jak90c].

Mixing conditions

At the first stage, we examine mixing properties of S_n's. It is proved in Theorem 2.1, that a p-stable limit theorem implies **Condition B**: For each $\lambda \in I\!R^1$,

$$\max_{\substack{1 \leq k,l \leq n \\ k+l \leq n}} |E\, e^{i\lambda(S_{k+l}/B_n)} - Ee^{i\lambda(S_k/B_n)} \cdot Ee^{i\lambda(S_l/B_n)}| \longrightarrow 0 \quad \text{as } n \to +\infty.$$

Conversely, if Condition B holds, $B_n \to \infty$ and

$$\frac{S_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu \neq \delta_0,$$

then μ must be strictly p-stable and B_n is a $1/p$-regularly varying sequence. This statement slightly improves Theorem 18.1.1 in [IbLi71], replacing ,

— *strong mixing* by Condition B,

— *non-degeneracy* of μ by $\mu \neq \delta_0$ and

— regular variation *on integers* by usual regular variation.

Under some technical assumptions (see Proposition 2.4) Condition B admits an alternative formulation:

$$\max_{\substack{1\leq k,l\leq n\\ k+l\leq n}} d\left(\mathcal{L}(\frac{S_{k+l}}{B_n}), \mathcal{L}(\frac{S_k}{B_n}) * \mathcal{L}(\frac{S_l}{B_n})\right) \longrightarrow 0 \quad \text{as } n \to +\infty,$$

where d is any metric, which metricizes the weak convergence of probability laws on $\mathbb{R}^1$. This means that under Condition B we can *break* sums into *seemingly* independent components and that any application of Condition B in limit theorems is, in fact, a variant of Bernstein's method. What should be emphasized, is that *Condition B does not imply any mixing (in the intuitive sense) properties and that it depends on the particular choice of* B_n. Perhaps Example 2.5 is here most striking: if X is Cauchy distributed and $X_k = X, \ k = 1, 2, \ldots, B_n = n$, then Condition B holds for this totally dependent sequence.

The most standard way of checking Condition B is based on separation of blocks: if $B_n \to +\infty$, one can find $m_n \to \infty$ such that Condition B is equivalent to

$$\max_{\substack{m_n\leq k,l,m\\ k+l+m\leq n}} |E\, e^{i\lambda(S_{k+l+m}-S_{k+l}+S_k)/B_n} - Ee^{i\lambda S_k/B_n} \cdot Ee^{i\lambda S_m/B_n}| \longrightarrow 0 \quad \text{as } n \to +\infty,$$

for every $\lambda \in \mathbb{R}^1$. Now we can use Rosenblatt's coefficient of strong mixing for estimation of the above covariances. It follows that strongly mixing sequences satisfy Condition B for *every* $B_n \to \infty$. Brief discussion of most important mixing coefficients and their mutual relations is given in Section 2.5.

Keeping in mind that the form with separated partial sums is the most applicable one, we prefer the original version of Condition B, for the latter provides us with what we want ("breaking") without all unpleasant technicalities. Moreover, there are tools such as Newman's inequality for associated random variables which allow us to check Condition B *directly* (see Section 2.4).

Regular variation in the limit

Suppose $Z_n \longrightarrow_{\mathcal{D}} \mu$, where μ is strictly p-stable. For each n, let $\{Y_{n,j}\}_{j\in\mathbb{N}}$ be a sequence of independent copies of Z_n. By *strict* stability of μ, for each $k \in \mathbb{N}$ we have

$$k^{-1/p}\sum_{j=1}^{k} Y_{n,j} \underset{\mathcal{D}}{\longrightarrow} \mu \quad \text{as } n \to +\infty.$$

Hence we can find a sequence $\{r_n\}$ of integers, $r_n \nearrow \infty$, such that if $k_n = o(r_n)$, then

$$k_n^{-1/p} \sum_{j=1}^{k_n} Y_{n,j} \underset{\mathcal{D}}{\longrightarrow} \mu \quad \text{as } n \to +\infty.$$

If $k_n \to \infty$, the array $\{Z_{n,j} = k_n^{-1/p} Y_{n,j}\,;\, 1 \le j \le k_n,\, n \in \mathbb{N}\}$ of row-wise independent random variables is *infinitesimal*. For such arrays we can use existing limit theorems for independent summands (see Theorem A.5) and find expressions involving $Z_{n,j}$'s (in fact: Z_n's), which are *necessary* for $Z_n \longrightarrow_{\mathcal{D}} \mu$. As a result we get Proposition 3.1.

Obtained this way conditions have a very special form: given a sequence of functions f_n on $\mathbb{R}^+$ (e.g. $f_n(x) = P(Z_n > x)$) we assume that there exists a sequence $r_n \nearrow \infty$ such that

$$x_n^p f_n(x_n) \longrightarrow c\,,$$

whenever $x_n \to \infty$, $x_n = o(r_n)$. If $c > 0$, we say that the sequence $\{f_n\}$ is $(-p)$-**regularly varying in the limit**. The name is motivated by the following natural example: take a $(-p)$-regularly varying function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and a sequence $a_n \to \infty$ of numbers. Then $\{f_n(x) = c \cdot f(a_n x)/f(a_n)\}_{n\in\mathbb{N}}$ is $(-p)$-regularly varying in the limit.

In the classical limit theory for sums of independent identically distributed summands, the notion of regular variation plays a fundamental role (see [Fel71, Chapter VIII.9]). In particular, Karamata's Theorem (see Theorem B.11), establishing a link between truncated moments and regularly varying tail probabilities, is a very useful tool. For *sequences* regularly varying in the limit we can prove a result only partially corresponding to the direct half of Karamata's Theorem. Nevertheless, our Theorem 3.6 is still sufficient to exhibit dependencies between conditions we obtained above as necessary for $Z_n \longrightarrow_{\mathcal{D}} \mu$. This enables us to reduce the number of conditions and to consider in Proposition 3.1 only the essential ones.

Tauberian Limit Theorems

Recall that by our convention, a p-stable limit theorem holds for $\{X_j\}$, if $S_n/B_n \longrightarrow_{\mathcal{D}} \mu$, where the limit law μ is non-degenerated and strictly p-stable and B_n varies $1/p$-regularly, for some p, $0 < p \le 2$. Our final result—Theorem 4.1—gives two conditions which are **necessary and sufficient** for a p-stable limit theorem to hold:

- There is a sequence $\{r_n\}$, $r_n \nearrow +\infty$, such that for every sequence $\{k_n\}$ of integers "tending to infinity slowly enough" (i.e. $k_n = o(r_n)$) we have

  $$k_n^{-1/p} \sum_{j=1}^{k_n} Y_{n,j} \underset{\mathcal{D}}{\longrightarrow} \mu \quad \text{as } n \to +\infty,$$

  where for each n, $Y_{n,1}, Y_{n,2}, \ldots,$ are independent copies of S_n/B_n.

- Condition B is satisfied for $\{S_k/B_n\}$.

The first condition is in the form *independent of* p; applying corresponding results for arrays of independent summands we may "translate" it into the form *specific* to the case $0 < p < 1$, $p = 1$, $1 < p < 2$ or $p = 2$. This is done in Theorems 4.2–4.5. Obtained this way criteria are improvements of [DeJa89] and [JaSz90].

The above results are of "Tauberian" type. Indeed, we deal with necessary conditions obtained by *averaging* independent copies and the extra information we need in order to get sufficiency, is just Condition B (playing here the role of a "Tauberian condition").

We have constructed a quite satisfactory theory of what we called "p-stable limit theorems". The traditional formulation of limit problems is, however, somewhat more general: instead of

$$S_n/B_n \underset{\mathcal{D}}{\longrightarrow} \mu$$

a convergence with centering

$$(S_n - A_n)/B_n \underset{\mathcal{D}}{\longrightarrow} \mu$$

is considered. Fortunately, if Condition B holds for $\{S_k/B_n\}$, there is no need to develop the theory parallelling the preceding one (for example, with limits which are stable, and *not strictly* stable). Theorem 4.9 asserts, that for $p \neq 1$ we can always find a number $A \in I\!R^1$ such that $(S_n - n \cdot A)/B_n \longrightarrow_{\mathcal{D}} \mu * \delta_{-a}$, where $\mu * \delta_{-a}$ is strictly stable. Since $X_j' = X_j - A$ is a strictly stationary sequence satisfying Condition B, Theorem 4.9 provides—in the case $p \neq 1$—a complete reduction of the apparently more general limit problem with centering to the restricted one considered in this paper.

Examples of p-stable limit theorems

Conditions appearing in Tauberian limit theorems are *tractable*.

We have already discussed Condition B (in Chapter 2), so now we are going to review some methods of checking

$$k_n^{-1/p} \sum_{j=1}^{k_n} Y_{n,j} \underset{\mathcal{D}}{\longrightarrow} \mu \quad \text{as } n \to +\infty,$$

where for each n, $Y_{n,1}, Y_{n,2}, \ldots,$ are independent copies of S_n/B_n and $k_n \to \infty$ increases slowly enough. Formally we solved the problem in Theorems 4.2–4.5 by means of corresponding limit theorems for triangular arrays. The point is that in these theorems we deal with expressions involving S_n/B_n and not individual summands. On the other hand the required information is reduced and we claim that *this is the proper level of reduction*: using our Tauberian theorems we can either prove most of existing results or at least indicate the essential step in their proof.

For example, applying the Lindeberg Central Limit Theorem we obtain Denker's criterion [Den86] stating that under strong mixing, *the uniform integrability* of $\{S_n^2/\mathrm{Var}\,(S_n)\}$ is necessary and sufficient for $S_n/\sqrt{\mathrm{Var}\,(S_n)} \longrightarrow_{\mathcal{D}} \mathcal{N}(0,1)$.

In Theorem 5.5 we derive a similar criterion operating with truncated random variables. Using this criterion it is possible to obtain recent CLT theorems for mixing random variables with infinite variance due to Bradley [Bra88] and Peligrad [Pel90].

In Section 5.3 we apply Theorems 4.2–4.4 in proving non-central limit theorems from [Dav83], [Sam84] and [JaKo89]. It is assumed in all the results that there are no clusters of big values of summands. And this makes our computations easy.

If clusters of big values exist, our technique works as well. This is demonstrated in the new, short proof of Theorem 5.16, being a one-dimensional refinement of limit theorems for m-dependent random variables obtained in [JaKo89].

Our theory, although general, cannot replace such traditional and powerful methods as "martingale approach" (see [HaHe80], [Jak86], [JaSh87]) or ASIP. Nevertheless, Tauberian limit theorems constitute a unified tool in a wide variety of problems.

Asymptotic Representations for Order Statistics

In the second part of the paper we study asymptotic representations for order statistics of sequences of random variables. The motivation is here the same as for sums: we are interested in possibly general limit theory extending the classical one for independent random variables and describing phenomena of "asymptotic independence". We stress, however, a different aspect of the theory: it is not only the tool for getting limit theorems but it helps to understand better the limit structure of order statistics.

This is important particularly for higher order statistics, where we suggest using a simple (although dependent) universal model instead of difficult in analysis limit distributions. But in the simplest case of maxima, we can refresh our point of view on such structural notion as extremal index, either.

It should be pointed out, that our approximating sequences may exist even if the original order statistics are not convergent under any linear normalization. This corresponds to the fact, that linear normalization is natural for sums rather than for maxima, and that in most cases limiting probability of exceedances over a given sequence is of interest only:

$$\lim_{n\to\infty} P(\max_{1\le k\le n} X_k > v_n) =?.$$

Therefore in our criteria we operate with conditions describing properties of order statistics with respect to suitably chosen, but only one sequence of boundaries $\{v_n\}$.

Asymptotic Independent Representations for Maxima

Let $\{X_j\}_{j\in\mathbb{N}}$ be a sequence of random variables. Define $M_n = \max_{1\le j\le n} X_j$ and $M_0 = -\infty$. We say that $\{X_j\}$ admits an asymptotic independent representation for maxima, if there exists a sequence $\{\widetilde{X}_j\}$ of independent random variables with partial maxima $\widetilde{M}_n$ such that

$$\sup_{x\in\mathbb{R}^1} |P(M_n \le x) - P(\widetilde{M}_n \le x)| \to 0 \quad \text{as} \quad n \to \infty.$$

If $\{X_j\}$ is stationary, it is quite natural to ask for $\{\widetilde{X}_j\}$ being an independent *identically* distributed sequence. If G is the common distribution function for $\widetilde{X}_j$'s, then we can rewrite the above definition in the form

$$\sup_{x\in\mathbb{R}^1} |P(M_n \le x) - G^n(x)| \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

which defines a *phantom distribution function* G for $\{X_j\}$—the notion introduced by O'Brien [OBr87]. O'Brien gave widely applicable sufficient conditions for existence of such G; an improvement of his results obtained by the author [Jak91a] states that a *stationary* sequence $\{X_j\}$ has a phantom distribution function G satisfying

$$G(G_*-) = 1 \quad \text{and} \quad \frac{1 - G(x)}{1 - G(x-)} \longrightarrow 1 \quad \text{as} \quad x \nearrow G_*,$$

where $G_* = \sup\{u; G(u) < 1\}$, if, and only if, there is a sequence $\{v_n\}$ of numbers such that

$$P(M_n \le v_n) \longrightarrow \alpha$$

for some $\alpha, 0 < \alpha < 1$, and **Condition** $\mathbf{B}_\infty(v_n)$ holds:

$$\sup_{j,k\in\mathbb{N}} |P(M_{j+k} \le v_n) - P(M_j \le v_n)P(M_k \le v_n)| \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

The restriction imposed on G is well known in the literature; it guarantees that for some (and then for any) $0 < \alpha < 1$, one can find a sequence v_n such that $G^n(v_n) \to \alpha$. By analogy to sums we say that such G determines a *regular* asymptotic independent representation.

We derive the above result as Theorem 6.17 being a consequence to more general Theorem 6.2 on existence of a.i.r. for maxima of nonstationary sequences. There are some reasons to consider here the nonstationary setup. First, Theorem 6.2 provides a criterion, which is very convenient for stationary sequences (e.g. Markov chains with stationary initial distribution—see Theorem 6.15). Further, using this criterion we can find a regular phantom distribution function even for nonstationary sequences (e.g. Markov chains with arbitrary initial distribution—Corollary 6.16). And last but not least—Theorem 6.2 is interesting by itself. Indeed, in this theorem we construct marginal laws of the approximating independent sequence using the limiting function

$$\alpha_t = \lim P(M_{[nt]} \le v_n), \quad t > 0.$$

The construction is possible if $\alpha_t > 0, \quad t > 0$, $\sup_{t>0} \alpha_t = 1$, $\inf_{t>0} \alpha_t = 0$ and there exists a *concave* function g_α such that

$$\alpha_t = \exp(g_\alpha(\log t)).$$

In particular, the limit $\alpha_t = \alpha^t$ for some $0 < \alpha < 1$ (or $g_\alpha = e^x \cdot \log \alpha$) "produces" an i.i.d. sequence with marginal distribution function G given by the formula

$$G(x) = \begin{cases} 0 & \text{if} \quad x < v_1, \\ \alpha^{1/n} & \text{if} \quad v_n \le x < v_{n+1}, \\ 1 & \text{if} \quad x \ge \sup_k v_k \end{cases}$$

(We may assume that the sequence $\{v_n\}$ is non-decreasing). Theorems 6.2 and 6.15 and Corollary 6.16 were originally proved in [Jak90a].

Theorem 6.17 and the explicit form of a phantom distribution function enables us to generalize the notion of the extremal index and to prove easily a criterion for its existence (Theorem 6.21). Let us give a sketch of the reasoning.

Let $\{X_j\}$ and $\{X'_j\}$ be two stationary sequences. Suppose that for some non-decreasing sequence $\{v_n\}$ Condition $\mathrm{B}_\infty(v_n)$ holds for both sequences, and that as $n \to \infty$

$$P(M_n \le v_n) \to \alpha, \qquad P(M'_n \le v_n) \to \alpha',$$

where $0 < \alpha, \alpha' < 1$. By Theorem 6.17, both $\{M_n\}$ and $\{M'_n\}$ admit a phantom distribution function G and G', respectively, and

$$G = {G'}^\theta,$$

where

$$\theta = \frac{\log \alpha}{\log \alpha'}.$$

It follows now by the very definition of an asymptotic independent representation, that

$$\sup_{x \in \mathbb{R}^1} \left| P(M_n \le x) - P(M'_n \le x)^\theta \right| \longrightarrow 0, \quad \text{as } n \to +\infty.$$

We call such number θ the relative extremal index of $\{X_j\}$ with respect to $\{X'_j\}$. This is a generalization of the (usual) extremal index, which in our terminology is the relative extremal index of $\{X_j\}$ with respect to the i.i.d. sequence $\{\widehat{X}_j\}$ with the same marginal distribution: $\mathcal{L}(X_j) = \mathcal{L}(\widehat{X}_j)$. The concept of extremal index was introduced by Leadbetter [Lea83], who perfected earlier ideas of Loynes [Loy65] and O'Brien [OBr74a]. Leadbetter's proofs and criteria of existence are, however, different from our Theorem 6.21.

The relative extremal index is not an artificial notion. In naturally arises, for instance, in limit theorems for regenerative sequences, as interesting Example 6.20 due to Rootzén [Roo88] shows.

Eventually, we point out that there are stationary processes *without* asymptotic independent representation, for which the relative extremal index can be defined as well (see Example 6.22).

Equivalent Forms of Mixing Conditions

Describing "asymptotic independence" in the form of Condition $B_\infty(v_n)$ is not a common practice. The tradition in the Extreme Value Limit Theory prefers Leadbetter's Condition D ([Lea74], [LLR83]) or its variants ([HHL88]) close to strong mixing.

We were partially inspired with O'Brien's ([OBr87]) Condition $AIM(u_n)$. The main difference is that we relate mixing properties to a single sequence $\{v_n\}$, while O'Brien (and others) used to consider "breaking probabilities" $P(M_{j+k} \leq v_n(\beta))$ for *a family* of boundaries $\{v_n(\beta)\,;\,\beta \in B\}$, but on *bounded* intervals only: $j + k \leq n$. Propositions 7.7 and 7.8 show, that both approaches are essentially equivalent.

In fact, for stationary sequences Condition $B_\infty(v_n)$ is nothing but asymptotic exponential form of the path $\mathbb{R}^+ \ni t \mapsto P(M_{[nt]} \leq v_n)$: by Proposition 7.5 it is equivalent to

$$\sup_{t>0}\left|P(M_{[nt]} \leq v_n) - P(M_n \leq v_n)^t\right| \longrightarrow 0 \quad \text{as } n \to +\infty,$$

provided $0 < \liminf_{n\to\infty} P(M_n \leq v_n) \leq \limsup_{n\to\infty} P(M_n \leq v_n) < 1$.

The above results are taken from [Jak91a].

Limiting Probabilities for Maxima

Existence of an a.i.r. and limit theorems for maxima require an effective tool for calculating $\lim_{n\to\infty} P(M_n \leq v_n)$ in the presence of some mixing assumptions.

O'Brien [OBr87] obtained the representation

$$P(M_n \leq v_n) - \exp(-nP(X_0 > v_n, M_{r_n} \leq v_n)) \;\longrightarrow\; 0, \quad \text{as } n \to +\infty,$$

where $\{r_n\}$ is a suitable chosen sequence of integers. However such formula is useless, if we want to calculate the limiting probability and r_n tends to infinity: the expression under exponent depends on increasing number of random variables X_j, hence is of the same type as the approximated probability.

Therefore we investigate in detail other approximations, which are based on the knowledge of asymptotic properties of *finite dimensional* joint distributions only:

$$|P(M_n \leq v_n) - \exp(-nP(X_0 > u_n, M_m \leq v_n))| \;\longrightarrow\; 0, \quad \text{as } n \to +\infty,$$

for $m \in \mathbb{N}$ *fixed*, or

$$\lim_{m\to\infty}\limsup_{n\to\infty}|P(M_n \leq v_n) - \exp(-nP(X_0 > v_n, M_m \leq v_n))| = 0.$$

For example, the first approximation holds for m- dependent random variables ([New64]), while the second one is valid for uniformly strong mixing (i.e. ϕ-mixing) sequences ([OBr74b]).

It follows that the above approximations allow the following calculation:

$$\lim_{n\to\infty} \log P(M_n \leq v_n) = -\lim_{m\to\infty} \left\{ \begin{array}{c} \limsup_{n\to\infty} \\ \liminf_{n\to\infty} \end{array} \right\} nP(X_0 > v_n, M_m \leq v_n).$$

In Chapter 10 we discuss several conditions, including generalizations of Leadbetter's Condition D' ([Lea74], [LLR83]), which enable us to apply this formula.

All the results are taken from [Jak90b].

Asymptotic (r-1) - dependent Representations for rth Order Statistics

Let $X_1, X_2, \ldots$ be a stationary sequence of random variables. Denote by $M_n^{(k)}$ the kth largest value of $X_1, X_2, \ldots, X_n$.

It is well known, that for i.i.d. $X_1, X_2, \ldots$ convergence in distribution of suitably normalized partial maxima:

$$P(M_n \leq v_n(x)) \longrightarrow G(x), \quad x \in \mathbb{R}^1,$$

implies convergence of all order statistics: for each $q \geq 2$

$$P(M_n^{(q)} \leq v_n(x)) \longrightarrow G(x) \left(1 + \sum_{k=1}^{q-1} \frac{(-\log G(x))^k}{k!} \right), \quad x \in \mathbb{R}^1, \quad \text{as } n \to +\infty.$$

(see e.g. [Gal78] or [LLR83]).

If we drop the assumption of independence, *preserving only strong mixing property*, higher order statistics may fail to converge or they may converge to different limits. Assuming they converge for *each* $q \in \mathbb{N}$, Dziubdziela [Dzi84] and Hsing, Hüsler & Leadbetter [HHL88] describe possible limits in terms of parameters of certain compound Poisson distributions. We prefer the description given by Hsing [Hsi88] (see also Theorem 9.2): the limit for $M_n^{(q)}$ is of the form

$$G(x) \left(1 + \sum_{k=1}^{q-1} \frac{(-\log G(x))^k}{k!} \cdot \gamma_{q,k} \right),$$

where $0 \leq \gamma_{q,k} \leq 1, \quad k = 1, 2, \ldots, q-1$, and G is the limit for maxima. However, complexity of formulas for $\gamma_{q,k}$'s quickly increases with q, what makes difficult the analysis of asymptotic properties of higher order statistics. Therefore **we suggest approximation by a simple model in place of limiting distribution**.

The model is simple, indeed: take $\beta_1, \beta_2, \ldots, \beta_r \geq 0$ such that $\sum_{q=1}^{r} \beta_q = 1$ and a regular distribution function G. For each $1 \leq q \leq r$, let $\{\widetilde{Y}_{q,j}\}_{j\in \mathbb{N}}$ be independent and identically distributed: $\widetilde{Y}_{q,j} \sim G^{\beta_q}$, and let sequences $\{\widetilde{Y}_{1,j}\}_{j\in \mathbb{N}}$, $\{\widetilde{Y}_{2,j}\}_{j\in \mathbb{N}}, \ldots, \{\widetilde{Y}_{r,j}\}_{j\in \mathbb{N}}$ be mutually independent. Define

$$\begin{aligned} \widetilde{X}_j \;=\; & \widetilde{Y}_{1,j} \\ & \vee (\widetilde{Y}_{2,j} \vee \widetilde{Y}_{2,j+1}) \\ & \vee (\widetilde{Y}_{3,j} \vee \widetilde{Y}_{3,j+1} \vee \widetilde{Y}_{3,j+2}) \\ & \vdots \\ & \vee (\widetilde{Y}_{r,j} \vee \widetilde{Y}_{r,j+1} \vee \ldots \vee \widetilde{Y}_{r,j+r-1}). \end{aligned}$$

Finally, let $\widetilde{M}_n^{(q)}$, $q = 1, 2, \ldots, r$ be order statistics of $\widetilde{X}_1, \widetilde{X}_2, \ldots$.

Then Theorem 9.1 asserts, that

$$\sup_{x\in \mathbb{R}^1} |P(M_n^{(q)} \leq x) - P(\widetilde{M}_n^{(q)} \leq x)| \;\longrightarrow\; 0 \quad \text{as } n \to +\infty,$$

for each q, $1 \leq q \leq r$, if, and only if, there is a non-decreasing sequence $\{v_n\}$ such that for each q, $1 \leq q \leq r$ we have

$$P(M_n^{(q)} \leq v_n) \;\longrightarrow\; \alpha_q, \quad \text{as } n \to +\infty,$$

where $0 < \alpha_1 < 1$, and a natural mixing condition similar to Condition $\mathrm{B}_\infty(v_n)$ holds. Practically: every time there exists the limit for r first order statistics, we can approximate these statistics by our $(r-1)$-dependent model built up from G and $\beta_1, \beta_2, \ldots, \beta_r$.

It is a natural question, whether we can approximate simultaneously *all* order statistics by order statistics of a sequence $\widetilde{Y}_j$ obtained formally like $\widetilde{X}_j$ but for $r = \infty$. This is not automatic. For example, $\widetilde{Y}_j$ can be trivial: $\widetilde{Y}_j = G_*$ a.s.. Some other possibilities are discussed in Theorems 9.13 and 9.17.

All these results were originally obtained in [Jak91b].

Part I

p - stable limit theorems

Chapter 1

Asymptotic Independent Representations for sums of random variables

1.1 Simple Asymptotic Independent Representation

Let $X_1, X_2, \ldots$ be a strictly stationary sequence of random variables. Denote $S_0 = 0,\ S_n = \sum_{j=1}^n X_j,\ n = 1, 2, \ldots$.

We will say that $\{X_j\}$ admits (or possesses) asymptotic independent representation (a.i.r.) for partial sums, if one can find independent, identically distributed random variables $\widetilde{X}_1, \widetilde{X}_2, \ldots$, such that

$$\sup_{x \in \mathbb{R}^1} |P(S_n \leq x) - P(\widetilde{S}_n \leq x)| \longrightarrow 0 \quad \text{as } n \to +\infty, \tag{1.1}$$

where $\widetilde{S}_n = \widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n$.

In general, such a representation is not unique; it precisely describes, however, asymptotic properties of distributions of sums S_n. Therefore we will be interested in *regular* a.i.r. only, i.e. sequences $\{\widetilde{X}_j\}$ such that

$$\frac{\widetilde{S}_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu,$$

for some normalizing constants $B_n \to +\infty$ and some *non-degenerate* law μ.

Theorem 1.1 *A regular asymptotic independent representation exists for S_n if, and only if,*

$$\frac{S_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu,$$

where, for some $p \in (0, 2]$, μ is a non-degenerate strictly p-stable distribution, B_n is a $1/p$-regularly varying sequence and, if $p = 2$, then B_n^2/n is equivalent to a non-decreasing sequence.

PROOF FOR $p \neq 1$.

Let $\widetilde{S}_n = \widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n$ be a regular a.i.r. for S_n. If $B_n \to +\infty$ and non-degenerate μ are such that $\widetilde{S}_n/B_n \longrightarrow_{\mathcal{D}} \mu$, then $S_n/B_n \longrightarrow_{\mathcal{D}} \mu$, either, and B_n and μ possess the required properties directly by Theorems A.2 and A.3. To prove the converse implication, it is sufficient to find an i.i.d. sequence $\{\widetilde{X}_j\}$ such that $\widetilde{S}_n/B_n \longrightarrow_{\mathcal{D}} \mu$. Indeed, knowing that also $S_n/B_n \longrightarrow_{\mathcal{D}} \mu$, we get by continuity of μ both

$$\sup_{x \in \mathbb{R}^1} |P(\frac{\widetilde{S}_n}{B_n} \leq x) - \mu((-\infty, x])| \longrightarrow 0$$

and

$$\sup_{x \in \mathbb{R}^1} |P(\frac{S_n}{B_n} \leq x) - \mu((-\infty, x])| \longrightarrow 0.$$

Hence (1.1) follows.

The construction of $\mathcal{L}(\widetilde{X}_j)$ is based on Theorem A.5 and Lemma B.10 and is standard for $p \neq 1$. The case $p = 1$, however, requires more delicate treatment.

• By Theorem B.7 we may and do assume, that B_n is non-decreasing. If $p = 2$, we need more: we assume that B_n^2/n is non-decreasing.

• **Case** $0 < p < 1$. By (A.6), $\mu = \text{Pois}\,(\nu(p, c_+, c_-))$ for some $c_+,\ c_- \geq 0,\ c_+ + c_- > 0$. Let n_0 be such that $c_+ + c_- < n_0$. Define

$$F(x) = \begin{cases} c_-/n & \text{if } -B_{n+1} \leq x < -B_n \text{ and } n \geq n_0, \\ 1/2(1 - (c_+ - c_-)/n_0) & \text{if } -B_{n_0} \leq x < B_{n_0}, \\ 1 - c_+/n & \text{if } B_n \leq x < B_{n+1} \text{ and } n \geq n_0 \end{cases} \tag{1.2}$$

If $\widetilde{X}_1 \sim F$, then $nP(\widetilde{X}_1 > B_n) = c_+$, $nP(\widetilde{X}_1 < -B_n) = c_-$ and by Lemma B.10, both $f_+(x) = P(\widetilde{X}_1 > x)$ and $f_-(x) = P(\widetilde{X}_1 < -x)$ are regularly varying with index $-p$. Now, using Theorem B.11, we can easy verify conditions (A.20)–(A.22) in Theorem A.5. Hence $\widetilde{S}_n \longrightarrow_{\mathcal{D}} \mu$.

• **Case** $1 < p < 2$. If $\widetilde{Y}_j,\ j = 1, 2, \ldots$ are independent and distributed according to F defined by (1.2), then $E\widetilde{Y}_j$ exists and $\widetilde{X}_j = \widetilde{Y}_j - E\widetilde{Y}_j,\ j = 1, 2, \ldots$ satisfy assumptions of Theorem A.5 (iii).

• **Case** $p = 2$. In this case $\mu = N(0, \sigma^2)$ with $\sigma^2 > 0$. Set $\mathcal{L}(\widetilde{S}_n 1)$ to be symmetric and such that

$$E\widetilde{X}_1^2 I(|\widetilde{X}_1| \leq B_n) = \sigma^2 \frac{B_n^2}{n}. \tag{1.3}$$

CASE $p = 1$.

Now μ is of the form $\text{Pois}\,(\nu(1, c, c)) * \delta_a$, where $c > 0$. Let $\widetilde{Y}_1 \sim F$, with $c_+ = c_- = c$. Since $\widetilde{Y}_1$ is symmetric,

$$\frac{\widetilde{Y}_1 + \widetilde{Y}_2 + \ldots + \widetilde{Y}_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \text{Pois}\,(\nu(1, c, c)).$$

If $a = 0$, nothing else is to be proved. If $a \neq 0$, we will construct another i.i.d. sequence $\{\tilde{Z}_j\}$, being independent of $\{\tilde{Y}_j\}$ and such that

$$\frac{\tilde{Z}_1 + \tilde{Z}_2 + \ldots + \tilde{Z}_n}{B_n} \underset{\mathcal{P}}{\longrightarrow} a.$$

Without loss of generality, we can take $a = 1$. A special representation for slowly varying functions is necessary.

Lemma 1.2 *Let $\ell(x)$ be a slowly varying function. Then there exists a random variable Z such that*

$$\ell(x) \sim EZI(|Z| \leq x) \tag{1.4}$$

and

$$\frac{xP(|Z| \geq x)}{EZI(|Z| \leq x)} \longrightarrow 0 \quad \text{as } x \to +\infty. \tag{1.5}$$

Proof. By Theorem B.6 one can find $a_\ell > 0$ and a C^∞–function h defined on $[a_\ell, +\infty)$ such that

$$\ell(x) \sim e^{h(\log x)} = \ell_1(x). \tag{1.6}$$

and for each $n \geq 1$

$$h^{(n)}(x) \quad \longrightarrow 0 \quad \text{as } x \to +\infty. \tag{1.7}$$

Let $x_0 > a_\ell$ be such that

$$h'(\log x_0) < 1. \tag{1.8}$$

For $x \geq x_0$, set

$$q(x) = (\ell_1''(x))^-, \quad q(-x) = (\ell_1''(x))^+. \tag{1.9}$$

Let us observe that

$$M(x_0) = \int_{x_0}^\infty (q(x) + q(-x))\, dx < +\infty. \tag{1.10}$$

Indeed,

$$M(x_0) = \int_{x_0}^\infty e^{h(\log x)} |h''(\log x) - h'(\log x)(1 - h'(\log x))| x^{-2}\, dx,$$

where $e^{h(\log x)}/x^{1/2}$ and $|h''(\log x) - h'(\log x)(1 - h'(\log x))|$ are bounded functions and $x^{-3/2}$ is integrable on $[x_0, +\infty)$. In particular, for $x \geq x_0$

$$\int_x^\infty (q(u) - q(-u))\, du = \int_x^\infty d(-\ell_1'(u)) = \ell_1'(x) - \lim_{y\to\infty} \ell_1'(y) = \ell_1'(x), \tag{1.11}$$

for $\ell'(y) = y^{-1} e^{h(\log y)} h'(\log y) \to 0$ as $y \to \infty$.

We complete the definition of q, setting $q(0) = 0$ and for $0 < x < x_0$,

$$q(x) = D \cdot \frac{1}{x_0}, \quad q(-x) = 0. \tag{1.12}$$

Here D is chosen in such a way that

$$\begin{aligned}\ell_1(x_0) &= \int_0^{x_0}\left(\int_x^\infty (q(u)-q(-u))\,du\right)dx \\ &= \int_0^{x_0}\left(\int_x^{x_0} (q(u)-q(-u))\,du\right)dx+ \\ &\quad + \int_0^{x_0}\left(\int_{x_0}^\infty (q(u)-q(-u))\,du\right)dx \\ &= (1/2)x_0 D + \ell_1'(x_0)x_0'\end{aligned} \tag{1.13}$$

i.e.

$$D = 2\ell_1(x_0)\frac{1-h'(\log x_0)}{x_0}.$$

Notice that $D > 0$ by (1.8). If

$$C := \int_0^\infty (q(x)+q(-x))\,dx = D + M(x_0)\ (>0),$$

then $p(x) = C^{-1}q(x),\ x \in \mathbb{R}^1$ is a probability density on $\mathbb{R}^1$. Let $\mathcal{L}(W)$ has the density $p(x)$; then by (1.11) and (1.12)

$$\begin{aligned}\ell(x) \sim \ell_1(x) &= \int_{x_0}^x \ell_1'(u)\,du + \ell_1(x_0) \\ &= C\int_{x_0}^x (P(W>u)-P(W<-u))\,du+ \\ &\quad + C\int_0^{x_0} (P(W>u)-P(W<-u))\,du \\ &= C\cdot EWI(|W|\le x) + Cx(P(W>x)-P(W<-x)).\end{aligned} \tag{1.14}$$

Further, for $x \ge x_0$

$$\begin{aligned}\frac{xP(|W|\ge x)}{\ell_1(x)} &= \frac{C^{-1}\int_x^\infty |\ell_1''(u)|\,du}{x^{-1}\ell_1(x)} \sim \\ &\sim \frac{|h''(\log x) - h'(\log x)(1-h'(\log x))|}{C(1-h'(\log x))} \longrightarrow 0 \quad \text{as } x \to +\infty.\end{aligned}$$

In particular, $x(P(W>x)-P(W<-x)) = o(\ell_1(x))$ and $\ell(x) \sim C\cdot EWI(|W|\le x)$. Hence $EWI(|W|\le x)$ is slowly varying and

$$\ell(x) \sim C\cdot EWI(|W|\le x) \sim C\cdot EWI(|W|\le x/C) = E(C\cdot W)I(|C\cdot W|\le x).$$

So $Z = C\cdot W$ has the desired properties (1.4) and (1.5). □

Now we are ready to construct independent $\widetilde{Z}_1, \widetilde{Z}_2, \ldots$ satisfying

$$\frac{\widetilde{Z}_1+\widetilde{Z}_2+\ldots+\widetilde{Z}_n}{B_n} \xrightarrow[\mathcal{P}]{} 1. \tag{1.15}$$

Let $\ell_B(x)$ be an (asymptotic) inverse to 1-regularly varying $B_{[x]}$. If $\ell(x) = [x]/\ell_B(x)$, then it varies slowly and $B_n \sim \ell_B(B_n)\ell(B_n) \sim n\ell(B_n)$. Now, set $\mathcal{L}(\widetilde{Z}_j) = \mathcal{L}(Z)$ for Z satisfying (1.4) and (1.5) and apply Theorem A.4.

1.2 A.I.R. in the Array Setting

The case $p = 2$ is exceptional: we impose on B_n the requirement that "$B_n^2/n \sim c_n > 0, c_n \nearrow$ ", which is equivalent to "$\liminf_{n\to\infty} B_n^2/n > 0$ and $B_n^2/n \sim \inf_{m\geq n} B_m^2/m$." This implies that B_n^2/n *must* converge to nonzero limit (perhaps to infinity). Let us consider the following example, due to Bradley [Bra80, Lemma 2]

Example 1.3 There exists a centered gaussian stationary sequence $\{X_j\}$ such that $B_n = \sqrt{\mathrm{Var}\,(S_n)}$ is (1/2)-regularly varying and $B_n^2/n \to 0$. □

Clearly, for centered gaussian sequences, $S_n/\sqrt{\mathrm{Var}\,(S_n)} \sim N(0,1)$, so still: "$B_n$ is $1/p$-regularly varying and μ is strictly p-stable", while **no a.i.r. exists**. One can find, however, an approximation via a stationary and independent in rows array.

We will say, that $\{S_n\}$ admits an asymptotic independent representation *in the array setting* if there is an array $\{\widetilde{X}_{n,j}; j \in \mathbb{N}, n \in \mathbb{N}\}$ of independent and stationary in rows random variables, such that for each $\theta > 1$

$$\max_{n\leq m\leq \theta\cdot n} \sup_{x\in\mathbb{R}^1} |P(S_m \leq x) - P(\sum_{j=1}^m \widetilde{X}_{n,j} \leq x)| \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{1.16}$$

Theorem 1.4 *Suppose $B_n \to \infty$ and $S_n/B_n \longrightarrow_{\mathcal{D}} \mu$, where μ is non-degenerate. Then the following conditions (i) and (ii) are equivalent:*

(i) *B_n is $1/p$-regularly varying and μ is strictly p-stable for some $p \in (0,2]$.*

(ii) *$\{S_n\}$ admits an asymptotic independent representation in the array setting.*

PROOF.
• **(i)** $\Rightarrow$ **(ii)**. If $0 < p < 2$, set

$$\widetilde{X}_{n,j} = \widetilde{X}_j, \quad j \in \mathbb{N}, \; n \in \mathbb{N},$$

where $\widetilde{X}_j$'s are given by Theorem 1.1. Then

$$\sup_{n\leq m} \sup_{x\in\mathbb{R}^1} |P(S_m \leq x) - P(\sum_{j=1}^m \widetilde{X}_{n,j} \leq x)| \longrightarrow 0 \quad \text{as } n \to +\infty,$$

and (1.16) is satisfied. So let $p = 2$ and let $\mu = N(0,\sigma^2)$. Set

$$P(\widetilde{X}_{n,j} = \pm\sigma B_n/\sqrt{n}) = 1/2.$$

If $n \leq k_n \leq \theta \cdot n$, then

$$k_n E\frac{\widetilde{X}_{n,1}}{B_{k_n}} I(\frac{|\widetilde{X}_{n,1}|}{B_{k_n}} \leq 1) = 0, \quad \forall n \in \mathbb{N}. \tag{1.17}$$

We have also for $\varepsilon > 0$ and n large enough

$$k_n P(\frac{|\widetilde{X}_{n,1}|}{B_{k_n}} > \varepsilon) = 0, \tag{1.18}$$

and by 1/2-regular variation of B_n, as $n \to +\infty$

$$k_n E\left(\frac{\widetilde{X}_{n,1}}{B_{k_n}}\right)^2 I(\frac{|\widetilde{X}_{n,1}|}{B_{k_n}} \le 1) = k_n \cdot \frac{\sigma^2}{n} \cdot \frac{B_n^2}{B_{k_n}^2} \longrightarrow \sigma^2. \tag{1.19}$$

By Theorem A.5 (iv), conditions (1.17) – (1.19) imply that

$$B_{k_n}{}^{-1} \sum_{j=1}^{k_n} \widetilde{X}_{n,j} \underset{\mathcal{D}}{\longrightarrow} N(0, \sigma^2).$$

Since $\{k_n\}$ was arbitrary, (1.16) follows.

• **(ii) ⇒ (i).** By (1.16), for each $t > 1$, $\{S_{[nt]}/B_n\}$ possesses the same limit in distribution as $\sum_{j=1}^{[nt]} \widetilde{X}_{n,j}/B_n$ (if there is any). But $S_n/B_n \longrightarrow_{\mathcal{D}} \mu$, so for each $\lambda \in I\!R^1$

$$Ee^{i\lambda B_n^{-1} \sum_{j=1}^{[nt]} \widetilde{X}_{n,j}} = \left(Ee^{i\lambda B_n^{-1} \widetilde{X}_{n,1}}\right)^{[nt]} = \left\{\left(Ee^{i\lambda B_n^{-1} \widetilde{X}_{n,1}}\right)^n\right\}^{[nt]/n} \longrightarrow \hat{\mu}(\lambda)^t.$$

Hence

$$\frac{B_{[nt]}}{B_n}\left(\frac{1}{B_{[nt]}} \sum_{j=1}^{[nt]} \widetilde{X}_{n,j}\right)$$

converges in distribution to a non-degenerate limit, which is *distinct from* μ. By the convergence to types theorem, for each $t > 1$

$$\frac{B_{[nt]}}{B_n} \longrightarrow \psi(t) \neq 1, \tag{1.20}$$

where $\psi(t)$ is finite and positive. By Theorem B.1, $\psi(t) = t^\rho$ for some $-\infty < \rho < +\infty$ and (1.20) holds for each $t > 0$. Since $B_n \to \infty$ and $\psi(t) \not\equiv 1$, we have $\rho = 1/p > 0$, for some $p > 0$, and B_n is $1/p$-regularly varying. In particular, setting $t = 1/k$ we get

$$\mathcal{L}((1/k^{1/p})X)^{*k} = \mathcal{L}(X),$$

if $\mathcal{L}(X) = \mu$. By Proposition A.1, μ is strictly p-stable and $0 < p \le 2$. □

Remark 1.5 In the proof of (ii) ⇒ (i) we used only the property that for each $t > 1$

$$\sup_{x \in I\!R^1} |P(S_{[nt]} \le x) - P(\sum_{j=1}^{[nt]} \widetilde{X}_{n,j} \le x)| \longrightarrow 0. \tag{1.21}$$

This is the alternative form of (1.16). We prefer, however, (1.16), for it shows the n-th row provides a "good" approximation on intervals $n \le m \le \theta \cdot n$.

Chapter 2

Discussion of mixing properties

2.1 Condition B

We introduced "p-stable limit theorems" as results on the weak convergence of sums to strictly p-stable limit laws with $1/p$-regularly varying normalizing constants. From the previous chapter we know that a p-stable limit theorem holds if and only if one can find a convergent (after normalizing) asymptotic independent representation in the array setting. But existence of an a.i.r. implies a kind of "asymptotic independence" or "mixing". We are able to describe the minimal form of such mixing properties.

Theorem 2.1 *Suppose $B_n \to \infty$ and*

$$\frac{S_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu \neq \delta_0 \ . \tag{2.1}$$

There exists $p \in (0,2]$ such that μ is a strictly p-stable distribution and B_n is a regularly varying sequence with index $1/p$ if and only if the following condition is satisfied:
CONDITION B *For each $\lambda \in I\!R^1$,*

$$\max_{\substack{1\le k,l\le n\\ k+l\le n}} |E\, e^{i\lambda(S_{k+l}/B_n)} - Ee^{i\lambda(S_k/B_n)} \cdot Ee^{i\lambda(S_l/B_n)}| \ \longrightarrow\ 0 \quad \text{as } n \to +\infty. \tag{2.2}$$

PROOF. To prove the sufficiency, we need a variant of the convergence to types theorem.

Lemma 2.2 *Suppose that $B_n \to \infty$ and that $S_n/B_n \ \longrightarrow_{\mathcal{D}} \mu$. Let $k_n \to \infty$, $k_n \le n$. If μ is non-degenerate and $\{S_{k_n}/B_n\}$ is shift-tight, then*

$$\sup_n \frac{B_{k_n}}{B_n} \le C < +\infty \ . \tag{2.3}$$

In particular, $\{S_{k_n}/B_n\}$ is tight and its every limit distribution is of the form $\mathcal{L}(C' \cdot X)$, where $\mathcal{L}(X) = \mu$ and $0 \le C' \le C$.

PROOF. We know that
$$\frac{S_n - \overline{S}_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu * \overline{\mu} \neq \delta_0,$$
where $\overline{S}_n$ is an independent copy of S_n and $\overline{\mu}(A) = \mu(-A),\ A \in \mathcal{B}^1$.

Suppose $B_{k_{n'}}/B_{n'} \to +\infty$ along some subsequence $\{n'\} \subset \mathbb{N}$. Hence
$$\frac{S_{k_{n'}} - \overline{S}_{k_{n'}}}{B_{n'}} = \frac{B_{k_{n'}}}{B_{n'}} \cdot \frac{S_{k_{n'}} - \overline{S}_{k_{n'}}}{B_{k_{n'}}}$$
is not tight, i.e. $\{S_{k_{n'}}/B_{n'}\}$ cannot be shift-tight. □

- SUFFICIENCY

First we will show that μ is a strictly p-stable distribution and that $B_{k\cdot n}/B_n \to k^{1/p}$ for each $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$ and observe that by (2.2)
$$\Big(\mathcal{L}(\frac{S_n}{B_{k\cdot n}})\Big)^{*k} \Longrightarrow \mu \quad \text{as } n \to +\infty.$$
If $\mu = \delta_a, a \neq 0$, then $S_n/B_{k\cdot n} \longrightarrow_{\mathcal{D}} a/k$ and $B_{k\cdot n}/B_n \to k$. If μ is non-degenerate, we can apply the above lemma and see that $\{B_n/B_{k\cdot n}\}$ is bounded. If c_k is any limit point of $B_n/B_{k\cdot n}$ and $\mathcal{L}(X) = \mu$, then
$$\mathcal{L}(c_k \cdot X)^{*k} = \mathcal{L}(X)\,,$$
and since $\mu \neq \delta_0$, we have $c_k \neq 0$. So for each k one can find a constant $c_k > 0$ such that
$$\mathcal{L}(X)^{*k} = \mathcal{L}((1/c_k)X)\,.$$
By Proposition A.1, μ is strictly p-stable for some $p \in (0, 2]$. Moreover, $c_k = k^{1/p}$, so
$$\frac{B_{k\cdot n}}{B_n} \longrightarrow k^{1/p} \quad \text{as } n \to +\infty. \tag{2.4}$$

It remains to prove that B_n is $1/p$-regularly varying. By Lemma B.4 we have to prove that $B_{k_n}/B_{k_n+l_n} \to 1$ whenever $k_n \to \infty$ and $l_n/k_n \to 0$.

Since $B_n \to \infty$, there exists $m_n \to \infty$ such that
$$\max_{1\leq j\leq m_n} S_j/B_{k_n+l_n} \underset{\mathcal{P}}{\longrightarrow} 0\,.$$
If $l_{n'} \leq m_{n'}$ along a subsequence $\{n'\} \subset \mathbb{N}$, then
$$\frac{S_{k_{n'}}}{B_{k_{n'}}} \cdot \frac{B_{k_{n'}}}{B_{k_{n'}+l_{n'}}} = \frac{S_{k_{n'}}}{B_{k_{n'}+l_{n'}}} \sim \frac{S_{k_{n'}+l_{n'}}}{B_{k_{n'}+l_{n'}}} \underset{\mathcal{D}}{\longrightarrow} \mu\,,$$
and $B_{k_{n'}}/B_{k_{n'}+l_{n'}} \to 1$ by the convergence to types theorem (if $\mu = \delta_a,\ a \neq 0$, we get $B_{k_{n'}}/B_{k_{n'}+l_{n'}} \to 1$ by direct arguments). So we can assume that $l_n > m_n$, in particular, that $l_n \to \infty$. By the asymptotic decomposition given by (2.2)
$$\mathcal{L}(\frac{S_{k_n}}{B_{k_n+l_n}}) * \mathcal{L}(\frac{S_{l_n}}{B_{k_n+l_n}}) \Longrightarrow \mu. \tag{2.5}$$

If μ is non-degenerate, we know by Lemma 2.2 that both $B_{k_n}/B_{k_n+l_n}$ and $B_{l_n}/B_{k_n+l_n}$ are bounded.

If $\mu = \delta_a$, say $a > 0$, then for each $\varepsilon > 0$

$$P(\frac{S_{l_n}}{B_{k_n+l_n}} > -\varepsilon) \longrightarrow 1 \quad \text{as } n \to +\infty \,.$$

This and (2.5) imply for each $\varepsilon > 0$

$$P(\frac{S_{k_n}}{B_{k_n+l_n}} < a + \varepsilon) \longrightarrow 1$$

or

$$P\left(\frac{S_{k_n}}{B_{k_n}} < \frac{B_{k_n+l_n}}{B_{k_n}}(a+\varepsilon)\right) \longrightarrow 1 \,.$$

Hence $\liminf B_{k_n+l_n}/B_{k_n} \geq 1$ and $B_{k_n}/B_{k_n+l_n}$ is a bounded sequence. Similarly $B_{l_n}/B_{k_n+l_n}$ is bounded.

Suppose that $B_{k_n}/B_{k_n+l_n} \to c_1$ and $B_{l_n}/B_{k_n+l_n} \to c_2$ along a subsequence. Then

$$\mathcal{L}(c_1 X) * \mathcal{L}(c_2 X) = \mathcal{L}(X) = \mu \,,$$

and it follows from strict stability of μ, that $c_1^p + c_2^p = 1$. Hence

$$\left(\frac{B_{k_n}}{B_{k_n+l_n}}\right)^p + \left(\frac{B_{l_n}}{B_{k_n+l_n}}\right)^p \longrightarrow 1 \quad \text{as } n \to +\infty \,. \tag{2.6}$$

This in turn implies that for $k_n, l_n \to \infty$

$$\limsup_{n\to\infty} \frac{B_{k_n}}{B_{k_n+l_n}} \leq 1 \,. \tag{2.7}$$

If $l_n/k_n \to 0$ then $l_n \leq k_n/k$ for n large enough and by (2.7) and (2.4)

$$\begin{aligned}
\limsup_{n\to\infty} \frac{B_{l_n}}{B_{k_n+l_n}} &\leq \limsup_{n\to\infty} \frac{B_{l_n}}{B_{k_n}} \\
&\leq \limsup_{n\to\infty} \frac{B_{l_n}}{B_{k[k_n/k]}} \\
&= \limsup_{n\to\infty} \frac{B_{l_n}}{B_{[k_n/k]}} \cdot \frac{B_{[k_n/k]}}{B_{k[k_n/k]}} \\
&\leq k^{-1/p} \longrightarrow 0 \quad \text{as } k \to \infty \,.
\end{aligned}$$

Hence (2.6) implies $B_{k_n}/B_{k_n+l_n} \to 1$.

- Necessity

By Theorem B.7 we can assume that B_n is non-decreasing. Further, if $k_n/n \to 0$ then for n large enough $k_n \le n/k$ (k fixed) and

$$\frac{B_{k_n}}{B_n} = \frac{B_{k_n}}{B_{[n/k]}} \cdot \frac{B_{[n/k]}}{B_{k[n/k]}} \cdot \frac{B_{k[n/k]}}{B_n} \le \frac{B_{[n/k]}}{B_{k[n/k]}} \longrightarrow k^{-1/p} \quad \text{as } n \to +\infty .$$

Since k is arbitrary, we conclude that $B_{k_n}/B_n \to 0$ and, in particular,

$$\frac{S_{k_n}}{B_n} \underset{\mathcal{P}}{\longrightarrow} 0 . \tag{2.8}$$

If (2.2) does not hold, one can find $\lambda \in I\!R^1$, sequences k_n and l_n and a subsequence $\{n'\} \subset I\!N$ such that along n'

$$\frac{k_{n'}}{n'} \to s, \quad \frac{l_{n'}}{n'} \to t, \qquad s + t \le 1 ,$$

and for some $\delta > 0$

$$\left| E e^{i\lambda(S_{k_{n'}+l_{n'}}/B_{n'})} - E e^{i\lambda(S_{k_{n'}}/B_{n'})} \cdot E e^{i\lambda(S_{l_{n'}}/B_{n'})} \right| \ge \delta . \tag{2.9}$$

If $s > 0$ and $t > 0$, then by $1/p$-regular variation of B_n, $E e^{i\lambda(S_{k_{n'}}/B_{n'})} \to E e^{i\lambda s^{1/p} X}$, $E e^{i\lambda(S_{l_{n'}}/B_{n'})} \to E e^{i\lambda t^{1/p} X}$ and $E e^{i\lambda(S_{k_{n'}+l_{n'}}/B_{n'})} \to E e^{i\lambda (s+t)^{1/p} X}$, where $\mathcal{L}(X) = \mu$. Since μ is strictly p-stable,

$$E e^{i\lambda s^{1/p} X} \cdot E e^{i\lambda t^{1/p} X} = E e^{i\lambda (s+t)^{1/p} X} ,$$

what is in contradiction with (2.9).

If $s = 0$ and $t > 0$, we have by (2.8), $S_{k_{n'}}/B_{n'} \longrightarrow_{\mathcal{P}} 0$, $S_{l_{n'}}/B_{n'} \longrightarrow_{\mathcal{D}} t^{1/p} X$ and $S_{k_{n'}+l_{n'}}/B_{n'} \longrightarrow_{\mathcal{D}} t^{1/p} X$, so again (2.9) is impossible. Similarly, if $s = 0$ and $t = 0$, the three limits are 0. □

2.2 Alternative versions of Condition B

Remark 2.3 If X_j's are non-negative, i.e. values of S_n/B_n lie in $[0, \infty)$, we can use the Laplace Transform instead of characteristic functions. Condition B takes then the form: *For each $\lambda > 0$,*

$$\max_{\substack{1 \le k, l \le n \\ k+l \le n}} \left| E e^{-\lambda(S_{k+l}/B_n)} - E e^{-\lambda(S_k/B_n)} \cdot E e^{-\lambda(S_l/B_n)} \right| \longrightarrow 0 . \tag{2.10}$$

Condition B means we can break sums into seemingly independent components. This property becomes even more transparent, if we consider an alternative version of Condition B.

Proposition 2.4 *Suppose $B_n \to \infty$, $\{S_n/B_n\}$ is tight and no limit point of $\{S_n/B_n\}$ is degenerated. Then Condition B holds if and only if for some (and then for any) metric d, which metricizes the weak convergence of distributions on $\mathbb{R}^1$, we have*

$$\max_{\substack{1\le k,l\le n\\ k+l\le n}} d\left(\mathcal{L}(\frac{S_{k+l}}{B_n}), \mathcal{L}(\frac{S_k}{B_n}) * \mathcal{L}(\frac{S_l}{B_n})\right) \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{2.11}$$

PROOF. Tightness of $\{S_n/B_n\}$ and either (2.11) or (2.2) imply shift tightness of $\{S_k/B_n\ ;\ 1 \le k \le n, n \in \mathbb{N}\}$. If no limit point of $\{S_n/B_n\}$ is degenerated, then we can apply a slightly modified Lemma 2.2: $\sup_{1\le k\le n,\, n\in\mathbb{N}} B_k/B_n < +\infty$, hence, consequently, $\{S_k/B_n\ ;\ 1 \le k \le n, n \in \mathbb{N}\}$ is tight.

Now suppose (2.11) is not satisfied. Then we can find sequences $k_n \le n$ and $l_n \le n$ and a subsequence $\{n'\} \subset \mathbb{N}$ such that along n'

$$\frac{S_{k_{n'}+l_{n'}}}{B_{n'}} \underset{\mathcal{D}}{\longrightarrow} \mu_1, \quad \frac{S_{k_{n'}}}{B_{n'}} \underset{\mathcal{D}}{\longrightarrow} \mu_2, \quad \frac{S_{l_{n'}}}{B_{n'}} \underset{\mathcal{D}}{\longrightarrow} \mu_3,$$

and $\mu_1 \ne \mu_2 * \mu_3$. So (2.2) cannot hold. The converse implication can be proved the same way. □

2.3 Some Examples

It should be pointed out that Condition B can be satisfied even by non-ergodic sequences and only for very particular choice of B_n. Examples (2.5) - (2.7) below provide three types of such phenomena.

Example 2.5 Let $X \sim \text{Pois}\,(\nu(1, c, c))$ (Cauchy distribution) and let for each $j \in \mathbb{N}$, $X_j = X$. Then it is well known, that $\mathcal{L}((m/n)X) = \mathcal{L}(X/n)^{*m}$, hence

$$\begin{aligned}\mathcal{L}\left(\frac{S_{k+l}}{n}\right) &= \mathcal{L}\left(\frac{k+l}{n}X\right) = \mathcal{L}\left(\frac{X}{n}\right)^{*(k+l)} \\ &= \mathcal{L}\left(\frac{X}{n}\right)^{*k} * \mathcal{L}\left(\frac{X}{n}\right)^{*l} = \mathcal{L}\left(\frac{S_k}{n}\right) * \mathcal{L}\left(\frac{S_l}{n}\right)\end{aligned}$$

and Condition B is fulfilled with $\{X_1, X_2, \ldots\}$ totally dependent. □

Example 2.6 Let $0 < p < 1$ and let μ belongs to the domain of attraction of $\text{Pois}\,(\nu(p, c_+, c_-))$. If $\{\widetilde{X}_j\}$ are i.i.d. and $\mathcal{L}(\widetilde{X}_j) = \mu$, then

$$\frac{\widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \text{Pois}\,(\nu(p, c_+, c_-))\,,$$

where $B_n = n^{1/p}\ell(n)$ varies regularly. Now take arbitrary random variable X and define

$$X_j = \widetilde{X}_j + X\,, \quad j = 1, 2, \ldots\,.$$

Then $\{X_j\}$ is a stationary sequence and

$$\frac{X_1+X_2+\ldots+X_n}{B_n}=\frac{\widetilde{X}_1+\widetilde{X}_2+\ldots+\widetilde{X}_n}{B_n}+\frac{n}{B_n}X \underset{\mathcal{D}}{\longrightarrow} \text{Pois}\,(\nu(p,c_+,c_-))\,,$$

for $n/B_n = n^{1-1/p}/\ell(n) \to 0$ as $n\to+\infty$. By Theorem 2.1, Condition B holds. □

Example 2.7 Let μ_1 and μ_2 be two *distinct* probability distributions on $I\!R^1$ with zero mean and variance 1. Let $X_1, X_2, \ldots$ be conditionally independent over σ-field $\mathcal{I} = \{A, A^c, \emptyset, \Omega\}$ $(0 < P(A) < 1)$ and such that for each j, the regular conditional distribution of X_j given $\mathcal{I}$ is of the form $\mu_1\chi_A + \mu_2\chi_{A^c}$. Then

$$E(\exp(it\sum_{j=1}^n X_j/\sqrt{n})|\mathcal{I}) = (E(\exp(itX_1/\sqrt{n})|\mathcal{I}))^n \longrightarrow e^{-(1/2)t^2} \quad \text{a.s.},$$

so

$$E(\exp(it\sum_{j=1}^n X_j/\sqrt{n})) = E(E(\exp(it\sum_{j=1}^n X_j/\sqrt{n})|\mathcal{I})) \longrightarrow e^{-(1/2)t^2}\,.$$

Here again $X_1, X_2, \ldots$ satisfies Condition B by Theorem 2.1. □

2.4 Associated random variables

The next example is more subtle.

Example 2.8 Suppose that $X_1, X_2, \ldots$ are associated, i.e. for each $n \in I\!N$ and each pair f, g of functions $f, g : I\!R^n \to I\!R^1$, which are bounded, measurable and non-decreasing in each coordinate separately,

$$\text{Cov}\,(f(X_1,X_2,\ldots,X_n), g(X_1,X_2,\ldots,X_n)) \geq 0. \tag{2.12}$$

This definition is due to [EPW67]. For associated random variables with finite variances, Newman ([New80], see also [New84]) proved an inequality which we will use in the form

$$|Ee^{i\lambda S_{k+l}/B_n} - Ee^{i\lambda S_k/B_n}\cdot Ee^{i\lambda S_l/B_n}| \leq \frac{\lambda^2}{B_n^2}\text{Cov}\,(S_k, S_{k+l}-S_k)\,. \tag{2.13}$$

Notice that if $k+l \leq n$ and $EX_j = 0$ we have

$$\text{Cov}\,(S_k, S_{k+l}-S_k) \leq \sum_{j=1}^{k+l-1} jEX_0X_j \leq \sum_{j=1}^{n-1} jEX_0X_j\,. \tag{2.14}$$

Corollary 2.9 *Suppose $X_1, X_2, \ldots$ are stationary associated with finite variances and zero mean.*

(i) *If $EX_0X_n = o(1/n)$ then Condition B holds for $B_n = \sqrt{n}$.*

(ii) *$\ell(n) = EX_0^2 + 2\sum_{j=1}^n EX_0X_j$ is a slowly varying sequence if and only if $B_n = \sqrt{\mathrm{Var}\, S_n}$ is 1/2-regularly varying. In such a case Condition B is satisfied for B_n.*

Proof. **(i)** is immediate from (2.13) and (2.14). Notice that here $S_n/\sqrt{n}$ need not be tight; nevertheless Condition B holds.
(ii) For associated random variables with zero mean,

$$EX_0X_j \geq EX_0EX_j = 0\,,$$

hence $\ell(0) > 0$ and $\ell(n)$ is a non-decreasing sequence. By Corollary B.16, $\ell(n)$ is slowly varying if and only if $ES_n^2 = \ell(0) + \ell(1) + \ldots + \ell(n-1) \sim n\ell(n)$, i.e. ES_n^2 is 1-regularly varying. But

$$ES_n^2 = nEX_0^2 + 2\sum_{j=1}^{n-1}(n-j)EX_0X_j = n\ell(n) - 2\sum_{j=1}^{n-1} jEX_0X_j\,.$$

Since $ES_n^2 \sim n\ell(n)$ we see that

$$\frac{\sum_{j=1}^{n-1} jEX_0X_j}{ES_n^2} \longrightarrow 0\,,$$

and Condition B follows by (2.13) and (2.14). □

It is possible to weaken the assumptions of Newman's inequality (2.13) at the cost of a constant factor on the right-hand-side. The following definition was proposed by Burton, Dabrowski and Dehling [BDD86]: Random variables $X_1, X_2, \ldots, X_m$ are said to be *weakly associated*, if whenever π is a permutation of $\{1, 2, \ldots, n\}$, $1 \leq k < m$, and $f : I\!R^k \to I\!R^1$, $g : I\!R^{(m-k)} \to I\!R^1$ are coordinatewise increasing then

$$\mathrm{Cov}\,(f(X_{\pi(1)}, \ldots, X_{\pi(k)}), g(X_{\pi(k+1)}, \ldots, X_{\pi(m)})) \geq 0\,.$$

Dabrowski and Dehling [DaDe88, Proposition 2.1] proved that

$$\left| E\exp(i(\sum_{j=1}^m \lambda_j X_j)/B_n) - \prod_{j=1}^N E\exp(i\lambda_j X_j/B_n)\right| \leq 2B_n^{-2} \sum_{1\leq j<k\leq m} |\lambda_j||\lambda_k|\mathrm{Cov}\,(X_j, X_k), \qquad (2.15)$$

provided $X_1, X_2, \ldots, X_m$ are weakly associated. In particular,

$$|Ee^{i\lambda S_{k+l}/B_n} - Ee^{i\lambda S_k/B_n} \cdot Ee^{i\lambda S_l/B_n}| \leq 2\frac{\lambda^2}{B_n^2}\mathrm{Cov}\,(S_k, S_{k+l} - S_k)\,, \qquad (2.16)$$

and we get

Corollary 2.10 *In Corollary 2.9 we may assume that $X_1, X_2, \ldots$ are weakly associated only.* □

Remark 2.11 An example on p.302 [BDD86] shows that there are weakly associated sequences, which are not associated. This means Corollary 2.10 essentially improves Corollary 2.9.

2.5 Strong mixing and Condition B

In the above examples we checked Condition B either by Theorem 3.1 or by means of special tools like Newman's inequality. It is not a traditional approach. The tradition, initiated by Rosenblatt's paper [Ros56], suggests describing mixing properties via "mixing coefficients", being specific measures of dependence between "future" and "past". Let $\mathcal{G}$ and $\mathcal{H}$ be σ-fields in a probability space $(\Omega, \mathcal{F}, P)$. Define:

$$\alpha(\mathcal{G},\mathcal{H}) = \sup\{|\,P(A\cap B) - P(A)P(B)\,|\;;\; A\in\mathcal{G},\; B\in\mathcal{H}\}. \tag{2.17}$$
$$\phi(\mathcal{G},\mathcal{H}) = \sup\{|\,P(B|A) - P(B)\,|\;;\; A\in\mathcal{G},\; P(A)>0,\; B\in\mathcal{H}\}. \tag{2.18}$$
$$\psi(\mathcal{G},\mathcal{H}) = \sup\{\left|\frac{P(A\cap B)}{P(A)P(B)} - 1\right|\;;\; A\in\mathcal{G},\; B\in\mathcal{H},\; P(A)P(B)>0\}. \tag{2.19}$$

Clearly

$$\alpha(\mathcal{G},\mathcal{H}) \le \min\{\,\phi(\mathcal{G},\mathcal{H}),\phi(\mathcal{H},\mathcal{G})\,\} \le \max\{\,\phi(\mathcal{G},\mathcal{H}),\phi(\mathcal{H},\mathcal{G})\,\} \le \psi(\mathcal{G},\mathcal{H}). \tag{2.20}$$

Mixing coefficients provide a useful estimation of covariances, e.g. by [BrBr85, Theorem 1.1] (see also [Pel83]):

Lemma 2.12 *Suppose $1 \le p, q \le \infty$ and $p^{-1} + q^{-1} \le 1$. If X and Y are complex random variables and $X \in L^p(\mathcal{G})$, $Y \in L^q(\mathcal{H})$, then*

$$|EXY - EX\cdot EY| \le 2\pi\alpha(\mathcal{G},\mathcal{H})^{1/r}\phi(\mathcal{G},\mathcal{H})^{1/p}\phi(\mathcal{H},\mathcal{G})^{1/q}\|X\|_p\|Y\|_q\,, \tag{2.21}$$

where $1 \le r \le \infty$ is such that $p^{-1} + q^{-1} + r^{-1} = 1$. □

Another useful estimation can be obtained, if we set

$$\rho(\mathcal{G},\mathcal{H}) = \sup\frac{|\mathrm{Cov}\,(X,Y)|}{\sqrt{\mathrm{Var}\,(X)\mathrm{Var}\,(Y)}}\,, \tag{2.22}$$

where supremum is over all real $X \in L^2(\mathcal{G})$, $Y \in L^2(\mathcal{H})$ such that $\mathrm{Var}\,(X) > 0$ and $\mathrm{Var}\,(Y) > 0$. The above "maximal correlation of $\mathcal{G}$ and $\mathcal{H}$" was first studied in [Hir35] and [Geb41]. By the very definition, for real X and Y,

$$|EXY - EX\cdot EY| \le \rho(\mathcal{G},\mathcal{H})\|X\|_2\|Y\|_2. \tag{2.23}$$

Further,

$$\alpha(\mathcal{G},\mathcal{H}) \le \rho(\mathcal{G},\mathcal{H}) \le 2\pi\sqrt{\phi(\mathcal{G},\mathcal{H})\phi(\mathcal{H},\mathcal{G})}\,, \tag{2.24}$$

where the second inequality follows from (2.21) with $r = +\infty$ and $p = q = 2$.

Now, for $k < m$ and stationary $\{X_j\}_{j\in\mathbb{N}}$, define

$$\mathcal{F}_k^m = \sigma(X_j\;;\; k\le j\le m), \quad \mathcal{F}_k^\infty = \sigma(X_j\;;\; j\ge k). \tag{2.25}$$

Set

$$\alpha(n) = \sup_m \alpha(\mathcal{F}_1^m, \mathcal{F}_{m+n}^\infty), \quad n \geq 1, \tag{2.26}$$

and say that $\{X_k\}$ is α-mixing (or, following [Ros56], strongly mixing), if $\alpha(n) \to 0$ as $n \to \infty$. Replacing coefficient $\alpha(\cdot,\cdot)$ in (2.23) by $\psi(\cdot,\cdot)$ or $\rho(\cdot,\cdot)$, we get definitions of ψ- or ρ-mixing sequences, respectively. If for some $m \geq 0$, $\alpha(m+1) = 0$, then $\{X_j\}$ is said to be m-dependent.

In α-, ψ- and ρ-coefficients we may change the role of "future" and "past"—the coefficients are symmetric in $\mathcal{G}$ and $\mathcal{H}$. This is not true for ϕ, so define

$$\phi(n) = \sup_m \phi(\mathcal{F}_1^m, \mathcal{F}_{m+n}^\infty) \tag{2.27}$$

and

$$\phi^*(n) = \sup_m \phi(\mathcal{F}_{m+n}^\infty, \mathcal{F}_1^m). \tag{2.28}$$

If $\phi(n) \to \infty$ as $n \to \infty$, then $\{X_j\}$ is said to be ϕ-mixing or, according to [Ibr59], "uniformly strongly mixing", while if $\phi^*(n) \to 0$ as $n \to \infty$, we deal with "reversed ϕ-mixing" or ϕ^*-mixing. ϕ-mixing and ϕ^*-mixing are really different notions—a suitable example can be found in [KeOB76].

Mixing conditions are useful tools in various respects. For checking Condition B, however, we need only the weakest one.

Proposition 2.13 *If $\{X_j\}_{j\in \mathbb{N}}$ is α-mixing, then Condition B is satisfied for every sequence $B_n \to \infty$.*

PROOF. Let $k_n + l_n \leq n$. We have to prove that

$$E\, e^{i\lambda(S_{k_n+l_n}/B_n)} - Ee^{i\lambda(S_{k_n}/B_n)} \cdot Ee^{i\lambda(S_{l_n}/B_n)} \longrightarrow 0 \quad \text{as } n \to +\infty.$$

Since $B_n \to \infty$, we can assume that both $k_n \to \infty$ and $l_n \to \infty$. Let $m_n \leq \min\{k_n, l_n\}$, $m_n \to \infty$ be such that $S_{m_n}/B_n \longrightarrow_{\mathcal{P}} 0$. Define $U_n = S_{k_n - m_n}$ and $V_n = S_{k_n+l_n} - S_{k_n+m_n}$. Then

$$\begin{aligned} Ee^{i\lambda S_{k_n+l_n}/B_n} - Ee^{i\lambda(U_n+V_n)/B_n} &\longrightarrow 0, \\ Ee^{i\lambda S_{k_n}/B_n} - Ee^{i\lambda U_n/B_n} &\longrightarrow 0, \\ Ee^{i\lambda S_{l_n}/B_n} - Ee^{i\lambda V_n/B_n} &\longrightarrow 0. \end{aligned}$$

And by (2.21) with $r = 1$, $p = q = +\infty$,

$$|E\, e^{i\lambda(U_n+V_n)/B_n} - Ee^{i\lambda U_n/B_n} \cdot Ee^{i\lambda V_n/B_n}| \leq 2\pi\alpha(2m_n) \longrightarrow 0 \quad \text{as } n \to +\infty.$$

□

The idea we used above is known as "separation of blocks"—for each sequence $B_n \to \infty$ we can find $m_n \to \infty$ such that Condition B is equivalent to

$$\max_{\substack{m_n \leq k,l,m \\ k+l+m \leq n}} |E\, e^{i\lambda(S_{k+l+m} - S_{k+l} + S_k)/B_n} - Ee^{i\lambda S_k/B_n} \cdot Ee^{i\lambda S_m/B_n}| \longrightarrow 0 \quad \text{as } n \to +\infty, \tag{2.29}$$

for every $\lambda \in I\!R^1$.

It is well-known, that an ergodic homogeneous Markov chain on a finite state space is not α-mixing, if it is periodic with period $r > 1$. Nevertheless, it satisfies Condition B for each $B_n \to \infty$, as we shall see below. Following O'Brien([OBr87, p.286]), say that $\{X_j\}$ is r-strongly mixing if

$$\alpha_r(n) = \sup \left| \frac{1}{r} \sum_{k=0}^{r-1} P(A \cap C_k) - P(A)P(C) \right| \longrightarrow 0 \quad \text{as } n \to +\infty\,, \tag{2.30}$$

where the supremum is taken over all positive integers m, all $A \in \mathcal{F}_1^m$, all $C \in \mathcal{F}_{m+n}^\infty$ and C_k is the shift of C for k steps (if $C = \{(X_1, X_2, \ldots) \in E\}$ a.s. for some $E \in \mathcal{B}^\infty$, then $C_k = \{(X_{k+1}, X_{k+2}, \ldots) \in E\}$ a.s.).

Proposition 2.14 *If $\{X_j\}$ is r-strongly mixing, then Condition B holds for every $B_n \to \infty$.*

Proof. We have to check (2.29). So take $\lambda \in I\!R^1$, $r \le m_n \le k, l, m$; $k + l + m \le n$, and observe that it is enough to find an estimation for

$$c^f_{k,l,m} = Ef(S_k)f(S_{k+l+m} - S_{k+m}) - Ef(S_k) \cdot Ef(S_l)\,,$$

where f is either $\sin \lambda B_n^{-1}(\cdot)$ or $\cos \lambda B_n^{-1}(\cdot)$. But

$$c^f_{k,l,m} = \frac{1}{r} \sum_{j=0}^{r-1} c^f_{k,l,m+j} + \frac{1}{r} \sum_{j=0}^{r-1} (c^f_{k,l,m} - c^f_{k,l,m+j}) = I_1 + I_2\,.$$

We know that

$$EXY - EXEY = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (P(X > x, Y > y) - P(X > x)P(Y > y))\, dx\, dy\,,$$

so

$$\begin{aligned} |I_1| &= \left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{ \frac{1}{r} \sum_{j=0}^{r-1} P(f(S_k) > x, f(S_{k+l+m+j} - S_{k+m+j}) > y) \right. \right. \\ &\qquad \left. \left. - P(f(S_k) > x)P(f(S_l) > y) \right\} dx\, dy \right| \\ &\le 4\alpha_r(m) \le 4\alpha_r(m_n) \longrightarrow 0 \quad \text{as } n \to +\infty\,, \end{aligned}$$

uniformly in k, l, m. And for each $\varepsilon > 0$ we have

$$|f(x) - f(y)| \le \varepsilon + 2\chi_{\{|x-y|>\varepsilon\}}(x, y)\,,$$

hence

$$\begin{aligned}|I_2| &= |r^{-1}\sum_{j=0}^{r-1} Ef(S_k)(f(S_{k+l+m+j} - S_{k+m+j}) - f(S_{k+m+l} - S_{k+m}))| \\ &\le \varepsilon + 2r^{-1}\sum_{j=0}^{r-1} P(|\lambda||(S_{k+l+m+j} - S_{k+m+j}) - (S_{k+m+l} - S_{k+m})| > \varepsilon B_n) \\ &\le \varepsilon + 4r^{-1}\sum_{j=0}^{r-1} P(|\lambda||S_j| > \varepsilon/2) \quad\longrightarrow \varepsilon \quad \text{as } n \to +\infty.\end{aligned}$$

□

O'Brien ([OBr87, Theorem 5.2]) observed, that sequences being instantaneous functions of a stationary Harris chain with period r, are r-strongly mixing. By the above proposition we see that such sequences satisfy our Condition B.

2.6 Decoupling Methods

Besides "separation of blocks" another operation can be useful while checking Condition B. Suppose we can decompose

$$\frac{S_k}{B_n} = S'_{n,k} + S''_{n,k}, \quad 1 \le k \le n, \ n \in I\!N,$$

where $S''_{n,k} \ \longrightarrow_{\mathcal{P}} \ 0$ as $n \to \infty$ uniformly in k, i.e.

$$\max_{1\le k\le n} P(|S''_{n,k} > \varepsilon) \ \longrightarrow \ 0 \quad \text{as } n \to +\infty.$$

Then, obviously, Condition B is equivalent to

$$\max_{k+l\le n} |Ee^{i\lambda S'_{n,k+l}} - Ee^{i\lambda S'_{n,k}} \cdot Ee^{i\lambda S'_{n,l}}| \ \longrightarrow \ 0 \quad \text{as } n \to +\infty, \tag{2.31}$$

and this may require much less information, than α-mixing or thereabout. Developing the idea we come to ε-approximation as suggested by Theorem 4.2 of [Bil68]: if

$$\frac{S_k}{B_n} = S'_{n,k}(\delta) + S''_{n,k}(\delta)\,, \tag{2.32}$$

where

$$\lim_{\delta\searrow 0}\limsup_{n\to\infty}\max_{1\le k\le n} P(|S''_{n,k}(\delta)| > \varepsilon) = 0, \quad \forall \varepsilon > 0\,, \tag{2.33}$$

then Condition B holds if and only if

$$\lim_{\delta\searrow 0}\limsup_{n\to\infty}\max_{k+l\le n} |Ee^{i\lambda S'_{n,k+l}(\delta)} - Ee^{i\lambda S'_{n,k}(\delta)} \cdot Ee^{i\lambda S'_{n,l}(\delta)}| = 0\,. \tag{2.34}$$

Example 2.15 Suppose $\{X_k\}$ is a stationary sequence with *one-dimensional* marginal distribution belonging to the domain of attraction of a strictly p-stable law $\mu = \mathrm{Pois}\,(\nu(p, c_+, c_-))$, $0 < p < 1$, $c_+ + c_- = 1$. Let B_n be such that

$$nP(|X_1| > B_n) \;\longrightarrow\; 1\,. \tag{2.35}$$

For $\delta > 0$, define

$$\begin{aligned}
X_{n,j}(\delta) &= \frac{X_j}{B_n} I(\delta < \frac{X_j}{B_n} \leq \delta^{-1})\,, \\
S'_{n,k}(\delta) &= \sum_{j=1}^{k} X_{n,j}(\delta)\,, \\
S''_{n,k}(\delta) &= \frac{S_k}{B_n} - S'_{n,k}(\delta) = \\
&= \sum_{j=1}^{k} \frac{X_j}{B_n} I(\frac{|X_j|}{B_n} \leq \delta) + \sum_{j=1}^{k} \frac{X_j}{B_n} I(\frac{|X_j|}{B_n} > \delta^{-1}) \\
&= \sum_{j=1}^{k} Y'_{n,j}(\delta) + \sum_{j=1}^{k} Y''_{n,j}(\delta)\,.
\end{aligned}$$

By Karamata's Theorem B.11

$$\lim_{\delta\searrow 0} \limsup_{n\to\infty} \sum_{j=1}^{n} E|Y'_{n,j}(\delta)| = 0\,,$$

and

$$\begin{aligned}
&\lim_{\delta\searrow 0} \limsup_{n\to\infty} \max_{1\leq k\leq n} P(|\sum_{j=1}^{k} Y''_{n,j}(\delta)| > \varepsilon) \\
&\quad\leq \lim_{\delta\searrow 0} \limsup_{n\to\infty} \max_{1\leq k\leq n} P(\max_{1\leq j\leq k} \frac{|X_j|}{B_n} > \delta^{-1}) \\
&\quad\leq \lim_{\delta\searrow 0} \limsup_{n\to\infty} P(\max_{1\leq j\leq n} \frac{|X_j|}{B_n} > \delta^{-1}) = 0\,.
\end{aligned}$$

Hence (2.33) holds and we can restrict our attention to $S'_{n,k}(\delta)$ only.

Now observe that

$$S'_{n,k}(\delta) = \int x I(\delta < |x| \leq \delta^{-1})\; N_{n,k}(d\,x)\,, \tag{2.36}$$

where $N_{n,k}(\cdot)$ are point processes on $\mathbb{R}^1 \setminus \{0\}$ defined by

$$N_{n,k}(A) = \sum_{j=1}^{k} I(\frac{X_j}{B_n} \in A)\,.$$

Representation (2.36) allows us to apply the whole power of the point processes theory, as described in the book [Kal83]. For more details we refer the reader to [JaKo89].

Chapter 3

Convergence to Strictly p-stable Laws: Regular Variation in the Limit

3.1 Necessary Conditions

Suppose $Z_n \longrightarrow_{\mathcal{D}} \mu$, where μ is strictly p-stable. For each n, let $\{Y_{n,j}\}_{j\in I\!N}$ be a sequence of independent copies of Z_n. By *strict* stability of μ, for each $k \in I\!N$ we have

$$k^{-1/p} \sum_{j=1}^{k} Y_{n,j} \underset{\mathcal{D}}{\longrightarrow} \mu \quad \text{as } n \to +\infty.$$

Hence we can find a sequence $\{r_n\}$ of integers, $r_n \nearrow \infty$, such that

$$\max_{1\le k\le r_n} d_L(\mathcal{L}(k^{-1/p} \sum_{j=1}^{k} Y_{n,j})\,,\, \mu) \longrightarrow 0 \quad \text{as } n \to +\infty,$$

where d_L is the Lévy metric. In particular, if $\{k_n\}$ is a sequence of positive integers such that $k_n \to \infty$ and $k_n = o(r_n)$, then

$$k_n^{-1/p} \sum_{j=1}^{k_n} Y_{n,j} \underset{\mathcal{D}}{\longrightarrow} \mu \quad \text{as } n \to +\infty, \tag{3.1}$$

and $\{Z_{n,j} = k_n^{-1/p} Y_{n,j}\,;\, 1 \le j \le k_n,\, n \in I\!N\}$ is an infinitesimal array of row-wise independent random variables. For such arrays we can use Theorem A.5 and find expressions involving Z_n's, which are necessary for (3.1) and so—necessary for $Z_n \longrightarrow_{\mathcal{D}} \mu$.

Proposition 3.1 *Suppose Z_n converges in distribution to a (possibly degenerated) strictly stable distribution μ. Then there exists a sequence $r_n \nearrow +\infty$ such that for each sequence $\{k_n\} \subset I\!N$ tending to infinity so slowly, that $k_n/r_n \to 0$, one of listed below statements (i)-(iv) holds:*

(i) *If* $0 < p < 1$ *and* $\mu = \mathrm{Pois}\,(\nu(p, c_+, c_-))$,

$$\begin{array}{rcl} k_n P(Z_n > k_n^{1/p}) & \longrightarrow & c_+\,, \\ k_n P(Z_n < -k_n^{1/p}) & \longrightarrow & c_-\,. \end{array} \tag{3.2}$$

(ii) *If* $p = 1$ *and* $\mu = \mathrm{Pois}\,(\nu(1, c, c)) * \delta_a$,

$$\begin{array}{rcl} k_n P(Z_n > k_n^{1/p}) & \longrightarrow & c\,, \\ k_n P(Z_n < -k_n^{1/p}) & \longrightarrow & c\,, \\ E Z_n I(|Z_n| \le k_n) & \longrightarrow & a\,. \end{array} \tag{3.3}$$

(iii) *If* $1 < p < 2$ *and* $\mu = c_\infty - \mathrm{Pois}\,(\nu(p, c_+, c_-))$,

$$\begin{array}{rcl} k_n P(Z_n > k_n^{1/p}) & \longrightarrow & c_+\,, \\ k_n P(Z_n < -k_n^{1/p}) & \longrightarrow & c_-\,, \\ k^{1-1/p} E Z_n I(|Z_n| \le k_n^{1/p}) & \longrightarrow & (c_+ - c_-)/(1-p)\,. \end{array} \tag{3.4}$$

(iv) *If* $p = 2$ *and* $\mu = N(0, \sigma^2)$,

$$\begin{array}{rcl} k_n P(|Z_n| > k_n^{1/2}) & \longrightarrow & 0\,, \\ k^{1/2} E Z_n I(|Z_n| \le k_n^{1/2}) & \longrightarrow & 0\,, \\ E Z_n^2 I(|Z_n| \le k_n^{1/2} & \longrightarrow & \sigma^2. \end{array} \tag{3.5}$$

PROOF. Conditions involving $k_n P((-1)^i Z_n > k_n^{1/p})\,, i = 0, 1$, are implied by (A.20) with $x = 1$, if $0 < p < 2$, and (A.25) with $\varepsilon = 1$, if $p = 2$. Conditions operating with variances and expectations of truncated Z_n's are exactly as stated in Theorem A.5. □

We shall examine in details consequences and structure of conditions (3.1)-(3.5).

Proposition 3.2 *Let (3.1) holds with strictly* p*-stable* μ *and let along some subsequence* $\{n'\}$, $Z_{n'}$ *converges to some* strictly p-stable *law* ν. *Then* $\nu = \mu$.

PROOF. Repeating the considerations from the beginning of the chapter, we see that whenever $k_{n'}$ tends to infinity slowly enough,

$$k_{n'}^{-1/p} \sum_{j=1}^{k_{n'}} Y_{n',j} \underset{\mathcal{D}}{\longrightarrow} \nu\,,$$

while by (3.1), it has to converge to μ. □

Proposition 3.3 *Suppose either*

- *(3.2) with* $c_+ + c_- > 0$ *or*
- *(3.3) with* $c > 0$ *or*
- *(3.4) with* $c_+ + c_- > 0$ *or*
- *(3.5) with* $\sigma^2 > 0$.

Then $\{Z_n\}_{n \in \mathbb{N}}$ *is tight and no limit point of* $\{Z_n\}$ *is degenerated.*

PROOF. If $\{Z_n\}$ is not tight, we may assume without loss of generality, that for some $\eta > 0$

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(Z_n > k) \geq \eta \, .$$

Take $\{r_n\}$ as in Proposition 3.1 and find $N_l > N_{l-1}$ such that

$$P(Z_{N_l} > l^{1/p}) > \eta/2 \, , \tag{3.6}$$

and $r_n > l^2$ for all $n \geq N_l$. Define $k_n = l$ if $N_l \leq n < N_{l+1}$. Then $k_n/r_n < 1/k_n \to 0$ and by (3.6)

$$P(Z_{N_l} > k_{N_l}^{1/p}) \not\to 0 \quad \text{as } l \to \infty \, ,$$

while we know that

$$\sup_l k_{N_l} \; P(Z_{N_l} > k_{N_l}^{1/p}) < +\infty \, .$$

Hence $\{Z_n\}$ is tight.

Now suppose that $0 < p < 2$, $c_+ > 0$, and along some subsequence $\{n'\} \subset I\!N$, $Z_{n'} \;\longrightarrow_{\mathcal{P}}\; a$ or $Z_{n'} - a \;\longrightarrow_{\mathcal{P}}\; 0$. By Proposition 3.1, (i)-(iii),

$$k_{n'} P(Z_{n'} - a > k_{n'}^{1/p}) \to 0 \, .$$

On the other hand, if $l_{n'} = [(a + k_{n'}^{1/p})^p]$, then $l_{n'} \sim k_{n'}$ and we have

$$k_{n'} P(Z_{n'} - a > k_{n'}^{1/p}) \geq \frac{k_{n'}}{l_{n'}} l_{n'} P(Z_{n'} > l_{n'}^{1/p}) \to c_+ > 0 \, .$$

So consider the remaining case: $p = 2$. If $Z_{n'} \;\longrightarrow_{\mathcal{P}}\; a$,

$$E Z_{n'} I(|Z_{n'}| \leq k_{n'}^{1/2}) \;\longrightarrow\; a \neq 0 \, ,$$

provided $k_{n'} \to \infty$ slowly enough. But then

$$k_{n'}^{1/2} E Z_{n'} I(|Z_{n'}| \leq k_{n'}^{1/2}) \not\to 0 \, .$$

□

Corollary 3.4 *In assumptions of the above proposition, suppose that*

$$b_{n'} Z_{n'} \;\underset{\mathcal{D}}{\longrightarrow}\; \mu$$

along some subsequence $\{n'\} \subset I\!N$, where μ is strictly p-stable defined by the corresponding condition among (3.2)-(3.5). Then $b_{n'} \to 1$ along $\{n'\}$.

PROOF. We shall show that every subsequence $\{b_{n''}\}$ contains a further subsequence converging to 1. Indeed, one can find a subsequence $\{n'''\}$ such that $Z_{n'''} \;\longrightarrow_{\mathcal{D}}\; \nu$, where ν is non-degenerate. Since μ is non-degenerate, too, the convergence to types theorem implies $b_{n'''} \to b > 0$. But then ν is strictly p-stable with parameters determined by (3.2)-(3.5), hence $\nu = \mu$. Consequently, $b = 1$. □

3.2 Regular Variation in the Limit

Conditions (3.2)-(3.5) have a very special form: given a sequence of functions f_n on $\mathbb{R}^+$ (e.g. $f_n(x) = P(Z_n > x)$) we assume that there exists a sequence $r_n \nearrow \infty$ such that

$$x_n^p f_n(x_n) \quad \longrightarrow \quad c > 0\,, \tag{3.7}$$

whenever $x_n \to \infty$, $x_n = o(r_n)$.

Example 3.5 Let f be $(-p)$-regularly varying. Take $a_n \to \infty$ and define

$$f_n(x) = c\frac{f(a_n x)}{f(a_n)}\,.$$

Then $\{f_n\}$ possesses property (3.7). Indeed, by $(-p)$-regular variation,

$$f_n(x) \quad \longrightarrow \quad cx^{-p}\,, \quad x > 0.$$

and this convergence is uniform on compact subsets of $(0, +\infty)$ (Theorem B.3). In particular,

$$x^p f_n(x) \quad \longrightarrow \quad c$$

uniformly on compacts in $(0, \infty)$, hence (3.7) follows.

In a trivial sense the above example describes *all* sequences satisfying (3.7): let $g(x) = x^{-p}$, $a_n \to \infty$ and define

$$g_n(x) = \frac{1}{c}\frac{g(a_n x)}{g(a_n)} = \frac{1}{c}x^{-p}\,.$$

Then (3.7) means that

$$\frac{f_n(x_n)}{g_n(x_n)} \quad \longrightarrow \quad 1$$

for all $x_n \to \infty$ slowly enough.

Hence it is natural to say that a sequence of functions $f_n : (a, \infty) \to \mathbb{R}^1$ satisfying (3.7) is $(-p)$-**regularly varying in the limit**. The analogies between regular variation and regular variation in the limit go further: we can prove a result corresponding to the direct half of Karamata's Theorem (Theorem B.11)

Theorem 3.6 *Let $p > 0$ and let for each $n \in \mathbb{N}$, $f_n : (a, +\infty) \to \mathbb{R}^+$ be measurable, $(-p)$-regularly varying in the limit (i.e. (8.1) holds for some $c > 0$) and such that for each $b > a$*

$$\sup_n \int_a^b s^q f_n(s)\, ds \leq K_b < +\infty. \tag{3.8}$$

If $q - p + 1 > 0$, then

$$x_n^{-(q-p+1)} \int_a^{x_n} s^q f_n(s)\, ds \quad \longrightarrow \quad \frac{c}{q - p + 1} \tag{3.9}$$

for all $x_n \to \infty$ slowly enough. In particular, functions $g_n^q(x) = \int_a^x s^q f_n(s)\, ds$ are $(q-p+1)$-regularly varying in the limit.

PROOF. Let $\{r_n\}$ be taken from (8.1). Let $a < y_n < x_n$ be such that $x_n = o(r_n)$, $x_n/y_n \to \infty$ but still $y_n \to \infty$. Consider

$$\begin{aligned} g_n^q(x_n) - g_n^q(y_n) &= \int_{y_n}^{x_n} s^q f_n(s)\, ds \\ &= x_n^{q-p+1} \int_{y_n/x_n}^{1} (ux_n)^p f_n(ux_n) u^{q-p}\, du\,. \end{aligned}$$

If $y_n/x_n \le u_n \le 1$, the sequence $\{u_n x_n\}$ is tending to infinity slowly enough (i.e. is $o(r_n)$). Since (8.1) holds for every such sequence, $(ux_n)^p f_n(ux_n) \to c$ *uniformly* in $u \in [y_n/x_n, 1]$. Hence for some $\varepsilon_n \to 0$,

$$\begin{aligned} g_n^q(x_n) - g_n^q(y_n) &= x_n^{q-p+1}(c+\varepsilon_n) \int_{y_n/x_n}^{1} u^{q-p}\, du \\ &= x_n^{q-p+1} \frac{c+\varepsilon_n}{q-p+1} \left(1 - \left(\frac{y_n}{x_n}\right)^{q-p+1}\right). \end{aligned} \tag{3.10}$$

But $c > 0$, so $g_n^q(x_n) - g_n^q(y_n) \to \infty$, and, in particular, $g_n^q(x_n) \to \infty$. By (3.8) y_n can be chosen in such a way, that $g_n^q(x_n)/g_n^q(y_n) \to \infty$, and then (3.9) follows from (3.10). □

In the classical limit theory for independent summands, Karamata's Theorem provides a link between truncated moments and tail probabilities, and so is one of the most basic tools (see [Fel71, Chapter VIII.9]). Our approach preserves only a part of the power of Karamata's results—but it is still enough to prove

Proposition 3.7 *Suppose μ is a non-degenerate strictly p-stable law. Then (3.1) is equivalent to the corresponding condition among (3.2)-(3.5).*

PROOF. Fix $p \in (0, 2]$ and consider the condition

$$k_n P(Z_n > k_n^{1/p}) \to c_+, \quad k_n P(Z_n < -k_n^{1/p}) \to c_-\,, \tag{3.11}$$

for each sequence $\{k_n\}$ of integers such, that $k_n \to \infty$ and $k_n = o(r_n)$. Let $x_n \to \infty$ be a sequence of reals such that $x_n = o(r_n)$. Let $k_n = [x_n]$. Then $k_n/r_n \to 0$ and

$$(k_n+1)P(Z_n > k_n^{1/p}) \ge x_n P(Z_n > x_n^{1/p}) \ge k_n P(Z_n > (k_n+1)^{1/p})\,. \tag{3.12}$$

Hence (3.11) implies $x_n P(Z_n > x_n^{1/p}) \to c_+$. In particular, for each $x \in \mathbb{R}^+$ and $k_n \to \infty$, $k_n = o(r_n)$,

$$k_n P\left(\frac{Z_n}{k_n^{1/p}} > x\right) \longrightarrow \frac{c_+}{x^p} \;, k_n P\left(\frac{Z_n}{k_n^{1/p}} < -x\right) \longrightarrow \frac{c_-}{x^p}\,. \tag{3.13}$$

Replacing (3.11) by (3.13) in each of (3.2)-(3.5), we get conditions (A.20) and (A.25) of Theorem A.5 describing convergence to the Lévy measure $\nu(p, c_+, c_-)$. It is enough in the case $p = 2$: all conditions of Theorem A.5 are satisfied and $k_n^{-1/2} \sum_{j=1}^{k_n} Y_{n,j} \longrightarrow_{\mathcal{D}} N(0, \sigma^2)$.

So let $0 < p < 2$. We have

$$\begin{aligned} & k_n E\left(\frac{Z_n}{k_n^{1/p}}\right)^2 I(\left|\frac{Z_n}{k_n^{1/p}}\right| \le \delta) \\ \le\ & k_n^{1-2/p} E(Z_n \wedge (\delta k_n^{1/p}))^2 \\ =\ & 2k_n^{1-2/p} \int_0^{\delta k_n^{1/p}} tP(|Z_n| > t)\,dt \qquad (3.14) \\ \sim\ & 2k_n^{1-2/p} \frac{c_+ + c_-}{2-p} (\delta k_n^{1/p})^{1-p+1} \quad \text{by Theorem 3.6} \\ =\ & 2\delta^{2-p}(c_+ + c_-)/(2-p) \ \longrightarrow\ 0 \quad \text{as } \delta \searrow 0\,. \end{aligned}$$

This implies condition (A.22) and ends the proof for $1 \le p < 2$.

Let $0 < p < 1$. Similarly, as above

$$\begin{aligned} & k_n \left| E\frac{Z_n}{k_n^{1/p}} I(\left|\frac{Z_n}{k_n^{1/p}}\right| \le \delta)\right| \\ \le\ & k_n^{1-1/p} E|Z_n| \wedge (\delta k_n^{1/p}) \\ =\ & k_n^{1-1/p} \int_0^{\delta k_n^{1/p}} P(|Z_n| > t)\,dt \qquad (3.15) \\ \sim\ & k_n^{1-1/p} \frac{c_+ + c_-}{1-p} (\delta k_n^{1/p})^{-p+1} \\ =\ & \delta^{1-p}(c_+ + c_-)/(1-p) \ \longrightarrow\ 0 \quad \text{as } \delta \searrow 0\,, \end{aligned}$$

so (A.21) holds and we have also proved the case $0 < p < 1$. □

The next lemma provides an equivalent form of (3.7), which is sometimes more useful.

Lemma 3.8 *Let $\{h_n\}$ be a sequence of functions on $(a, +\infty)$. The following are equivalent:*

(i) *There are: a constant $c \in \mathbb{R}^1 \cup \{+\infty\} \cup \{-\infty\}$ and a sequence $r_n \nearrow +\infty$ such that*

$$h_n(x_n) \ \longrightarrow\ c\,, \qquad (3.16)$$

for each sequence $x_n \to \infty$, $x_n = o(r_n)$.

(ii) *For some (and then for any) $\delta > 0$*

$$\overline{c} := \limsup_{x\to\infty} \limsup_{n\to\infty} \overline{h}_n^{\delta}(x) = \liminf_{x\to\infty} \liminf_{n\to\infty} \underline{h}_n^{\delta}(x) =: \underline{c}\,, \qquad (3.17)$$

where for $x \in [m\delta, (m+1)\delta)$, $\overline{h}_n^{\delta}(x)$ and $\underline{h}_n^{\delta}(x)$ are, respectively, supremum and infimum of h_n on $[m\delta, (m+1)\delta)$.

If (i) or (ii) holds, then $c = \overline{c} = \underline{c}$.

PROOF. If $\overline{c} > \underline{c}$, we can find sequences $\overline{x}_{n'}$ and $\underline{x}_{n''}$, tending to infinity as slowly as desired and such that $h_{n'}(\overline{x}_{n'}) \to \overline{c}$ and $h_{n''}(\underline{x}_{n''}) \to \underline{c}$. So implication (i) $\Rightarrow$ (ii) follows. To prove the converse, set

$$\overline{b}_{n,m} = \overline{h}_n^{\delta}(m\delta), \quad \underline{b}_{n,m} = \underline{h}_n^{\delta}(m\delta),$$

and observe that if $x_n \in [m_n\delta, (m_n+1)\delta)$, then $x_n/m_n \to \delta > 0$ and

$$\underline{b}_{n,m_n} = \underline{h}_n^{\delta}(x_n) \le h_n(x_n) \le \overline{h}_n^{\delta}(x_n) = \overline{b}_{n,m_n}.$$

Hence it is sufficient to prove that if $\underline{b}_{n,m} \le \overline{b}_{n,m}$ and

$$c = \limsup_{m\to\infty} \limsup_{n\to\infty} \overline{b}_{n,m} = \liminf_{m\to\infty} \liminf_{n\to\infty} \underline{b}_{n,m}, \tag{3.18}$$

then there exists $r_n \nearrow +\infty$ such that

$$\lim_{n\to\infty} \overline{b}_{n,m_n} = \lim_{n\to\infty} \underline{b}_{n,m_n} = c, \tag{3.19}$$

for every sequence $m_n \to \infty$, $m_n = o(r_n)$.

Suppose that (3.18) is fulfilled. Let $|c| < \infty$. For every $p \in I\!N$ there exists $M_p > M_{p-1}$ ($M_0 = 1$), such that for each $m \ge M_p$ one can find $N_{p,m}$ satisfying

$$b - p^{-1} < \underline{b}_{n,m} \le \overline{b}_{n,m} < b + p^{-1}, \qquad \text{for all } n \ge N_{p,m}.$$

Define $N_0 = 0$ and for $p \ge 1$

$$N_p = (\max_{M_p \le m < M_{p+1}} N_{p,m}) \vee (N_{p-1} + 1). \tag{3.20}$$

Let

$$r_n = \begin{cases} M_p, & \text{if } N_p \le n < N_{p+1}; \\ 1, & \text{if } n < N_1. \end{cases} \tag{3.21}$$

Take $m_n \le r_n$, $m_n \to +\infty$. Let q_n be such, that $M_{q_n} \le m_n < M_{q_n+1}$. Then $r_n \ge m_n \ge M_{q_n}$ and by (3.21), $n \ge N_{q_n}$. Moreover, $q_n \to +\infty$, and by definition (3.20) we have $n \ge N_{q_n} \ge N_{q_n,m_n}$, hence

$$b - q_n^{-1} < \underline{b}_{n,m_n} \le \overline{b}_{n,m_n} < b + q_n^{-1}.$$

The proof of the cases $c = +\infty$ and $c = -\infty$ is similar. □

Regularly varying in the limit functions, which we consider in the paper, are mostly of the form

$$f_n(x) = b_{n,[x]} \tag{3.22}$$

where $\{b_{n,m}\}$ is an array of numbers. The other functions can be reduced to the above form by reasoning given in (3.12). For functions (3.22) we have

Corollary 3.9 (Lemma 1, [JaSz90]) *Let $\{b_{n,m}\}$ be a double array of real numbers. Then*

$$\limsup_{m\to\infty}\limsup_{n\to\infty} b_{n,m} = \liminf_{m\to\infty}\liminf_{n\to\infty} b_{n,m} = c,$$

if and only if there exists $\{r_n\}_{n\in N}$, $\lim_n r_n = +\infty$, such that

$$b_{n,m_n} \to b \quad \text{as} \quad n \to +\infty$$

for every sequence $\{m_n\}_{n\in N}$ of positive integers satisfying $\lim_n m_n = +\infty$ and $m_n = o(r_n)$.

Remark 3.10 It should be pointed out that in general

$$\limsup_{x\to\infty}\limsup_{n\to\infty} h_n(x) = \liminf_{x\to\infty}\liminf_{n\to\infty} h_n(x) \tag{3.23}$$

is not sufficient for (3.16) to hold. Consider the following simple example: In each interval $[m, m+1)$ choose a sequence of distinct numbers: $\{a_{m,1}, a_{m,2}, \ldots\} \subset [m, m+1)$. Let $A_n = \{a_{m,n}\ ;\ m \in I\!N\}$ and let $f_n(x) = \chi_{A_n}(x)$. Then for each x, $\limsup_{n\to\infty} f_n(x) = 0$, so (3.23) is satisfied. On the other hand, $f_n(x_n) = 1$ if $x_n = a_{m_n,n}$ for some $m_n \in I\!N$. Now, if $m_n = o(r_n)$, $m_n \to \infty$, we see that (3.16) does not hold.

Chapter 4

Tauberian p - stable Limit Theorems

4.1 Tauberian Limit Theorems

Let, as usually, S_n , $n \in \mathbb{N}$, be partial sums of a strictly stationary sequence $\{X_j\}_{j\in\mathbb{N}}$ and let $B_n \to +\infty$.

In Chapters 2 and 3 we derived several *necessary* conditions for S_n/B_n to converge to a strictly p-stable law μ. Let us summarize:

- Proposition 3.1 provides four sets of necessary conditions in the form specific to p.
- Proposition 3.7 asserts, that if μ is non-degenerate, those conditions admit a unified form, namely: there exists a sequence $r_n \nearrow +\infty$ such, that for every sequence $\{k_n\} \subset \mathbb{N}$, $k_n \to +\infty$, $k_n = o(r_n)$, we have

$$k_n^{-1/p} \sum_{j=1}^{k_n} Y_{n,j} \underset{\mathcal{D}}{\longrightarrow} \mu , \tag{4.1}$$

  where for each n, $Y_{n,1}, Y_{n,2}, \ldots$ are independent copies of S_n/B_n.
- If $\mu \neq \delta_0$ and B_n is $1/p$-regularly varying, then by Theorem 3.1, Condition B holds.

We aim at proving, that in the presence of Condition B, (4.1) is also *sufficient* for $S_n/B_n \longrightarrow_{\mathcal{D}} \mu$.

Theorem 4.1 *Suppose μ is a* non-degenerate *strictly p-stable distribution. Then S_n/B_n converges in distribution to μ and B_n is $1/p$-regularly varying if and only if Condition B is satisfied and (4.1) holds.*

Before proving the theorem, it seems to be useful to rewrite it, using Corollary 3.9 and in each of the cases $0 < p < 1$, $p = 1$, $1 < p < 2$ and $p = 2$ separately.

Recall, that according to our convention, the p-stable limit theorem holds for S_n/B_n if

— S_n/B_n converges in distribution to some strictly p-stable law μ,

— B_n is $1/p$-regularly varying.

Theorem 4.2 *Let $0 < p < 1$. Then the p-stable limit theorem holds with a non-degenerate limit μ if and only if Condition B is satisfied and*

$$\begin{aligned} \limsup_{m\to\infty}\limsup_{n\to\infty} m^p P(S_n/B_n > m) &= \\ = \liminf_{m\to\infty}\liminf_{n\to\infty} m^p P(S_n/B_n > m) &=: c_+\,, \\ \limsup_{m\to\infty}\limsup_{n\to\infty} m^p P(S_n/B_n < -m) &= \\ = \liminf_{m\to\infty}\liminf_{n\to\infty} m^p P(S_n/B_n < -m) &=: c_-\,, \end{aligned} \tag{4.2}$$

where $0 < c_+ + c_- < +\infty$.

The above conditions imply $\mu = \mathrm{Pois}\,(\nu(p, c_+, c_-))$. □

Theorem 4.3 *The 1-stable limit theorem holds with a non-degenerate limit μ if and only if Condition B is satisfied, (4.2) holds with $c_+ = c_- = c > 0$, and*

$$\begin{aligned} \limsup_{m\to\infty}\limsup_{n\to\infty} B_n^{-1} E S_n I(|S_n| \le m B_n) &= \\ = \liminf_{m\to\infty}\liminf_{n\to\infty} B_n^{-1} E S_n I(|S_n| \le m B_n) &=: a\,, \end{aligned} \tag{4.3}$$

for some $a \in I\!R^1$.

The above conditions imply $\mu = \mathrm{Pois}\,(\nu(1, c, c)) * \delta_a$. □

Theorem 4.4 *Let $1 < p < 2$. Then the p-stable limit theorem holds with a non-degenerate limit μ if and only if Condition B is satisfied, (4.2) holds and*

$$\begin{aligned} \limsup_{m\to\infty}\limsup_{n\to\infty} m^{p-1} E\frac{S_n}{B_n} I(\frac{|S_n|}{B_n} \le m) &= \\ = \liminf_{m\to\infty}\liminf_{n\to\infty} m^{p-1} E\frac{S_n}{B_n} I(\frac{|S_n|}{B_n} \le m) &=: (c_+ - c_-)/(1-p)\,. \end{aligned} \tag{4.4}$$

The above conditions imply

$$\mu = c_1 - \mathrm{Pois}\,(\nu(p, c_+, c_-)) * \delta_{(c_+ - c_-)/(1-p)} = c_\infty - \mathrm{Pois}\,(\nu(p, c_+, c_-)).\ \square$$

Theorem 4.5 *The Central (= 2-stable) Limit Theorem holds with a non-degenerate limit μ if and only if Condition B is satisfied and*

$$\lim_{m\to\infty}\limsup_{n\to\infty} m^2 P(\frac{|S_n|}{B_n} > m) = 0\,, \tag{4.5}$$

$$\lim_{m\to\infty}\limsup_{n\to\infty} m \left| E\frac{S_n}{B_n} I(\frac{|S_n|}{B_n} \le m)\right| = 0\,, \tag{4.6}$$

$$\begin{aligned} \limsup_{m\to\infty}\limsup_{n\to\infty} E\frac{S_n^2}{B_n^2} I(\frac{|S_n|}{B_n} \le m) &= \\ = \liminf_{m\to\infty}\liminf_{n\to\infty} E\frac{S_n^2}{B_n^2} I(\frac{|S_n|}{B_n} \le m) &=: \sigma^2 > 0\,. \end{aligned} \tag{4.7}$$

The above conditions imply $\mu = N(0, \sigma^2)$.

There are two prototypes for Theorems 4.2-4.5, both proved under the extra assumption of α-mixing. Theorem 4.5 improves (weakening α-mixing to Condition B) Theorem 1 in [JaSz90]. Similarly, Theorems 4.2-4.4 improve Theorem 1 in [DeJa89]. In addition, the latter result deals only with symmetric limits in the cases $p = 1$ and $1 < p < 2$, while in Theorems 4.3 and 4.4 general strictly p-stable limits are considered.

The above results are of "Tauberian" type. Indeed, we deal with necessary conditions obtained by *averaging* independent copies and the extra information we need in order to get sufficiency, is just Condition B (playing here the role of a "Tauberian condition").

Proof of Theorem 4.1 By the remarks preceding Theorem 4.1 and in the view of Theorem 2.1, it remains to be proved that **Condition B and (4.1) imply** $S_n/B_n \longrightarrow_{\mathcal{D}} \mu$.

The general line of the proof is similar to that of [JaSz90]; the details are, however, different, since we use Condition B instead of α-mixing and μ is a general strictly p-stable distribution.

First of all we shall find a sequence $s_n \nearrow +\infty$ such that $n \cdot s_n^{-1} \nearrow +\infty$ and for *every* sequence $k_n \to +\infty$, $k_n = o(s_n)$, we have

$$\frac{S_{k_n \cdot n}}{B_{k_n \cdot n}} \underset{\mathcal{D}}{\longrightarrow} \mu\,. \tag{4.8}$$

Suppose (4.1) holds. By Proposition 3.3, we know that $\{S_n/B_n\}$ is a tight sequence with no degenerate limiting distribution. This in turn implies, via Proposition 2.4, that Condition B is equivalent to (2.11). In particular, one can find a sequence $\tilde{r}_n \nearrow +\infty$, $\tilde{r}_n = o(n)$ such that

$$\max_{1\le k\le \tilde{r}_n} d_L\left(\mathcal{L}\left(\frac{S_{k\cdot n}}{B_{k\cdot n}}\right),\ \mathcal{L}\left(\frac{S_n}{B_{k\cdot n}}\right)^{*k}\right) \longrightarrow 0 \quad \text{as } n \to +\infty, \tag{4.9}$$

where, as previously, d_L is the Lévy metric.

Suppose, that $k_n \to +\infty$, $k_n = o(r_n \wedge \tilde{r}_n)$ (where r_n is taken from (4.1)), and

$$\frac{S_{k_{n'} \cdot n'}}{B_{k_{n'} \cdot n'}} \underset{\mathcal{D}}{\longrightarrow} Y$$

along a subsequence $\{n'\} \subset I\!N$. It follows from (4.9) that

$$\frac{B_{n'}}{B_{k_{n'} \cdot n'}} \sum_{j=1}^{k_{n'}} Y_{n',j} \underset{\mathcal{D}}{\longrightarrow} Y\,.$$

On the other hand, by (4.1), $\sum_{j=1}^{k_{n'}} Y_{n',j}$ converges to μ, when normalized by $k_{n'}^{1/p}$. Since Y is non-degenerate, by the convergence to types theorem,

$$B_{n'}(k_{n'})^{1/p}/B_{k_{n'} \cdot n'} \to c\,, \quad 0 < c < +\infty\,.$$

But then $c^{-1}Y \sim \mu$, so $\mathcal{L}(Y)$ is strictly p-stable. By Proposition 3.2, $Y \sim \mu$ and we have proved (4.8) with the only exception that $s_n = r_n \wedge \tilde{r}_n$ may not satisfy $n \cdot s_n^{-1} \nearrow$. To get this property, let us define $s_1 = r_1 \wedge \tilde{r}_1$ and for $n \ge 1$, $s_{n+1} = r_{n+1} \wedge \tilde{r}_{n+1} \wedge ((1 + n^{-1})s_n)$.

Now, let

$$k_n = o(s_{[ns_n^{-1}]}),\ k_n \to +\infty\,. \tag{4.10}$$

Then for large n's

$$\frac{k_n}{s_{[(n+k_n)k_n^{-1}]}} \leq \frac{k_n}{s_{[n\cdot k_n^{-1}]}} \leq \frac{k_n}{s_{[n\cdot s_n^{-1}]}} \quad \longrightarrow \quad \text{as } n \to +\infty$$

and the growth of k_n is slow enough to get from (4.8)

$$\frac{S_{[(n+k_n)k_n^{-1}]\,k_n}}{B_{[(n+k_n)k_n^{-1}]\,k_n}} \underset{\mathcal{D}}{\longrightarrow} \mu\,. \tag{4.11}$$

Observe that

$$P(B^{-1}_{[(n+k_n)k_n^{-1}]\,k_n}|S_{[(n+k_n)k_n^{-1}]\,k_n} - S_n| > \varepsilon) \leq P((B_n^*)^{-1}\max_{1\leq k\leq k_n}|S_k| \geq \varepsilon)\,,$$

where $B_n^* = \inf_{m\geq n} B_m \nearrow +\infty$. If, in addition to (4.10) , k_n is such, that

$$\max_{1\leq k\leq k_n}|S_k|/B_n^* \underset{\mathcal{P}}{\longrightarrow} 0,$$

then by (4.11)

$$\frac{S_n}{B_{[(n+k_n)k_n^{-1}]\,k_n}} \underset{\mathcal{D}}{\longrightarrow} \mu$$

and by Corollary 3.4 also $S_n/B_n \longrightarrow_{\mathcal{D}} \mu$. □

Remark 4.6 We used Condition B only in the weak form (4.9), which, under assumptions of Proposition 3.3, is implied by

Condition B'. For each $\lambda \in \mathbb{R}^1$ and each $k \in \mathbb{N}$

$$Ee^{i\lambda S_{k\cdot n}/B_{k\cdot n}} - \left(Ee^{i\lambda S_n/B_{k\cdot n}}\right)^k \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{4.12}$$

□

A review of methods of verifying conditions (4.2)-(4.7) is contained in Chapter 5.

4.2 Limit theorems with centering

We conclude our considerations with discussion of the general limit problem. From now onwards suppose that S_n **normalized by** B_n **satisfies Condition B** and that there are constants $\{A_n\}$ such that

$$\frac{S_n - A_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu\,, \tag{4.13}$$

where μ is a non-degenerate p-stable distribution (not necessarily *strictly* p-stable).

Taking in (4.13) symmetrizations and applying Theorem 2.1, we obtain

Lemma 4.7 *$\{B_n\}$ is a $1/p$-regularly varying sequence.*

We are not going to develop the theory of convergence (4.13). Instead we suggest reducing it, when possible, to the restricted case considered above. More precisely, we are looking for constants A and a such that

$$\frac{S_n - n \cdot A}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu * \delta_{-a}\,. \tag{4.14}$$

If A and a exist, they provide a complete reduction: $X_j' = X_j - A\,,\ j = 1, 2, \ldots$, form a new *stationary* sequence satisfying Condition B with the same normalizing constants B_n, hence $\mu * \delta_{-a}$ must be *strictly* p-stable.

In general, such A and a do not exist.

Example 4.8 Let $X_1, X_2, \ldots$ be i.i.d. with $\mathcal{L}(X_1) = c_1 - \mathrm{Pois}\,(\nu(1, c_+, c_-))$. Then

$$X_1 \sim \frac{X_1 + X_2 + \ldots + X_n}{n} - (c_+ - c_-)\log n = \frac{1}{n}\sum_{j=1}^{n}(X_j - (c_+ - c_-)\log n)\,.$$

If $c_+ \neq c_-$, then $\mathcal{L}(X_1)$ is a shift of no strictly 1-stable distribution. On the other hand, by the convergence to types theorem, no essentially different centering exists and the centered sums cannot be replaced by partial sums of a stationary sequence.

Fortunately, the case $p = 1$ is exceptional.

Theorem 4.9 *Suppose S_n/B_n satisfies Condition B and for some constants $\{A_n\}$*

$$(S_n - A_n)/B_n \underset{\mathcal{D}}{\longrightarrow} \mu,$$

where μ is a p-stable distribution, $p \neq 1$.

Let $a \in \mathbb{R}^1$ be such, that $\mu * \delta_{-a}$ is strictly p-stable.

(i) *If $0 < p < 1$, then $A_n/B_n \to -a$ and*

$$\frac{S_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu * \delta_{-a}\,. \tag{4.15}$$

(ii) *If $1 < p \leq 2$, then A_n/n converges to some $A \in \mathbb{R}^1$ and*

$$\frac{n \cdot A - A_n}{B_n} \longrightarrow a \quad \text{as } n \to +\infty. \tag{4.16}$$

Further,

$$\frac{S_n - n \cdot A}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu * \delta_{-a}\,. \tag{4.17}$$

Lemma 4.10 *In assumptions of the above theorem, if $p \neq 1$, then for all $\{k_n\}$, $\{l_n\} \subset I\!N$ such that $k_n + l_n \to +\infty$ we have as $n \to +\infty$*

$$\left(\frac{A_{k_n+l_n}}{B_{k_n+l_n}} + a\right) - \frac{B_{k_n}}{B_{k_n+l_n}}\left(\frac{A_{k_n}}{B_{k_n}} + a\right) - \frac{B_{l_n}}{B_{k_n+l_n}}\left(\frac{A_{l_n}}{B_{l_n}} + a\right) \longrightarrow 0\,. \tag{4.18}$$

Proof of the lemma follows by the convergence to types theorem.

Proof.Part (i). Set

$$h(n) = \frac{A_n}{B_n} + a\,.$$

Suppose that $0 < h_\infty = \limsup_{n\to\infty} h(n) < +\infty$. Let $m_n \nearrow +\infty$ be such that

$$\lim_{n\to\infty} h(m_n) = h_\infty.$$

If $k_n = [m_n/2]$, $l_n = m_n - k_n$, then

$$\limsup_{n\to\infty} \frac{B_{k_n}}{B_{m_n}} h(k_n) + \frac{B_{l_n}}{B_{m_n}} h(l_n) \leq 2^{1-1/p} h_\infty < h_\infty\,,$$

and (4.18) cannot hold. If $h_\infty = +\infty$, take m_n such that

$$h(m_n) = \max_{1\leq k\leq m_n} h(k)\,.$$

With the same choice of k_n and l_n as above, we have for n large enough

$$\frac{B_{k_n}}{B_{m_n}} h(k_n) + \frac{B_{l_n}}{B_{m_n}} h(l_n) \leq 2^{1-1/p}(1+\varepsilon) h(m_n)\,. \tag{4.19}$$

If $2^{1-1/p}(1+\varepsilon) < 1$, the gap between $h(m_n)$ and the sum on the left-hand-side of (4.19) tends to infinity, hence (4.18) cannot hold, again. So $\limsup_{n\to\infty} h(n) \leq 0$.

The same way we prove that $\liminf_{n\to\infty} h(n) \geq 0$. □

Proof.Part (ii). Set

$$f(n) = (A_n + aB_n)/n\,, \quad g(n) = B_n/n.$$

It is enough to prove, that $f(n)$ converges to some A (i.e. $A_n/n \to A$), and that $A - f(n) = o(g(n))$ (i.e. $(n\cdot A - A_n)/B_n \to a$).

Let $k_n = l_n = n$. Then by (4.18)

$$\frac{2B_n}{B_{2n}} \frac{f(2n) - f(n)}{g(n)} \longrightarrow 0 \quad \text{as } n \to +\infty$$

and, since $2B_n/B_{2n} \to 2^{1-1/p} \neq 0$,

$$\frac{f(2n) - f(n)}{g(n)} \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{4.20}$$

Proceeding with induction, we get

$$\frac{f(k\cdot n)-f(n)}{g(n)} \longrightarrow 0 \quad \text{as } n\to+\infty, \forall k\in I\!N\,. \tag{4.21}$$

Further, it follows from (4.21) that for all $k,\ l\in I\!N$

$$\frac{f(k\cdot n)-f(l\cdot n)}{g(k\cdot n)} \longrightarrow 0 \quad \text{as } n\to+\infty. \tag{4.22}$$

We will mimic the proof of [BGT87, Theorem 3.1.10c, p.134]. Since $1/p-1<0$ and g is $(1/p-1)$-regularly varying, one can find m_0' such that $g(2m)/g(m)\le\delta<1$ and $g(n)/g(m)\le C<+\infty$ whenever $n\ge m\ge m_0'$.

Let $m_0\ge m_0'$ be such that for $m\ge m_0$, $|f(2m)-f(m)|/g(m)\le\varepsilon$. If $l>k\ge m_0$, then for every $n\in I\!N$

$$\begin{aligned}\frac{|f(l)-f(k)|}{g(k)} &\le \frac{g(l)}{g(k)}\sum_{j=1}^{n}\frac{|f(2^j l)-f(2^{j-1}l)|}{g(2^{j-1}l)}\frac{g(2^{j-1}l)}{g(l)}\\ &\quad+\frac{|f(2^n l)-f(2^n k)|}{g(2^n k)}\frac{g(2^n k)}{g(k)}\\ &\quad+\sum_{j=1}^{n}\frac{|f(2^j k)-f(2^{j-1}k)|}{g(2^{j-1}k)}\frac{g(2^{j-1}k)}{g(k)}\\ &\le (B+1)\left(\sum_{j=1}^{n}\varepsilon\delta^{j-1}\right)+\delta^n\frac{|f(2^n l)-f(2^n k)|}{g(2^n k)}\,.\end{aligned}$$

The last term tends to zero as $n\to\infty$ by (4.22). Finally, we have

$$\frac{|f(l)-f(k)|}{g(k)}\le\varepsilon(B+1)\frac{1}{1-\delta}\,. \tag{4.23}$$

Since $g(k)\to 0$, $\{f(n)\}$ is a Cauchy sequence, so converges to some A. Letting $l\to\infty$ in (4.23), we get $A-f(k)=o(g(k))$. □

Remark 4.11 Subtraction of A in the case $1<p\le 2$ corresponds to centering by expectation (if exists).

Chapter 5

Examples of p - stable Limit Theorems

In the previous chapter we have found necessary and sufficient conditions for p - stable limit theorems. Below we are going to show that the conditions are *tractable*.

Some methods of checking Condition B were already presented in Chapter 2. So we will mainly concentrate on examining

$$k_n^{-1/p} \sum_{j=1}^{k_n} Y_{n,j} \underset{\mathcal{D}}{\longrightarrow} \mu \tag{5.1}$$

for every sequence k_n increasing to infinity slowly enough (i.e. $k_n = o(r_n)$ for some "rate sequence" $\{r_n\}$), where for each n, $Y_{n,1}, Y_{n,2}, \ldots$ are independent copies of S_n/B_n and μ is a strictly stable distribution.

5.1 Uniform Integrability and the Central Limit Theorem

Let us apply the simplest limit theorem for triangular arrays: the Lindeberg-Feller Central Limit Theorem. By this theorem, in order to check (5.1) with $\mu = \mathcal{N}(0,1)$ (standard normal), we need the following assumptions:

- $\forall n \in \mathbb{N}$, $EY_{n,1} = ES_n/B_n = 0$, i.e. $EX_1 = 0$.
- $\forall n \in \mathbb{N}$, $EY_{n,1}^2 = E(S_n/B_n)^2 < +\infty$, i.e. $EX_1^2 < +\infty$.
- $\forall k_n \to \infty,\ k_n = o(r_n)$, $k_n E(Y_{n,1}/k_n^{(1/2)})^2 = ES_n^2/B_n^2 \to 1$, i.e. $ES_n^2 \sim B_n^2$.
- $\forall k_n \to \infty,\ k_n = o(r_n)$ and $\forall \varepsilon > 0$, $k_n E(Y_{n,1}/k_n^{(1/2)})^2 I(Y_{n,1}^2 > \varepsilon k_n) \to 0$, i.e.

$$E\Big(\frac{S_n}{B_n}\Big)^2 I\Big(\Big(\frac{S_n}{B_n}\Big)^2 > \varepsilon k_n\Big) \longrightarrow 0. \tag{5.2}$$

Using Corollary 3.9 we may rewrite the above relation as

$$\lim_{m\to\infty}\limsup_{n\to\infty} E\Big(\frac{S_n}{B_n}\Big)^2 I\Big(\Big(\frac{S_n}{B_n}\Big)^2 > m\Big) = 0, \tag{5.3}$$

what is nothing but *uniform integrability* of S_n^2/B_n^2.

On the other hand, if $B_n = \sqrt{ES_n^2}$, then $S_n/B_n \longrightarrow_{\mathcal{D}} \mathcal{N}(0,1)$ implies uniform integrability of S_n^2/B_n^2 by [Bil68, Theorem 5.4]. So we have proved

Theorem 5.1 *Suppose that $EX_1^2 < +\infty$ and $EX_1 = 0$. Let $\sigma_n^2 = ES_n^2 \to +\infty$. Then*

$$\mathcal{L}(\sigma_n^{-1}S_n) \underset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0,1) \quad \text{as } n \to +\infty$$

and σ_n^2 is 1-regularly varying if, and only if, Condition B holds and $\{\sigma_n^{-2}S_n^2\}_{n\in\mathbb{N}}$ is a uniformly integrable sequence.

The above theorem is an improvement of Theorem 3 in [Den86], where only strongly mixing sequences were considered. Although stated formally so late, the theorem was used implicitly in most central limit theorems obtained by Bernstein's method, starting with Ibragimov's pioneer works [Ibr59] (ϕ-mixing) and [Ibr75] (ρ-mixing):

Lemma 5.2 *Suppose that $\{X_j\}$ is ρ-mixing, $EX_1 = 0$, $E|X_1|^{2+\delta} < +\infty$ for some $\delta > 0$ and $\sigma_n^2 \to +\infty$. Then*

$$E\Big|\sum_{j=1}^n X_j\Big|^{2+\delta} \le C|\sigma_n|^{2+\delta} \tag{5.4}$$

for some $C > 0$. (In particular, $\{\sigma_n^{-2}S_n^2\}$ is a uniformly integrable sequence).

Observe, that "by the way" we obtained representation $\sigma_n^2 = n\ell(n)$, where $\ell(x)$ is a slowly varying function on $\mathbb{R}^+$. This property, however, does not require moments higher than 2 and very strong mixing properties like ρ-mixing: it is easy to prove (using e.g. Lemma B.4) that σ_n^2 varies 1-regularly if $ES_n^2 \to \infty$ and

$$\tau(n) := \sup\Big\{\frac{ES_k(S_{k+r+m} - S_{k+r})}{\sqrt{ES_k^2\cdot ES_m^2}} \;:\; k,r,m \in \mathbb{N},\ r \ge n\Big\} \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{5.5}$$

When only second moments are finite, uniform integrability is not easily verifiable. One may try, for example, to truncate random variables and then use some special tools, like the following inequality due to Peligrad [Pel82]

$$ES_n^2 \le K\cdot n\cdot EX_1^2, \tag{5.6}$$

valid for centered ρ-mixing random variables with the rate of mixing

$$\sum_{i=1}^\infty \rho(2^i) < +\infty, \tag{5.7}$$

and with K depending on coefficients $\{\rho(k)\}_{k\in\mathbb{N}}$ only.

Using this inequality, one can prove the best possible result for ρ-mixing sequences, when only second moments are assumed to exist:

Theorem 5.3 ([Ibr75]) *Suppose $\{X_j\}$ is stationary, $EX_1 = 0$, $EX_1^2 < +\infty$. If (5.7) holds, then there exists*

$$\lim_{n\to\infty} \frac{ES_n^2}{n} = \sigma^2 \geq 0, \tag{5.8}$$

and if $\sigma^2 > 0$, then

$$\frac{S_n}{\sqrt{n}} \underset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \sigma^2).$$

In fact, in most cases we have $\sigma^2 > 0$ in (5.8): Bradley [Bra81] proved that

$$\sum_{i=1}^{\infty} \tau(2^i) < +\infty$$

(where $\tau(n) \leq \rho(n)$ is defined by (5.5)), together with $ES_n^2 \to \infty$ imply (5.8) with *positive* limit (see also [Pel82]).

Examples in [Bra87] show that the rate of mixing (5.7) cannot be weakened.

The situation is different, when we pass to ϕ-mixing sequences. Peligrad [Pel85] proved that under ϕ-mixing, uniform integrability of $\{\max_{1\leq j\leq n}(S_j^2/\sigma_n^2)\}$ is equivalent to uniform integrability of $\{\max_{1\leq j\leq n}(X_j^2/\sigma_n^2)\}$. But the latter sequence converges to 0 in L^2, *provided*

$$\liminf_{n\to\infty} \frac{ES_n^2}{n} > 0. \tag{5.9}$$

So we get

Theorem 5.4 ([Pel85]) *Suppose $\{X_j\}$ is a stationary and ϕ-mixing sequence, $EX_1 = 0$, $EX_1^2 < +\infty$. If (5.9) holds, then*

$$\frac{S_n}{\sigma_n} \underset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, 1).$$

Eventually, let us mention the paper [DDP86], where interesting equivalent (under strong mixing) expressions for uniform integrability of $\{S_n^2/\sigma_n^2\}$ were formulated: e.g.

$$\limsup_{n\to\infty} \frac{\sigma_n}{E|S_n|} \leq \sqrt{\pi/2}.$$

5.2 Central Limit Theorem for stationary sequences with infinite variances

Central limit theorems for stationary sequences without finite second moment were considered by several authors, starting with the early eighties. We can mention here papers by Lin [Lin81], Samur [Sam85], Heinrich [Hei82], [Hei85]. These papers generalized the independent case, but their assumptions were either technical or too restrictive.

On the contrary, assumptions of recent papers by Bradley [Bra88], Shao [Sha86], Szewczak [Sze88] and Peligrad [Pel90] are probabilistic in nature and close to what we know from the finite variance case.

For example, Theorem 1 in [Bra88] says that if $\ell(x) := EX_1^2 I(|X_1| \leq x)$ is slowly varying as $x \to \infty$, $EX_1 = 0$ and

$$\rho(1) < 1 \quad \text{and} \quad \sum_{i=1}^{\infty} \rho(2^i) < +\infty, \tag{5.10}$$

then $S_n/B_n \longrightarrow_{\mathcal{D}} \mathcal{N}(0,1)$ for some sequence $B_n \to \infty$.

Similarly, [Pel90] shows that under ϕ-mixing with $\phi(1) < 1$, the regular variation of tail probabilities

$$P(|X_1| > x) = x^{-2}\ell(x) \tag{5.11}$$

is sufficient for CLT to hold.

Theoretically, all those theorems are contained in our Theorem 4.5. Conditions (4.5)–(4.7) are, however, not the easiest in direct handling and therefore we suggest following [JaSz90] and using a criterion similar to Theorem 5.1.

For $b_n > 0$ define

$$X_{n,j} = X_j I\left(|X_j| < b_n\right) - EX_j I\left(|X_j| < b_n\right) \tag{5.12}$$

$$T_n = \sum_{j=1}^{n} X_{n,j}, \qquad \tau_n^2 = VarT_n. \tag{5.13}$$

Theorem 5.5 *Let $\{X_j\}$ be a strongly mixing stationary sequence. Suppose we can find $b_n \to +\infty$ such that $\tau_n^2 \to +\infty$ and*

$$\tau_n^{-1} \sum_{j=1}^{n} X_j I\left(|X_j| \geq b_n\right) \underset{\mathcal{P}}{\longrightarrow} 0 \quad \text{as} \quad n \to +\infty. \tag{5.14}$$

Then

$$\mathcal{L}(\tau_n^{-1}(S_n - nEX_1 I(|X_1| < b_n))) \underset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0,1), \quad \text{as} \quad n \to +\infty, \tag{5.15}$$

if and only if $\{\tau_n^{-2}T_n^2\}_{n\in\mathbb{N}}$ is a uniformly integrable sequence of random variables.

PROOF. NECESSITY. From (5.14) and (5.15) we have $\mathcal{L}(\tau_n^{-1}T_n) \longrightarrow_{\mathcal{D}} \mathcal{N}(0,1)$. Uniform integrability of $\{\tau_n^{-2}T_n^2\}$ follows then by [Bil68, Theorem 5.4].

SUFFICIENCY. Let $\{\widehat{X}_j\}_{j\in\mathbb{N}}$ and $\{\widehat{X}_{n,j}\}_{j=1,\ldots,n;\ n\in\mathbb{N}}$ denote independent copies of $\{X_j\}$ and $\{X_{n,j}\}$, respectively. Set

$$\widehat{S}_n = \sum_{j=1}^{n} \widehat{X}_j; \qquad U_n = S_n - \widehat{S}_n;$$

$$\widehat{T}_n = \sum_{j=1}^{n} \widehat{X}_{n,j}; \quad Z_n = T_n - \widehat{T}_n; \quad \zeta_n^2 = \operatorname{Var} Z_n = 2\tau_n^2.$$

First, we shall prove that conditions (4.5)-(4.6) hold when B_n and S_n are replaced by ζ_n and U_n, i.e. that

$$\mathcal{L}(\zeta_n^{-1}U_n) \underset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0,1) \quad \text{as} \quad n \to +\infty. \tag{5.16}$$

Condition (4.6) is satisfied trivially, for U_n's are symmetric. To prove (4.5), observe that by (5.14) $\zeta_n^{-1}(U_n - Z_n) \longrightarrow_{\mathcal{P}} 0$. Hence for every $\varepsilon > 0$ and $m \in I\!N$

$$\begin{aligned}\limsup_{n\to\infty} m^2 P(\zeta_n^{-1}U_n > m) &\le \limsup_{n\to\infty} m^2 P(\zeta_n^{-1}Z_n > m - \varepsilon)\\ &\le \limsup_{n\to\infty} 2m^2 P(\tau_n^{-1}T_n > (m-\varepsilon)/\sqrt{2})\\ &\le \limsup_{n\to\infty} 4\frac{m^2}{(m-\varepsilon)^2}\tau_n^{-2}E(T_n^2 I(\tau_n^{-1}T_n > (m-\varepsilon)/\sqrt{2}))\\ &\longrightarrow 0 \quad \text{as} \quad m \to +\infty\end{aligned}$$

by the uniform integrability of $\tau_n^{-2}T_n^2$. Notice, that we have also checked condition (4.5) for $B_n = \zeta_n$ and $S_n = Z_n$. The sequence $\zeta_n^{-1}Z_n$ satisfies condition (4.7), too:

$$\begin{aligned}\zeta_n^{-2}EZ_n^2 I(|Z_n| \ge m\zeta_n) &= m^2 P(|Z_n| > m\zeta_n) + 2\int_m^\infty y\, P(|Z_n| > \zeta_n y)\, dy\\ &\le 2m^2 P(|T_n| > 2^{-1}m\zeta_n) + 4\int_m^\infty y\, P(|T_n| > 2^{-1}\zeta_n y)\, dy\\ &= 4\tau_n^{-2}ET_n^2 I(|T_n| \ge m\tau_n/\sqrt{2})\\ &\to 0 \quad \text{as} \quad m \to +\infty.\end{aligned}$$

So it is enough to compare $\zeta_n^{-2}EU_n^2 I(|U_n| \le m\zeta_n)$ and $\zeta_n^{-2}EZ_n^2 I(|Z_n| \le m\zeta_n)$. But

$$\zeta_n^{-2}EU_n^2 I(|U_n| \le m\zeta_n) = 2\int_0^m y P(|U_n| > y\zeta_n)\, dy - m^2 P(|U_n| > m\zeta_n)$$

and we have already known that the last term is negligible for large m's (condition (4.5) !). Since $(U_n - Z_n)/\zeta_n \longrightarrow_{\mathcal{P}} 0$, we can replace the integral on the right-hand side by another one, involving Z_n instead of U_n. This proves condition (4.7) and we conclude that (5.16) holds. In particular

$$\mathcal{L}(\zeta_n^{-1}Z_n) = \mathcal{L}((\sqrt{2}\tau_n)^{-1}(T_n - \widehat{T}_n)) \underset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0,1) \quad \text{as} \quad n \to +\infty. \tag{5.17}$$

Since $\{\tau_n^{-1}T_n\}_{n\in I\!N}$ is obviously tight, we may assume that

$$\mathcal{L}((\sqrt{2}\tau_n)^{-1}T_n) \underset{\mathcal{D}}{\longrightarrow} \mathcal{L}(X)$$

along a subsequence $n \in Q \subseteq I\!N$. Then also

$$\mathcal{L}(-(\sqrt{2}\tau_n)^{-1}\widehat{T}_n) \underset{\mathcal{D}}{\longrightarrow} \mathcal{L}(-X)$$

along $n \in Q$ and by (5.17) $\mathcal{L}(X) * \mathcal{L}(-X) = \mathcal{N}(0,1)$. According to the Cramer Theorem (e.g. [Loé77, p.283]), there exist $a \in I\!R$ and $\sigma > 0$, such that $\mathcal{L}(X) = \mathcal{N}(a, \sigma^2)$. By our main assumption $\{\tau_n^{-2} T_n^2\}_{n \in I\!N}$ is uniformly integrable, so $a = 0$ and $\sigma^2 = 1/2$. Hence the only possible limit point for tight sequence $\mathcal{L}(\tau_n^{-1} T_n)$ is $\mathcal{N}(0,1)$, i.e. we have weak convergence to the standard normal law. Returning to non-truncated sums S_n via (5.14) completes the proof. □

Let us discuss briefly the assumptions of Theorem 5.5. The i.i.d. case suggests the choice of $\{b_n\}$: if $\mathcal{L}(X_1)$ is being attracted by the normal law ($\mathcal{L}(X_1) \in \mathcal{DA}(2)$), define

$$b_n := \inf\{x;\ x^{-2} E X_1^2(|X_1| < x) \leq 1/n\}. \tag{5.18}$$

So take b_n's as above and consider the next assumption of Theorem 5.5, i.e.

$$\tau_n^2 = \mathrm{Var}\,(\sum_{j=1}^n X_j I(|X_j| \leq b_n)) \to +\infty.$$

It is satisfied if, for example, $\varrho_1 < 1$ — see [Bra88, Lemma 2.2]. In the class of m-dependent sequences the latter condition is rather restrictive, as the following example shows:

Example 5.6 Let $\{\xi_j\}_{j \in Z}$ be an i.i.d. sequence such that $\mathcal{L}(\xi_1) \in \mathcal{DA}(2)$ and let $\eta_j = \xi_j + \xi_{j+1}$. Then $\varrho_1(\{\eta_j\}) = 1$. To see this, let us introduce $\vartheta_j = \xi_{2j} - \xi_{2j+2}$ and observe that by Theorem 1,[Bra88], $\varrho_1(\{\vartheta_j\}) = 1$. Thus

$$\begin{aligned}
1 = \varrho_1(\{\vartheta_j\}) &= \varrho((\ldots, \xi_{-2} - \xi_0, \xi_0 - \xi_2); (\xi_2 - \xi_4, \xi_4 - \xi_6, \ldots)) \\
&\leq \varrho((\ldots, \xi_{-2} - \xi_0, \xi_{-1} - \xi_1, \xi_0 - \xi_2); (\xi_2 - \xi_4, \xi_3 - \xi_5, \xi_4 - \xi_6, \ldots)) \\
&= \varrho((\ldots, \eta_{-2} - \eta_{-1}, \eta_{-1} - \eta_0, \eta_0 - \eta_1); (\eta_2 - \eta_3, \eta_3 - \eta_4, \eta_4 - \eta_5, \ldots)) \\
&\leq \varrho_1(\{\eta_j\}).
\end{aligned}$$

For m-dependent sequences estimation of the rate of growth of τ_n can provide a very useful information:

Theorem 5.7 *Let $\{X_j\}_{j \in I\!N}$ be an m-dependent, strictly stationary sequence such that $\mathcal{L}(X_1) \in \mathcal{DA}(2)$ and $E X_1^2 = \infty$. Assume for simplicity that $E X_1 = 0$. Take b_n as in (5.18) and let τ_n be defined by (5.13). Then*

$$\liminf_{n \to \infty} \tau_n b_n^{-1} > 0 \tag{5.19}$$

implies $\mathcal{L}(\tau_n^{-1} S_n) \ \longrightarrow_{\mathcal{D}}\ \mathcal{N}(0,1)$.

PROOF. First notice that $\tau_n \to +\infty$. By the choice of b_n

$$b_n^{-1} \sum_{j=1}^n X_j I(|X_j| \geq b_n) \ \underset{\mathcal{P}}{\longrightarrow}\ 0,$$

so (5.19) implies condition (5.14). Further, it is easy to see that $\{b_n^{-2}T_n^2\}_{n\in \mathbb{N}}$ is uniformly integrable, hence by (5.19) we have also uniform integrability of $\{\tau_n^{-2}T_n^2\}_{n\in \mathbb{N}}$ and all assumptions of Theorem 5.5 are satisfied. Consequently (5.15) holds and we have to prove only, that

$$n\tau_n^{-1}|EX_1I(|X_1| < b_n)| \longrightarrow 0 \quad \text{as} \quad n \to +\infty. \tag{5.20}$$

But by Theorem 2, VIII, §9 in [Fel71] we have

$$nb_n^{-1}E|X_1|I(|X_1| \geq b_n) \longrightarrow 0 \quad \text{as} \quad n \to +\infty.$$

Now (5.20) follows from the above formula, assumption (5.19) and the fact that $ES_n = 0$. □

The above theorem, although operating with ϱ-mixing sequences, is not contained in Bradley's Theorem 1 [Bra88], since we have no restrictions on ϱ_1.

It is worth noticing, that checking condition (5.19) is the key step in the proof of the corresponding result in [Lin81] (for further discussion of m-dependent case we refer to [Sze88]). On the other hand (5.19) is far from necessity:

Example 5.8 Let

$$P(Y > x) = P(Y < -x) = \frac{1}{2x^2},$$

for $x > 1$ and

$$P(Z > x) = P(Z < -x) = \frac{e^2\ln x}{2x^2},$$

for $x \geq e$. Let $\{Y_j\}_{j\in \mathbb{N}}$ and $\{Z_j\}_{j\in \mathbb{N}}$ be *i.i.d.* with $\mathcal{L}(Y_1) = \mathcal{L}(Y)$ and $\mathcal{L}(Z_1) = \mathcal{L}(Z)$. Define $X_j = Y_j + Z_j - Z_{j+1}$, then by Theorem, VIII, §8, [Fel71]

$$P(|X_1| > x) \sim \frac{e^2 \ln x}{x^2} \quad \text{as} \quad x \longrightarrow +\infty,$$

so

$$EX_1^2I(|X_1| < x) \sim 2e^2(\ln x)^2, \quad \text{as} \quad x \longrightarrow +\infty,$$

and

$$b_n \sim \frac{e\sqrt{n}\ln n}{\sqrt{2}}, \quad \text{as} \quad n \longrightarrow +\infty.$$

On the other hand it is not difficult to see that

$$EY_1^2I(|Y_1| < x) \sim 2\ln x, \quad \text{as} \quad x \longrightarrow +\infty,$$

so $\tau_n^2 \sim n\ln n$ and $\lim_n \tau_n b_n^{-1} = 0$.

5.3 Non-central Limit Theorems

When "second order" methods are useless, the analysis of the characteristic function of a sum of dependent random variables can be very difficult. Maybe this is the reason that the first general p-stable limit theorem with $p < 2$ was published only in 1983.In

[Dav83], developing some ideas of the Extreme Value Limit Theory and using special representation for stable laws, Davis proved two theorems, separately for cases $0 < p < 1$ and $1 \leq p < 2$. Especially his Theorem 2, corresponding to $0 < p < 1$, is very interesting, since its assumptions are flexible enough to cover existing examples (e.g. continued fraction expansions) as well as the following

Example 5.9 Let $\{Y_j\}_{j\in I\!N}$ be a stationary Gaussian sequence with zero mean, unit variance and covariance function $r_n = EY_1Y_n$ and let $\widehat{Y}_1, \widehat{Y}_2, \ldots$ be the independent imitation of $\{Y_j\}$, i.e. i.i.d. with the same marginal distributions $\mathcal{L}(Y_j) = \mathcal{L}(\widehat{Y}_j)$. Let $H : I\!R^1 \to I\!R^1$ be such that the law of $H(Y_1)$ belongs to the domain of attraction of a stable law μ with index $0 < p < 1$. Finally, let B_n be such that

$$\frac{H(\widehat{Y}_1) + H(\widehat{Y}_2) + \ldots + H(\widehat{Y}_n)}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu.$$

If $r_n \cdot \log n \to 0$ or $\sum_{k=1}^{\infty} r_n^2 < +\infty$, then also

$$\frac{H(Y_1) + H(Y_2) + \ldots + H(Y_n)}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu.$$

Davis' result was rederived in a particular case by Aaronson [Aar86] (see Corollary 5.12 below) and then applied in studying properties of f-expansions. Later, using a point processes technique, Jakubowski and Kobus [JaKo89] generalized it to several dimensions and for nonstationary sequences.

Here we shall join our general Theorem 4.2 and computations of Theorem 2 and Proposition 3 [DeJa89] in order to prove a slight generalization of Davis' Theorem.

Theorem 5.10 *Let $\{X_j\}$ be a stationary sequence and let $\{\widehat{X}_j\}$ be its independent imitation.*

Suppose there exist constants B_n such that

$$\frac{\widehat{X}_1 + \widehat{X}_2 + \ldots + \widehat{X}_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu, \tag{5.21}$$

where μ is strictly p-stable, $0 < p < 1$.

Further, suppose the following Condition D′ *holds:*

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \cdot \sum_{j=2}^{[n/k]} P(|X_1| > \varepsilon B_n, |X_j| > \varepsilon B_n) = 0, \quad \forall \varepsilon > 0. \tag{5.22}$$

Then

$$\frac{X_1 + X_2 + \ldots + X_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu$$

if, and only if, Condition B is satisfied.

PROOF. By (5.21) we know that $\mu = \text{Pois}\,(p, c_+, c_-)$ and that for each $x > 0$

$$\begin{array}{lcl} n \cdot P(X_1 > x \cdot B_n) & \longrightarrow & c_+/x^p, \\ n \cdot P(X_1 < (-x) \cdot B_n) & \longrightarrow & c_-/x^p. \end{array} \tag{5.23}$$

In particular, if $x_n \to \infty$ is increasing slowly enough, then

$$x_n^p \cdot n \cdot P(X_1 > x_n B_n) \;\longrightarrow\; c_+, \tag{5.24}$$

and similarly for c_-.

Lemma 5.11 *If Condition* D′ *holds, then there exists a sequence $r_n \to \infty$ such that for all sequences $0 < x_n \to \infty$ with $x_n = o(r_n)$,*

$$\lim_{n\to\infty} x_n^p \sum_{1\le i<j\le n} P(|X_i| > x_n B_n, |X_j| > x_n B_n) = 0. \tag{5.25}$$

PROOF. Condition D′ implies

$$k_n \cdot \sum_{1\le i<j\le n} P(|X_i| > \varepsilon B_{k_n\cdot n}, |X_j| > \varepsilon B_{k_n\cdot n}) \;\longrightarrow\; 0, \tag{5.26}$$

for $k_n \to \infty$, $k_n = o(\tilde{r}_n)$. By regular variation of B_n, there exists a sequence $r_n \to \infty$, $r_n \le \tilde{r}_n$ such that

$$\lim_{n\to\infty} \sup_{1\le s\le r_n} s^{1/p} B_n / B_{[sn]} = 1. \tag{5.27}$$

Let $x_n = k_n^{1/p}\varepsilon_n$ be such that $x_n \to \infty$, $\varepsilon_n \to 1$ and $k_n = o(r_n)$. Then, for $0 < \varepsilon' < 1$ and n large enough,

$$\begin{aligned} & x_n^p \sum_{1\le i<j\le n} P(|X_i| > x_n B_n, |X_j| > x_n B_n) \\ &\quad = \varepsilon_n^p k_n \sum_{1\le i<j\le n} P(|X_i| > k_n^{1/p}\varepsilon_n B_n, |X_j| > k_n^{1/p}\varepsilon_n B_n) \\ &\quad = \varepsilon_n^p k_n \sum_{1\le i<j\le n} P\Big(|X_i| > \varepsilon_n \frac{k_n^{1/p} B_n}{B_{k_n\cdot n}} B_{k_n\cdot n}, |X_j| > \varepsilon_n \frac{k_n^{1/p} B_n}{B_{k_n\cdot n}} B_{k_n\cdot n}\Big) \\ &\quad \le 2^p k_n \cdot \sum_{1\le i<j\le n} P(|X_i| > \varepsilon B_{k_n\cdot n}, |X_j| > \varepsilon B_{k_n\cdot n}) \;\longrightarrow\; 0. \end{aligned}$$

□

In order to apply Theorem 4.2 we have to prove that

$$x_n^p P(S_n > x_n B_n) \;\longrightarrow\; c_+ \tag{5.28}$$

for all increasing slowly enough $x_n \to \infty$ (and similar relation for c_-). Let us observe that by Karamata's Theorem, function $f(x) = E|X_1| \wedge x$ varies regularly with index $1 - p$. Hence for $\delta > 0$ and x_n increasing slowly enough

$$x_n^p P\Big(\Big|\sum_{j=1}^n X_j I(|X_j| \le \delta B_n x_n)\Big| > B_n x_n\Big) \le x_n^p \frac{nE(|X_1| \wedge \delta B_n x_n)}{B_n x_n} \sim \frac{\delta^{1-p}}{1-p}. \tag{5.29}$$

So

$$\lim_{\delta\searrow 0}\limsup_{n\to\infty} x_n^p P\Big(\Big|\sum_{j=1}^n X_j I(|X_j|\le \delta B_n x_n)\Big| > B_n x_n\Big) = 0,$$

and to prove (5.28) it is enough to show that

$$\lim_{n\to\infty} x_n^p P\Big(\sum_{j=1}^n X_j I(|X_j| > \delta B_n x_n) > B_n x_n\Big) = c_+. \tag{5.30}$$

So fix $0<\delta<1$. We have

$$\begin{aligned}&\Big\{\sum_{j=1}^n X_j I(|X_j|>\delta B_n x_n) > B_n x_n\Big\} \triangle \Big\{\sum_{j=1}^n X_j I(X_j > B_n x_n) > B_n x_n\Big\}\\ &\quad\subset\ \{\exists i\ne j : |X_i|>\delta B_n x_n, |X_j|>\delta B_n x_n\} = \bigcup_{1\le i<j\le n}\{|X_i|>\delta B_n x_n, |X_j|>\delta B_n x_n\},\end{aligned}$$

so using (5.25) for δx_n we obtain that

$$\lim_{n\to\infty} x_n^p \left| P\Big(\sum_{j=1}^n X_j I(|X_j|>\delta B_n x_n) > B_n x_n\Big) - P\Big(\sum_{j=1}^n X_j I(X_j > B_n x_n) > B_n x_n\Big)\right| = 0.$$

Moreover, since

$$\Big\{\sum_{j=1}^n X_j I(X_j > B_n x_n) > B_n x_n\Big\} = \bigcup_{j=1}^n \{X_j > B_n x_n\},$$

it follows from Bonferroni's inequality and again by (5.25) that

$$\begin{aligned}&\lim_{n\to\infty} x_n^p \left| P\Big(\sum_{j=1}^n X_j I(X_j > B_n) > B_n x_n\Big) - nP(X_1 > B_n x_n)\right|\\ &\qquad\le\ \lim_{n\to\infty} x_n^p \sum_{1\le i<j\le n} P(|X_i|>x_n B_n, |X_j| > x_n B_n) = 0.\end{aligned}$$

Now (5.30) follows by (5.24). □

Corollary 5.12 *Condition* D′ *holds, if for each* $\varepsilon>0$

$$\sup_{n\in\mathbb{N}} n\cdot P(|X_1|>\varepsilon B_n) \le C < +\infty, \tag{5.31}$$

and

$$\psi_\infty(1) := \sup_{j\in\mathbb{N}}\sup_{x>0}\frac{P(|X_1|>x, |X_j|>x)}{(P(|X_1|>x))^2} < +\infty. \tag{5.32}$$

In particular, if $\{X_j\}$ *is a stationary,* ψ*-mixing sequence with* $\psi(1)<+\infty$ *and* B_n *is such that (5.21) is satisfied, then*

$$S_n/B_n \ \underset{\mathcal{D}}{\longrightarrow}\ \mu.$$

PROOF.

$$n\sum_{j=2}^{[n/k]} P(|X_1|>\varepsilon B_n, |X_j|>\varepsilon B_n) \le \psi_\infty(1) n[n/k] P(|X_1|>\varepsilon B_n)^2 \le C^2\psi_\infty(1)\cdot k^{-1}.$$

□

Let us note that we used the assumption $0<p<1$ in (5.29) only. One may use other tools for this estimation. For example Peligrad's inequality (5.6) gives us:

Theorem 5.13 *Let $\{X_j\}$ be a stationary ρ-mixing sequence with*

$$\sum_{k=1}^{\infty}\rho(2^k) < +\infty.$$

If B_n is such that

$$\frac{\widehat{X}_1+\widehat{X}_2+\ldots+\widehat{X}_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu,$$

where μ is a non-degenerate strictly p-stable distribution, $1\le p<2$, and if Condition D′ *holds, then*

$$\frac{X_1+X_2+\ldots+X_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu.$$

PROOF. Let us assume, for the sake of simplicity, that random variables X_j have symmetric laws. We have to check (5.29). By inequality (5.6) and by Karamata's Theorem

$$x_n^p P\Big(\Big|\sum_{j=1}^{n}\Big(X_j I(|X_j|\le\delta B_n x_n)\Big)\Big| > B_n x_n\Big) \le x_n^p\frac{KnE(X_1^2\wedge(\delta B_n x_n)^2)}{B_n^2 x_n^2} \sim K\frac{\delta^{2-p}}{2-p},$$

and this approaches 0 as $\delta\searrow 0$. □

The above theorem is a (poor) counterpart of Ibragimov's CLT and was proved (in the nonstationary setting) in [JaKo89]. It improves Corollary 5.10 in [Sam84], where ϕ-mixing with $\phi(1)<1$ and $\sum_{k=1}^{\infty}\phi^{1/2}(k)<+\infty$ is considered.

5.4 Limit Theorems for m-dependent Sequences

Recall, that $\{X_j\}$ is m-dependent if $\alpha(m+1)=0$, i.e. for each $n\in I\!N$, $X_1,X_2,\ldots,X_n$ and $X_{n+m+1},X_{n+m+2},\ldots$ are independent. This notion was introduced by Hoeffding and Robbins in [HoRo48], together with some, by now classic, statistical applications. Outside of statistics, 1-dependent sequences arise naturally in regeneration theory of certain Markov processes—see [Asm87, Chapter VI].

In [HoRo48] a central limit theorem involving finiteness of third moments was proved. Final form of the Central Limit Theorem for m- dependent sequences was found by Diananda [Dia55, Theorem 4]:

Theorem 5.14 *If $\{X_j\}_{j\in\mathbb{N}}$ is a strictly stationary sequence of m-dependent random variables with zero mean and finite variance, then*

$$\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}} \underset{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \sigma^2), \tag{5.33}$$

where

$$\sigma^2 = EX_1^2 + \sum_{j=2}^{m+1} EX_1X_j. \tag{5.34}$$

(When $\sigma^2 = 0$ we set $\mathcal{N}(0,0) = \delta_0$).

Heinrich [Hei82] and [Hei85] proposed a general method of derivation of limit theorems for m-dependent sequences, for both $p = 2$ and $0 < p < 2$ (including estimations of the rate of convergence). His conditions are, however, very technical and far from being minimal.

The conditions in [JaKo89, Theorem 5.3] are very natural.

Theorem 5.15 *Let $\{X_j\}$ be a strictly stationary and m-dependent sequence. Assume that the distribution of random vector $Y_0 = (X_1, \ldots, X_{m+1})$ belongs to the domain of attraction of a non-degenerate $(m+1)$-dimensional p-stable law μ with the Lévy measure ν, i.e. there exist constants B_n such that sums $\widehat{Y}_1 + \widehat{Y}_2 + \ldots + \widehat{Y}_n$ normalized by B_n and suitably centered are convergent in distribution to μ.*

Let

$$\nu_0 := \nu_{x_1+\ldots+x_{m+1}} - \nu_{x_1+x_2+\ldots+x_m}, \tag{5.35}$$

where $\nu_{x_1+\ldots+x_{m+1}}(A) := \nu(\{(x_1, x_2, \ldots, x_{m+1}) \in \mathbb{R}^{(m+1)} : x_1 + x_2 + \ldots + x_{m+1} \in A\})$ and $\nu_{x_1+x_2+\ldots+x_m}$ is defined similarly.

(i) *If $0 < p < 1$, then $S_n/B_n \longrightarrow_{\mathcal{D}} \text{Pois}\,(\nu_0)$.*

(ii) *If $1 < p < 2$, then $(S_n - ES_n)/B_n \longrightarrow_{\mathcal{D}} c_\infty - \text{Pois}\,(\nu_0)$.*

(iii) *If $p = 1$, then $(S_n - A_n)/B_n \longrightarrow_{\mathcal{D}} c_1 - \text{Pois}\,(\nu_0)$, where*

$$A_n = n\Big(E(X_1 + X_2 + \ldots + X_{m+1})I(|X_1 + X_2 + \ldots + X_{m+1}| \le B_n) - E(X_1 + X_2 + \ldots + X_m)I(|X_1 + X_2 + \ldots + X_m| \le B_n)\Big).$$

Examples 5.6 and 5.7, [JaKo89, pp.237–239], show how formula (5.35) works. We turn the attention to the fact that ν_0 is determined by asymptotic properties of $\mathcal{L}(X_1 + X_2 + \ldots + X_{m+1})$ and $\mathcal{L}(X_1 + X_2 + \ldots + X_m)$ only, and not by the whole $\mathcal{L}(Y_0)$. The situation in Theorem 5.14 is the same:

$$\sigma^2 = E(X_1 + \ldots + X_{m+1})^2 - E(X_1 + \ldots + X_m)^2.$$

Szewczak [Sze88] ($p = 2$, infinite variances) and Kobus [Kob90] ($0 < p < 2$) using completely different tools proved that limit theorems for stationary m-dependent random variables possess structure independent of p.

Let for each $k \in \mathbb{N}$, $\widehat{U}_1^{(k)}, \widehat{U}_2^{(k)}, \ldots$ be independent copies of $S_k = X_1 + X_2 + \ldots + X_k$.

Theorem 5.16 *Let $X_1, X_2, \ldots$ be a stationary m-dependent sequence.*

Suppose $\{B_n\}$ is such that

$$\widehat{U}_1^{(m+1)} + \widehat{U}_2^{(m+1)} + \ldots + \widehat{U}_n^{(m+1)}/B_n \underset{\mathcal{D}}{\longrightarrow} \mu_{m+1} \tag{5.36}$$

and

$$\widehat{U}_1^{(m)} + \widehat{U}_2^{(m)} + \ldots + \widehat{U}_n^{(m)}/B_n \underset{\mathcal{D}}{\longrightarrow} \mu_m, \tag{5.37}$$

where μ_{m+1} and μ_m are non-degenerate strictly p-stable distributions.

Then $(X_1 + X_2 + \ldots + X_n)/B_n$ converges in distribution to the strictly p-stable law μ with the characteristic function

$$\widehat{\mu}(t) = \frac{\widehat{\mu}_{m+1}(t)}{\widehat{\mu}_m(t)}. \tag{5.38}$$

Proof. We will give a short proof based on our Theorem 4.2 for the case $0 < p < 1$ only.

At first, let us observe that for each $x > 0$

$$P(X_1 > x \cdot B_n) \le P(S_{m+1} > (x/2) \cdot B_n) + P(S_m > (x/2) \cdot B_n). \tag{5.39}$$

Estimating the same way $P(X_1 < (-x) \cdot B_n)$ we get

$$\sup_{n \in \mathbb{N}} nP(|X_1| > xB_n) < +\infty. \tag{5.40}$$

As in the proof of Lemma 5.11, we obtain from m-dependence and (5.40)

$$\lim_{n\to\infty} x_n^p \sum_{\substack{1\le i<j\le n\\ j-i>m}} P(|X_i| > x_n B_n, |X_j| > x_n B_n) = 0, \tag{5.41}$$

for all increasing slowly enough $x_n \to \infty$.

Similarly, as in Theorem 5.10, it is sufficient to prove

$$x_n^p \Big| P(S_{\delta,n,n} > B_n x_n) - n(P(S_{\delta,n,m+1} > B_n x_n) - P(S_{\delta,n,m} > B_n x_n)) \Big| \longrightarrow 0, \tag{5.42}$$

where $\{x_n\}$ is increasing slowly enough and

$$S_{\delta,n,k} = \sum_{j=1}^{k} X_j I(|X_j| > \delta B_n x_n), \quad k = 1, 2, \ldots, n.$$

Define

$$\begin{aligned} \tau_n &= \min\{j : |X_j| > \delta B_n x_n\}, \\ S^*_{\delta,n,k} &= \sum_{j=1}^{n} X_j I(|X_j| > \delta B_n x_n) \cdot I(\tau_n + m \ge j) = S_{\delta,n,k\wedge(\tau_n+m)}. \end{aligned}$$

Since

$$x_n^p P(S_{\delta,n,n} \ne S^*_{\delta,n,n}) \le x_n^p P\Big(\bigcup_{\substack{1\le i<j\le n\\ j-i>m}} \{|X_i| > \delta x_n B_n, |X_j| > \delta x_n B_n\}\Big) \longrightarrow 0$$

by (5.41), it is enough to deal with $x_n^p P(S^*_{\delta,n,n} > B_n x_n)$.

Lemma 5.17 *Let $Z_1, Z_2, \ldots, Z_n$ be random variables with partial sums $T_k = \sum_{j=1}^k Z_j$, $T_0 = 0$. Then for any $C > 0$,*

$$I(T_n > C) = \sum_{k=1}^n I(T_k > C, T_{k-1} \le C) - \sum_{k=1}^n I(T_k \le C, T_{k-1} > C). \tag{5.43}$$

□

Setting in the above lemma $Z_j = X_j I(|X_j| > \delta B_n x_n) \cdot I(\tau_n + m \ge j)$ we obtain

$$\begin{aligned} I(S^*_{\delta,n,n} > B_n x_n) &= \sum_{k=1}^n I(S^*_{\delta,n,k} > B_n x_n, S^*_{\delta,n,k-1} \le B_n x_n) \\ &\quad - \sum_{k=1}^n I(S^*_{\delta,n,k} \le B_n x_n, S^*_{\delta,n,k-1} > B_n x_n). \end{aligned} \tag{5.44}$$

But $k > \tau_n + m$ implies $S^*_{\delta,n,k} = S^*_{\delta,n,\tau_n+m}$, so

$$\sum_{k=1}^n I(S^*_{\delta,n,k} > B_n x_n, S^*_{\delta,n,k-1} \le B_n x_n) = \sum_{k=1}^{\tau_n+m} I(S^*_{\delta,n,k} > B_n x_n, S^*_{\delta,n,k-1} \le B_n x_n).$$

On the other hand, if $m < k \le \tau_n + m$ then

$$S^*_{\delta,n,k} = S_{\delta,n,k} - S_{\delta,n,k-m-1} \quad \text{and} \quad S^*_{\delta,n,k-1} = S_{\delta,n,k-1} - S_{\delta,n,k-m-1}.$$

Moreover,

$$\begin{aligned} 0 \le\ & \sum_{k=1}^n I(S_{\delta,n,k} - S_{\delta,n,k-m-1} > B_n x_n, S_{\delta,n,k-1} - S_{\delta,n,k-m-1} \le B_n x_n) \\ & - \sum_{k=1}^{\tau_n+m} I(S_{\delta,n,k} - S_{\delta,n,k-m-1} > B_n x_n, S_{\delta,n,k-1} - S_{\delta,n,k-m-1} \le B_n x_n) \\ \le\ & \sum_{\substack{1 \le i < j \le n \\ j-i > m}} I(|X_i| > \delta x_n B_n, |X_j| > \delta x_n B_n). \end{aligned}$$

Taking into account the three last relations and (5.41), we obtain

$$\begin{aligned} & \lim_{n\to\infty} x_n^p \Big| E \sum_{k=1}^n I(S^*_{\delta,n,k} > B_n x_n, S^*_{\delta,n,k-1} \le B_n x_n) - nP(S_{\delta,n,m+1} > B_n x_n, S_{\delta,n,m} \le B_n x_n) \Big| \\ &\le \lim_{n\to\infty} x_n^p E \Big| \sum_{k=1}^n I(S^*_{\delta,n,k} > B_n x_n, S^*_{\delta,n,k-1} \le B_n x_n) \\ &\qquad - \sum_{k=1}^n I(S_{\delta,n,k} - S_{\delta,n,k-m-1} > B_n x_n, S_{\delta,n,k-1} - S_{\delta,n,k-m-1} \le B_n x_n) \Big| \\ &= 0. \end{aligned}$$

An analogous formula can be obtained for the second term on the right-hand side of (5.44).

Eventually, observe that

$$\begin{aligned} P(S_{\delta,n,m+1} > B_n x_n, S_{\delta,n,m} \leq B_n x_n) - P(S_{\delta,n,m+1} \leq B_n x_n, S_{\delta,n,m} > B_n x_n) \\ = \quad P(S_{\delta,n,m+1} > B_n x_n) - P(S_{\delta,n,m} > B_n x_n). \end{aligned}$$

□

Remark 5.18 Condition D′ considered in the previous section excludes clustering of big values in the sequence $\{X_j\}$. In general, Condition D′ need not be satisfied by m-dependent sequences.

Example 5.19 Let Y_j's be i.i.d. with $\mathcal{L}(Y_j) \in \mathcal{D}(\mu)$ for some p-stable μ and set $X_j = Y_j \vee Y_{j-1}$. Clearly, in this sequence big values "go in pairs", hence D′ cannot hold and point processes technique cannot be applied, at least directly. This is the reason for hard technical proof of Theorem 5.15 in [JaKo89]. The new approach using Tauberian theorems seems to be much better tool.

This example illustrates also another phenomenon. Observe that finite dimensional distributions of the process $[0,1] \ni t \mapsto S_n(t) = \sum_{j=1}^{[nt]} X_j$ converge and that the limit is stable. But if $p < 2$, we do not have functional convergence! Indeed, the jumps of the limit are produced by pairs of jumps of processes S_n. And this is impossible in the Skorokhod topology. Even if the limit is Gaussian, but Y_j has infinite variance, the functional convergence may fail. A suitable sequence is provided by Example 5.8: this time $\max_{1\leq j\leq n} X_j \not\to 0$ in probability—see [Sze89].

We refer to [Sam87] and [Dab87], where necessary and sufficient conditions for functional convergence in some classes of mixing processes are given.

Part II

Order statistics

Chapter 6

Asymptotic Independent Representations for Maxima

6.1 Main Criterion

Let $\{X_j\}_{j\in\mathbb{N}}$ be a sequence of random variables. Define $M_{m:n} = \max_{m<j\leq n} X_j$ for $m < n$, $M_{n:m} = -\infty$ for $m \geq n$ and $M_n = M_{0:n}$.

The concept of asymptotic independent representation for maxima is the same as for sums: We say that $\{X_j\}_{j\in\mathbb{N}}$ admits an a.i.r. for maxima if there exists a sequence $\{\widetilde{X}_j\}_{j\in\mathbb{N}}$ of independent random variables such that

$$\sup_{x\in\mathbb{R}^1} |P(M_n \leq x) - P(\widetilde{M}_n \leq x)| \to 0 \quad \text{as } n \to +\infty, \tag{6.1}$$

where $\widetilde{M}_n$'s are partial maxima for $\{\widetilde{X}_j\}$.

Existence of independent asymptotic representation reduces many problems on asymptotic properties of laws of $\{M_n\}_{n\in\mathbb{N}}$ to the easily computable independent case. For example, possible limit laws for suitably centered and normalized M_n's can be identified with those found by Meizler ([Mei56]), see also [Gal78, Chapter 3].

In the Extreme Value Limit Theory, the idea of replacement of the "original" sequence by an independent one, being equivalent from some point of view, goes back to paper by Watson([Wat54]). Loynes([Loy65]) considered the "associated" sequence for $\{X_j\}$—an i.i.d. sequence $\{\widehat{X}_j\}$ with the same one-dimensional marginals: $\mathcal{L}(X_j) = \mathcal{L}(\widehat{X}_j)$. We will say that $\{\widehat{X}_j\}$ is the "independent imitation" of $\{X_j\}$. Leadbetter ([Lea74]) proved that in a wide class of stationary sequences the limit behaviour of all order statistics is the same for both $\{X_j\}$ and $\{\widehat{X}_j\}$. Even if the correspondence between higher order statistics breaks, the maxima of $\{X_j\}$ and $\{\widehat{X}_j\}$ can remain closely related. This holds, for example, if so called extremal index of $\{X_j\}$ exists—see [Lea83], [LLR83, Chapter 3] and Section 6.5 below.

If $\{X_j\}$ is stationary, $\{\widetilde{X}_j\}$ are i.i.d. and G is the distribution function of $\widetilde{X}_1$, then

(6.1) can be rewritten as

$$P(M_n \leq u_n) - G(u_n)^n \to 0 \quad \text{as } n \to +\infty, \tag{6.2}$$

for every sequence $\{u_n\} \subset I\!R^1$. O'Brien [OBr87] calls any distribution function G satisfying (6.2) a *phantom* distribution function for $\{X_j\}$ (in view of our considerations on sums, a "max-phantom" in place of "phantom" would be more appropriate here).

We shall show, how to construct an a.i.r. knowing limit of paths

$$I\!R^+ \ni t \mapsto P(M_{[nt]} \leq v_n)$$

for some sequence $\{v_n\} \subset I\!R^1$. If the limit is of the form $e^{-t\beta}$, where $\beta > 0$, this construction gives an *i.i.d.* sequence, i.e. we obtain a phantom distribution function. This means we are going to find a universal tool for both stationary and nonstationary cases.

At the very beginning, let us observe that we may consider non-decreasing sequences $\{v_n\}$ only.

Lemma 6.1 *Suppose for some sequence $\{v_n\}$ and some subset $D \subset I\!R^+$,*

$$P(M_{[nt]} \leq v_n) \longrightarrow \alpha_t, \quad t \in D.$$

If $\sup_{t\in D} \alpha_t = 1$ and $\alpha_{t_0} < 1$ for some $t_0 \in D$, then one can find a non-decreasing *sequence $k_n \subset I\!N$, $k_n \to +\infty$ such that $\{v_n^* = v_{k_n}\}$ is* non-decreasing *and satisfies*

$$P(M_{[nt]} \leq v_n^*) \longrightarrow \alpha_t, \quad t \in D.$$

PROOF. Define $(F_\infty)_* = \sup_j (F_j)_*$, where $(F_j)_* = \sup\{x\,;\ F_j(x) < 1\}$. Since $\alpha_{t_0} < 1$ for some $t_0 \in D$, we have $v_n < (F_\infty)_*$ for n large enough. Hence we can define

$$v_n^* = \begin{cases} \inf\{v_l : l \in I\!N\} & \text{if} \quad v_k \geq (F_\infty)_* \text{ for } 1 \leq k \leq n \\ \max\{v_k : v_k < (F_\infty)_*, 1 \leq k \leq n\} & \text{otherwise.} \end{cases} \tag{6.3}$$

In particular, for large n

$$v_n \leq v_n^* < (F_\infty)_*. \tag{6.4}$$

Set $k_0 = \min\{l : v_l = v_l^*\}$ and

$$k_n = \begin{cases} k_0 & \text{if} \quad n < k_0 \\ \min\{k \leq n : v_k = v_n^*\} & \text{otherwise.} \end{cases} \tag{6.5}$$

Clearly, $\{k_n\}$ is non-decreasing. If $k_n = k_\infty$ for $n \geq n_0$, then for every $j \in I\!N$ and $t \in D$,

$$\alpha_t = \lim_{n\to\infty} P(M_{[nt]} \leq v_n) \leq \lim_{n\to\infty} P(X_j \leq v_n) \leq \lim_{n\to\infty} P(X_j \leq v_n^*) = P(X_j \leq v_{k_\infty}),$$

and, consequently, $P(X_j \leq v_{k_\infty}) = 1$ for each $j \in I\!N$, i.e. $v_{k_\infty} \geq (F_\infty)_*$. But this is impossible by (6.4) and thus $k_n \to \infty$. If so, for $\varepsilon > 0$, $t \in D$ and n large enough

$$\begin{aligned} \alpha_t - \varepsilon &< P(M_{[nt]} \leq v_n) \leq P(M_{[nt]} \leq v_n^*) \\ &= P(M_{[nt]} \leq v_{k_n}) \leq P(M_{[k_n t]} \leq v_{k_n}) < \alpha_t + \varepsilon \end{aligned}$$

i.e. $P(M_{[nt]} \leq v_n^*) \longrightarrow \alpha_t$. □

Theorem 6.2 *Assume there is a sequence* $\{v_n\}$ *such that for each* t *in some dense subset* $D \subset I\!R^+ = (0, +\infty)$

$$P(M_{[nt]} \leq v_n) \longrightarrow \alpha_t, \tag{6.6}$$

where

$$\begin{aligned} \alpha_t &> 0, \quad t \in D & (6.7)\\ \sup_{t\in D} \alpha_t &= 1, & (6.8)\\ \inf_{t\in D} \alpha_t &= 0. & (6.9) \end{aligned}$$

Set

$$\widetilde{\alpha}_t = \sup_{D \ni u > t} \alpha_u.$$

Then the following statements (i) - (iv) are equivalent.

(i) $\{X_j\}$ *admits an asymptotic independent representation.*

(ii) $\{X_j\}$ *admits an asymptotic independent representation defined by marginal distribution functions*

$$\widetilde{X}_j \sim F_j(x) = \begin{cases} 0 & \text{if} \quad x < v_1^*, \\ \widetilde{\alpha}_{j/n}/\widetilde{\alpha}_{j-1/n} & \text{if} \quad v_n^* \leq x < v_{n+1}^*, \\ 1 & \text{if} \quad x \geq \sup_k v_k^* \end{cases} \tag{6.10}$$

where numbers v_n^* *are defined by (6.3).*

(iii) *For each* $u \geq 1$ *the function* $f_u(t) = \widetilde{\alpha}_{ut}/\widetilde{\alpha}_t$ *is non-increasing on* $(0, \infty)$.

(iv) *The function* $g_\alpha = \log \circ \widetilde{\alpha} \circ \exp$ *is concave.*

Corollary 6.3 *Suppose* $\{X_j\}_{j\in I\!N}$ *are independent and (6.6)-(6.9) hold for some dense* $D \subset I\!R^+$. *Then* $\lim_{n\to\infty} P(M_{[nt]} \leq v_n) = \alpha_t$ *uniformly in* $t \geq 0$ *and* $\alpha_{(\cdot)} = \exp(g_\alpha(\log(\cdot)))$ *for some concave* g_α.

Corollary 6.4 *Suppose that (6.6) is satisfied with* $\alpha_t = \exp(-t \cdot \beta)$, *where* $\beta > 0$. *Then* $\{X_j\}$ *admits a phantom distribution function* G *given by formula*

$$G(x) = \begin{cases} 0 & \text{if} \quad x < v_1^*, \\ \exp(-\beta)^{1/n} & \text{if} \quad v_n^* \leq x < v_{n+1}^*, \\ 1 & \text{if} \quad x \geq \sup_k v_k^* \end{cases} \tag{6.11}$$

where numbers v_n^* *are defined by (6.3).*

Corollary 6.5 *Assume, in addition to (6.6)-(6.9), that*

$$\sup_{k\leq l} |P(M_l \leq v_n) - P(M_k \leq v_n)P(M_{k:l} \leq v_n)| \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{6.12}$$

Then $\{X_j\}$ *admits an asymptotic independent representation.*

Remark 6.6 If (6.8) and (6.9) are not satisfied, the limit function may contain no information.

Example 6.7 Let

$$F(x) = \begin{cases} 1 - x^{-\beta} & \text{for } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $\{Y_j\}$ are i.i.d. with $Y_j \sim F$, define $X_j = j^{-1/\beta} Y_j$ and $v_n = \log^{1/\beta} n$. Then for *every* $t > 0$,

$$P(M_{[nt]} \leq v_n) \longrightarrow e^{-1}.$$

6.2 Proofs

We divide the proof of Theorem 6.2 into several steps. First we shall adapt the scheme of getting uniform closeness, developed in [Jak90a] and [Jak91a]. The formulation of Lemma 6.8 below is more complicated than we need for the present purposes; this is motivated by future applications.

Lemma 6.8 *Let* $\{Z_n\}_{n\in\mathbb{N}}$ *and* $\{\widetilde{Z}_n\}_{n\in\mathbb{N}}$ *be two non-decreasing sequences of random variables and let* $\{v_n\}$ *be a non-decreasing sequence of numbers. Let* $g, \widetilde{g} : [0,1] \to [0,1]$ *be non-decreasing,* $g(0) = \widetilde{g}(0) = 0$, $g(1) = \widetilde{g}(1) = 1$.

Suppose that for each t *in some dense subset* $D \subset \mathbb{R}^+$, *as* $n \to \infty$

$$f_n(t) = g(P(Z_{[nt]})) \to f(t), \quad \widetilde{f}_n(t) = \widetilde{g}(P(\widetilde{Z}_{[nt]} \leq v_n)) \to f(t), \tag{6.13}$$

where $f : \mathbb{R}^+ \to [0,1]$ *is non-increasing and continuous and* $f(0) = 1$, $\lim_{t\to\infty} f(t) = 0$. *Then*

$$\sup_{x\in\mathbb{R}^1} |g(P(Z_n \leq x)) - \widetilde{g}(P(\widetilde{Z}_n \leq x))| \to 0, \quad \text{as } n \to +\infty. \tag{6.14}$$

Proof. By properties of f, convergence (6.13) is *uniform* in $t \in \mathbb{R}^+$. Hence

$$g(P(Z_n \leq v_{m_n})) = f(n/m_n) + o(1), \quad \widetilde{g}(P(\widetilde{Z}_n \leq v_{m_n})) = f(n/m_n) + o(1), \tag{6.15}$$

provided $m_n \to \infty$. If $m_{n'} \leq M$ along a subsequence $\{n'\} \subset \mathbb{N}$, take another subsequence $k_{n'} \geq M$ such that $k_{n'} \to \infty$ so slowly that $n'/k_{n'} \to \infty$. Since $\{v_n\}$ is monotone,

$$g(P(Z_{n'} \leq v_{m_{n'}})) \leq g(P(Z_{n'} \leq v_{k_{n'}})) = f(n'/k_{n'}) + o(1) \quad \longrightarrow \quad 0, \tag{6.16}$$

and similarly $\tilde{g}(P(\tilde{Z}_{n'} \leq v_{m_{n'}})) \to 0$. Hence (6.15) holds for *every* sequence $\{m_n\}$.

Let $\{u_n\}$ be any sequence of numbers. Define integers $m_n = m_n(\{u_n\})$:

$$m_n = \begin{cases} 1, & \text{if } \ u_n < v_1 \\ m, & \text{if } \ v_m \leq u_n < v_{m+1} \\ n^2, & \text{if } \ u_n \geq \sup\{v_m : m \in \mathbb{N}\}. \end{cases} \tag{6.17}$$

If $m_{n'} = 1$ along $\{n'\}$, then as in (6.16)

$$g(P(Z_{n'} \leq u_{n'})) \to 0 = g(P(Z_{n'} \leq v_{m_{n'}})) + o(1).$$

If $m_{n'} = (n')^2$, then

$$1 \geq g(P(Z_{n'} \leq u_{n'})) \geq g(P(Z_{n'} \leq v_{m_{n'}})) = f(1/n') + o(1) \to 1.$$

In the remaining case, when $v_{m_{n'}} \leq v_{m_{n'}+1}$,

$$\begin{aligned} f(n'/m_{n'}) = g(P(Z_{n'} \leq v_{m_{n'}})) = o(1) \leq g(P(Z_{n'} \leq u_{n'})) + o(1) \\ \leq \ g(P(Z_{n'} \leq v_{m_{n'}+1})) + o(1) = f(n'/(m_{n'}+1)) + o(1) = f(n'/m_{n'}) + o(1). \end{aligned}$$

So

$$g(P(Z_n \leq u_n)) = g(P(Z_n \leq v_{m_n})) + o(1) = f(n/m_n) + o(1)$$

and, similarly,

$$\tilde{g}(P(\tilde{Z}_n \leq u_n)) = f(n/m_n) + o(1),$$

i.e. for *every* $\{u_n\}$

$$g(P(Z_n \leq u_n)) - \tilde{g}(P(\tilde{Z}_n \leq u_n)) \to 0. \tag{6.18}$$

This is exactly (6.14). □

Lemma 6.9 *Properties (iii) and (iv) are equivalent.*

Proof. Take $u \geq 1$ and $t > s > 0$. Write $h = \log u, h' = \log(t/s)$ and $x = \log s$. Let $g_\alpha = \log \circ \tilde{\alpha} \circ \exp$. Then property (iii) can be rewritten as

$$g_\alpha(x+h) - g_\alpha(x) \geq g_\alpha(x+h+h') - g_\alpha(x+h')$$

or

$$g_\alpha(x+h) + g_\alpha(x+h') \geq g_\alpha(x+h+h') + g_\alpha(x)$$

for every $x \in \mathbb{R}^1$ and $h, h' \geq 0$. The last inequality is nothing but concavity of function g_α. □

Lemma 6.10 *Suppose $\{v_n\}$ is* non-decreasing *and conditions (6.6) - (6.9) hold. If the limit function $\tilde{\alpha}_{(\cdot)}$ has property (iii), then $\{X_j\}$ admits an a.i.r. given by formula (6.10) .*

PROOF. Notice that $v_n = v_n^*$. Let $F_j(x)$ be given by (6.10). For each k, F_j is a distribution function. Indeed, if $\lim_{n\to\infty} v_n < +\infty$, then $\lim_{x\to\infty} F_j(x) = 1$ trivially. If $\lim_{n\to\infty} v_n = +\infty$, then $\lim_{x\to\infty} F_j(x) = \lim_{t\to 0}\widetilde{\alpha}_t / \lim_{t\to 0}\widetilde{\alpha}_t = 1$, in this case also. So we have to check only monotonicity, i.e.

$$\frac{\widetilde{\alpha}_{j/n}}{\widetilde{\alpha}_{j-1/n}} \leq \frac{\widetilde{\alpha}_{j/n+1}}{\widetilde{\alpha}_{j-1/n+1}}$$

or, equivalently

$$\frac{\widetilde{\alpha}_{j/n}}{\widetilde{\alpha}_{j/n+1}} \leq \frac{\widetilde{\alpha}_{j-1/n}}{\widetilde{\alpha}_{j-1/n+1}}.$$

Setting $u = n+1/n > 1, s = j-1/n+1, t = j/n+1$, we see that the last inequality is just property (ii) of function $\widetilde{\alpha}_{(\cdot)}$.

Let $\{\widetilde{X}_j\}$ be independent with $\widetilde{X}_j$ distributed according to F_j and let $\widetilde{M}_n = \max_{j\leq n}\widetilde{X}_j$. We have for each $t > 0$

$$P(\widetilde{M}_{[nt]} \leq v_n) = \prod_{j=1}^{[nt]} \frac{\widetilde{\alpha}_{j/n}}{\widetilde{\alpha}_{j-1/n}} = \widetilde{\alpha}_{[nt]/n}.$$

Since (iv) (equivalent to (iii) by Lemma 6.9) provides continuity of $\widetilde{\alpha}_{(\cdot)}$, we get (6.6) with $D = I\!R^+$ and $M_{[nt]}$ replaced by $\widetilde{M}_{[nt]}$.

Set $g(x) = \widetilde{g}(x) = x,\ f(t) = \widetilde{\alpha}_t$ and observe that all assumptions of Lemma 6.8 are satisfied. But (6.14) means that $\{\widetilde{X}_j\}$ constructed above is an a.i.r. for $\{X_j\}$. □

Lemma 6.11 *Suppose* $\{v_n\}$ *is* non-decreasing *and conditions (6.6) and (6.7) hold. If for every* $0 < s < t$

$$P(M_{[nt]} \leq v_n) - P(M_{[ns]} \leq v_n)P(M_{[ns]:[nt]} \leq v_n) \longrightarrow 0, \tag{6.19}$$

then the limit function $\widetilde{\alpha}_{(\cdot)}$ *has property (iii).*

PROOF. Note that (6.6) implies convergence of $P(M_{[n(\cdot)]} \leq v_n)$ to $\widetilde{\alpha}_{(\cdot)}$ in every point of continuity of the limit. We shall prove that

$$\frac{\widetilde{\alpha}_{us}}{\widetilde{\alpha}_s} \geq \frac{\widetilde{\alpha}_{ut}}{\widetilde{\alpha}_t}, \tag{6.20}$$

provided $\widetilde{\alpha}_{(\cdot)}$ is continuous at s, us and t and then we shall derive from (6.20) continuity of $\widetilde{\alpha}_{(\cdot)}$ in the entire half-line. This will give us property (iii).

So let $u > 1$ and let $t > s > 0$ and us be continuity points of $\widetilde{\alpha}_{(\cdot)}$. By (6.19), the fact that $\widetilde{\alpha}_t > 0, t \geq 0$ and right continuity of $\widetilde{\alpha}_{(\cdot)}$, it is enough to prove

$$\begin{aligned}\lim_{n\to\infty} P(M_{[ns]:[n(us)]} \leq v_n) &= \widetilde{\alpha}_{us}/\widetilde{\alpha}_s \geq \\ \geq \widetilde{\alpha}_{ut+\delta}/\widetilde{\alpha}_t &= \lim_{n\to\infty} P(M_{[nt]:[n(ut+\delta)]} \leq v_n),\end{aligned}$$

where δ is such that $ut+\delta$ is a point of continuity of $\widetilde{\alpha}_{(\cdot)}$. Let us observe, that for large n

$$\begin{aligned} P(M_{[ns]:[n(us)]} \le v_n) &= P(M_{[(ns/t)t]:[(ns/t)ut]} \le v_{[(ns/t)(t/s)]}) \\ &\ge P(M_{[(ns/t)t]:[(ns/t)ut]} \le v_{[ns/t]}) \\ &\ge P(M_{[[ns/t]t]:[[ns/t](ut+\delta)]} \le v_{[ns/t]}). \end{aligned}$$

The last expression approaches $\widetilde{\alpha}_{ut+\delta}/\widetilde{\alpha}_t$, while the first one — $\widetilde{\alpha}_{us}/\widetilde{\alpha}_s$, as desired.

We conclude the proof while showing continuity of $\widetilde{\alpha}_{(\cdot)}$ on $(0,\infty)$. If not, suppose $\widetilde{\alpha}_{(\cdot)}$ has a jump at t_0: for some $\eta > 0, 1-\eta > \widetilde{\alpha}_{t_0+}/\widetilde{\alpha}_{t_0-}$. Let $s < t_0 < (1+\varepsilon)s$ and let $s, (1+\varepsilon)s, (1+\varepsilon)^2 s, (1+\varepsilon)^3 s, \ldots$ be points of continuity of $\widetilde{\alpha}_{(\cdot)}$. Applying consecutively (6.20) we get

$$\frac{\widetilde{\alpha}_{(1+\varepsilon)s}}{\widetilde{\alpha}_s} \ge \frac{\widetilde{\alpha}_{(1+\varepsilon)^2 s}}{\widetilde{\alpha}_{(1+\varepsilon)s}} \ge \frac{\widetilde{\alpha}_{(1+\varepsilon)^3 s}}{\widetilde{\alpha}_{(1+\varepsilon)^2 s}} \ge \ldots,$$

and so on. In particular, for each $k = 1, 2, \ldots$

$$\widetilde{\alpha}_{(1+\varepsilon)^k s} \le \left(\frac{\widetilde{\alpha}_{(1+\varepsilon)s}}{\widetilde{\alpha}_s}\right)^k \widetilde{\alpha}_s.$$

Choosing s close enough to t_0 and ε sufficiently small, we get $\widetilde{\alpha}_{(1+\varepsilon)s}/\widetilde{\alpha}_s \le 1-\eta < 1$ hence $\widetilde{\alpha}_{t_0+} = \widetilde{\alpha}_{t_0} = 0$. This contradicts (6.7). □

Now we are ready to complete the Proof of Theorem 6.2. By Lemma 6.1, we can assume that $\{v_n\}$ is non-decreasing. Next, Lemma 6.10 gives us implication (iii)⇒(ii). Since (ii)⇒(i) is trivial and (iii)⇔(iv) is proved in Lemma 6.9, the only remaining implication is (i)⇒(iii).

Let $\{\widetilde{X}_j\}$ be a max-phantom sequence for $\{X_j\}$. By (6.1) , $\widetilde{M}_n$'s satisfy (6.6) and (6.7). And condition (6.19) is obviously satisfied by an independent sequence. Hence we can apply Lemma 6.11 in order to get property (iii) for $\widetilde{\alpha}_{(\cdot)}$.

The Proof of Corollary 6.5 is similar:

— Reduction to non-decreasing $\{v_n\}$. Take $\{v_n^*\}$ defined in by (6.3). Observe that condition (6.12) remains to be true with v_n replaced by $v_n^* = v_{k_n}$.

— Application of Lemma 6.11. Condition (6.19) is implied by (6.12).

— Construction of an a.i.r. by Lemma 6.10.

Corollaries 6.3 and 6.4 are obvious consequences of Theorem 6.2.

6.3 Phantom Distribution Functions for Markov Chains

Corollary 6.5 suggests verifying condition 6.12 as a method for checking property (iii) of function $\widetilde{\alpha}_t$. The procedure is standard: by (6.9) we can restrict our attention to maxima of length at most $[nT]$, say. Next, if X_j's satisfy for each $T > 0$

$$\max_{j \le nT} P(X_j \ge v_n) \longrightarrow 0, \quad \text{as } n \to \infty,$$

we can reduce the problem to proving

$$\begin{aligned} P(M_{k_n} \le v_n, M_{(k_n+r_n):l_n} \le v_n) \\ -P(M_{k_n} \le v_n) P(M_{(k_n+r_n):l_n} \le v_n) \longrightarrow 0, \end{aligned} \tag{6.21}$$

for all $k_n, l_n \to \infty, k_n + l_n \le [nT]$, where $r_n \to \infty$ is such that

$$r_n \max_{j \le [nT]} P(X_j > v_n) \longrightarrow 0.$$

The form of (6.21) is already "typical" for mixing conditions and similar to O'Brien's condition AIM(u_n) [OBr87].

It may happen, however, that direct checking property (iii) is possible without explicit invoking arguments of "mixing". For example, one can use a martingale approach in a similar way as for sums.

Recall, that $\{\mathcal{F}_k\}_{k \in I\!N \cup \{0\}}$ is a *filtration* if $\mathcal{F}_k$'s form a non-decreasing sequence of σ-algebras and that sequence $\{X_k\}$ is *adapted* to $\{\mathcal{F}_k\}$ if X_k is $\mathcal{F}_k$-measurable for each $k \in I\!N$.

We will follow idea of the "Principle of Conditioning" due to [Jak86], being a heuristic rule for derivation of limit theorems for dependent summands from results proved in independent case only. We are going to show that this idea works in limit theorems for maxima as well.

The heart of what follows is a lemma corresponding to Lemma 1.2 in [Jak86].

Lemma 6.12 *Let $\{X_j\}$ be adapted to $\{\mathcal{F}_j\}$ and suppose that*

$$\prod_{j=1}^{k_n} P(X_j \le v_n | \mathcal{F}_{j-1}) \underset{P}{\longrightarrow} \alpha > 0, \tag{6.22}$$

where α is a constant. Then also

$$P(M_{k_n} \le v_n) \longrightarrow \alpha.$$

□

Proof. One can get this lemma immediately from Lemma 2, p.66,[JaSl86]. □

Here is well-known example of how to check assumption (6.22) of the above lemma.

Corollary 6.13 *If*

$$\max_{1 \le j \le k_n} P(X_j > v_n | \mathcal{F}_{j-1}) \underset{P}{\longrightarrow} 0, \tag{6.23}$$

$$\sum_{j=1}^{k_n} P(X_j > v_n | \mathcal{F}_{j-1}) \underset{P}{\longrightarrow} \beta, \tag{6.24}$$

then

$$\prod_{j=1}^{k_n} P(X_j \le v_n | \mathcal{F}_{j-1}) \quad \underset{P}{\longrightarrow} \quad e^{-\beta} \tag{6.25}$$

and

$$P(M_{k_n} \le v_n) \longrightarrow e^{-\beta}.$$

PROOF. If $X_1, X_2, \ldots$ are *independent* and $\mathcal{F}_j = \sigma(X_1, X_2, \ldots, X_j)$ then (6.23) and (6.24) become

$$\max_{1 \le j \le k_n} P(X_j > v_n) \quad \longrightarrow \quad 0,$$
$$\sum_{j=1}^{k_n} P(X_j > v_n) \quad \longrightarrow \quad \beta.$$

Simple computations (see e.g. [Gal78, Chapter 3]) show that

$$P(M_n \le v_n) = \prod_{j=1}^{k_n} P(X_j \le v_n) \longrightarrow \exp(-\beta).$$

So our corollary is true in this particular case.

By usual arguments we can assume that both (6.23) and (6.24) hold pointwise for each ω in a set of full measure. But in such ω we can mimic the proof of the independent case in order to get $\prod_{j=1}^{k_n} P(X_j \le v_n | \mathcal{F}_{j-1})(\omega) \longrightarrow e^{-\beta}$. □

Now we are ready to state our criterion based on the martingale approach:

Theorem 6.14 *Suppose that $\{X_j\}$ is adapted to $\{\mathcal{F}_j\}$ and the following two conditions hold for each $t > 0$:*

$$\max_{1 \le j \le [nt]} P(X_j > v_n | \mathcal{F}_{j-1}) \quad \underset{P}{\longrightarrow} \quad 0, \tag{6.26}$$
$$\sum_{j=1}^{[nt]} P(X_j > v_n | \mathcal{F}_{j-1}) \quad \underset{P}{\longrightarrow} \quad \beta_t, \tag{6.27}$$

where $\{\beta_t\}_{t \ge 0}$ are finite constants and

$$\lim_{t \to 0+} \beta_t = 0, \qquad \lim_{t \to +\infty} \beta_t = +\infty. \tag{6.28}$$

Then $\{X_j\}$ has an asymptotic independent representation.

PROOF. By Corollary 6.13 we know that (6.6)–(6.9) hold with

$$\alpha_t = e^{-\beta_t}.$$

Moreover, by the same corollary

$$\prod_{j=1}^{[nt]} P(X_j \leq v_n | \mathcal{F}_{j-1}) \underset{\mathcal{P}}{\longrightarrow} \alpha_t \tag{6.29}$$

for each $t \geq 0$. For every $j \in \mathbb{N}$ choose a version of the regular conditional distribution of X_j with respect to $\mathcal{F}_{j-1}$ and denote it by $\mu_j(A, \omega)$. Fix $\omega \in \Omega$ and let $X_1^{(\omega)}, X_2^{(\omega)}, X_3^{(\omega)}, \ldots$ be independent and distributed according to $\mu_1(\cdot, \omega), \mu_2(\cdot, \omega), \mu_3(\cdot, \omega), \ldots$, respectively. Then

$$\begin{aligned} P(X_j \leq v_n | \mathcal{F}_{j-1})(\omega) &= P(X_j^{(\omega)} \leq v_n) \quad \text{a.s. and} \\ \prod_{j=1}^{[nt]} P(X_j \leq v_n | \mathcal{F}_{j-1})(\omega) &= P(M_{[nt]}^{(\omega)} \leq v_n) \quad \text{a.s.} \end{aligned}$$

By (6.27) and (6.29) there exists a subsequence $\{n'\} \subset \mathbb{N}$ such that

$$\max_{1 \leq j \leq [n't]} P(X_j^{(\omega)} > v_{n'}) \longrightarrow 0 \quad \text{and} \quad P(M_{[n't]}^{(\omega)} \leq v_{n'}) \longrightarrow \alpha_t, \ t \geq 0, t\text{–rational},$$

for every ω in a set Ω' of probability 1. Fix $\omega \in \Omega'$. Both Lemma 6.10 and Lemma 6.11 remain valid if we consider convergence along a *subsequence* $\{n'\}$ instead of a full $\mathbb{N}$. Hence we can assume that v_n is non-decreasing and condition (6.19) holds (since $P(M_{[nt]}^{(\omega)} \leq v_n) = P(M_{[ns]}^{(\omega)} \leq v_n) P(M_{[ns]:[nt]}^{(\omega)} \leq v_n)$). So $\tilde{\alpha}_s$ has property (iii) of Theorem 6.2 and by this theorem $\{X_j\}$ admits an a.i.r.. □

Compiling Theorem 6.14 and Corollary 6.4 seems to be most fruitful for Markov chains.

Theorem 6.15 *Suppose $\{Z_j\}$ is a homogeneous Markov chain with state space $(\mathcal{S}, \mathcal{B}_\mathcal{S})$, transition probabilities $P(x, A)$ and a unique stationary initial distribution ν. Let $f : (\mathcal{S}, \mathcal{B}_\mathcal{S}) \to (\mathbb{R}^1, \mathcal{B}^1)$ be a measurable function such that for some sequence $\{v_n\}$ we have*

$$nP(\,\cdot\,, f > v_n) \longrightarrow U(\cdot) \quad \text{in } L^1(\mathcal{S}, \mathcal{B}_\mathcal{S}, \nu) \tag{6.30}$$

If $EU \neq 0$, then $\{X_j = f \circ Z_j\}_{j \in \mathbb{N}}$ has a phantom distribution function.

Proof. By Corollary 6.4, it is enough to check assumptions of Theorem (6.2) with $\beta_t = t \cdot (EU)$. Set $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_j = \sigma(Z_1, Z_2, \ldots, Z_j)$. Then for each $t > 0$

$$\begin{aligned} E \max_{1 \leq j \leq [nt]} & \left(P(X_j > v_n | \mathcal{F}_{j-1})\right)^2 \\ &\leq (\nu(f > v_n))^2 + \sum_{j=2}^{n} E\left(P(Z_{j-1}, f > v_n)\right)^2 \\ &\leq nE\left(P(Z_1, f > v_n)\right)^2 \longrightarrow 0, \end{aligned}$$

since $n\,(P(Z_1, f > v_n))^2 \longrightarrow 0$ in probability and is dominated by the uniformly integrable sequence $\{nP(Z_1, f > v_n)\}$. Checking (6.27) is a little bit more complicated. First, we may neglect the term $P(X_1 > v_n|\mathcal{F}_0) = \nu(f > v_n)$. Then, by (6.30),

$$\begin{aligned} E\left|\sum_{j=2}^{[nt]} P(X_j > v_n|\mathcal{F}_{j-1}) - (1/n)\sum_{j=1}^{[nt]-1} U(Z_j)\right| & \\ \leq \frac{[nt]-1}{n} E|nP(Z_1, f > v_n) - U(Z_1)| \longrightarrow 0. \end{aligned}$$

But the ergodic theorem gives

$$(1/n)\sum_{j=1}^{[nt]-1} U(Z_j) \underset{L^1}{\longrightarrow} t \cdot E(U(Z_1))$$

and our theorem follows. □

Using assumptions stronger then (6.30) we are able to work independently of whether a stationary initial distribution for $\{Z_j\}$ exists or not.

Corollary 6.16 *Suppose $\{Z_j\}$ is a homogeneous Markov chain on $(\mathcal{S}, \mathcal{B}_\mathcal{S})$, with transition probabilities $P(x, A)$ and initial distribution ν. If f is such that*

$$\gamma_n = \sup_{x\in\mathcal{S}} |\beta - nP(x, f > v_n)| \longrightarrow 0, \tag{6.31}$$

for some $\beta > 0$ and

$$\nu(f > v_n) \longrightarrow 0, \tag{6.32}$$

then $\{X_j = f \circ Z_j\}_{j\in\mathbb{N}}$ has a phantom distribution function.

Proof. Following the notations from the proof of Theorem 6.15 we have

$$\begin{aligned} \max_{1\leq j\leq [nt]} P(X_j > v_n|\mathcal{F}_{j-1}) & \\ \leq \ & \nu(f > v_n) + \max_{2\leq j\leq n} P(Z_{j-1}, f > v_n) \\ \leq \ & \nu(f > v_n) + \beta/n + \gamma_n/n \longrightarrow 0. \end{aligned}$$

Similarly

$$\begin{aligned} \left|\sum_{j=1}^{[nt]} P(X_j > v_n|\mathcal{F}_{j-1}) - \frac{([nt]-1)}{n}\beta\right| & \\ = \ & |\nu(f > v_n) + n^{-1}\sum_{j=2}^{[nt]}(nP(Z_{j-1}, f > v_n) - \beta)| \\ \leq \ & \nu(f > v_n) + n^{-1} \cdot n \cdot \gamma_n \longrightarrow 0. \end{aligned}$$

Hence $\beta_t = \lim_{n\to\infty}([nt]-1)\beta/n = t\beta$. □

6.4 Regular Phantom Distribution Functions for Stationary Sequences

Recall, that any distribution function G satisfying

$$P(M_n \leq u_n) - G(u_n)^n \longrightarrow 0 \tag{6.33}$$

for all sequences $\{u_n\}$, is a phantom distribution function for $\{X_j\}$. Notice that G is not uniquely determined.

From the point of view of limit theorems we are interested in sequences $\{X_j\}$ for which $P(M_n \leq v_n)$ converges to non-trivial limit at least for a single $\{v_n\} \subset I\!R^1$; this means that also

$$G(v_n)^n \longrightarrow \alpha, \tag{6.34}$$

where α, $0 < \alpha < 1$. By a well-known observation due to O'Brien [OBr74a] (see also [LLR83, p.24]), such sequence v_n exists for some (and then for any) $0 < \alpha < 1$ iff

$$G(G_*-) = 1 \quad \text{and} \quad \lim_{x \to G_*-} \frac{1 - G(x)}{1 - G(x-)} = 1, \tag{6.35}$$

where $G_* = \sup\{x : G(x) < 1\}$. Say that G is **regular** (in the sense of O'Brien) if (6.35) is fulfilled.

We are able to give a complete description of sequences possessing regular phantom distribution functions.

Theorem 6.17 *A stationary sequence $\{X_j\}$ has a regular phantom distribution function if and only if there is a sequence $\{v_n\}$ such that for some α, $0 < \alpha < 1$,*

$$P(M_n \leq v_n) \longrightarrow \alpha, \tag{6.36}$$

and the following Condition $B_\infty(v_n)$ holds:

$$\sup_{p,q \in I\!N} |P(M_{p+q} \leq v_n) - P(M_p \leq v_n)P(M_q \leq v_n)| \longrightarrow 0, \quad \text{as } n \to +\infty. \tag{6.37}$$

Moreover, given a sequence $\{v_n\}$ satisfying both (6.36) and $B_\infty(v_n)$, a regular phantom distribution function G can be constructed explicitly: If $F(x) = P(X_1 \leq x)$ and

$$v_n^* = \begin{cases} \inf\{v_l : l \in I\!N\} & \text{if} \quad v_k \geq F_* \text{ for } 1 \leq k \leq n \\ \max\{v_k : v_k < F_*, 1 \leq k \leq n\} & \text{otherwise.} \end{cases} \tag{6.38}$$

we can set

$$G(x) = \begin{cases} 0 & \text{if} \quad x < v_1^*, \\ \alpha^{1/n} & \text{if} \quad v_n^* \leq x < v_{n+1}^*, \\ 1 & \text{if} \quad x \geq \sup_k v_k^* \end{cases} \tag{6.39}$$

PROOF. Given Corollary 6.4, sufficiency is an easy task. Indeed, we have to prove only that for every pair of integers $p, q > 0$

$$P(M_{[(p/q)n]} \leq v_n) \longrightarrow \alpha^{(p/q)}, \quad \text{as } n \to +\infty. \tag{6.40}$$

Observe, that $P(X_1 \leq v_n) \to 1$ (if not, we would have $P(M_{n'} \leq v_{n'}) \to 0$ by Condition $\mathrm{B}_\infty(v_n)$, at least along a subsequence $\{n'\} \subset I\!N$). Hence also $P(M_{k_n} \leq v_n) \to 1$ for *bounded* k_n. In particular, $P(M_{[(p/q)n]} \leq v_n) = P(M_{p\cdot[n/q]} \leq v_n) + o(1)$ and Condition $\mathrm{B}_\infty(v_n)$ implies that

$$P(M_{[(p/q)n]} \leq v_n) = (P(M_{[n/q]} \leq v_n))^p + o(1).$$

This proves (6.40).

So let us prove necessity of (6.36) and (6.37). Suppose that $\{X_j\}$ has a regular phantom distribution function G. Fix $\alpha \in (0,1)$. (6.36) holds by *regularity* of G. In order to check $\mathrm{B}_\infty(v_n)$ for $\{X_j\}$, it is enough to prove that

$$P(M_{p_n+g_n} \leq v_n) - P(M_{p_n} \leq v_n)P(M_{q_n} \leq v_n) \longrightarrow 0, \tag{6.41}$$

for every pair p_n, q_n of sequences of positive integers. Observe, that $G(v_n) \to 1$, hence one can find a sequence $\{k_n\}$ tending to infinity so slowly that still $G^{k_n}(v_n) \to 1$. If $p_{n'} \leq k_{n'}$ along a subsequence $\{n'\} \subset I\!N$, then

$$P(M_{p_{n'}} \leq v_{n'}) \geq P(M_{k_{n'}} \leq v_{n'}) = G^{k_{n'}}(v_{n'}) + o(1) \longrightarrow 1,$$

hence also

$$P(M_{p_{n'}+q_{n'}} \leq v_{n'}) - P(M_{q_{n'}} \leq v_{n'}) \longrightarrow 0,$$

and (6.41) holds along $\{n'\}$.

So without loss of generality we can assume that $p_n > k_n$ and $q_n > k_n$ for every $n \in I\!N$. In particular, both p_n and q_n tend to infinity. By the definition of a phantom distribution function,

$$\begin{aligned} P(M_{p_n+q_n} \leq v_n) &= G^{p_n+q_n}(v_n) + o(1) \\ &= G^{p_n}(v_n)G^{q_n}(v_n) + o(1) \\ &= (P(M_{p_n} \leq v_n) + o(1))\,(P(M_{q_n} \leq v_n) + o(1)) + o(1) \\ &= P(M_{p_n} \leq v_n)P(M_{q_n} \leq v_n) + o(1), \end{aligned}$$

i.e. (6.41) holds, either. □

In fact, we have proved

Corollary 6.18 *If $\{X_j\}$ has a phantom distribution function and $\{v_n\}$ is such that $P(M_n \leq v_n) \longrightarrow \alpha$, for some $0 < \alpha < 1$, then $B_\infty(v_n)$ holds for $\{X_j\}$.* □

Let us define

$$\widetilde{v}_n = \inf\{x \,:\, P(M_n \leq x) \geq e^{-1}\}. \tag{6.42}$$

Under different mixing assumptions, O'Brien [OBr87, Theorem 4.1] proved that G defined by (6.39) with $\alpha = e^{-1}$ and $v_n^* = \widetilde{v}_n$ is a regular phantom distribution function for $\{X_j\}$.

Our theorem improves O'Brien's result in two aspects:

- we turn the attention to necessity
- we deal with arbitrary sequence $\{v_n\}$ satisfying (6.36) and (6.37)

In two subsequent chapters we will study equivalent forms of Condition $\mathrm{B}_\infty(v_n)$ (Chapter 7) and possible ways of effective checking the convergence $P(M_n \leq v_n) \to \alpha$ (Chapter 8). The rest of the present chapter will be devoted to a theoretical application of the criterion obtained above.

Remark 6.19 Let G be a regular distribution function (e.g. continuous) and $\{v_n\}$ be such that $G^n(v_n) \to \alpha$, $0 < \alpha < 1$. If $\{X_j\}_{j\in\mathbb{N}}$ are i.i.d. with distribution function G, then for each $t \in \mathbb{R}^+$

$$P(M_{[nt]} \leq v_n) = (G^n(v_n))^{[nt]/n} \;\longrightarrow\; \alpha^t, \quad \text{as } n \to +\infty,$$

i.e. (6.6) holds.

If G does not belong to the domain of attraction of a max-stable distribution (see [LLR83, Theorem 1.4.1, p. 16]) then no linear normalization $a_n x + b_n$ exists such that

$$G^n(a_n x + b_n) \;\longrightarrow\; H(x), \quad \text{as } n \to +\infty, \quad x \in \mathbb{R}^1,$$

where $H(x)$ is non-degenerate.

It follows that the convergence (6.6) is much weaker, than the classical convergence in distribution of linearly normalized maxima.

6.5 Relative Extremal Index of Two Stationary Sequences

Let us begin with an example, essentially due to Rootzén [Roo88]

Example 6.20 Let $\{X_j\}_{j\in\mathbb{N}}$ be max-regenerative, i.e. there exist integer-valued random variables $0 < S_0 < S_1 < \ldots$ such that

- $Y_0 = S_0, Y_1 = S_1 - S_0, Y_2 = S_2 - S_1, \ldots$ are independent with $Y_1, Y_2, \ldots$ – identically distributed.
- $X_0' = \max_{0<j\leq S_0} X_j, X_1' = \max_{S_0<j\leq S_1} X_j, \ldots$ are independent, with $X_1', X_2', \ldots$ – identically distributed.

There are naturally arising examples of such sequences, including instantaneous functions of certain Harris recurrent Markov chains, or, more generally, of regenerative sequences — see Asmussen (1987), Chapter VI, for details.

Suppose we may neglect the influence of the null cycle:

$$P(X_0' > M_n') \longrightarrow 0, \quad \text{as } n \to +\infty, \tag{6.43}$$

and that the regeneration occurs after a finite average time:

$$\mu = EY_1 < +\infty. \tag{6.44}$$

Then, by the law of large numbers

$$\frac{Y_1 + Y_2 + \ldots + Y_n}{n} \longrightarrow \mu \quad \text{a.s.},$$

hence, heuristically, $M_{[n\mu]}$ can be replaced by M_n'. In fact, Theorem 3.1, [Roo88] shows that

$$\sup_{x \in \mathbb{R}^1} |P(M_n \leq x) - P(M_n' \leq x)^{1/\mu}| \longrightarrow 0.$$

Generalizing the above example, we will say that a stationary sequence $\{X_j\}$ has the relative extremal index θ with respect to another stationary sequence $\{X_j'\}$, if

$$\sup_{x \in \mathbb{R}^1} |P(M_n \leq x) - P(M_n' \leq x)^{\theta}| \ \longrightarrow\ 0 \quad \text{as } n \to +\infty, \tag{6.45}$$

where M_n and M_n' are partial maxima for $\{X_j\}$ and $\{X_j'\}$, respectively. Write

$$\{X_n\} \sim^{\theta} \{X_n'\}.$$

In order to explain the meaning of this relation, we may repeat the remarks on asymptotic independent representations: if $\{X_n\} \sim^{\theta} \{X_n'\}$, then the asymptotic properties of laws of M_n are completely determined by those for M_n'. Further, if $\{X_n'\}$ is an i.i.d. or exchangeable sequence, then $\{X_n\} \sim^{\theta} \{X_n'\}$ provides information about necessary and sufficient conditions for the convergence in law of suitably normalized and centered M_n's and about possible limit laws.

In general, formula (6.45) does not determine uniquely the value of θ. The relative extremal index of $\{X_n\}$ with respect to $\{X_n'\}$ is well-defined by (6.45) iff one can find a subsequence $\{n'\} \subset \mathbb{N}$ and real numbers $\{v_{n'}\}$ such that

$$P(M_{n'}' \leq v_{n'}) \longrightarrow \alpha, \tag{6.46}$$

for some α, $0 < \alpha < 1$. Moreover, for any such a sequence $\{v_{n'}\}$, (6.45) implies

$$\theta = \lim_{n' \to \infty} \frac{\log P(M_{n'} \leq v_{n'})}{\log P(M_{n'}' \leq v_{n'})}. \tag{6.47}$$

The relative extremal index generalizes the notion of the extremal index due to Leadbetter.

Perfecting the ideas of Loynes [Loy65] and O'Brien [OBr74a], Leadbetter [Lea83] defined the extremal index of a stationary sequence $\{X_n\}$ as a number θ, $0 \leq \theta \leq 1$, such that for all $\tau > 0$,

$$P(M_n \leq u_n(\tau) \longrightarrow e^{-\theta\tau} \tag{6.48}$$

whenever

$$nP(X_1 > u_n(\tau)) \longrightarrow \tau. \tag{6.49}$$

Let $\{\widehat{X}_j \,:\, n \in I\!\!N\}$ be the independent imitation of $\{X_j\}$, i.e. $\widehat{X}_j$'s are i.i.d. with the same marginal distributions as X_j: $\mathcal{L}(\widehat{X}_j) = \mathcal{L}(X_j)$. Then (6.49) means $P(\widehat{M}_n \leq u_n(\tau)) \to e^{-\tau}$ and (6.48) and (6.49) imply

$$P(M_n \leq u_n) - P(\widehat{M}_n \leq u_n)^{\theta} \longrightarrow 0 \tag{6.50}$$

at least for sequences $u_n = u_n(\tau)$ defined by (6.49).

In fact, Leadbetter [Lea83] proved (6.50) for *all* sequences $\{u_n\}$, provided $\theta > 0$. It follows that (6.45) is satisfied and the extremal index $\theta > 0$ is our relative extremal index of $\{X_j\}$ with respect to its independent imitation $\{\widehat{X}_j\}$.

Now suppose that $\{X_j\}$ admits a regular phantom distribution function G. If $\{X'_j\}$ is an i.i.d. sequence with marginals G, then (6.45) holds with $\theta = 1$. But more important is for our purposes, that we may redefine G according to formula (6.39).

Theorem 6.21 *Assume there is a sequence $\{v_n\}$ such that $P(M'_n \leq v_n) \longrightarrow \alpha'$, where $0 < \alpha' < 1$, and Condition $B_\infty(v_n)$ is satisfied for $\{X'_j\}$.*

Then there exists θ, $0 < \theta < \infty$, such that $\{X_j\}$ has the relative extremal index θ with respect to $\{X'_j\}$ if, and only if, $\{X_j\}$ satisfies $B_\infty(v_n)$ and for some α, $0 < \alpha < 1$, one has $P(M_n \leq v_n) \longrightarrow \alpha$.

In such a case

$$\theta = \log\alpha / \log\alpha'. \tag{6.51}$$

Proof. By Theorem 6.17, $\{X'_j\}$ has a regular phantom distribution function, say G'. If $\{X_j\} \sim^{\theta} \{X'_j\}$, then by definition (6.45), $G = (G')^{\theta}$ is a regular phantom distribution function for $\{X_j\}$. By (6.45), $P(M_n \leq v_n) \longrightarrow \alpha = (\alpha')^{\theta}$. And Condition $B_\infty(v_n)$ holds for $\{X_j\}$ by Corollary 6.18.

To prove the converse part, assume that $P(M_n \leq v_n) \longrightarrow \alpha$ for some $\alpha \in (0,1)$ and that $B_\infty(v_n)$ holds for $\{X_j\}$. By Theorem 6.17 both $\{X_j\}$ and $\{X'_j\}$ admit phantom distribution functions G and G', respectively, given by formula (6.39), with α replaced by α' in the latter case. If θ is defined by (6.51), then $G = (G')^{\theta}$, and (6.45) follows by the definition of a phantom distribution function. □

Theorem 1.5 contains existing in the area results. We refer to [Lea83], [LeRo88] and [Roo88] for standard examples of calculation of the extremal index.

It should be pointed out, that there are classes of stationary sequences with no phantom distribution functions for which the relative extremal index can be calculated, as well.

Example 6.22 Let Y be a positive (with probability one) and non-degenerate random variable. Let $Z = Y_1 + Y_2$, where Y_1 and Y_2 are independent copies of Y. Fix $\gamma > 0$ and consider two random probability distribution functions $F(\omega, x)$ and $G(\omega, x)$ such that as $x \to \infty$

$$\begin{aligned} 1 - G(\omega, x) &\sim Y(\omega) x^{-\gamma} \quad \text{a.s.} \\ 1 - F(\omega, x) &\sim Z(\omega) x^{-\gamma} \quad \text{a.s.} \end{aligned}$$

If $\{X_n\}$ and $\{X'_n\}$ are exchangeable sequences given by the kernels $\omega \mapsto \bigotimes^\infty F(\omega, \cdot)$ and $\omega \mapsto \bigotimes^\infty G(\omega, \cdot)$, respectively, then $\{X_n\}$ has the relative extremal index $\theta = 2$ with respect to $\{X'_n\}$. To see this, set $v_n = n^{1/\gamma}$ and observe that for $t > 0$

$$P(M_{[nt]} \leq v_n) = E(F(\omega, v_n)^{[nt]}) \longrightarrow E\, e^{-t \cdot Z}$$

and that

$$P(M'_{[nt]} \leq v_n)^2 \longrightarrow \left(E\, e^{-t \cdot Y} \right)^2 = E\, e^{-t \cdot Z}.$$

Hence the assumptions of Lemma (6.8) are satisfied and

$$\sup_{x \in \mathbb{R}^1} |P(M_n \leq x) - P(M'_n \leq x)^2| \longrightarrow 0 \quad \text{as } n \to +\infty.$$

Observe that both $\{X_n\}$ and $\{X'_n\}$ do not satisfy Condition $\mathrm{B}_\infty(v_n)$, for Y is non-degenerate, so they cannot have a phantom distribution function.

Chapter 7

Equivalent Forms of Mixing Conditions

7.1 Exponential Forms

In this section we establish some equivalent forms of mixing properties of the sequence $\{M_n\}$ with respect to a given sequence $\{v_n\}$ of numbers. For $T > 0$ let us introduce **Condition** $\mathrm{B}_T(v_n)$

$$\max_{j+k \leq [T\cdot n]} |P(M_{j+k} \leq v_n) - P(M_j \leq v_n)P(M_k \leq v_n)| \longrightarrow 0, \quad \text{as } n \to +\infty. \tag{7.1}$$

Setting formally $T = \infty$ we get Condition $\mathrm{B}_\infty(v_n)$ defined in the previous chapter.

Recently O'Brien [OBr87] has considered stationary sequences having "asymptotic independence of maxima" with respect to a sequence $\{v_n\}$ ($\mathbf{AIM}(v_n)$) :

$$\max \left| P(\max_{i\leq j} X_i \leq v_n, \max_{j+q_n < i \leq j+k+q_n} X_i \leq v_n) \quad - \quad P(M_j \leq v_n)P(M_k \leq v_n) \right| \tag{7.2}$$
$$\longrightarrow 0, \quad \text{as } n \to +\infty,$$

where the maximum is taken over all j and k with the properties $j \geq q_n, k \geq q_n$ and $j + k + q_n \leq n$ and $\{q_n\}$ is a sequence of non-negative integers, $q_n = o(n)$.

Notice that $\mathrm{B}_1(v_n)$ is a little bit stronger than $\mathrm{AIM}(v_n)$. In most applications, however, q_n is such that $P(M_{q_n} \leq v_n) \longrightarrow 1$. Under this condition $\mathrm{AIM}(v_n)$ and $\mathrm{B}_1(v_n)$ are equivalent. Bearing in mind that $\mathrm{AIM}(v_n)$ is the verifiable form of our mixing assumption, we prefer $\mathrm{B}_1(v_n)$, for it effects in breaking probabilities into products without inconvenient separation of blocks.

An example on p.287, [OBr87], shows that $\mathrm{AIM}(v_n)$ (hence practically: $\mathrm{B}_1(v_n)$) is weaker than commonly used in Extreme Value Limit Theory Leadbetter's [Lea74] **Condition** $\mathbf{D}(v_n)$: there are constants $\{\alpha_{n,l}\}$ with $\alpha_{n,[n\lambda]} \to 0$ as $n \to \infty$, for all $\lambda > 0$, such that

$$|P(AB) - P(A)P(B)| \leq \alpha_{n,l},$$

for all sets A of the form $\{X_{i_1} \leq v_n, \ldots, X_{i_p} \leq v_n\}$ and sets B of the form $\{X_{j_1} \leq v_n, \ldots, X_{j_{p'}} \leq v_n\}$ with $1 \leq i_1 < \ldots < i_p < j_1 < \ldots < j_{p'} \leq n$ and $j_1 - i_p \geq l$. So for checking $B_1(v_n)$ one can use all tools developed in [Gal78], [Lea83] and [OBr87].

For sequences $\{X_j\}$ with the AIM(v_n) property (and satisfying some additional conditions, e.g. $\sup nP(X_1 > v_n) < +\infty$), O'Brien found an asymptotic representation of $P(M_n \leq v_n)$ in the exponential form:

$$P(M_n \leq v_n) - \exp(-nP(X_0 > v_n, M_{r_n} \leq v_n)) \longrightarrow 0, \quad \text{as } n \to +\infty, \tag{7.3}$$

where $\{r_n\}$ is a suitably chosen sequence of non-negative integers (if $r_n = 0$, we set $M_0 = -\infty$).

For our purposes a stronger result is necessary.

Proposition 7.1 *There exists a sequence $\{r_n\}$ of non-negative integers such that $r_n \leq T \cdot n$ and*

$$\max_{1 \leq k \leq [T \cdot n]} |P(M_k \leq v_n) - \exp\left(-kP(X_0 > v_n, M_{r_n} \leq v_n)\right)| \longrightarrow 0, \quad \text{as } n \to +\infty, \tag{7.4}$$

if and only if $P(X_0 > v_n) \longrightarrow 0$ and $B_T(v_n)$ holds.

Moreover, if $\{r_n\}$ satisfies (7.4) then necessarily

$$P(M_{r_n} \leq v_n) \longrightarrow 1. \tag{7.5}$$

PROOF. NECESSITY Condition (7.4) implies $B_T(v_n)$, easily. Substituting in (7.4) $k = 1$ we see that

$$1 - P(X_0 > v_n) \geq \exp(-P(X_0 > v_n)) + o(1).$$

This implies $P(X_0 > v_n) \longrightarrow 0$. Further, observe that for each $k \leq r_n$

$$\begin{aligned} P(M_k > v_n) &\geq P\left(\bigcup_{1 \leq j \leq k} \{X_j > v_n, \max_{j < i \leq j + r_n} X_i \leq v_n\}\right) \\ &= kP(X_0 > v_n, M_{r_n} \leq v_n) \end{aligned} \tag{7.6}$$

Hence, letting in (7.4) $k = r_n$ we get

$$\begin{aligned} 1 - P(M_{r_n} > v_n) &= \exp(-r_n P(X_0 > v_n, M_{r_n} \leq v_n)) + o(1) \\ &\geq \exp(-P(M_{r_n} > v_n)) + o(1), \end{aligned}$$

and, again, $P(M_{r_n} > v_n) \longrightarrow 0$.

SUFFICIENCY: Assume $P(X_0 > v_n) \longrightarrow 0$ and $B_T(v_n)$. First we shall prove that whenever $P(M_{r_n} \leq v_n) \longrightarrow 1$ then for every sequence $k_n \leq T \cdot n$

$$P(M_{k_n} \leq v_n) \geq \exp(-k_n \cdot P(X_0 > v_n, M_{r_n} \leq v_n)) + o(1) \tag{7.7}$$

Set $M_{k:l} = \max_{k<j\le l} X_j$ for $k < l$ and $M_{k:l} = -\infty$ for $k \ge l$. Let us introduce events $A_{n,i} = \{X_i \le v_n\}$ and

$$\begin{aligned} A'_{n,i} &= \{X_i \le v_n\} \cup \{X_{i+1} > v_n\} \cup \ldots \cup \{X_{i+r_n} > v_n\} \\ &= A_{n,i} \cup A^c_{n,i+1} \cup \ldots \cup A^c_{n,i+r_n} \end{aligned}$$

Observe that for every $0 \le j < k$

$$\bigcap_{i=j+1}^{k} A'_{n,i} \setminus \bigcap_{i=j+1}^{k} A_{n,i} \subset \bigcup_{i=k+1}^{k+r_n} A^c_{n,i} = \{\max_{k<i\le k+r_n} X_i > v_n\} = \{M_{k:k+r_n} > v_n\}.$$

It follows that

$$\max_{k<l} \left| P\left(\bigcap_{i=k+1}^{l} A'_{n,i}\right) - P\left(\bigcap_{i=k+1}^{l} A_{n,i}\right)\right| \le \max_{l} P(M_{l:l+r_n} > v_n) = P(M_{r_n} > v_n) \to 0.$$

This in turn implies that for $0 \le j_n \le k_n \le l_n \le [nT]$

$$\begin{aligned} P\left(\bigcap_{i=j_n+1}^{l_n} A'_{n,i}\right) &= P\left(\bigcap_{i=j_n+1}^{l_n} A_{n,i}\right) + o(1) \\ &= P(M_{l_n-j_n} \le v_n) + o(1) \\ &= P(M_{l_n-k_n} \le v_n) \cdot P(M_{k_n-j_n} \le v_n) + o(1) \\ &= P\left(\bigcap_{i=j_n+1}^{k_n} A'_{n,i}\right) P\left(\bigcap_{i=k_n+1}^{l_n} A'_{n,i}\right) + o(1), \end{aligned}$$

i.e. $\mathrm{B}_T(v_n)$ being valid for $\{A_{n,i}\}$ is transformed into

$$\max_{0\le j\le k\le l\le T\cdot n} \left| P(\bigcap_{i=j+1}^{l} A'_{n,i}) - P(\bigcap_{i=j+1}^{k} A'_{n,i}) P(\bigcap_{i=k+1}^{l} A'_{n,i})\right| \longrightarrow 0, \quad \text{as } n \to +\infty. \tag{7.8}$$

Similarly

$$P(M_{k_n} \le v_n) = P(\bigcap_{i=1}^{k_n} A_{n,i}) = P(\bigcap_{i=1}^{k_n} A'_{n,i}) + o(1)$$

Since also $P((A'_{n,i})^c) = P(X_i > v_n, M_{i:i+r_n} \le v_n) = P(X_0 > v_n, M_{r_n} \le v_n)$, we can rewrite (7.7) as

$$P(\bigcap_{i=1}^{k_n} A'_{n,i}) \ge \exp\left(-\sum_{i=1}^{k_n} P((A'_{n,i})^c)\right) + o(1). \tag{7.9}$$

Noticing that $P(A'_{n,i}) \ge P(A_{n,i}) = P(X_0 \le v_n) \to 1$, we get the above inequality from the following lemma.

Lemma 7.2 *Let $\{A'_{n,i} : 1 \le i \le k_n, n \in \mathbb{N}\}$ be an array of events satisfying 7.8 and such that $\min_{1\le i\le k_n} P(A'_{n,i}) \longrightarrow 1$. Then (7.9) holds.*

PROOF. Our assumptions constitute a part of what is presumed in Lemma 3.2 of [JaKo89]. An inspection of the proof of this lemma (inequality (3.21) on p. 226) shows that it is just enough for (7.9) to hold. For completeness, we restate here this computation replacing $A'_{n,i}$ by $A_{n,i}$ without the "prime" sign.

It is sufficient to show that the convergence

$$\sum_{i=1}^{k_{n'}} P((A^c_{n',i}) \longrightarrow C,$$

along a subsequence $\{n'\} \subset I\!N$, where $C \in [0, +\infty)$, implies

$$\liminf_{n'\to\infty} P\left(\bigcap_{i=1}^{k_{n'}} A_{n',i}\right) \geq e^{-C}. \tag{7.10}$$

In the sequel we will write for simplicity n instead of n'.

If $C = 0$, then (7.10) holds trivially. So assume that $0 < C < +\infty$. Fix $r \in I\!N$ and define

$$\begin{aligned} j^r_{n,0} &= 0; \\ j^r_{n,p} &= \begin{cases} \inf\{k\,;\, \sum_{i=1}^k P(A^c_{n,i}) \geq (p/r)C\} & \text{if this set is non-empty} \\ k_n & \text{otherwise;} \end{cases} \\ j^r_{n,r} &= k_n. \end{aligned}$$

By the uniform infinitesimality of events $P(A^c_{n,i})$, we have for every $1 \leq p \leq r$

$$\sum_{i=j^r_{n,p-1}+1}^{j^r_{n,p}} P(A^c_{n,i}) \longrightarrow C/r, \quad \text{as } n \to +\infty,$$

and by (7.8), for r fixed,

$$P\left(\bigcap_{i=1}^{k_n} A_{n,i}\right) - \prod_{p=1}^{r} P\left(\bigcap_{i=j^r_{n,p-1}+1}^{j^r_{n,p}} A_{n,i}\right) \longrightarrow 0, \quad \text{as } n \to +\infty.$$

Let $\{N_r, r \in I\!N\}$ be such that $N_r > N_{r-1}$ and for $n \geq N_r$,

$$\max_{1\leq p\leq r} \left| \sum_{i=j^r_{n,p-1}+1}^{j^r_{n,p}} P(A^c_{n,i}) - C/r \right| \leq 1/r, \tag{7.11}$$

$$\left| P\left(\bigcap_{1\leq i\leq k_n} A_{n,i}\right) - \prod_{p=1}^{r} P\left(\bigcap_{i=j^r_{n,p-1}+1}^{j^r_{n,p}} A_{n,i}\right) \right| \leq 1/r. \tag{7.12}$$

For natural n, define

$$r_n := r \quad \text{iff} \quad N_r \leq n < N_{r+1}.$$

Clearly, $r_n \to \infty$ as $n \to \infty$. Let

$$m_{n,p} := j^{r_n}_{n,p}, \quad p = 0, 1, \ldots, r_n.$$

Divide the intersection $\cap_{1\le i\le k_n} A_{n,i}$ into r_n blocks

$$B_{n,p} = \bigcap_{i=m_{n,p-1}+1}^{m_{n,p}} A_{n,i}, \quad p = 1, 2, \ldots, r_n,$$

(here $\cap_\emptyset = \Omega$). We have by (7.11) and (7.12)

$$\begin{aligned} \max_{1\le p\le r_n} P(B^c_{n,p}) \quad &\longrightarrow \quad 0, \quad \text{as } n \to +\infty \\ P\left(\bigcap_{p=1}^{r_n} B_{n,p}\right) - \prod_{p=1}^{r_n} P(B_{n,p}) \quad &\longrightarrow \quad 0, \quad \text{as } n \to +\infty. \end{aligned}$$

Since $|\exp(-x) - 1 + x| \le (1/2)x^2$ for $x \ge 0$, so

$$\left|\prod_{p=1}^{r_n} P(B_{n,p}) - \exp\left(-\sum_{p=1}^{r_n} P(B^c_{n,p})\right)\right| \le \frac{1}{2}\max_{1\le p\le r_n} P(B^c_{n,p})\left(\sum_{p=1}^{r_n} P(B^c_{n,p})\right) \to 0, \quad \text{as } n \to +\infty.$$

Hence

$$\begin{aligned} \liminf_{n\to\infty} P\left(\bigcap_{i=1}^{k_n} A_{n,i}\right) &= \exp\left(-\limsup_{n\to\infty}\sum_{p=1}^{r_n} P(B^c_{n,p})\right) \\ &\ge \exp\left(-\limsup_{n\to\infty}\sum_{i=1}^{k_n} P(A^c_{n,i})\right) = \exp(-C). \end{aligned}$$

□

The proof of Proposition 7.1 will be complete if we are able to derive the converse inequality to (7.7) for *some* sequence $\{r_n\}$ satisfying (7.5).

It suffices to find for every $Q \in I\!N$ a sequence $\{r_n = r_n(Q)\}$ such that

$$P(M_{r_n} > v_n) \le Q^{-1} + o(1)$$

and for any $k_n \le T \cdot n$

$$P(M_{k_n} \le v_n) \le \exp(-k_n \cdot P(X_0 > v_n, M_{r_n} \le v_n)) + 1/Q + o(1). \tag{7.13}$$

To do this let us define

$$r_n = \min\{k : P(M_k > v_n) > 1/Q\} \wedge [T \cdot n]. \tag{7.14}$$

Suppose $P(M_{r_{n'}} > v_{n'}) \le 1/Q$ along a subsequence $\{n'\} \subset I\!N$. Then $r_{n'} = [n'T]$ and by (7.6) for any $k_{n'} \le [n'T]$,

$$1/Q \ge P(M_{[n'T]} > v_{n'}) \ge k_{n'} P(X_0 > v_{n'}, M_{[n'T]} \le v_{n'}).$$

Hence

$$\begin{aligned} P(M_{k_{n'}} \le v_{n'}) &\le 1 \le e^{-1/Q} + 1/Q \\ &\le \exp\left(-k_{n'} P(X_0 > v_{n'}, M_{[n'T]} \le v_{n'})\right) + 1/Q, \end{aligned}$$

i.e. (7.13) holds along $\{n'\}$.

So we may assume that $P(M_{r_n} > v_n) > 1/Q,\ n \in \mathbb{N}$. Since $P(X_1 > v_n) \longrightarrow 0$, we have $\lim_{n\to\infty} P(M_{r_n} > v_n) = 1/Q$. Choose an integer W such that $e^{-W} \le 1/Q$. Let $\mathbb{N}_1 \subset \mathbb{N}$ consists of those numbers n for which $k_n > QWr_n$. If $n \in \mathbb{N}_1$, let $q_n = k_n - QWr_n$. Suppose $\mathbb{N}_1$ is infinite and assume for notational convenience that $\mathbb{N}_1 = \mathbb{N}$. Then by $\mathrm{B}_T(v_n)$

$$\begin{aligned} P(M_{k_n} \le v_n) &= P(M_{r_n} \le v_n)^{QW} \times P(M_{q_n} \le v_n) + o(1) \\ &\le P(M_{r_n} \le v_n)^{QW} + o(1) \\ &\longrightarrow (1 - 1/Q)^{QW} \le e^{-W} \le 1/Q, \end{aligned}$$

and inequality (7.13) holds along $\mathbb{N}_1$. If $\mathbb{N}_1$ is finite, we have $k_n \le QWr_n$ for n large enough. For such n denote $U_n = [k_n/r_n],\quad q_n = k_n - U_n \cdot r_n$. Then we can estimate similarly as O'Brien ([OBr87, Corollary 2.2]):

$$\begin{aligned} P(M_{k_n} \le v_n) &= P(M_{r_n} \le v_n)^{U_n} \cdot P(M_{q_n} \le v_n) + o(1) && \text{by } \mathrm{B}_T(v_n) \\ &\le \exp(-U_n \cdot P(M_{r_n} > v_n) - P(M_{q_n} > v_n)) + o(1) \\ &\le \exp(-(U_n \cdot r_n + q_n) P(X_0 > v_n, M_{r_n} \le v_n)) + o(1) && \text{by (7.6).} \end{aligned}$$

This proves Proposition (7.1). □

Remark 7.3 It is easy to see that in definition (7.14) of r_n we could use any $0 < T' \le T$ instead of T. So if $\mathrm{B}_\infty(v_n)$ is satisfied one can define r_n as for $T = 1$ and the proof still works.

The asymptotic uniform representation given by (7.4) has consequences which are especially useful in our considerations.

Proposition 7.4 *If for some $T > 0$*

$$\liminf_{n\to\infty} P(M_{[nT]} \le v_n) > 0$$

then

$$P(M_{[nt]} \le v_n) - P(M_{[nT]} \le v_n)^{t/T} \longrightarrow 0 \tag{7.15}$$

uniformly in $t \in [0, T]$ if and only if Condition $B_T(v_n)$ holds.

Proposition 7.5 *Suppose*

$$0 < \liminf_{n\to\infty} P(M_n \le v_n) \le \limsup_{n\to\infty} P(M_n \le v_n) < 1.$$

Then the following items (i)–(iii) *are equivalent:*

(i) *For each $T > 0$ Condition $B_T(v_n)$ holds.*

(ii) *Condition $B_\infty(v_n)$ is satisfied.*

(iii) *$P(M_{[nt]} \le v_n)$ is asymptotically exponential on $[0,\infty)$:*

$$\sup_{t>0} |P(M_{[nt]} \le v_n) - P(M_n \le v_n)^t| \longrightarrow 0, \quad \text{as } n \to +\infty. \tag{7.16}$$

PROOF. Necessity of $\mathrm{B}_T(v_n)$ in Proposition 7.4 and the chain of implications (iii)⇒(ii)⇒(i) in Proposition 7.5 are obvious.

Let us assume $\mathrm{B}_T(v_n)$. We claim that $P(X_0 > v_n) \to 0$. Indeed, by $\mathrm{B}_T(v_n)$ we have for each q

$$\liminf_{n\to\infty} P(M_{[nT]} \le v_n) \le \liminf_{n\to\infty} P(M_{[[nT]/q]} \le v_n)^q \le \liminf_{n\to\infty} P(X_0 \le v_n)^q.$$

Now $\liminf_{n\to\infty} P(M_{[nT]} \le v_n) > 0$ implies $\liminf_{n\to\infty} P(X_0 \le v_n) = 1$. Hence we can apply Proposition 7.1. Let $\{r_n\}$ be as in this proposition. Define

$$c_n = \exp(-P(X_0 > v_n, M_{r_n} \le v_n)).$$

By (7.4)

$$P(M_{[nt_n]} \le v_n) - c_n^{[nt_n]} = P(M_{[nt_n]} \le v_n) - \left(c_n^{[nt_0]}\right)^{[nt_n]/[nt_0]} \longrightarrow 0,$$

for every sequence $\{t_n\}$ of numbers, $0 \le t_n \le T$, and fixed $t_0 > 0$. If $t_0 = T$, we have $c_n^{[nT]} = P(M_{[nT]} \le v_n) + o(1)$ and since $P(M_{[nT]} \le v_n) \ge \eta > 0$ for n large enough, we get an equivalent form of (7.15), i.e.

$$P(M_{[nt_n]} \le v_n) - P(M_{[nT]} \le v_n)^{t_n/T} \longrightarrow 0$$

for every sequence $\{t_n\}, 0 \le t_n \le T$.

We have already proved Proposition 7.4. To prove the remaining implication (i)⇒(iii) in Proposition 7.5, let us repeat the above considerations for $t_0 = 1$ and observe that

$$P(M_{[nt]} \le v_n) - P(M_n \le v_n)^t \longrightarrow 0$$

uniformly on each interval $[0,T]$. Take $\varepsilon > 0$ and let $Q \in \mathbb{N}$ be such that

$$\limsup_{n\to\infty} P(M_n \le v_n)^Q < \varepsilon.$$

By $\mathrm{B}_Q(v_n)$, if $t > Q$,

$$P(M_{[nt]} \le v_n) \le P(M_{nQ} \le v_n) = P(M_n \le v_n)^Q + o(1) \le \varepsilon$$

for n large enough. For such n,

$$P(M_n \le v_n)^t \le P(M_n \le v_n)^Q < \varepsilon,$$

either, and we see that as $n \to \infty$

$$\sup_{t\geq 0} \left| P(M_{[nt]} \leq v_n) - P(M_n \leq v_n)^t \right|$$
$$\leq \sup_{0\leq t\leq Q} \left| P(M_{[nt]} \leq v_n) - P(M_n \leq v_n)^t \right| + 2\varepsilon \longrightarrow 2\varepsilon.$$

□

7.2 Families of Mixing Conditions

Remark 7.6 In some problems (e.g. convergence in law of maxima) it is more natural to consider mixing conditions allowing "to break probabilities" for a family of levels $\{v_n(\beta) : \beta \in B\}$ but on bounded intervals only. In fact this situation is covered by the preceding theory as we can see from the proposition below.

Proposition 7.7 *The following conditions* (i), (ii) *and* (iii) *are equivalent.*

- **(i)** *For some* $0 < \alpha < 1$, *there exists a sequence* $\{v_n = v_n(\alpha)\}$ *such that* $P(M_n \leq v_n) \longrightarrow \alpha$ *and* $B_\infty(v_n)$ *holds for* $\{X_j\}$.
- **(ii)** *One can find a sequence* $\{v_n(\alpha)\}$ *as in* (i) *for each* $\alpha \in (0,1)$.
- **(iii)** *There exist: a decreasing to zero sequence* $\{\alpha_q : q \in \mathbb{N}\}$ *of positive numbers and an array* $\{v_n(q) : n, q \in \mathbb{N}\}$ *of numbers such that for each* $q \in \mathbb{N}$ *Condition* $B_1(v_n(q))$ *holds and*
$$\lim_{n\to\infty} P(M_n \leq v_n(q)) = \alpha_q \tag{7.17}$$

Proof. To see (i) ⇒ (ii), choose $0 < \alpha' < 1$ and set $t = \log \alpha / \log \alpha'$. By representation (7.16)

$$P(M_n \leq v_{[nt]}) = P(M_{[nt]} \leq v_{[nt]})^{(n/[nt])} + o(1) \longrightarrow \alpha^{(1/t)} = \alpha'.$$

Eventually, we note that $B_\infty(v_{[nt]}) \equiv B_\infty(v_n)$. The implication (ii) ⇒ (iii) is obvious, so let us suppose (iii). We may assume $\alpha_1 < 1$. Choose $\alpha, 1 > \alpha > \alpha_1$ and define

$$t_q = \log \alpha / \log \alpha_q.$$

Clearly, $1 > t_q \searrow 0$. By Proposition 7.4, condition (7.17) implies for each $q \in \mathbb{N}$

$$P(M_{[nt]} \leq v_n(q)) - \alpha_q^t \longrightarrow 0$$

uniformly in $t \in [0,1]$. Hence one can find integers $N_1 < N_2 < \ldots < N_q < \ldots$ such that $\beta_q(n,t) := P(M_{[nt]} \leq v_n(q)) - \alpha_q^t$ satisfy

$$\sup_{n\geq N_q} \sup_{t\in[0,1]} |\beta_q(n,t)| \leq 1/q.$$

For $n \in \mathbb{N}$, let $q = q(n)$ be such that $N_q \leq n < N_{q+1}$. We have

$$P(M_{[nt]} \leq v_{[n/t_q]}(q)) = \left(\alpha_q^{t_q}\right)^{[nt]/(t_q \cdot [n/t_q])} + \beta_q([n/t_q], [nt]/[n/t_q]).$$

When $n \to \infty$, the first summand on the right-hand side tends to α^t uniformly in $t \in [0, +\infty)$.

Choose $T > 0$ and let Q be so large that $T \cdot t_Q \leq 1$. If n is large enough, $q(n) \geq Q$ and $[nt]/[n/t_q] \leq 1$ whenever $t \leq T$. Since also $[n/t_q] \geq N_q$, we conclude that the absolute value of the second term does not exceed $1/q(n) \to 0$, provided $t \leq T$. Hence, setting

$$v_n = v_{[n/t_q]}(q) \quad \text{if} \quad N_q \leq n < N_{q+1}$$

we get

$$P(M_{[nt]} \leq v_n) - \alpha^t \longrightarrow 0$$

uniformly in $t \in [0, T]$, for every $T > 0$. In particular, $P(M_n \leq v_n) \longrightarrow \alpha$ and we may apply Proposition 7.5 (i) in order to get $\mathrm{B}_\infty(v_n)$. □

The above proposition can be used to clarify the connections between our preferred Condition $\mathrm{B}_\infty(v_n)$ and conditions used in the literature (e.g. [LLR83]).

First, it is mentioned in [Lea83], p. 293ff, that the minimal property we need in limit theorems for maxima is "breaking":

$$P(M_n \leq v_n) - P(M_{[n/k]} \leq v_n)^k \longrightarrow 0,$$

for each $k = 2, 3, \ldots$. Hence in all proofs it is enough to assume $\mathrm{B}_1(v_n)$ instead of conditions like $D_1(v_n)$ — as far as we deal with maxima only. This remark allows us to discuss two known results using B_1-type conditions, while originally they were proved under D_1's.

In analysis of both results we aim at proving (via Proposition 7.7), that *a family* of mixing conditions can be replaced by a *single* condition of the form $\mathrm{B}_\infty(v_n)$, for some $\{v_n\}$.

Suppose that $\{M_n\}$'s suitably centered and normalized are convergent in law to some distribution function H:

$$P(M_n \leq a_n x + b_n) \longrightarrow H(x)$$

on some dense subset $D \subset \mathbb{R}^1$. Further, let for each $x \in D_H = D \cap \{x : 0 < H(x) < 1\}$, Condition $\mathrm{B}_1(a_n x + b_n)$ is fulfilled. If H has no atom in its left end (i.e. $H({}_*H) = 0$, where ${}_*H = \inf\{x : H(x) > 0\}$), then (7.17) is satisfied and $\mathrm{B}_\infty(a_n x + b_n)$ holds for all $x \in D_H$. The relation $H({}_*H) = 0$ can be derived directly, but we may use a result due to [Lea74] asserting that H must be *max-stable* (hence—continuous) whenever it is non-degenerate.

The assumptions of the next proposition are motivated by Leadbetter's [Lea83] criterion for the existence of the (ordinary) extremal index. Recall that the marginal distribution function of $\{X_j\}$ is regular—in the sense of (6.35)—if for each $\tau > 0$ one can find a sequence $u_n(\tau)$ such that

$$nP(X_1 > u_n(\tau)) \longrightarrow \tau. \tag{7.18}$$

Proposition 7.8 *Suppose X_1 has a regular distribution function. Let $\{u_n(\tau)\}$ denotes a sequence satisfying (7.18).*

(i) *If for each τ from some dense subset $S \subset I\!R^+ \setminus \{0\}$ Condition $B_1(u_n(\tau))$ holds, then $B_T(u_n(\tau))$ is satisfied for all $\tau > 0$ and $T > 0$.*

(ii) *If, in addition, for some $\tau_0 > 0$*

$$\limsup_{n\to\infty} P(M_n \le u_n(\tau_0)) < 1,$$

then for each $\tau > 0$, Condition $B_\infty(u_n(\tau))$ holds.

Proof. Notice that by Lemma 7.2, $\liminf_{n\to\infty} P(M_n \le u_n(\tau)) \ge e^{-\tau} > 0$. Hence we can use Proposition 7.4 for checking Condition $\mathrm{B}_T(u_n(\tau))$. Fix $T > 0$ and $\tau_0 > 0$ and choose $\tau', \tau'' \in S$ in such a way that

$$\frac{\tau'}{T} < \tau_0 < \frac{\tau''}{T}.$$

By definition (7.18)

$$n(1 - F(u_{[nT]}(\tau'))) \longrightarrow \frac{\tau'}{T},$$

so at least for n large enough, $u_n(\tau_0) \le u_{[nT]}(\tau')$. Similarly, we may assume that $u_{[nT]}(\tau'') \le u_n(\tau_0)$. This implies

$$P(M_k \le u_{[nT]}(\tau'')) \le P(M_k \le u_n(\tau_0)) \le P(M_k \le u_{[nT]}(\tau')), \quad k \in I\!N.$$

Moreover, if $k \le [nT]$,

$$\begin{aligned} P(M_k \le u_{[nT]}(\tau')) &- P(M_k \le u_{[nT]}(\tau'')) \\ &\le [nT] P(u_{[nT]}(\tau'') < X_1 \le u_{[nT]}(\tau')) = \tau'' - \tau' + o(1) \end{aligned}$$

Let $t_n \in [0,T], n \in I\!N$. Choosing τ'' and τ' as close as desired, we see that both $P(M_{[nt_n]} \le u_n(\tau_0))$ and $P(M_{[nT]} \le u_n(\tau_0))^{(t_n/T)}$ can be approximated by $P(M_{[nT(t_n/T)]} \le u_{[nT]}(\tau'))$ and $P(M_{[nT]} \le u_{[nT]}(\tau'))^{(t_n/T)}$, respectively. By $\mathrm{B}_1(u_n(\tau'))$, the difference between the two last expressions tends to zero. So $\mathrm{B}_T(u_n(\tau_0))$ holds.

To prove (ii) observe that by the first part and by Proposition 7.5 it is enough to prove that $\limsup_{n\to\infty} P(M_n \le u_n(\tau)) < 1$ for every $\tau > 0$. Let $T > 0$ be such that $\tau_0/T < \tau$. Similarly as above we get $u_n(\tau) \le u_{[nT]}(\tau_0)$ for n large enough, hence

$$\begin{aligned} \limsup_{n\to\infty} P(M_n \le u_n(\tau)) &\le \limsup_{n\to\infty} P(M_n \le u_{[nT]}(\tau_0)) \\ &\le \limsup_{n\to\infty} P(M_n \le u_n(\tau_0))^{(1/T)} < 1. \end{aligned}$$

□

Chapter 8

Limiting Probabilities for Maxima

8.1 The Problem

Let $X_0, X_1, X_2, \ldots$ be a stationary sequence of random variables, and, as before, $M_0 = -\infty$ and for $n \geq 1$, $M_n = \max_{1 \leq k \leq n} X_j$. Let $\{v_n\}$ be a sequence of numbers.

If X_j's are i.i.d., then the convergence

$$P(M_n \leq v_n) - \exp(-nP(X_0 > v_n)) \longrightarrow 0 \quad \text{as } n \to \infty \tag{8.1}$$

holds. The above relation is no longer true if we drop the assumption of independence: there are simple examples of 1-dependent sequences not satisfying (8.1). On the other hand, for m-dependent sequences a modification of (8.1) is valid:

$$P(M_n \leq v_n) - \exp(-nP(X_0 > v_n, M_m \leq v_n)) \longrightarrow 0. \tag{8.2}$$

This was proved by Newell [New64] under the additional assumption

$$\sup_n nP(X_0 > v_n) < +\infty.$$

O'Brien [OBr87] has considered stationary sequences having "asymptotic independence of maxima" and obtained the representation

$$P(M_n \leq v_n) - \exp(-nP(X_0 > v_n, M_{r_n} \leq v_n)) \longrightarrow 0, \tag{8.3}$$

where $\{r_n\}$ is a suitably chosen sequence of integers.

Formulas like (8.3) are useful tools in structural problems, e.g. existence of phantom distribution functions or extremal indices (see [Jak91a],[OBr87]). They are useless, however, if r_n tends to infinity and we want to calculate the limit for $P(M_n \leq v_n)$: the expression under exponent depends on increasing number of random variables X_j, hence it is of the same type as the approximated quantity $P(M_n \leq v_n)$. This inconvenience disappears if we are able to approximate $P(M_n \leq v_n)$ with m *possibly large*, but *fixed*:

$$\lim_{m\to\infty} \limsup_{n\to\infty} |P(M_n \leq v_n) - \exp(-nP(X_0 > v_n, M_m \leq v_n))| = 0. \tag{8.4}$$

Say that $\{X_j\}$ satisfies **Condition C(v_n)**, if the above relation holds.

In the paper we study a version of C(v_n), which allows us to approximate $P(M_{k_n} \leq v_n)$ for every sequence $k_n \leq T \cdot n$:

Condition $\mathbf{C}^*_T(v_n)$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \max_{1\leq k\leq T\cdot n} |P(M_k \leq v_n) - \exp(-kP(X_0 > v_n, M_m \leq v_n))| = 0.$$

By the above condition, if $k_n \leq T \cdot n$, then

$$P(M_{k_n} \leq v_n) = \exp(-k_n P(X_0 > v_n, M_m \leq v_n)) + \alpha_{m,n}, \tag{8.5}$$

where $\lim_{m\to\infty} \limsup_{n\to\infty} |\alpha_{m,n}| = 0$. In particular, denoting

$$\begin{array}{rcl} \overline{\Lambda} & = & \lim_{m\to\infty} \limsup_{n\to\infty} k_n P(X_0 > v_n, M_m \leq v_n), \\ \underline{\Lambda} & = & \lim_{m\to\infty} \liminf_{n\to\infty} k_n P(X_0 > v_n, M_m \leq v_n), \end{array} \tag{8.6}$$

we get

$$\liminf_{n\to\infty} P(M_{k_n} \leq v_n) = e^{-\overline{\Lambda}}$$

and

$$\limsup_{n\to\infty} P(M_{k_n} \leq v_n) = e^{-\underline{\Lambda}}.$$

This leads to the most transparent (and immediate) consequence of $\mathrm{C}^*_T(v_n)$:

Theorem 8.1 *Suppose $C^*_T(v_n)$ holds. Let k_n be a sequence of integers, $k_n \leq T \cdot n$, and let $\underline{\Lambda}$ and $\overline{\Lambda}$ be defined by (8.6).*

Then there exists $\lambda = \lim_{n\to\infty} P(M_{k_n} \leq v_n)$ if and only if $\underline{\Lambda} = \overline{\Lambda} =: \Lambda$. In such a case, $\lambda = \exp(-\Lambda)$. □

Note once again, that only *finite-dimensional* asymptotic properties of $\{X_j\}$ are involved in checking the equality $\underline{\Lambda} = \overline{\Lambda}$.

Let j_n and k_n be non-negative integers, such that $j_n + k_n \leq T \cdot n$, $n \in I\!N$. If $\mathrm{C}^*_T(v_n)$ holds, then by (8.5)

$$\begin{aligned} P(M_{j_n+k_n} \leq v_n) &= \exp(-(j_n + k_n)P(X_0 > v_n, M_m \leq v_n)) + \alpha'_{m,n} \\ &= \exp(-j_n P(X_0 > v_n, M_m \leq v_n)) \cdot \exp(-k_n P(X_0 > v_n, M_m \leq v_n)) + \alpha'_{m,n} \\ &= (P(M_{j_n} \leq v_n) + \alpha''_{m,n}) \cdot (P(M_{k_n} \leq v_n) + \alpha'''_{m,n}) + \alpha'_{m,n} \\ &= P(M_{j_n} \leq v_n) \cdot P(M_{k_n} \leq v_n) + \alpha_{m,n} \end{aligned}$$

where $\lim_{m\to\infty} \limsup_{n\to\infty} |\alpha_{m,n}| = 0$, and the same relation holds for $\alpha'_{m,n}$, $\alpha''_{m,n}$ and $\alpha'''_{m,n}$.

Hence $\mathrm{C}^*_T(v_n)$ is of "mixing" type. In particular, $\mathrm{C}^*_T(v_n)$ implies

Condition $\mathbf{B}_T(v_n)$.

$$\lim_{m\to\infty} \max_{j+k\leq T\cdot n} |P(M_{j+k} \leq v_n) - P(M_j \leq v_n)P(M_k \leq v_n)| = 0. \tag{8.7}$$

This condition is close to mixing assumptions considered in the literature—a brief discussion of its relations to Leadbetter's Condition $D(v_n)$ (see [LLR83]) and O'Brien's [OBr87] Condition $AIM(v_n)$ can be found in the previous chapter.

Mutual connections between $\mathrm{C}(v_n)$, $\mathrm{C}_T^*(v_n)$ and $B_T(v_n)$ are examined in the next section.

A collection of conditions, which are sufficient for $\mathrm{C}_T^*(v_n)$ is given in Section 8.3.

The last section of this chapter contains some illustrating the theory examples.

This section is concluded with discussion of the existence of the extremal index due to Leadbetter [Lea83], when Condition $\mathrm{C}_T(v_n)$ holds. (see Section 6.5, p. 82 for definitions). For convenience, recall only that$\{X_n\}$ has the extremal index $\theta, 0 \le \theta \le 1$, iff (i) and (ii) below hold:

(i) The distribution function $F(x) = P(X_0 \le x)$ is **regular** in the sense of O'Brien, i.e. for each $\tau > 0$ there are numbers $u_n(\tau)$ such that

$$nP(X_0 > u_n(\tau)) \longrightarrow \tau. \tag{8.8}$$

(ii) For each $\tau > 0$,

$$P(M_n \le u_n(\tau)) \longrightarrow e^{-\theta\tau}.$$

Our result is a generalization of Theorem 1 [OBr74b].

Theorem 8.2 *Suppose the marginal distribution function F of X_0 is regular in the sense of (8.8). Define*

$$\underline{\theta} = \lim_{m\to\infty} \liminf_{x\to F_*-} P(M_m \le x | X_0 > x),$$

$$\overline{\theta} = \lim_{m\to\infty} \limsup_{x\to F_*-} P(M_m \le x | X_0 > x),$$

where $F_* = \sup\{x : F(x) < 1\}$.

Let $\tau_0 > 0$ and let $\{u_n(\tau_0)\}$ be numbers satisfying (8.8).

If $C(u_n(\tau_0))$ holds and $B_T(u_n(\tau_0))$ is satisfied for each $T > 0$, then $\{X_j\}$ has the extremal index θ if and only if $\underline{\theta} = \overline{\theta}$. In such a case, $\theta = \underline{\theta} = \overline{\theta}$.

Proof. Necessity. By $\mathrm{C}(u_n(\tau_0))$ and part (ii) of the definition of the extremal index,

$$\begin{aligned} \theta\tau &= \lim_{m\to\infty} \limsup_{n\to\infty} nP(X_0 >_n (\tau_0), M_m \le u_n(\tau_0)) \\ &= \tau \lim_{m\to\infty} \limsup_{n\to\infty} P(M_m \le u_n(\tau_0) | X_0 > u_n(\tau_0)). \end{aligned}$$

By Lemma 1, [OBr74b],

$$\limsup_{n\to\infty} P(M_m \le u_n(\tau_0) | X_0 > u_n(\tau_0)) = \limsup_{x\to F_*-} P(M_m \le x | X_0 > x),$$

i.e. $\theta = \overline{\theta}$. We can check $\theta = \underline{\theta}$ similarly.

SUFFICIENCY. Assume that $\theta = \underline{\theta} = \overline{\theta} > 0$. By $\mathrm{C}(u_n(\tau_0))$ and Theorem 8.1

$$P(M_n \le u_n(\tau_0)) \longrightarrow e^{-\theta\tau_0},$$

where $0 < e^{-\theta\tau_0} < 1$. By Proposition 7.5, validity of Conditions $B_T(u_n(\tau_0))$ for each $T > 0$ is equivalent to Condition $B_\infty(u_n(\tau_0))$:

$$\lim_{n\to\infty} \max_{j,k} |P(M_{j+k} \le u_n(\tau_0)) - P(M_j \le u_n(\tau_0))P(M_k \le u_n(\tau_0))| = 0. \tag{8.9}$$

Hence all assumptions of our Theorem 6.21 are satisfied and $\{X_j\}$ has the extremal index θ.

In the case $\theta = 0$ we have no ready tools, but the situation is much simpler. First, let us remark that if the sequence $\{u_n(\tau)\}$ is given for *some* $\tau > 0$, one can find such sequences for *each* $\tau' > 0$: it is enough to define $u_n(\tau') = u_{[n\tau/\tau']}(\tau)$. We claim, that for sequences defined this way

$$\lim_{m\to\infty} \limsup_{n\to\infty} nP(X_0 > u_n(\tau'), M_m \le u_n(\tau')) = 0. \tag{8.10}$$

Indeed, this is true for $\tau' = \tau_0$ by $\mathrm{C}(u_n(\tau_0))$ and for other $\tau' > 0$ by the definition of $u_n(\tau')$.

Eventually, the inclusion

$$\begin{aligned}&\{M_n > u_n(\tau')\} \subset \\ &\quad \{\max_{1\le j\le m} X_{n+j} > u_n(\tau')\} \cup \bigcup_{k=1}^{n} \{X_k > u_n(\tau'),\ \max_{1\le k\le m} X_{k+j} \le u_n(\tau')\}\end{aligned}$$

shows that (8.10) and $P(X_0 > u_n(\tau')) \to 0$ imply $P(M_n \le u_n(\tau')) \to 1$ for each $\tau' > 0$. Hence $\{X_j\}$ has the extremal index 0.

Note that in the case $\theta = 0$ we do not use Condition $B_T(u_n(\tau_0))$. □

8.2 Around Condition $\mathrm{C}_T^*(v_n)$

We have already checked that $\mathrm{C}_T^*(v_n)$ implies $B_T(v_n)$. Further, setting in (8.5) $k_n = 1$ we get

$$\begin{aligned} P(X_0 \le v_n) &= P(X_1 \le v_n) \\ &= \exp(-P(X_0 > v_n, M_m \le v_n)) + \alpha_{m,n} \\ &\ge \exp(-P(X_0 > v_n)) + \alpha_{m,n}, \end{aligned}$$

where $\lim_{m\to\infty} \limsup_{n\to\infty} |\alpha_{m,n}| = 0$. Hence

$$P(X_0 > v_n) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{8.11}$$

By Proposition 7.1, $B_T(v_n)$ and (8.11) are satisfied if and only if a uniform version of (8.3) holds, i.e. there is a sequence r_n of integers such that

$$P(M_{r_n} \le v_n) \longrightarrow 1 \tag{8.12}$$

and

$$\lim_{n\to\infty} \max_{1\le k\le T\cdot n} |P(M_k \le v_n) - \exp(-kP(X_0 > v_n, M_{r_n} \le v_n))| = 0. \tag{8.13}$$

It follows, that if $C^*_T(v_n)$ holds, then (8.12) and (8.13) are fulfilled by *some* sequence $\{r_n\}$. Can we say anything more on such sequence $\{r_n\}$? An informal answer is—yes, $\{r_n\}$ is *increasing slowly enough.* The formal statement is given in

Proposition 8.3 *Condition $C^*_T(v_n)$ holds if and only if (8.13) is fulfilled by every sequence $\{r_n\}$ of integers increasing to infinity so slowly that*

$$r_n P(X_0 > v_n) \longrightarrow 0 \quad \text{as } n \to \infty. \tag{8.14}$$

PROOF. Condition (8.13) is equivalent to

$$P(M_{k_n} \le v_n) = \exp(-k_n P(X_0 > v_n, M_{r_n} \le v_n)) + o(1) \tag{8.15}$$

for every sequence $k_n \le T \cdot n$. So let $k_n \le T \cdot n$ and assume $C^*_T(v_n)$. Let m_ε be such that for $n \ge N_\varepsilon$

$$|\alpha_{m_\varepsilon,n}| = |P(M_{k_n} \le v_n) - \exp(-k_n P(X_0 > v_n, M_{m_\varepsilon} \le v_n))| < \varepsilon.$$

If $n \ge N_\varepsilon$ and $r_n \ge m_\varepsilon$, then

$$P(M_{k_n} \le v_n) \le \exp(-k_n P(X_0 > v_n, M_{r_n} \le v_n)) + \varepsilon.$$

Hence $C^*_T(v_n)$ implies

$$P(M_{k_n} \le v_n) \le \exp(-k_n P(X_0 > v_n, M_{r_n} \le v_n)) + o(1) \tag{8.16}$$

for every $k_n \le T \cdot n$ and every $r_n \to \infty$.

To proceed further we need inequality (7.7), which we restate here in

Lemma 8.4 *If $B_T(u_n)$ holds and $\{r_n\}$ satisfies (8.12), then for every sequence $k_n \le T\cdot n$*

$$P(M_{k_n} \le u_n) \ge \exp(-k_n P(X_0 > u_n, M_{r_n} \le u_n)) + o(1). \tag{8.17}$$

□

By the above lemma, if

$$0 = \lim_{n\to\infty} r_n P(X_0 > v_n) \ge \lim_{n\to\infty} P(M_{r_n} > v_n),$$

then the converse to (8.16) inequality holds, too, and (8.15) follows for all $\{r_n\}$ satisfying (8.14).

If $C^*_T(v_n)$ does not hold, one can find a sequence $\{r_n\}$ increasing to infinity as slowly as desired (for instance: satisfying (8.14)) and such that (8.13) fails. □

We have proved that $C^*_T(v_n)$ is stronger, than the uniform version of (8.3). The uniform version of (8.2) is, in turn, stronger than $C^*_T(v_n)$.

Proposition 8.5 *Fix $m_0 \in \mathbb{N}$. Condition $C_T^*(v_n)$ is implied by*

$$\lim_{n\to\infty} \max_{1\le k\le T\cdot n} |P(M_k \le v_n) - \exp(-kP(X_0 > v_n, M_{m_0} \le v_n))| = 0. \tag{8.18}$$

Proof. By (8.18), for every $k_n \le T \cdot n$ and $m \ge m_0$,

$$\begin{aligned} P(M_{k_n} \le v_n) &= \exp(-k_n P(X_0 > v_n, M_{m_0} \le v_n)) + o(1) \\ &\le \exp(-k_n P(X_0 > v_n, M_m \le v_n)) + o(1). \end{aligned}$$

The converse inequality is given by Lemma 8.4. □

Finally, we shall find additional assumptions which allow us to deduce from $\mathrm{C}(v_n)$ its uniform version $\mathrm{C}_T^*(v_n)$.

Proposition 8.6 *Suppose that*

$$\liminf_{n\to\infty} P(M_n \le v_n) > 0. \tag{8.19}$$

Then Condition $C_T^*(v_n)$ is satisfied if and only if both $C(v_n)$ and $B_T(v_n)$ hold.

Proof. Only sufficiency has to be proved. By Proposition 7.4, under (8.19), Condition $B_T(v_n)$ is equivalent to

$$P(M_{k_n} \le v_n) = P(M_{[nT]} \le v_n)^{k_n/[nT]} + o(1)$$

for every $k_n \le T \cdot n$. In particular,

$$P(M_{k_n} \le v_n) = P(M_n \le v_n)^{k_n/n} + o(1). \tag{8.20}$$

But $\mathrm{C}(v_n)$ implies

$$\begin{aligned} P(M_n \le v_n)^{k_n/n} &= (\exp(-nP(X_0 > v_n, M_m \le v_n)) + \alpha_{m,n})^{k_n/n} \\ &= \exp(-k_n P(X_0 > v_n, M_m \le v_n)) + \alpha'_{m,n}, \end{aligned}$$

where $\lim_{m\to\infty} \limsup_{n\to\infty} |\alpha_{m,n}| = 0$ and similarly for $\alpha'_{m,n}$. Hence $\mathrm{C}_T^*(v_n)$ holds. □

Remark 8.7 Condition (8.19) can be verified using Lemma (8.4). It is satisfied if, for example, $P(X_0 > v_n) \longrightarrow 0$, $B_T(v_n)$ holds and

$$\lim_{m\to\infty} \limsup_{n\to\infty} nP(X_0 > v_n, M_m \le v_n) < +\infty. \tag{8.21}$$

8.3 Sufficient Conditions for $\mathbf{C}^*_T(v_n)$

Consider again the representation

$$P(M_{k_n} \le v_n) = \exp(-k_n P(X_0 > v_n, M_{r_n} \le v_n)) + o(1) \tag{8.22}$$

for $k_n \le T \cdot n$, valid when $B_T(v_n)$ and $P(X_0 > v_n) \to 0$ hold. The expression under exponent is the average number of gaps of length at least r_n between two subsequent exceedances of X_n's over v_n. Heuristically, it depends on the way the exceedances are grouping: longer clusters—longer gaps, small clusters—small gaps.[1] Hence controlling the size of clusters allows us to get the desired length of r_n's. These remarks find their formal counterpart in

Proposition 8.8 *For Condition $C^*_T(v_n)$ to hold it is enough, that $B_T(v_n)$ is satisfied, $P(X_0 > v_n) \to 0$ and*

$$\lim_{m\to\infty} \lim_{k\to\infty} \limsup_{n\to\infty} \; n \sum_{j=m+1}^{[n/k]} P(X_0 > v_n, X_j > v_n) = 0. \tag{8.23}$$

Lemma 8.9 *Condition (8.23) implies*

$$\sup_n \; nP(X_0 > v_n, M_{m_0} \le v_n) \le K < +\infty \tag{8.24}$$

for some $m_0 \in I\!N$.

Proof. As usually, let $M_{k:l} = \max_{k<j\le l} X_j$ for $k < l$ and $M_{k:l} = -\infty$ for $k \ge l$. We have

$$\begin{aligned}
&\sum_{k=1}^{[n/k]} I(X_j > v_n, M_{k:k+m} \le v_n) \\
&\qquad \le \; I(M_{[n/k]} > v_n) + \sum_{\substack{1\le i<j\le[n/k] \\ j-i>m}} I(X_i > v_n, X_j > v_n) \; .
\end{aligned}$$

Hence

$$\begin{aligned}
&nP(X_0 > v_n, M_m \le v_n) \\
&\qquad \le \; (k+1)\left(1 + \sum_{\substack{1\le i<j\le[n/k] \\ j-i>m}} P(X_i > v_n, X_j > v_n)\right) \\
&\qquad \le \; (k+1) + (k+1)[n/k]\left(\sum_{j=m+1}^{[n/k]} P(X_0 > v_n, X_j > v_n)\right) \\
&\qquad \le \; (k+1) + 2n\left(\sum_{j=m+1}^{[n/k]} P(X_0 > v_n, X_j > v_n)\right)
\end{aligned}$$

[1] Notice, that by the ergodic theorem, the number of exceedances over v_n among $X_1(\omega), X_2(\omega), \ldots, X_n(\omega)$ is approximately $nP(X_0 > v_n)$ and thus does not depend on the particular configuration of clusters.

and by (8.23) the last expression is finite for some $k \in I\!N$ and $m \in I\!N$. □

Proof of Proposition 3.1. Let $\{r_n\}$ satisfies (8.22). By Lemma 8.4 we can and do assume that $r_n \to \infty$ but still $P(M_{r_n} \leq v_n) \to 1$. By the latter and (8.22), $r_n P(X_0 > v_n, M_{r_n} \leq v_n) \to 0$. Combining this with (8.24) we see that $r_n = o(n)$.

Let m and k be such that for $n \geq N_{m,k}$, $r_n \leq [n/k]$ and

$$n \sum_{j=m+1}^{[n/k]} P(X_0 > v_n, X_j > v_n) < \varepsilon.$$

Then

$$\begin{aligned} 0 &\leq \exp(-k_n P(X_0 > v_n, M_{r_n} \leq v_n)) - \exp(-k_n P(X_0 > v_n, M_m \leq v_n) \\ &= \exp(-k_n P(X_0 > v_n, M_m \leq v_n)) \\ &\quad \times (\exp(k_n P(X_0 > v_n, M_m \leq v_n, M_{m:r_n} > v_n)) - 1) \qquad (8.25) \\ &\leq \exp(k_n \sum_{j=m+1}^{r_n} P(X_0 > v_n, X_j > v_n)) - 1 \leq e^{T\cdot\varepsilon} - 1 = \varepsilon' \,. \end{aligned}$$

Hence for n large enough

$$\begin{aligned} \exp(-k_n P(X_0 > v_n, M_m \leq v_n)) &\leq P(M_{k_n} \leq v_n) + o(1) \\ &\leq \exp(-k_n P(X_0 > v_n, M_m \leq v_n)) + \varepsilon' + o(1) \end{aligned}$$

and $\mathrm{C}_T^*(v_n)$ follows. □

With almost identical proof we get

Corollary 8.10 *If Condition $B_T(v_n)$ holds, $P(X_0 > v_n) \to 0$, and for some $m_0 \geq 0$*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \; n \sum_{j=m_0+1}^{[n/k]} P(X_0 > v_n, X_j > v_n) = 0, \qquad (8.26)$$

then condition (8.18) is satisfied. □

Notice, that in the case $m_0 = 0$, (8.26) becomes the well-known Leadbetter's Condition $D'(v_n)$, while (8.18) takes the form

$$P(M_{k_n} \leq v_n) - \exp(-k_n P(X_0 > v_n)) \longrightarrow 0. \qquad (8.27)$$

So our Corollary 8.10 contains Theorem 3.4.1 of [LLR83]. Note that the latter result is derived under stronger than $B_T(v_n)$ Condition $D_T(v_n)$. But the original proof can be easily adapted to our assumptions—it is a typical situation as far as we deal with limit theorems for *maxima* only. This observation is due to Leadbetter [Lea83], p. 293 ff.

Conditions like (8.23) and (8.26) depend on properties of two-dimensional distributions, so they are easier in checking than $\mathrm{C}_T^*(v_n)$ itself.

Leadbetter [Lea74], using Berman's results, shows $D'_T(u_n(\tau))$ for normal sequences with covariances $\gamma_n = EX_jX_{j+n}$ satisfying $\gamma_n \log n \to 0$ (here $u_n(\tau)$ is defined by (8.8)).

Now suppose that $X_0, X_1, X_2, \ldots$ are m-dependent, i.e. for each $k \in I\!N$, $\sigma(X_0, \ldots, X_j)$ and $\sigma(X_{j+m+1}, X_{k+m+2}, \ldots)$ are independent. If

$$\sup_n \; nP(X_0 > v_n) \leq K < +\infty,$$

then

$$n \sum_{j=m_0+1}^{[n/k]} P(X_0 > v_n, X_j > v_n) =$$

$$= \; n \sum_{j=m_0+1}^{[n/k]} P(X_0 > v_n)^2 \leq \frac{1}{k}(nP(X_0 > v_n))^2 \leq \frac{K^2}{k} \longrightarrow 0$$

as $k \to \infty$, hence (8.26) holds for m_0 and, by Corollary 8.10, condition (8.18) is satisfied. This provides the uniform version of Newell's result [New64].

It is also possible to derive (8.23) introducing "two-dimensional" mixing coefficients. Suppose, for example, that $\sup_n \; nP(X_0 > v_n) \leq K < +\infty$ and for each $u \in I\!R^1$

$$|P(X_0 > u, X_j > u) - P(X_0 > u)^2| \leq \rho(j)P(X_0 > u),$$

where $\sum_{j=1}^{\infty} \rho(j) < +\infty$. Then (8.23) holds:

$$\begin{aligned} & n \sum_{j=m+1}^{[n/k]} P(X_0 > v_n, X_j > v_n) \\ & \leq \left(n \sum_{j=m+1}^{[n/k]} \rho(j) P(X_0 > v_n) \right) + \left(n \sum_{j=m+1}^{[n/k]} P(X_0 > v_n)^2 \right) \\ & \leq \left(K \cdot \sum_{j=m+1}^{+\infty} \rho(j) \right) + \frac{K^2}{k} \longrightarrow 0 \end{aligned}$$

as $k \to \infty$ and $m \to \infty$.

If $\{X_n\}$ has very strong mixing properties, then $\mathrm{C}^*_T(v_n)$ holds without extra rate of mixing.

Recall, that $\{X_n\}$ is ϕ-mixing (or: uniformly strongly mixing) iff as $n \to \infty$

$$\phi(n) = \sup_m \sup\{|P(B|A) - P(B)| \, : \, A \in \sigma(X_j : j \leq m), B \in \sigma(X_j : j \geq m+n)\} \to 0.$$

Lemma 8.11 (O'Brien [OBr74b], Lemma 3)

If $\{X_n\}$ is ϕ-mixing, then for integers $p, q \geq 1$,

$$|P(\max_{1 \leq i \leq p} X_{i \cdot q} \leq v_n) - P(X_1 \leq v_n)^p| \leq \phi(q).$$

□

Corollary 8.12 *For ϕ-mixing $\{X_n\}$, $\liminf_{n\to\infty} P(M_n \le v_n) > 0$ if and only if $\sup_n nP(X_0 > v_n) \le K < +\infty$.*

PROOF. By Lemma 8.4, $\liminf_{n\to\infty} P(M_n \le v_n) \ge e^{-K} > 0$. To get the converse implication, observe that by Lemma 8.11

$$\begin{aligned} P(M_n \le v_n) &\le P(\max_{1\le i\le [n/q]} X_{i\cdot q} \le v_n) \\ &\le P(X_0 \le v_n)^{[n/q]} + \phi(q) \\ &= \exp(-[n/q]P(X_0 > v_n)) + \phi(q) + o(1). \end{aligned}$$

If $\phi(q) < \liminf_{n\to\infty} P(M_n \le v_n)$, then

$$\sup_n \; nP(X_0 > v_n) \le q\left(\sup_n \; [n/q]P(X_0 > v_n) + 1\right) < +\infty.$$

□

Proposition 8.13 *If $\{X_n\}$ is ϕ-mixing and*

$$\liminf_{n\to\infty} P(M_n \le v_n) > 0, \tag{8.28}$$

then $C_T^*(v_n)$ holds.

PROOF. Inspecting the proof of Proposition 8.8—and, especially, the chain of inequalities (8.25)—we see, that it is enough to check

$$\lim_{m\to\infty} \limsup_{n\to\infty} \; nP(X_0 > v_n, M_m \le v_n, M_{m:r_n} > v_n) = 0, \tag{8.29}$$

where r_n satisfies (8.22) and $r_n \to \infty$.

By ϕ-mixing, condition (8.28) is equivalent to $nP(X_0 > v_n) \le K < +\infty$ and we have

$$\begin{aligned} & nP(X_0 > v_n, M_m \le v_n, M_{m:r_n} > v_n) \\ &\quad \le \quad nP(X_0 > v_n, M_{m:r_n} > v_n) \\ &\quad \le \quad n|P(X_0 > v_n, M_{m:r_n} > v_n) - P(X_0 > v_n)P(M_{m:r_n} > v_n)| \\ &\qquad\quad + nP(X_0 > v_n)P(M_{m:r_n} > v_n) \\ &\quad \le \quad K(\phi(m) + P(M_{r_n} > v_n)) \\ &\quad \longrightarrow \quad K\phi(m) \quad \text{as } n \to \infty \quad \text{by (8.12)} \\ &\quad \longrightarrow \quad 0 \quad \text{as } m \to \infty. \end{aligned}$$

□

The above proofs show that given sequence $\{r_n\}$ satisfying (8.22), we can check $C_T^*(v_n)$ by showing (8.29).

One can prove, that under $\liminf_{n\to\infty} P(M_n \leq v_n) > 0$, conditions (8.29) and $C^*_T(v_n)$ are equivalent (note that via $\{r_n\}$, Condition $B_T(v_n)$ is implicitly involved by (8.29)). We prefer $C^*_T(v_n)$, since it contains all useful information that is necessary in proving limit theorems like our Theorems 8.1 and 8.2, while to (8.29) we can relate all remarks on formula (8.3): (8.29) is not based on asymptotic properties of finite dimensional distribution and thus, is difficult in direct checking.

Condition (8.29) is an easy form of Condition C introduced in [OBr74b]. Hence in some sense, our Proposition 8.13 restates an observation due to O'Brien, [OBr74b, p. 58].

8.4 Some examples

All examples we are going to generate, are constructed the same way, with various parameters only. As parameters we take
- a number $C \in (0,1]$.
- a function $h : [C,\infty) \to [0,1]$ satisfying

$$1 \geq h(x) \geq 1/x \quad \text{for } x \geq C. \tag{8.30}$$

- a random variable Y taking values in $[1/C,\infty)$.

Given h and Y we define a regular conditional distribution function

$$F(\omega,x) = \begin{cases} 0 & \text{if} \quad x < 1/C \\ 1-h(Y(\omega)) & \text{if} \quad 1/C \leq x < Y(\omega) \\ 1-1/x & \text{if} \quad x \geq Y(\omega) \end{cases} \tag{8.31}$$

For each ω, $F(\omega,\cdot)$ is a distribution function by property (8.30). Let $X_0, X_1, \dots$ be an exchangeable sequence, which is conditionally independent over $\sigma(Y)$ and such that the regular conditional distribution of X_j given $\sigma(Y)$ equals to $F(\omega,\cdot)$.

By conditional independence we can make explicit calculations, taking into account that $P(Y \leq n) \to 1$:

$$\begin{aligned} P(M_{[nt]} \leq n) &= E((F(\omega,n))^{[nt]}) \\ &= (1-1/n)^{[nt]} P(Y \leq n) \\ &\quad + E((1-h(Y(\omega)))^{[nt]} I(Y(\omega) > n)) \\ &= (1-1/n)^{[nt]} + o(1) \\ &= e^{-t} + o(1). \end{aligned}$$

Since both $f(t) = e^{-t}$ and the path $t \mapsto P(M_{[nt]} \leq n)$ are monotonic,

$$P(M_{[nt]} \leq n) \to e^{-t} \quad \text{uniformly in } t \in [0,+\infty).$$

In particular

$$P(M_{k_n} \leq n) - P(M_n \leq n)^{k_n/n} \longrightarrow 0. \tag{8.32}$$

This shows that $C_T^*(v_n = n)$ holds if and only if

$$\lim_{m\to\infty} \limsup_{n\to\infty} |e^{-1} - \exp(-nP(X_0 > n, M_m \le n))| = 0,$$

or, equivalently,

$$\lim_{m\to\infty} \limsup_{n\to\infty} |nP(X_0 > n, M_m \le n) - 1| = 0. \tag{8.33}$$

But

$$\begin{aligned} nP(X_0 > n, X_1 \le n, X_2 \le n, \dots, X_m \le n) \\ &= (1 - 1/n)^m P(Y \le n) \\ &\quad + nE(h(Y)(1 - h(Y))^m I(Y > n)) \\ &= V_{m,n} + W_{m,n}. \end{aligned}$$

Since for m fixed, $\lim_{n\to\infty} V_{m,n} = 1$, (8.33) is equivalent to

$$\lim_{m\to\infty} \limsup_{n\to\infty} W_{m,n} = 0. \tag{8.34}$$

But asymptotic properties of $W_{m,n}$ essentially depend on h and the law of Y.

Example 8.14 Set $h(u) = 1$. Then for $m = 0$

$$W_{0,n} = nP(Y > n)$$

and this may have an arbitrary asymptotics (convergence to zero, to infinity or to a finite limit, oscillations, ...), while for $m = 1$, $W_{1,n} = 0$, i.e. condition (8.18) holds.

Example 8.15 Let $m > 0$. Take $\lambda = 1 + m^{-1}$, $\sigma = (2m)^{-1}$ and

$$h(u) = \begin{cases} 1 & \text{if } \ 1 \le 2^{1/\sigma}, \\ 1 - u^{-\sigma} & \text{if } \ u \ge 2^{1/\sigma}, \end{cases}$$

$$P(Y \le x) = K \cdot \int_1^x u^{-\lambda}\, du \quad \text{for } x \ge 1.$$

Then integration gives

$$W_{m,n} \sim K \cdot n \int_n^\infty u^{-\sigma m - \lambda}\, du = K' \cdot n^{2 - \sigma m - \lambda}.$$

Here $\lim_{n\to\infty} W_{m,n} = 0$, while $\lim_{n\to\infty} W_{m-1,n} = K' > 0$. Thus (8.18) holds for $m_0 = m$, but not for $m_0 = m - 1$.

Example 8.16 Let $h(u) = C$, $0 < C < 1$ and $Y \ge C^{-1}$ a.s. Then

$$W_{m,n} = C(1 - C)^m nP(Y > n).$$

If $nP(Y > n) \to K$, $0 < K < +\infty$, then $C_T^*(v_n = n)$ holds, but no m_0 exists such that (8.18) is satisfied.

If $\lim_{n\to\infty} nP(Y > n) = +\infty$, then $C_T^*(v_n = n)$ does not hold. Nevertheless, $W_{n,r_n} \to 0$ for some $r_n \to \infty$, $r_n = o(1)$, i.e. (8.22) is fulfilled.

Chapter 9

Asymptotic (r-1) - dependent Representations for rth Order Statistics

9.1 Convergence of Order Statistics

Let $X_1, X_2, \ldots$ be a stationary sequence of random variables. Denote by $M_{k:l}^{(q)}$ the qth largest value of $X_{k+1}, X_{k+2}, \ldots, X_l$ (if $k \geq l$ or $q > l-k$, then *by convention* $M_{k:l}^{(q)} = -\infty$). For simplicity write $M_n = M_{0:n}^{(1)}$ and $M_n^{(q)} = M_{0:n}^{(q)}$. It is well known, that for i.i.d. $X_1, X_2, \ldots$ convergence in distribution of suitably normalized partial maxima:

$$P(M_n \leq v_n(x)) \longrightarrow G(x), \quad x \in I\!R^1, \tag{9.1}$$

implies convergence of all order statistics: for each $q \geq 2$

$$P(M_n^{(q)} \leq v_n(x)) \longrightarrow G(x) \left(1 + \sum_{k=1}^{q-1} \frac{(-\log G(x))^k}{k!} \right), \quad x \in I\!R^1, \quad \text{as } n \to +\infty. \tag{9.2}$$

(see e.g. [Gal78] or [LLR83]).

If we drop the assumption of independence, *preserving only strong mixing property*, convergence (9.2) may fail in two ways:

—higher order statistics do not converge at all

—they converge, but to different limits

Mori [Mor76] demonstrates the first possibility: he gives an example of 1-*dependent* sequence $X_1, X_2, \ldots$ satisfying (9.1) and such that $M_n^{(2)}$'s do not converge in distribution.

Unexpectedly, convergence in law of M_n's and $M_n^{(r)}$'s for *some* $r \geq 3$ implies convergence of *all other* $M_n^{(q)}$'s, $2 \leq q \leq r-1$. This was proved by Hsing [Hsi88, Theorem 3.3].

Assuming that $\{M_n^{(q)}\}$ converge weakly *for each* $q \in I\!N$—what is almost the convergence in law of the corresponding point processes of exceedances—Dziubdziela [Dzi84] and Hsing, Hüsler & Leadbetter [HHL88] describe possible limits in terms of parameters of certain compound Poisson distributions. We prefer the description given by Hsing [Hsi88] (see also Theorem 9.2 below): the limit for $M_n^{(q)}$ is of the form

$$G(x)\left(1+\sum_{k=1}^{q-1}\frac{(-\log G(x))^k}{k!}\cdot\gamma_{q,k}\right), \tag{9.3}$$

where $0 \leq \gamma_{q,k} \leq 1$, $k = 1,2,\ldots,q-1$, and G is the limit for maxima. However, complexity of formulas for $\gamma_{q,k}$'s quickly increases with q, what makes difficult the analysis of asymptotic properties of higher order statistics. Therefore **we suggest taking into account simple approximating models in place of limiting distributions**.

9.2 Asymptotic representations

Let $\beta_1,\beta_2,\ldots,\beta_r \geq 0$ be such that

$$\sum_{q=1}^{r}\beta_q = 1, \tag{9.4}$$

and let G be a distribution function. For each $1 \leq q \leq r$, let $\{\widetilde{Y}_{q,j}\}_{j\in\mathbb{N}}$ be independent, identically distributed:

$$\widetilde{Y}_{q,j} \sim G^{\beta_q}, \tag{9.5}$$

and let sequences $\{\widetilde{Y}_{1,j}\}_{j\in\mathbb{N}},\{\widetilde{Y}_{2,j}\}_{j\in\mathbb{N}},\ldots,\{\widetilde{Y}_{r,j}\}_{j\in\mathbb{N}}$ be mutually independent.

Define new, this time $(r-1)$-dependent sequence:

$$\begin{aligned}\widetilde{X}_j \;=\; & \widetilde{Y}_{1,j}\\ & \vee(\widetilde{Y}_{2,j}\vee\widetilde{Y}_{2,j+1})\\ & \vee(\widetilde{Y}_{3,j}\vee\widetilde{Y}_{3,j+1}\vee\widetilde{Y}_{3,j+2})\\ & \vdots\\ & \vee(\widetilde{Y}_{r,j}\vee\widetilde{Y}_{r,j+1}\vee\ldots\vee\widetilde{Y}_{r,j+r-1}).\end{aligned} \tag{9.6}$$

We will say, that order statistics $M_n^{(1)},M_n^{(2)},\ldots,M_n^{(r)}$ of a stationary sequence $X_1,X_2,\ldots$ **possess (or admit) asymptotic** $(G,\beta_1,\beta_2,\ldots,\beta_r)$**-representation**, if for each $1 \leq q \leq r$

$$\sup_{x\in\mathbb{R}^1}|P(M_n^{(q)} \leq x) - P(\widetilde{M}_n^{(q)} \leq x)| \;\longrightarrow\; 0 \quad \text{as } n\to+\infty, \tag{9.7}$$

where $\widetilde{M}_n^{(q)}$, $q=1,2,\ldots,r$ are order statistics of $\widetilde{X}_1,\widetilde{X}_2,\ldots$ defined by (9.6)

The representation $(G,\beta_1,\beta_2,\ldots,\beta_n)$ is **regular**, if G is regular in the sense of O'Brien ([OBr74a]), i.e. satisfies (6.35). Recall, that this regularity means for some (and then for all) $0<\alpha<1$ one can find a sequence $v_n = v_n(\alpha)$ of numbers satisfying

$$G^n(v_n) \;\longrightarrow\; \alpha, \quad \text{as } n\to+\infty. \tag{9.8}$$

Clearly, if $r=1$, then $(G,1)$-representation coincides with *asymptotic independent representation for maxima* and G itself is *a phantom distribution function* (see Chapter 6).

We know when a $(G,1)$ - representation exists: Theorem 6.17 states, that $\{X_j\}$ admits a *regular* asymptotic independent representation for maxima if and only if there is a sequence $\{v_n\}$ such that for some α, $\quad 0<\alpha<1$,

$$P(M_n \le v_n) \ \longrightarrow\ \alpha, \tag{9.9}$$

and the following condition holds:
Condition $\mathbf{B}_\infty(v_n)$:

$$\sup_{k,l\in \mathbb{N}} |P(M_{k+l}\le v_n) - P(M_k\le v_n)P(M_l\le v_n)| \to 0 \quad \text{as } n\to+\infty. \tag{9.10}$$

We aim at extending the above result to the case $r\ge 2$. Let us introduce
Condition $\mathbf{B}_\infty^{(r)}(v_n)$: for all $p,q\ge 0, p+q\le r-1, \quad$ as $n\to+\infty$

$$\sup_{k,l\in \mathbb{N}} |P(M_k^{(p+1)}\le v_n, M_{k:k+l}^{(q+1)}\le v_n) - P(M_k^{(p+1)}\le v_n)P(M_l^{(q+1)}\le v_n)| \to 0. \tag{9.11}$$

Observe that for each $q\in \mathbb{N}$

$$E|I(M_{k-j_n}^{(q)}\le v_n) - I(M_k^{(q)}\le v_n)| \le j_n P(X_1>v_n)$$

uniformly in k. So if $\{j_n\}$ and $\{v_n\}$ are such that $j_nP(X_1>v_n)\to 0$, then Condition $\mathrm{B}_\infty^{(r)}(v_n)$ is equivalent to

$$\sup_{k,l\ge j_n} |P(M_{k-j_n}^{(p+1)}\le v_n, M_{k+j_n:k+l}^{(q+1)}\le v_n) - P(M_{k-j_n}^{(p+1)}\le v_n)P(M_{l-j_n}^{(q+1)}\le v_n)| \ \longrightarrow\ 0 \tag{9.12}$$

as $n\to\infty$, for all $p,q,p+q\le r-1$. Now the blocks are separated and we can use "standard" mixing arguments for checking Condition $\mathrm{B}_\infty^{(r)}(v_n)$, e.g. strong mixing or slightly modified condition $\Delta(v_n)$ defined on p.99 [HHL88].

It is intuitively clear (and can be proved rigorously following the line of the proof of Corollary 6.18), that every sequence $\{X_j\}$ admitting a regular $(G,\beta_1,\ldots,\beta_r)$ - representation satisfies Condition $\mathrm{B}_\infty^{(r)}(v_n)$, *provided* $G(v_n)\to 1$. For the converse we need more information on properties of order statistics with respect to $\{v_n\}$:

Theorem 9.1 *Order statistics $M_n^{(1)}, M_n^{(2)},\ldots,M_n^{(r)}$ of stationary $X_1,X_2,\ldots,$ admit a regular $(G,\beta_1,\beta_2,\ldots,\beta_r)$ - representation if and only if for some non-decreasing sequence $\{v_n\}$ Condition $B_\infty^{(r)}(v_n)$ holds and for each q, $1\le q\le r$*

$$P(M_n^{(q)}\le v_n) \ \longrightarrow\ \alpha_q \qquad \text{as } n\to+\infty, \tag{9.13}$$

where $0<\alpha_1<1$.

Theorem 9.2 *Suppose that $M_n^{(1)}, M_n^{(2)},\ldots,M_n^{(r)}$ admit a regular $(G,\beta_1,\beta_2,\ldots,\beta_r)$ - representation.*

Then for each $2 \leq q \leq r$ *there are numbers* $\gamma_{q,1}, \gamma_{q,2}, \ldots, \gamma_{q,q-1} \in [0,1]$ *such that as* $n \to \infty$

$$\sup_{x \in \mathbb{R}^1} \Big| P(M_n^{(k)} \leq x) - P(M_n^{(1)} \leq x)\Big(1 + \sum_{j=1}^{k-1} \frac{(-\log P(M_n^{(1)} \leq x))^j}{j!} \cdot \gamma_{k,j}\Big)\Big| \longrightarrow 0. \quad (9.14)$$

More precisely

$$\gamma_{q,k} = \sum_{\substack{j_1, j_2, \ldots, j_{q-1} \geq 0 \\ k \leq \sum_{l=1}^{q-1} l j_l \leq q-1 \\ \sum_{l=1}^{q-1} j_l = k}} \frac{k!}{j_1! j_2! \ldots j_{q-1}!} {\beta_1}^{j_1} {\beta_2}^{j_2} \cdot \ldots \cdot {\beta_{q-1}}^{j_{q-1}}. \quad (9.15)$$

Proofs of both theorems are deferred to the next section.

We conclude this section with several comments.

Remark 9.3 We do not know any stationary sequence satisfying Condition $\mathrm{B}_\infty^{(q)}(v_n)$ and not Condition $\mathrm{B}_\infty^{(r)}(v_n)$, with $q < r$.

Remark 9.4 A $(G, \beta_1, \beta_2, \ldots, \beta_r)$ - representation may exist even if $M_n, M_n^{(2)}, \ldots$ does not converge in distribution to a non-degenerate limit under any linear normalization: the trivial example is provided by an *i.i.d.* sequence with *regular* marginal distribution function G which does not belong to the domain of attraction of any max-stable distribution.

Remark 9.5 Welsch [Wel72] described all possible two-dimensional limits for *joint distribution* of suitably normalized and centered maxima and second maxima of strongly mixing stationary sequences. However, Welsch' representation is not easy in handling and becomes very involved for higher order statistics. So again, limiting distributions do not seem to be the best tool for analysis of asymptotic properties of *joint* laws of $(M_n^{(1)}, M_n^{(2)}, \ldots, M_n^{(r)})$. It would be much more interesting to have a simple approximating model, similar to what we introduced above (for one-dimensional distributions only).

Example 1 of [Mor76] gives some hope for the existence of $(r-1)$ - dependent representation in the following sense:

$$\sup_{x_1 \geq x_2 \geq \ldots \geq x_r} \Big| P(M_n^{(1)} \leq x_1, M_n^{(2)} \leq x_2, \ldots, M_n^{(r)} \leq x_r) - P(\widetilde{M}_n^{(1)} \leq x_1, \widetilde{M}_n^{(2)} \leq x_2, \ldots, \widetilde{M}_n^{(r)} \leq x_r)\Big| \to 0. \quad (9.16)$$

Remark 9.6 It is not difficult to identify numbers $\beta_1, \beta_2, \ldots, \beta_r$ considered above with numbers $\pi(1), \pi(2), \ldots, \pi(r)$ defined by Hsing in [Hsi88]. In particular, our formula (9.15) is a "condensed" form of formula defining $\pi^{*l}(i)$ in Theorem 3.3 of [Hsi88].

Remark 9.7 In Theorem 9.1, convergence of *all* $P(M_n^{(q)} \leq v_n)$, $q = 1, 2, \ldots, r$, is necessary, as the following example shows:

Example 9.8 Let $\{X_{1,j}\}_{j\in\mathbb{N}}$, $\{X_{2,j}\}_{j\in\mathbb{N}}$, $\{X_{3,j}\}_{j\in\mathbb{N}}$ be mutually independent sequences of i.i.d. random variables with one-dimensional marginals given by distribution functions F, G and H, respectively. Define

$$X_j = X_{1,j} \vee (X_{2,j} \vee X_{2,j+1}) \vee (X_{3,j} \vee X_{3,j+1} \vee X_{3,j+2}) \tag{9.17}$$

Suppose numbers $v_n \nearrow \infty$ are such that $\sup_n n(1-F(v_n)) < +\infty$, $\sup_n n(1-G(v_n)) < +\infty$ and $H(v_n) \nearrow 1$. Then it is easy to see that

$$P(M_n \le v_n) = \exp\Big(-n(1-F(v_n)+1-G(v_n)+1-H(v_n)\Big) + o(1) \tag{9.18}$$

$$P(M_n^{(2)} \le v_n) = P(M_n \le v_n)\Big(1+n(1-F(v_n))\Big) + o(1) \tag{9.19}$$

$$P(M_n^{(3)} \le v_n) = P(M_n \le v_n)\Big(1+n(1-F(v_n)) + \binom{n}{2}(1-F(v_n))^2 + n(1-G(v_n))\Big) + o(1) \tag{9.20}$$

We shall find F, G and H such, that $\{P(M_n \le n)\}$ and $\{P(M_n^{(3)} \le n)\}$ converge to some limits different from 0 and 1 while $\{P(M_n^{(2)} \le n)\}$ does not converge. By (9.18)–(9.20) it is enough to find non-negative functions $f(x)$, $g(x)$ and $h(x)$, $x \in \mathbb{R}^+$, such that for some $D, E > 0$

$$n\int_n^\infty (f(u)+g(u)+h(u))\,du \longrightarrow D \tag{9.21}$$

$$n\int_n^\infty (f(u)+g(u))\,du + \binom{n}{2}\Big(\int_n^\infty f(u)\,du\Big)^2 \longrightarrow E \tag{9.22}$$

but

$$n\int_n^\infty f(u)\,du \quad \text{does not converge.} \tag{9.23}$$

Set

$$f(x) = C\sum_{k=1}^\infty x^{-2} I(2^{2k} \le x < 2^{2k+1}).$$

Then $2^{2k}\int_{2^{2k}}^\infty f(u)\,du = 2C/3$ while $2^{2k+1}\int_{2^{2k+1}}^\infty f(u)\,du = C/3$, hence (9.23) holds. Now take $E > C + C^2$. Then $Ex^{-2} > f(x)\Big(1 + x\int_x^\infty f(u)\,du\Big)$, and

$$g(x) = Ex^{-2} - f(x)\Big(1 + x\int_x^\infty f(u)\,du\Big) + (1/2)\Big(\int_x^\infty f(u)\,du\Big)^2 > 0.$$

Integration by parts gives (9.22). And (9.21) is satisfied, if we set

$$h(x) = Dx^{-2} - f(x) - g(x)$$

with $D > C + (1/2)C^2 + E$.

9.3 Proofs

9.3.1 Convergence under mixing conditions

Under assumptions of Theorem 9.1, we shall identify limits for r first coefficients of generating functions of $N_n = \sum_{j=1}^n I(X_j > v_n)$.

We know that for $0 \leq q \leq r-1$,

$$P(N_n = q) = P(M_n^{(q+1)} \leq v_n) - P(M_n^{(q)} \leq v_n) \quad \longrightarrow \alpha_{q+1} - \alpha_q,$$

(where $\alpha_0 = 0$). But numbers $\alpha_1, \alpha_2, \ldots, \alpha_r$ are not arbitrary, for Condition $\mathrm{B}_\infty^{(r)}(v_n)$ holds.

Lemma 9.9 *Condition $B_\infty^{(r)}(v_n)$ is equivalent to*

$$\max_{k,l} \Big| P\Big(\sum_{j=1}^k I(X_j > v_n) = p, \sum_{j=k+1}^{k+l} I(X_j > v_n) = q\Big) \tag{9.24}$$

$$- P\Big(\sum_{j=1}^k I(X_j > v_n) = p\Big) \cdot P\Big(\sum_{j=1}^l I(X_j > v_n) = q\Big) \Big| \quad \longrightarrow 0$$

as $n \to \infty$, for every $p,\ q \geq 0,\ p+q \leq r-1$. □

Hence for $0 \leq q \leq r-1$

$$\begin{aligned} P(N_n = q) &= \sum_{q_1+q_2=q} P(N_{[n/2]} = q_1, N_n - N_{[n/2]} = q_2) \\ &= \sum_{q_1+q_2=q} P(N_{[n/2]} = q_1) P(N_n - N_{[n/2]} = q_2) + o(1) \\ &= \sum_{q_1+q_2=q} P(N_{[n/2]} = q_1) P(N_{[n/2]} = q_2) + o(1) \end{aligned}$$

and

$$\begin{aligned} \sum_{q=0}^{r-1} s^q P(N_n = q) &= \sum_{q=0}^{r-1} \sum_{q_1+q_2=q} s^{q_1} P(N_{[n/2]} = q_1) s^{q_2} P(N_{[n/2]} = q_2) + o(1) \\ &= L_r\left(\Big(\sum_{q=0}^{\infty} s^q P(N_{[n/2]} = q) \Big)^2 \right) + o(1), \end{aligned}$$

where for the series $w(s) = \sum_{j=0}^\infty a_j s^j$, $L_r(w(s))$ is the polynomial obtained by taking only r first coefficients:

$$L_r(w(s)) = a_0 + a_1 s + \ldots + a_{r-1} s^{r-1} \tag{9.25}$$

Similar reasoning shows that for each $k \in \mathbb{N}$

$$\sum_{q=0}^{r-1} s^q P(N_n = q) = L_r\left(\Big(\sum_{q=0}^{\infty} s^q P(N_{[n/k]} = q) \Big)^k \right) + o(1), \tag{9.26}$$

It follows that for every $0 \leq q \leq r-1$ and $k \in I\!N$, $\{P(N_{[n/k]} = q)\}_{n \in I\!N}$ converges:

$$\lim_{n\to\infty} P(N_{[n/k]} = q) = \alpha_{q+1,k} - \alpha_{q,k},$$

and that $w(s) = \sum_{q=0}^{r-1}(\alpha_{q+1} - \alpha_q)s^q$ is infinitely divisible at the level r, i.e. for each k there is a generating function $w_k(s)$ such that

$$L_r(\left(w_k(s)\right)^k) = L_r(w(s)). \tag{9.27}$$

Lemma 9.10 *If $w(s)$ is infinitely divisible at the level r and $a_0 > 0$, then there are numbers $\beta_1, \beta_2, \ldots, \beta_r \geq 0$, $\sum_{q=1}^r \beta_q = 1$ (depending only on $a_0, a_1, \ldots, a_{r-1}$), such that*

$$L_r(w(s)) = L_r\Big(\exp(-\log a_0 \cdot \sum_{q=1}^r \beta_q(s^q - 1))\Big). \tag{9.28}$$

Proof. If $a_0 = 1$, set $\beta_1 = \beta_2 = \ldots = \beta_{r-1} = 0$, $\beta_r = 1$. So let $a_0 < 1$ and let $w_k(s) = a_{k,0} + a_{k,1}s + \ldots$. Clearly,

$$a_0 = a_{k,0}^k,$$

hence $a_{k,0} \nearrow 1$. We have also

$$a_1 = k \cdot a_{k,1} \cdot a_{k,0}^{k-1}$$

and for $2 \leq q \leq r-1$

$$a_q = \sum_{\substack{j_0, j_1, \ldots, j_q \geq 0 \\ \sum_{l=0}^q l j_l = q \\ \sum_{l=0}^q j_l = k}} \frac{k!}{j_0! j_1! \ldots j_q!} a_{k,0}^{j_0} \cdot a_{k,1}^{j_1} \cdot \ldots \cdot a_{k,q}^{j_q}. \tag{9.29}$$

If $k \geq q$ and $\sum_{l=1}^q l j_l = q$, then $\sum_{l=1}^q j_l \leq k$, and the above formula can be rewritten as

$$a_q = a_0 \sum_{\substack{j_1, j_2, \ldots, j_q \geq 0 \\ \sum_{l=1}^q l j_l = q}} \frac{1}{j_1! j_2! \ldots j_q!} \times \prod_{l=1}^q (k \cdot a_{k,l})^{j_l} \times \tag{9.30}$$

$$\times\, a_{k,0}^{-(j_1+\ldots+j_q)} \frac{k!}{(k-(j_1+\ldots+j_q))! k^{j_1+\ldots+j_q}}$$

Notice that in each such expression there is a term of the form $a_0 \cdot (k \cdot a_{k,q})/a_{k,0}$ and all other summands depend on $a_{k,m}$ with $m \leq q-1$. It follows that for each $1 \leq q \leq r-1$ there exists

$$\beta_q = -\lim_{k\to\infty} k \cdot a_{k,q}/\log a_0 \geq 0. \tag{9.31}$$

and

$$\sum_{q=1}^{r-1} \beta_q = (-\log a_0)^{-1} \lim_{k\to\infty} k \sum_{q=1}^{r-1} a_{k,q} \leq (-\log a_0)^{-1} k(1 - a_{k,0}) \to 1.$$

Hence

$$\beta_r = 1 - \sum_{q=1}^{r-1} \beta_q \geq 0.$$

Further, going with k to infinity in (9.30), we get

$$a_q = a_0 \sum_{\substack{j_1, j_2, \ldots, j_q \geq 0 \\ \sum_{l=1}^{q} l j_l = q}} \frac{1}{j_1! j_2! \ldots j_q!} \times \prod_{l=1}^{q} {\beta_l}^{j_l} (-\log a_0)^{j_l} \tag{9.32}$$

for $q = 0, 1, \ldots, r-1$, i.e. (9.28). □

By the above lemma and by (9.26)

$$\sum_{q=0}^{r-1} s^q P(N_n = q) = L_r \left(\exp(-\log \alpha_1 \cdot \sum_{q=1}^{r} \beta_q (s^q - 1)) \right) + o(1).$$

for some $\beta_1, \beta_2, \ldots, \beta_r \geq 0$, $\beta_1 + \ldots + \beta_r = 1$. Again by (9.26), for each $k \in \mathbb{N}$

$$\sum_{q=0}^{r-1} s^q P(N_{[n/k]} = q) = L_r \left(\exp\Big(-\frac{1}{k} \log \alpha_1 \cdot \sum_{q=1}^{r} \beta_q (s^q - 1)\Big) \right) + o(1).$$

Repeating arguments we get for all integers $k \geq 1$, $l \geq 0$

$$\sum_{q=0}^{r-1} s^q P(N_{[n(l/k)]} = q) = L_r \left(\exp\Big(-\frac{l}{k} \log \alpha_1 \cdot \sum_{q=1}^{r} \beta_q (s^q - 1)\Big) \right) + o(1). \tag{9.33}$$

Fix q, $1 \leq q \leq r$, and set in Lemma 6.8 $Z_n = M_n^{(q)}$, $g(x) = x$. Consider functions

$$\mathbb{R}^+ \ni t \mapsto f_{q,n}(t) = P(M_{[nt]}^{(q)} \leq v_n) \tag{9.34}$$

and let $f_q(t)$ equals to sum of q first coefficients of the generating function

$$w_t(s) = \exp\Big(-t \log \alpha_1 \sum_{q=1}^{r} \beta_q (s^q - 1)\Big).$$

Then (9.33) means that functions $f_{q,n}$ converge to f_q on the dense set of rationals and a half of assumptions of Lemma 6.8 holds.

9.3.2 Proof of Theorem 9.1

Consider the $(G, \beta_1, \beta_2, \ldots, \beta_r)$-representation defined by (9.6). Define also

$$\widetilde{X}_{q,j} = \widetilde{Y}_{q,j} \vee \widetilde{Y}_{q,j+1} \vee \ldots \vee \widetilde{Y}_{q,j+q-1}. \tag{9.35}$$

We will find the limit for the generating functions of $\widetilde{N}_n = \sum_{j=1}^{n} I(\widetilde{X}_j > v_n)$.

Lemma 9.11 *Let $\{v_n\}$ be such that $G^n(v_n) \to \alpha_1, \quad \alpha_1 > 0$. Then*

$$\lim_{n\to\infty} Es^{\widetilde{N}_n} = \exp\Big(-\log\alpha_1 \cdot \sum_{q=1}^{r} \beta_q (s^q - 1)\Big). \tag{9.36}$$

PROOF. First note that random variables $\widetilde{N}_n$ can be replaced by r independent components:

$$\begin{aligned} E\left|\sum_{j=1}^{n}\Big(I(\widetilde{X}_j > v_n) - \sum_{q=1}^{r} I(\widetilde{X}_{q,j} > v_n)\Big)\right| & \\ \leq \quad & n \sum_{1\leq p<q\leq r} P(\widetilde{X}_{p,1} > v_n) P(\widetilde{X}_{q,1} > v_n) \\ \leq \quad & \sum_{1\leq p<q\leq r} pq\Big(nP(\widetilde{Y}_{p,1} > v_n)\Big) P(\widetilde{Y}_{q,1} > v_n) \\ \longrightarrow \quad & \sum_{1\leq p<q\leq r} pq(-\beta_p \log\alpha_1)\cdot 0 = 0. \end{aligned}$$

Further, each of $\widetilde{N}_{q,n} = \sum_{j=1}^{n} I(\widetilde{X}_{q,j} > v_n)$ is asymptotically equivalent to $q \cdot \widetilde{N}^{(0)}_{q,n} = \sum_{j=1}^{n} q \cdot I(\widetilde{Y}_{q,j} > v_n)$:

$$\begin{aligned} 0 \quad \leq & \quad \sum_{j=1}^{n+q-1} qI(\widetilde{Y}_{q,j} > v_n) - \sum_{j=1}^{n} I(\widetilde{X}_{q,j} > v_n) \\ \leq & \quad q \sum_{j=n+1}^{n+q-1} I(\widetilde{Y}_{q,j} > v_n) + q(q-1) \sum_{\substack{1\leq j<k\leq n \\ k-j\leq q-1}} I(\widetilde{Y}_{q,j} > v_n, \widetilde{Y}_{q,k} > v_n) \end{aligned}$$

and, as $n \to +\infty$

$$E|\widetilde{N}_{q,n} - q\cdot \widetilde{N}^{(0)}_{q,n}| \leq q^2 P(\widetilde{Y}_{q,1} > v_n) + q^3\Big(nP(\widetilde{Y}_{q,1} > v_n)\Big) P(\widetilde{Y}_{q,1} > v_n) \longrightarrow 0.$$

Finally

$$\begin{aligned} Es^{\widetilde{N}_n} \quad & = \quad \prod_{q=1}^{r} Es^{\widetilde{N}_{q,n}} + o(1) \\ & = \quad \prod_{q=1}^{r} E\Big(s^q\Big)^{\widetilde{N}^{(0)}_{q,n}} + o(1) \\ & = \quad \prod_{q=1}^{r} \Big(s^q\Big(1 - G^{\beta_q}(v_n)\Big) + G^{\beta_q}(v_n)\Big)^n + o(1) \\ & = \quad \prod_{q=1}^{r} \left(1 + \frac{n\Big(1 - G^{\beta_q}(v_n)\Big)\Big(s^q - 1\Big)}{n}\right)^n + o(1) \\ & \longrightarrow \quad \prod_{q=1}^{r} e^{-\beta_q \log\alpha_1 (s^q - 1)} \qquad \square \end{aligned}$$

Define functions

$$\widetilde{f}_{q,n}(t) = P(\widetilde{M}^{(q)}_{[nt]} \leq t), \quad q = 1, 2, \ldots, r.$$

$\{\widetilde{X}_j\}$ is an $(r-1)$ - dependent sequence and so satisfies Condition $\mathrm{B}_\infty^{(\infty)}(v_n)$. Hence for $\widetilde{f}_{q,n}$ we can repeat all considerations we did for $f_{q,n}$ defined by (9.34). In particular, $\widetilde{f}_{q,n}(t)$ converge on rationals to the same limit f_q. Applying the "tilde" - part of Lemma 6.8 we see that $\widetilde{M}^{(1)}_n, \widetilde{M}^{(2)}_n, \ldots, \widetilde{M}^{(r)}_n$ form a regular asymptotic representation for $M^{(1)}_n, M^{(2)}_n, \ldots, M^{(r)}_n$. □

9.3.3 Proof of Theorem 9.2

It is enough to prove (9.14) for the $(G, \beta_1, \beta_2, \ldots, \beta_r)$ - representation $\{\widetilde{X}_j\}$.

Fix $2 \leq q \leq r$ and set

$$\begin{aligned} f_n(t) &= P(\widetilde{M}^{(q)}_{[nt]} \leq v_n), \\ \widetilde{f}_n(t) &= P(M_{[nt]} \leq v_n)\Big(1 + \sum_{k=1}^{q-1} \frac{(-\log P(M_{[nt]} \leq v_n))^k}{k!} \cdot \gamma_{q,k}\Big). \end{aligned}$$

By Lemma 6.8 we have to prove that both f_n and $\widetilde{f}_n$ converge on rationals to the same limit $f_q(t)$. It is easy to find limit for $\widetilde{f}_n(t)$:

$$\alpha_1^t\Big(1 + \sum_{k=1}^{q-1} \frac{(-\log \alpha_1^t)^k}{k!}\gamma_{q,k}\Big). \tag{9.37}$$

We know also the limit for $f_n(t)$: by Lemma 9.11 it is the sum of q first coefficients of the generating function $g_t(s) = \exp(-t \log \alpha_1(\sum_{q=1}^r \beta_q s^q - 1))$, i.e.

$$\begin{aligned} f_q(t) &= \alpha_1^t \sum_{m=0}^{q-1} \sum_{\substack{j_1,\ldots,j_m \geq 0 \\ \sum_{l=1}^m l j_l = m}} \prod_{l=1}^m \frac{{\beta_l}^{j_l}}{j_l!}(-\log \alpha_1^t)^{j_l} \\ &= \alpha_1^t \sum_{k=0}^{q-1} \frac{(-\log \alpha_1^t)^k}{k!} \times \sum_{\substack{j_1,j_2,\ldots,j_{q-1} \geq 0 \\ k \leq \sum_{l=1}^{q-1} l j_l \leq q-1 \\ \sum_{l=1}^{q-1} j_l = k}} \frac{k!}{j_1! j_2! \ldots j_{q-1}!} {\beta_1}^{j_1} {\beta_2}^{j_2} \cdot \ldots \cdot {\beta_{q-1}}^{j_{q-1}} \\ &= \alpha_1^t \sum_{k=0}^{q-1} \frac{(-\log \alpha_1^t)^k}{k!}\gamma_{q,k}. \end{aligned}$$

Notice that $\gamma_{q,k} \leq 1$, for the summands are elements of the multinomial expansion of $(\beta_1 + \ldots + \beta_r)^k = 1$. □

Corollary 9.12 *If a regular $(G, \beta_1, \ldots, \beta_r)$ - representation exists and $\{v_n\}$ is such that $P(M_n \leq v_n) \to \alpha_1$, $0 < \alpha_1 < 1$, then for each q, $2 \leq q \leq r$, and $T \geq 1$*

$$\lim_{n\to\infty} P(M^{(q)}_{T\cdot n} \leq v_n) \leq \alpha_1^T\Big(1 + \sum_{k=1}^{q-1} \frac{(-T \log \alpha_1)^k}{k!}\Big). \tag{9.38}$$

□

9.4 Convergence of *all* order statistics

Now assume that for *every* $q \in I\!N$

$$P(M_n^{(q)} \leq v_n) \quad \longrightarrow \quad \alpha_q \quad \text{as } n \to +\infty, \tag{9.39}$$

where $0 < \alpha_1 < 1$, and that Condition $\mathrm{B}_\infty^{(r)}(v_n)$ holds for all $r \in I\!N$ (shortly: **Condition** $\mathbf{B}_\infty^{(\infty)}(v_n)$ holds), i.e. for all $p, q \geq 0$, as $n \to +\infty$

$$\sup_{k,l \in I\!N} |P(M_k^{(p+1)} \leq v_n, M_{k:k+l}^{(q+1)} \leq v_n) - P(M_k^{(p+1)} \leq v_n) P(M_l^{(q+1)} \leq v_n)| \to 0. \tag{9.40}$$

By Theorem 9.2, for every $q \in I\!N$, $P(M_n^{(q)} \leq x)$ admits an asymptotic uniform representation in the form

$$P(M_n \leq x)\Big(1 + \sum_{j=1}^{k-1} \frac{(-\log P(M_n \leq x))^j}{j!} \cdot \gamma_{k,j}\Big), \tag{9.41}$$

where $\gamma_{q,k}$ can be expressed as functions of $\beta_1, \beta_2, \ldots \geq 0$ such that

$$\sum_{q=1}^{\infty} \beta_q \leq 1. \tag{9.42}$$

(See (9.15) for explicit formulas for $\gamma_{q,k}$).

It is not difficult to exclude the possibility $\sum_{q=1}^{\infty} \beta_q < 1$.

Theorem 9.13 *Suppose that conditions (9.39) and (9.40) hold. Then numbers $\gamma_{q,k}$ in (9.41) are built on the base of $\beta_1, \beta_2, \ldots \geq 0$ such that*

$$\sum_{q=1}^{\infty} \beta_q = 1 \tag{9.43}$$

if and only if

$$\lim_{q \to \infty} \alpha_q = 1. \tag{9.44}$$

Proof. Let $N_n = \sum_{k=1}^{n} I(X_k > v_n)$. By the well-known relation

$$\{M_n^{(q+1)} \leq v_n\} = \{N_n \leq q\} \tag{9.45}$$

we may interpret (9.44) as *tightness* of $\{N_n\}_{n \in I\!N}$. Hence (9.39) and (9.44) imply that N_n converges in law to some integer-valued a.s. finite random variable N. Clearly:

$$Es^N = \sum_{q=0}^{\infty} (\alpha_{q+1} - \alpha_q) s^q,$$

where $\alpha_0 = 0$. So by Lemma 9.11

$$Es^N = \lim_{r\to\infty} \exp\Big(-\log\alpha_1\Big(\sum_{q=1}^{r-1}\beta_q s^q + (1-\sum_{q=1}^{r-1}\beta_q)s^r - 1\Big)\Big). \tag{9.46}$$

Consider the Lévy measures

$$\nu_r(\{q\}) = \beta_q, \quad q = 1,2,\ldots,r-1, \quad \nu_r(\{r\}) = 1 - \sum_{q=1}^{r-1}\beta_q$$

and

$$\nu(\{q\}) = \beta_q, \quad q = 1,2,\ldots.$$

By (9.46), $\sum_{q=1}^{\infty}\beta_q = \nu(I\!N) = \lim_{r\to\infty}\nu_r(I\!N) = 1$.

Conversely, if $\sum_{q=1}^{\infty}\beta_q = 1$, then the limit on the right-hand-side of (9.46) is a generating function. So N_n converges in distribution, hence is tight and by (9.45) condition (9.44) holds. □

The $(r-1)$-dependent sequences considered in Section 2, satisfy assumptions of the above theorem. Another example can be obtained by taking in $(G,\beta_1,\beta_2,\ldots,\beta_r)$-representations formal limit over r:

Example 9.14 Let

$$\begin{aligned}\widetilde{Y}_j &= \widetilde{Y}_{1,j}\\ &\quad\vee(\widetilde{Y}_{2,j}\vee\widetilde{Y}_{2,j+1})\\ &\quad\vee(\widetilde{Y}_{3,j}\vee\widetilde{Y}_{3,j+1}\vee\widetilde{Y}_{3,j+2})\\ &\quad\vdots\end{aligned} \tag{9.47}$$

where, as in (9.6), $\{\widetilde{Y}_{1,j}\}_{j\in I\!N}$, $\{\widetilde{Y}_{2,j}\}_{j\in I\!N},\ldots$ are mutually independent and $\widetilde{Y}_{q,j}\sim G^{\beta_q}$ with $\beta_q\geq 0$.

The problem is that $\widetilde{Y}_j$ can be trivial, i.e. $\widetilde{Y}_j = G_*$ a.s. and that G may be not a phantom distribution function for $\widetilde{Y}_1,\widetilde{Y}_2,\ldots$.

Lemma 9.15
(i) *$\widetilde{Y}_j < G_*$ a.s. if and only if*

$$\sum_{q=1}^{\infty} q\beta_q < \infty. \tag{9.48}$$

(ii) *G is a phantom distribution function for $\widetilde{Y}_1,\widetilde{Y}_2,\ldots$ iff (9.48) holds and $\sum_{q=1}^{\infty}\beta_q = 1$.*
□

Lemma 9.16 *If $\sum_{q=1}^{\infty} q\beta_q < +\infty$, G is regular and $\{v_n\}$ is such that $G^n(v_n)\to\alpha_1$, $0<\alpha_1<1$, then $\{\widetilde{Y}_j\}$ satisfies both (9.39) and Condition $B_{\infty}^{(\infty)}(v_n)$.*

Proof. To prove Condition $B_\infty^{(\infty)}(v_n)$, take $p, q \geq 1, \quad p \leq q$, say, and consider the difference

$$\Delta_{p,q}(k,l) = |P(\widetilde{M}_k^{(p)} \leq v_n, \widetilde{M}_{k:k+l}^{(q)} \leq v_n) - P(\widetilde{M}_k^{(p)} \leq v_n)P(\widetilde{M}_l^{(q)} \leq v_n)|.$$

Let $R_0 \in I\!N$ be such that $\Theta_{R_0} = \sum_{m=1}^{R_0} \beta_m > 0$. Let $\varepsilon > 0$. Take N_q such that

$$\alpha_1^{\Theta_{R_0} N_q}\Big(1 + \sum_{k=1}^{q-1} \frac{(-\Theta_{R_0} N_q \log \alpha_1)^k}{k!}\Big) < \varepsilon. \tag{9.49}$$

Now choose $R \geq R_0$ so large that

$$N_q \sum_{r=R+1}^{\infty} < \varepsilon \tag{9.50}$$

and consider sequences:

$$\begin{aligned}
\widetilde{Y}_j' &= \widetilde{Y}_{1,j} \\
&\quad \vee (\widetilde{Y}_{2,j} \vee \widetilde{Y}_{2,j+1}) \\
&\quad \vdots \\
&\quad \vee (\widetilde{Y}_{R,j} \vee \widetilde{Y}_{R,j+1} \vee \ldots \vee \widetilde{Y}_{R,j+R-1}) \\
\\
\widetilde{Y}_j'' &= (\widetilde{Y}_{R+1,j} \vee \widetilde{Y}_{R+1,j+1} \vee \ldots \vee \widetilde{Y}_{R+1,j+R}) \\
&\quad \vee (\widetilde{Y}_{R+2,j} \vee \widetilde{Y}_{R+2,j+1} \vee \ldots \vee \widetilde{Y}_{R+2,j+R} \vee \widetilde{Y}_{R+2,j+R+1}) \\
&\quad \vdots
\end{aligned}$$

with order statistics $\widetilde{M}_n'^{(q)}$ and $\widetilde{M}_n''^{(q)}$, respectively.

Notice the difference between $\widetilde{Y}_j'$ and $\widetilde{X}_j$ defined by (9.6): if we denote

$$\Theta_R = \sum_{m=1}^{R} \beta_m,$$

then $\widetilde{Y}_j'$ is the $(G^{\Theta_R}, \beta_1/\Theta_R, \ldots, \beta_R/\Theta_R)$-representation, for Θ_R may be different from 1.

We have

$$\{\widetilde{M}_{k:k+l}^{(q)} \leq v_n\} \subset \{\widetilde{M}'^{q}_{k:k+l} \leq v_n\} \subset \{\widetilde{M}_{k:k+l}^{(q)} \leq v_n\} \cup \{\widetilde{M}''_{k:k+l} > v_n\}, \tag{9.51}$$

hence

$$E|I(\widetilde{M}'^{(q)}_{k:k+l} \leq v_n) - I(\widetilde{M}_{k:k+l}^{((q))} \leq v_n)| \leq \min(P(\widetilde{M}_l'^{(q)} \leq v_n), P(\widetilde{M}_l'' > v_n)). \tag{9.52}$$

$\{\widetilde{Y}_j'\}$ being $(R-1)$-dependent sequence, satisfies Condition $B_\infty^{(\infty)}(v_n)$. In particular, by Lemma 9.11

$$P(\widetilde{M}_n' \leq v_n) \quad \longrightarrow \quad \alpha_1', \tag{9.53}$$

$$P(\widetilde{M}_n'^{(q)} \leq v_n) \quad \longrightarrow \quad \alpha_q', \tag{9.54}$$

where $\alpha_1' = \alpha_1^{\Theta_R}$. By Corollary 9.12, if $l \geq N_q \cdot n$, then

$$\begin{aligned}
P(\widetilde{M}_{k:k+l}^{(q)} \leq v_n) &\leq P(\widetilde{M}_{N_q \cdot n}^{(q)} \leq v_n) \\
&\leq P(\widetilde{M}'^{(q)}_{N_q \cdot n} \leq v_n) \\
&\leq {\alpha_1'}^{N_q}\Big(1 + \sum_{k=1}^{q-1} \frac{(-N_q \log \alpha_1')^k}{k!}\Big) + o(1) \\
&\leq \alpha_1^{\Theta_{R_0} N_q}\Big(1 + \sum_{k=1}^{q-1} \frac{(-\Theta_{R_0} N_q \log \alpha_1)^k}{k!}\Big) + o(1) \\
&< \varepsilon + o(1).
\end{aligned}$$

Similarly, if $k \geq N_q \cdot n$, then

$$P(\widetilde{M}_k^{(p)} \leq v_n) < \varepsilon + o(1).$$

Consequently, by (9.52)

$$\Delta_{p,q}(k,l) < 4\varepsilon \tag{9.55}$$

for large n, provided $k > N_q \cdot n$ or $l > N_q \cdot n$. So we can restrict our attention to $k, l \leq N_q \cdot n$. In this case, (9.52) gives the estimation

$$\Delta_{p,q}(k,l) \leq 4P(\widetilde{M}''_{N_q \cdot n} > v_n). \tag{9.56}$$

And using the inequality

$$1 - (1-\delta)^x \leq 2\delta x \quad \text{if} \quad 0 \leq \delta \leq 1/2,$$

we obtain for large n

$$\begin{aligned}
P(\widetilde{M}''_{N_q \cdot n} > v_n) &= 1 - P(\widetilde{M}''_{N_q \cdot n} \leq v_n) \\
&= 1 - \prod_{r=R+1}^{\infty} (G^{\beta_r}(v_n))^{N_q \cdot n + r - 1} \\
&= 1 - \Big(1 - (1 - G(v_n))\Big)^{N_q \cdot n \sum_{r=R+1}^{\infty} \beta_r + \sum_{r=R+1}^{\infty} (r-1)\beta_r} \\
&\leq 2\Big(n(1 - G(v_n))\Big) \cdot N_q \sum_{r=R+1}^{\infty} \beta_r + 2(1 - G(v_n)) \sum_{r=R+1}^{\infty} (r-1)\beta_r \\
&\leq 2\varepsilon(-\log \alpha_1 + \varepsilon + \varepsilon).
\end{aligned}$$

By (9.55) and (9.56), Condition $\mathrm{B}_{\infty}^{(\infty)}(v_n)$ holds for $\{\widetilde{Y}_j\}$.

Condition (9.39) follows similarly: by (9.52)

$$|P(\widetilde{M}_n^{(q)} \leq v_n) - P(\widetilde{M}_n'^{(q)} \leq v_n)| \leq P(\widetilde{M}_n'' > v_n),$$

while by (9.54) $P(\widetilde{M}_n'^{(q)} \leq v_n)$ converges. □

It follows that at least for some stationary sequences, $\{\widetilde{Y}_j\}_{j\in\mathbb{N}}$ provides a universal model for *all* order statistics. If for each $q \in \mathbb{N}$

$$\sup_{x\in\mathbb{R}^1} |P(M_n^{(q)} \leq x) - P(\widetilde{M}_n^{(q)} \leq x)| \;\longrightarrow\; 0 \quad \text{as } n \to +\infty, \tag{9.57}$$

where $\widetilde{M}_n^{(q)}$'s are order statistics for $\{\widetilde{Y}_j\}$, then we say that $\{X_j\}_{j\in\mathbb{N}}$ admits an asymptotic $(G, \beta_1, \beta_2, \ldots)$ - representation. The term **"regular $(G, \beta_1, \beta_2, \ldots)$ - representation"** means that G is regular, $\sum_{q=1}^{\infty} \beta_q = 1$ and $\sum_{q=1}^{\infty} q\beta_q < +\infty$.

We are not able to give necessary and sufficient conditions for the existence of a regular $(G, \beta_1, \beta_2, \ldots)$ - representation. However, we have found simple and natural *sufficient* conditions:

Theorem 9.17 *If for some non-decreasing sequence $\{v_n\}$ of numbers, a stationary sequence $\{X_j\}$ satisfies (9.39), Condition $B_{\infty}^{(\infty)}(v_n)$ and*

$$\sup_{n\in\mathbb{N}} nP(X_1 > v_n) < +\infty, \tag{9.58}$$

then there exists a regular $(G, \beta_1, \beta_2, \ldots)$ - representation for all order statistics of $\{X_j\}$.

Proof. Let, as in the proof of Theorem 9.13, $N_n = \sum_{k=1}^{n} I(X_k > v_n)$. By (9.58), $\{N_n\}$ is a tight sequence, hence (9.44) holds and by Theorem 9.13

$$Es^{N_n} \;\longrightarrow\; Es^N = \exp(-\log\alpha_1(\sum_{q=1}^{\infty}\beta_q s^q - 1)),$$

where $\sum_{q=1}^{\infty} \beta_q = 1$. Moreover, by Theorem 5.3 [Bil68],

$$EN \leq \liminf_{n\to\infty} EN_n \leq \sup_{n\in\mathbb{N}} nP(X_1 > v_n) < +\infty.$$

This implies that $\sum_{q=1}^{\infty} q\beta_q < +\infty$ and using $\beta_1, \beta_2, \ldots$ we can construct a non-trivial sequence $\{\widetilde{Y}_j\}$. By Lemma 9.16, $\widetilde{N}_n = \sum_{k=1}^{n} I(\widetilde{Y}_j > v_n)$ converges in distribution to N, either. Now we can check (9.57) the same way as in Section 3 for the finite case $r < +\infty$. □

Remark 9.18 When all order statistics converge in distribution, the case $\sum_{q=1}^{\infty} q\beta_q < +\infty$ gives limits considered in [Dzi84], while $\sum_{q=1}^{\infty} \beta_q = 1$ corresponds to the convergence of point processes treated in [HHL88]. However, we cannot find any sequence $\{X_j\}$ with limits (for $M_n^{(q)}$'s) computed from $\beta_1, \beta_2, \ldots$ such that $\sum_{q=1}^{\infty} \beta_q = 1$ but $\sum_{q=1}^{\infty} q\beta_q = +\infty$. The theory would also be completed, if we could find such $\{X_j\}$ for the case $\sum_{q=1}^{\infty} \beta_q < 1$.

Appendix A

Stable distributions

A.1 Definitions

A probability measure μ on $\mathbb{R}^1$ is said to be *stable* if for every $b_1 > 0$ and $b_2 > 0$ one can find $b > 0$ and $a \in \mathbb{R}^1$ such that

$$b_1 X_1 + b_2 X_2 \sim bX + a, \tag{A.1}$$

where $\mathcal{L}(X_1) = \mathcal{L}(X_2) = \mathcal{L}(X) = \mu$, and X_1 and X_2 are independent. Following Feller [Fel71], we say that μ is *strictly* stable, if no additional centering is required:

$$b_1 X_1 + b_2 X_2 \sim bX.$$

In P. Lévy's monograph [Lév54], stable, but not strictly stable distributions are called *quasistable*, while the name *stable* is used for what we have defined as strictly stable.

The following well-known fact (see [Bre68, p.199], [Zol83, p.25]) can be used as an alternative definition.

Proposition A.1

(i) *μ is stable if and only if for each $k \in \mathbb{N}$ there are $b_k > 0$ and $a_k \in \mathbb{R}^1$ such that*

$$X_1 + X_2 + \ldots + X_k = b_k X + a_k, \tag{A.2}$$

where $\mathcal{L}(X_1) = \ldots = \mathcal{L}(X_k) = \mathcal{L}(X) = \mu$, and $X_1, X_2, \ldots, X_k$ are independent.

Moreover, μ is strictly stable, if $a_k = 0$, $k = 2, 3, \ldots$ in (A.2).

(ii) *If (A.2) holds for each $k = 2, 3, \ldots$, then there exists $p \in (0, 2]$ such that*

$$b_k = k^{1/p}, \quad k = 2, 3, \ldots. \tag{A.3}$$

□

The number $p \in (0,2]$ mentioned in the above proposition is called the exponent of μ, and we say that μ is p-stable or, respectively, strictly p-stable. In particular, if b_1, b_2 and b satisfy (A.1), then

$$b_1^p + b_2^p = b^p. \tag{A.4}$$

(A.2) implies that μ is infinitely divisible; its characteristic function is of the form

$$\hat{\mu}(t) = \begin{cases} \exp\left(ita + \int_{\mathbb{R}^1} (e^{itx} - 1)\,\nu(dx)\right) & \text{if} \quad 0 < p < 1, \\ \exp\left(ita + \int_{\mathbb{R}^1} (e^{itx} - 1 - itxI(|x| \le 1))\,\nu(dx)\right) & \text{if} \quad p = 1, \\ \exp\left(ita + \int_{\mathbb{R}^1} (e^{itx} - 1 - itx)\,\nu(dx)\right) & \text{if} \quad 1 < p < 2, \\ \exp\left(ita - (1/2)t^2\sigma^2\right) & \text{if} \quad p = 2, \end{cases} \tag{A.5}$$

where $a \in \mathbb{R}^1$, $\sigma^2 \ge 0$ and $\nu = \nu(p, c_+, c_-)$ is an absolutely continuous measure on $\mathbb{R}^1$ with density

$$f_\nu(x) = (c_+ I(x > 0) + c_- I(x < 0))|x|^{-(1+p)}.$$

If μ is strictly stable, then using the notation introduced by [ArGi80, Chapter 2], we have

- if $0 < p < 1$, then $\mu = \text{Pois}\,(\nu(p, c_+, c_-))$, i.e.

$$\hat{\mu}(t) = \exp\left(\int_{\mathbb{R}^1} (e^{itx} - 1)\,\nu(dx)\right). \tag{A.6}$$

- if $p = 1$, then $\mu = \text{Pois}\,(\nu(1, c, c)) * \delta_a$, i.e.

$$\hat{\mu}(t) = \exp\left(ita + \int_{\mathbb{R}^1} (e^{itx} - 1)\,\nu(dx)\right). \tag{A.7}$$

- if $1 < p < 2$, then $\mu = c_\infty - \text{Pois}\,(\nu(p, c_+, c_-)) = c_1 - \text{Pois}\,(\nu(p, c_+, c_-)) * \delta_{(c_+ - c_-)/(1-p)}$, i.e.

$$\hat{\mu}(t) = \exp\left(\int_{\mathbb{R}^1} (e^{itx} - 1 - itx)\,\nu(dx)\right). \tag{A.8}$$

- if $p = 2$, then $\mu = N(0, \sigma^2)$, i.e.

$$\hat{\mu}(t) = \exp(-(1/2)t^2\sigma^2). \tag{A.9}$$

In particular, if $p \ne 1$ and μ is p-stable, then there exists $a \in \mathbb{R}^1$ such that $\mu * \delta_{-a}$ is strictly p-stable.

It is possible to calculate the above integrals explicitly (see [Zol83] for discussion of various representations). For our purposes, however, the Lévy-Khintchine representation is quite satisfactory, since it operates with quantities a, σ^2 and ν admitting an interpretation in a much wider class than stable distributions only.

A.2 Domains of attraction

Stable distributions coincide with the class of possible (weak) limits for suitably normalized and centered sums of independent and identically distributed summands. Strictly stable distributions are limits for normalized sums (without centering). More precisely, we have

Theorem A.2 *Let $\{X_j\}_{j\in\mathbb{N}}$ be an i.i.d. sequence. If there exist constants $B_n > 0$ and $A_n \in \mathbb{R}^1$ such that*

$$\frac{X_1 + X_2 + \ldots + X_n - A_n}{B_n} \xrightarrow[\mathcal{D}]{} \mu, \tag{A.10}$$

then μ is stable. If

$$\frac{X_1 + X_2 + \ldots + X_n}{B_n} \xrightarrow[\mathcal{D}]{} \mu, \tag{A.11}$$

then μ is strictly stable.

In both cases, if μ is non-degenerated and p-stable, then B_n is a $1/p$-regularly varying sequence.

Conversely, if μ is stable (strictly stable), then one can find $\{X_j\}_{j\in\mathbb{N}}$, $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ such that (A.10) ((A.11)) holds. □

If (A.10) is satisfied for some $\{A_n\}$ and $\{B_n\}$, we say that $\mathcal{L}(X_1)$ is in the domain of attraction of μ ($\mathcal{L}(X_1) \in \mathcal{D}(\mu)$). Necessary and sufficient conditions for $\mathcal{L}(X_1)$ to be in the domain of attraction of a non-degenerated μ, can be found in many textbooks and monographs, starting with [GnKo54, Chapter VII]. We follow [Fel71, Chapter IX].

Theorem A.3 *Suppose that $\mathcal{L}(X_1)$ is non-degenerated. Then*

(i) $\mathcal{L}(X_1) \in \mathcal{D}(N(a,\sigma^2))$ *iff*

$$g(x) = EX_1^2 I(|X_1| \le x) \tag{A.12}$$

is slowly varying.

(ii) $\mathcal{L}(X_1) \in \mathcal{D}(\mu)$, *where μ is p-stable, $0 < p < 2$, iff*

$$h(x) = P(|X_1| > x) \tag{A.13}$$

is $(-p)$-regularly varying and

$$\frac{P(X_1 > x)}{P(|X_1| > x)} \longrightarrow \alpha, \quad \frac{P(X_1 < -x)}{P(|X_1| > x)} \longrightarrow \beta. \tag{A.14}$$

(iii) *Up to a constant factor, normalizing constants can be chosen to satisfy*

$$\frac{n}{B_n^2} EX_1^2 I(|X_1| \le B_n) \longrightarrow 1. \tag{A.15}$$

□

Results due to Rogozin [Rog76] and Maller [Mal78] show that laws δ_a, $a \neq 0$, (strictly 1-stable!) posses "domain of strict attraction", as well.

Theorem A.4 *Suppose $P(|X_1| > x) > 0)$ for every $x > 0$. Then one can find $B_n \to \infty$ such that*

$$\frac{X_1 + X_2 + \ldots + X_n}{B_n} \quad \longrightarrow_{\mathcal{P}} 1, \tag{A.16}$$

if and only if $EX_1 I(|X_1| \leq x) > 0$ for x large enough and

$$\frac{xP(|X_1| \geq x)}{EX_1 I(|X_1| \leq x)} \quad \longrightarrow 0 \quad \text{as } x \to \infty. \tag{A.17}$$

The sequence B_n is then 1-regularly varying and satisfies

$$B_n \sim nEX_1 I(|X_1| \leq B_n). \tag{A.18}$$

□

A.3 Convergence of sums of independent random variables to strictly stable laws

All criteria on convergence to stable laws are based on the general limit theory for independent summands. We restate here a result of this type, being of central importance for the whole paper.

Theorem A.5 (An adaptation of [ArGi80, Theorem 4.7, p.61]) *Let $\{Z_{n,i}; i, n \in \mathbb{N}\}$ be an array of random variables, which are independent and identically distributed in each row. Let μ be a strictly p-stable law.*

In order that

$$\sum_{j=1}^{k_n} Z_{n,j} \quad \underset{\mathcal{D}}{\longrightarrow} \mu, \tag{A.19}$$

it is necessary and sufficient that:

(i)

$$\left.\begin{array}{lcl} k_n P(Z_{n,1} > x) & \longrightarrow & c_+/x^p \\ k_n P(Z_{n,1} < -x) & \longrightarrow & c_-/x^p \end{array}\right\} \forall x > 0 \tag{A.20}$$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} k_n |EZ_{n,1} I(|Z_{n,1}| \leq \delta)| = 0, \tag{A.21}$$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} k_n EZ_{n,1}^2 I(|Z_{n,1}| \leq \delta) = 0, \tag{A.22}$$

provided $0 < p < 1$ and $\mu = \text{Pois}\,(\nu(p, c_+, c_-))$.

(ii) *(A.20) with $c_+ = c_- = c$, (A.22) and*

$$k_n E Z_{n,1} I(|Z_{n,1}| \leq \delta) \longrightarrow a, \tag{A.23}$$

provided $p = 1$ and $\mu = \mathrm{Pois}\,(\nu(1,c,c)) * \delta_a$.

(iii) *(A.20), (A.22) and*

$$k_n E Z_{n,1} I(|Z_{n,1}| \leq 1) \longrightarrow (c_+ - c_-)/(1-p), \tag{A.24}$$

provided $1 < p < 2$ and $\mu = c_1 - \mathrm{Pois}\,(\nu(p, c_+, c_-)) * \delta_{(c_+ - c_-)/(1-p)}$.

(iv)

$$k_n P(|Z_{n,1}| > \varepsilon) \longrightarrow 0, \quad \forall \varepsilon > 0, \tag{A.25}$$

$$k_n E Z_{n,i} I(|Z_{n,1}| \leq 1) \longrightarrow 0, \tag{A.26}$$

$$k_n E Z_{n,1}^2 I(|Z_{n,1}| \leq 1) \longrightarrow \sigma^2, \tag{A.27}$$

provided $p = 2$ and $\mu = N(0, \sigma^2)$. □

All above sets of conditions are equivalent in the case $\mu = \delta_0$ (strictly stable for each p!):

Corollary A.6

$$\sum_{j=1}^{k_n} Z_{n,j} \xrightarrow[\mathcal{P}]{} 0$$

if and only if (A.25),(A.26) and (A.27) with $\sigma^2 = 0$ hold. □

Appendix B

Regularly varying functions

B.1 Definitions and basic properties

A measurable positive function f, defined on some neighbourhood $[x_f, \infty)$ of infinity and such that for some $-\infty < \rho < +\infty$

$$\frac{f(\lambda x)}{f(x)} \longrightarrow \lambda^\rho \quad \text{as } x \to +\infty, \forall \lambda > 0, \tag{B.1}$$

is called *regularly varying* of *index* ρ.

If $\rho = 0$, i.e.

$$\frac{f(\lambda x)}{f(x)} \longrightarrow 1 \quad \text{as } x \to +\infty, \forall \lambda > 0, \tag{B.2}$$

then f is said to be slowly varying. Clearly, if f varies regularly with index ρ then it is of the form $f(x) = x^\rho \ell(x)$, where $\ell(x)$ is slowly varying.

It is well-known that (B.1) can be apparently relaxed:

Theorem B.1 (see e.g. [Sen76, Theorem 1.3]) *If f is measurable positive and defined on $(x_f, +\infty)$ and if*

$$\frac{f(\lambda x)}{f(x)} \longrightarrow \psi(\lambda) \quad \text{as } x \to +\infty, \forall \lambda \in S, \tag{B.3}$$

where $S \subset (0, +\infty)$ is a set of positive Lebesgue measure, then $\psi(\lambda) = \lambda^\rho$ for some $-\infty < \rho < +\infty$ and f is ρ-regularly varying. □

If f is monotone, condition (B.3) can be considerably weakened:

Theorem B.2 (see e.g. [Fel71, Lemma 3, VIII.8]) *If f is positive and monotone on some neighbourhood of infinity, $a_n/a_{n+1} \longrightarrow 1$, $x_n \longrightarrow \infty$ and*

$$a_n f(\lambda x_n) \longrightarrow \psi(\lambda) \in (0, \infty), \quad \text{as } n \to +\infty, \tag{B.4}$$

for each λ in some dense subset Λ of $(0, \infty)$, then f varies regularly. □

The next result is known as **the Uniform Convergence Theorem**:

Theorem B.3 (see e.g. [BGT87, Theorem 1.5.2, p.22]) *The convergence in (B.1) holds uniformly for λ in every compact subset of* $(0, \infty)$. □

Throughout the paper we deal with functions determined by a sequence $\{c_n\}$ of positive numbers:

$$f_{\{c_n\}}(x) = c_{[x]}, \quad x \geq 1, \tag{B.5}$$

where $[x]$ is the integer part of x. If $f_{\{c_n\}}(x)$ is ρ-regularly varying, we say that $\{c_n\}$ is a regularly varying sequence of index ρ.

It can be shown (see [BoSe73], also [BGT87, Theorem 1.9.5, p.52]), that $\{c_n\}$ is regularly varying iff

$$\frac{c_{[\lambda n]}}{c_n} \longrightarrow \psi(\lambda) \in (0, \infty), \quad \text{as } n \to +\infty, \forall \lambda > 0. \tag{B.6}$$

Direct checking of (B.6) is, however, not easy and we prefer the following

Lemma B.4 *$\{c_n\}$ varies ρ-regularly if and only if*

$$\frac{c_{k\cdot n}}{c_n} \longrightarrow k^\rho, \quad \text{as } n \to +\infty, \forall k \in I\!N, \tag{B.7}$$

and for all sequences $\{k_n\}$, $\{l_n\} \subset I\!N$, such that $k_n \to \infty, l_n/k_n \longrightarrow 0$,

$$\frac{c_{k_n}}{c_{k_n+l_n}} \longrightarrow 1, \quad \text{as } n \to +\infty. \tag{B.8}$$

Proof. By (B.7) and (B.8), we have for $p, q \in I\!N$

$$\begin{aligned} \frac{c_{[(p/q)\cdot n]}}{c_n} &= \frac{c_{[p\cdot(n/q)]}}{c_{p\cdot[n/q]}} \cdot \frac{c_{p[n/q]}}{c_{[n/q]}} \cdot \frac{c_{[n/q]}}{c_{q[n/q]}} \cdot \frac{c_{q[n/q]}}{c_n} \\ &\longrightarrow 1 \cdot p^\rho \cdot q^{-\rho} \cdot 1 \quad \text{as } n \to +\infty. \end{aligned}$$

Fix $\lambda > 0$ and let $p_n/q_n \searrow \lambda$ so slowly that still $c_{[(p_n/q_n)\cdot n]}/c_n \to \lambda^\rho$. Since also

$$c_{[\lambda n]}/c_{[(p_n/q_n)\cdot n]} \to 1$$

by (B.8), we see that

$$\frac{c_{[\lambda n]}}{c_n} = \frac{c_{[\lambda n]}}{c_{[(p_n/q_n)\cdot n]}} \cdot \frac{c_{[(p_n/q_n)\cdot n]}}{c_n} \longrightarrow \lambda^\rho,$$

i.e. (B.6) holds. □

Remark B.5 "Regular variation on integers", i.e. relation (B.7), is a weaker property, than regular variation "on positive reals": sequence $c_n = \omega(n) + \sqrt{\log\log n}$, where $\omega(n)$ is the number of prime divisors of n, satisfies (B.7), but not $c_n/c_{n-1} \to 1$ (see [GaSe73]).

B.2 Smooth and monotone equivalents

We say that functions f_1 and f_2 are asymptotically equivalent ($f_1 \sim f_2$), if

$$\frac{f_1(x)}{f_2(x)} \longrightarrow 1 \quad \text{as } x \to +\infty.$$

Theorem B.6 (see e.g. [BGT87, Theorem 1.3.3, p.14]) *Let ℓ be slowly varying. Then $\ell \sim \ell_1$, where $\ell_1 \in C^\infty[a,\infty)$, and $h_1(x) := \log \ell_1(e^x)$ has the property*

$$h_1^{(n)}(x) \quad \longrightarrow \quad 0, \quad \text{as } x \to +\infty, \quad n = 1, 2, \ldots. \tag{B.9}$$

□

Theorem B.7 (see e.g. [BGT87, Theorem 1.5.3, p.23]) *If f is ρ-regularly varying, $\rho \neq 0$, then $f \sim f_1$, where f_1 is non-decreasing if $\rho > 0$, and non-increasing, if $\rho < 0$.* □

Now, let $\rho > 0$ and let f be ρ-regularly varying on $[x_f, \infty)$. Then

$$f^{\leftarrow}(x) := \inf\{y \geq x_f; f(y) > x\} \tag{B.10}$$

is defined on $[f(x_f), \infty)$ and is monotone increasing to $+\infty$. Further,

$$f(f^{\leftarrow}(x)) \sim f^{\leftarrow}(f(x)) \sim x \quad \text{as } x \to +\infty.$$

$f^{\leftarrow}$ is an example of an "asymptotic inverse" of f.

Theorem B.8 (see e.g. [BGT87, Theorem 1.5.12, p.28]) *If f is regularly varying with index $\rho > 0$, there exists $1/\rho$-regularly varying g such that*

$$f(g(x)) \sim g(f(x)) \sim x \quad \text{as } x \to +\infty. \tag{B.11}$$

Here g is determined uniquely to within asymptotic equivalence, and one version of g is $f^{\leftarrow}$. □

Corollary B.9 *Let $a, b > 0$ and let $f(x) \sim x^{ab}(\ell(x^b))^a$, where ℓ is slowly varying. If g is an asymptotic inverse of f, then*

$$g(x) \sim x^{\frac{1}{ab}} \left(\ell^{\#}(x^{\frac{1}{a}})\right)^{\frac{1}{b}}, \tag{B.12}$$

where $\ell^{\#}$ is the de Bruijn conjugate of ℓ, i.e. the unique (up to asymptotic equivalence) slowly varying function satisfying

$$\ell(x)\ell^{\#}(x\ell(x)) \to 1, \quad \ell^{\#}(x)\ell(x\ell^{\#}) \to 1, \quad \text{as } x \to +\infty. \tag{B.13}$$

□

In most cases, explicit calculation of the de Bruijn conjugate (or the asymptotic inverse) is not an easy task. On the other hand, it is trivial, if f is monotone.

Lemma B.10 *Let f be positive and monotone on some neighbourhood of infinity. Let $\{B_n\}_{n\in\mathbb{N}}$ and $\{C_n\}_{n\in\mathbb{N}}$ vary regularly with index $\beta > 0$ and $\gamma \in \mathbb{R}^1$, respectively. If*

$$C_n f(B_n) \quad \longrightarrow \quad a \in (0,\infty), \tag{B.14}$$

then f varies regularly with index $\rho = -\gamma/\beta$.

PROOF. Suppose f is non-increasing (then $\gamma \geq 0$). Fix $\lambda > 0$ and let $0 < \lambda' < \lambda < \lambda'' < +\infty$. we know, that for each $\theta > 0$,

$$B_{[\theta^{(1/\beta)}\cdot n]} \sim \theta \cdot B_n,$$

hence for n large enough,

$$C_n f(B_{[(\lambda'')^{1/\beta}.B_n]}) \leq C_n f(\lambda \cdot B_n) \leq C_n f(B_{[(\lambda')^{1/\beta}.B_n]}).$$

Now, $C_n/C_{[(\lambda')^{1/\beta}.B_n]} \to ((\lambda')^{1/\beta})^{\gamma}$ and the expression on the right approaches $a \cdot (\lambda')^{-\gamma/\beta}$. Similarly, left-hand-side converges to $a \cdot (\lambda'')^{-\gamma/\beta}$ and, consequently, the middle term tends to $a \cdot (\lambda)^{-\gamma/\beta}$, for every $\lambda > 0$. This implies that f varies regularly (see Theorem B.2), hence

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lim_{n\to\infty} \frac{C_n f(\lambda B_n)}{C_n f(B_n)} = \lambda^{-\gamma/\beta}.$$

□

B.3 Karamata's Theorem

Theorem B.11 (The Direct Half) (see Theorem 1.5.11, p.28, in [BGT87]) *Let f varies regularly with index ρ, and let x_f be such that f is locally bounded in $[x_f,\infty)$. Then for any $\sigma \geq -(\rho+1)$*

$$\frac{x^{\sigma+1} f(x)}{\int_{x_f}^{x} t^{\sigma} f(t)\,dt} \quad \longrightarrow \quad \sigma+\rho+1 \quad \text{as } x \to +\infty, \tag{B.15}$$

and for any $\sigma < -(\rho+1)$,

$$\frac{x^{\sigma+1} f(x)}{\int_{x}^{\infty} t^{\sigma} f(t)\,dt} \quad \longrightarrow \quad -(\sigma+\rho+1) \quad \text{as } x \to +\infty. \tag{B.16}$$

Condition (B.16) holds also if $\sigma = -(\rho+1)$ and $\int_{x_f}^{\infty} t^{-(\rho+1)} f(t)\; dt < +\infty$. □

Theorem B.12 (The Converse Half) (see [BGT87, Theorem 1.6.1, p.30]) *Let f be positive and locally integrable on $[x_f,\infty)$.*

If for some $\sigma > -(\rho+1)$ condition (B.15) holds, then f varies regularly with index ρ. Similarly, (B.16) for some $\sigma < -(\rho+1)$ implies ρ-regular variation of f. □

B.4 The Hardy-Littlewood-Karamata Theorem

Theorem B.13 (see e.g. [BGT87, Theorem 1.7.1, p.37]) *Let $U : \mathbb{R}^1 \to \mathbb{R}^1$ be a non-decreasing right-continuous function with $U(x) = 0$ for all $x < 0$. If ℓ varies slowly and $c > 0,\ \rho \geq 0$, the following are equivalent*

$$U(x) \sim cx^\rho \ell(x)/?(1+\rho) \quad \text{as } x \to +\infty, \tag{B.17}$$

$$\hat{U}(s) \sim cs^{-\rho}\ell(\frac{1}{s}) \quad \text{as } s \to 0+. \tag{B.18}$$

where $\hat{U}(s) = \int_{[0,\infty)} e^{-sx}\, dU(x)$. □

Remark B.14 The following also holds: $U(x) = o(x^\rho \ell(x))$ is equivalent to $\hat{U}(s) = o(s^{-\rho}\ell(1/s))$.

Remark B.15 Implication (B.17) ⇒ (B.18) is usually called "Abelian theorem", while the converse one is "Tauberian". The next result is known as **"Karamata's Tauberian Theorem for power series"**

Corollary B.16 (see e.g. [BGT87, Corollary 1.7.3, p.40]) *Suppose that $a_n \geq 0$ and the power series $A(s) = \sum_{n=0}^\infty a_n s^n$ converges for $s \in [0, 1)$.*

If $c, \rho \geq 0$, then

$$\sum_{k=0}^n a_k \sim cn^\rho \ell(n)/?(1+\rho) \quad \text{as } n \to +\infty \tag{B.19}$$

if and only if

$$A(s) \sim c\ell(\frac{1}{1-s})/(1-s)^\rho \quad \text{as } s \nearrow 1-. \tag{B.20}$$

If $c\rho > 0$ and a_n is ultimately monotone, both (B.19) and (B.20) are equivalent to

$$a_n \sim cn^{\rho-1}\ell(n)/?(\rho) \quad \text{as } n \to +\infty. \tag{B.21}$$

□

Bibliography

[Aar86] Aaronson, J., *Random f-expansions*, Ann. Probab., **14** (1986) 1037–1057.

[Asm87] Asmussen, S., **Applied Probability and Queues**, Wiley, New York 1987.

[ArGi80] Araujo, A. & Giné, E., **The Central Limit Theorem for Real and Banach Valued Random Variables**, Wiley, New York 1980.

[Bil68] Billingsley, P., **Convergence of Probability Measures**, Wiley, New York 1968.

[BGT87] Bingham, N.H., Goldie, C.M. & Teugels, J.L., **Regular Variation**, Cambridge University Press, Cambridge 1987.

[BoSe73] Bojanic, R. & Seneta, E., *A unified theory of regularly varying sequences*, Math. Zeitschrift, **134** (1973) 91-106.

[Bra80] Bradley, R.C., *A remark on the Central Limit Question for dependent random variables*, J. Appl. Prob., **17** (1980) 94-101.

[Bra81] Bradley, R.C., *A sufficient condition for linear growth of variances in a stationary sequence*, Proc. Amer. Math. Soc. **83** (1981) 586–589.

[Bra86] Bradley, R.C., *Basic Properties of strong mixing conditions*, *in:* Eberlein, E. & Taqqu, M.S., Eds., **Dependence in Probability and Statistics** pp. 165–192, *Progress in Probab. and Statist.*, Vol. 11., Birkhäuser, Boston 1986.

[Bra87] Bradley, R.C., *The central limit question under ρ-mixing*, Rocky Mountain J. Math., **17** (1987) 95–114.

[Bra88] Bradley, R. C., *A Central Limit Theorem for Stationary ϱ-mixing Sequences with Infinite Variances*, Ann. Proba., **16** (1988), 313–333.

[BrBr85] Bradley, R.C. & Bryc, W., *Multilinear forms and measures of dependence between random variables*, J. Multivariate Anal., **16** (1985) 335-367.

[Bre68] Breiman, L., **Probability**, Addison-Wesley, Reading, Mass. 1968.

[BDD86] Burton, R.M., Dabrowski, A.R. & Dehling, H., *An Invariance Principle for weakly associated random vectors*, Stoch. Proc. Appl., **23** (1986) 301-306.

[Dab87] Dabrowski, A. R., *Invariance principles in probability for stable processes generated by a class of dependent sequences*, Canad. J. Stat., **15** (1987) 253–267.

[DaDe88] Dabrowski, A. R. & Dehling, H., *A Berry-Esséen Theorem and a Functional law of the Iterated Logarithm for weakly associated random vectors*, Stoch. Proc. Appl., **30** (1988) 277-289.

[Dav83] Davis, R. A., *Stable limits for partial sums of dependent random variables*, Ann. Probab., **11** (1983) 262–269.

[DDP86] Dehling, H., Denker, M. & Philipp, W., *Central limit theorems for mixing sequences of random variables under minimal conditions*, **14** (1986) 1359–1370.

[Den86] Denker, M., *Uniform integrability and the central limit theorem for strongly mixing processes*, *in:* Eberlein, E. & Taqqu, M.S., Eds., **Dependence in Probability and Statistics** pp. 269–274, *Progress in Probab. and Statist.*, Vol. 11., Birkhäuser, Boston 1986.

[DeJa89] Denker, M. & Jakubowski, A., *Stable limit distributions for strongly mixing sequences*, Stat. and Proba. Letters, **8** (1989) 477-483.

[Dia55] Diananda, P.H., *The central limit theorem for m-dependent variables*, Proc. Cambridge Phil. Soc., **51** (1955) 92–95.

[Dzi84] Dziubdziela, W., *Limit laws for kth order statistics from strong-mixing processes*, J. Appl. Prob., **21** (1984) 720–729.

[EPW67] Esary, J., Proschan, F. & Walkup, D., *Association of random variables with applications*, Ann. Math. Statist., **38** (1967) 1466-1474.

[Fel71] Feller, W., **An Introduction to Probability Theory and Its Applications, Vol. II, 2nd Ed.**, Wiley, New York 1971.

[Gal78] Galambos, J., **The asymptotic theory of extreme order statistics**, Wiley, New York 1987.

[GaSe73] Galambos, J. & Seneta, E., *Regularly varying sequences*, Proc. Amer. Math. Soc., **41** (1973) 110-116.

[Geb41] Gebelain, H., *Das Statistische Problem der Korrelation als Zusammenhang mit der Ausgleichungsrechnung*, Z. Angew. Math. Mech., **21** (1941) 364-379.

[GnKo54] Gnedenko, B.V. & Kolmogorov, A.N., **Limit Distributions for Sums of Independent Random Variables**, Addison-Wesley, Reading, Mass. 1954.

[HaHe80] Hall, P. & Heyde, C., **Martingale Limit Theory and its Applications**, Academic Press, New York 1980.

[Hei82] Heinrich, L., *A method for the derivation of limit theorems for sums of m-dependent random variables*, Z. Wahr. verw. Gebiete, **60** (1982) 501–515.

[Hei85] Heinrich, L., *Stable limits for sums of m-dependent random variables*, Serdica, **11** (1985) 189–199.

[Hei86] Heinrich, L., *Stable Limit theorems for sums of multiply indexed m-dependent random variables*, Math. Nachr., **127** (1986) 193–210.

[Hir35] Hirschfeld, O., *A connection between correlation and contigency*, Proc. Camb. Phil. Soc., **31** (1935) 520-524.

[HoRo48] Hoeffding, W. & Robbins, H., *The central limit theorem for dependent variables*, Duke Math. J., **15** (1948) 773–780.

[Hsi88] Hsing, T., *On the extreme order statistics for a stationary sequence*, Stoch. Proc. and Appl., **29** (1988) 155–169.

[HHL88] Hsing, T., Hüsler, J. & Leadbetter, M.R., *On the Exceedance Point Process for a Stationary Sequence*, Probab. Th. Rel. Fields, **78** (1988) 97–112.

[Hüs86] Hüsler, J., *Extreme values of nonstationary random sequences*, J.Appl.Prob., **23** (1986) 937–950.

[Ibr59] Ibragimov, I.A., *Some limit theorems for strictly stationary stochastic processes*, Dokl. Akad. Nauk SSSR, **125** (1959) 711-714.

[Ibr75] Ibragimov, I. A., *A note on the central limit theorem for dependent random variables*, Theory Probab. Appl., **20** (1975) 135–140.

[IbLi71] Ibragimov, I.A. & Linnik Y.V., **Independent and Stationary Sequences of Random Variables**, Walters-Nordhoff, Gröningen 1971.

[Ios77] Iosifescu, M., *Limit theorems for ϕ-mixing sequences. A survey. in:* **Proc. of the Fifth Conf. on Probab. Theory, Brasov 1974**, pp. 51–57, Editura Acad. R.S.R., Bucuresti 1977.

[JaSh87] Jacod, J. & Shiryaev, A. N., **Limit Theorems for Stochastic Processes**, Springer, Berlin 1987.

[Jak86] Jakubowski, A., *Principle of Conditioning in limit theorems for sums of random variables*, Ann. Probab., **14** (1986) 902–915.

[Jak90a] Jakubowski, A., *An asymptotic independent representation in limit theorems for maxima of nonstationary random sequences*, *tentatively accepted in* The Annals of Probab., (1990+).

[Jak90b] Jakubowski, A., *Limits for maxima in terms of joint distributions of a fixed dimension*, *submitted*, (1990+).

[Jak90c] Jakubowski, A., *Minimal Conditions in p-stable limit theorems*, *submitted*, (1990+).

[Jak91a] Jakubowski, A., *Relative extremal index of two stationary sequences*, Stoch. Proc. and their Appl., **37** (1991) 281-297.

[Jak91b] Jakubowski, A., *Asymptotic $(r-1)$-dependent representation for rth order statistic from a stationary sequence*, *submitted*, (1991+).

[JaKo89] Jakubowski, A. & Kobus, M., *α-stable limit theorems for sums of dependent random vectors*, J. Multivariate Anal., **29** (1989) 219–251.

[JaSł86] Jakubowski, A. & Słomiński, L., *Extended Convergence to Continuous in Probability Processes with Independent Increments*, Probab. Th. Rel. Fields, **72** (1986) 55–82.

[JaSz90] Jakubowski, A. & Szewczak, Z. S., *A normal convergence criterion for strongly mixing stationary sequences*, *in:* Coll. Math. Soc. János Bolyai **57.** *Limit Theorems in Probability and Statistics, Pécs (Hungary) 1989),* pp. 281–292, North-Holland, Amsterdam 1990.

[Kal83] Kallenberg, O., **Random measures**, Akademie-Verlag, Berlin 1983.

[KeOB76] Kesten, H. & O'Brien, G.L., *Examples of mixing sequences*, Duke Math. J., **43** (1976) 405-415.

[Kob90] Kobus, M., *Generalized Poisson distributions as limits for arrays of dependent random vectors*, *preprint*, (1990+).

[Lea74] Leadbetter, M.R., *On extreme values in stationary sequences*, Z. Wahr. verw. Gebiete, **28** (1974) 289-303.

[Lea83] Leadbetter, M.R., *Extremes and local dependence in stationary sequences*, Z. Wahr. verw. Gebiete, **65** (1983) 291-306.

[LLR83] Leadbetter, M.R., Lindgren, G. & Rootzén, H., **Extremes and related properties of random sequences and processes**, Springer, Berlin 1983.

[LeRo88] Leadbetter, M.R. & Rootzén, H., *Extremal theory for stochastic processes*, Ann. Probab., **16** (1983) 431-478.

[Lév54] Lévy, P., **Théorie de l'addition des variables aleatoires, 2-me ed.**, Gauthier-Villars, Paris 1954.

[Lin81] Lin, Z., *Limit Theorem for a Class of Sequences of Weakly Dependent Random Variables*, Chinese Ann. Math., **2** (1981), 181–185.

[Loé77] Loève, M., **Probability Theory I, 4th Ed.**, Springer, Berlin 1977.

[Loè78] Loève, M., **Probability Theory II., 4th Ed.**, Springer, Berlin 1978.

[Loy65] Loynes, R.M., *Extreme values in uniformly mixing stationary processes*, Ann. Math. Statist., **36** (1965) 993-999.

[Mal78] Maller, R.A., *Relative stability and the Strong Law of Large Numbers*, Z. Wahr. verw. Gebiete, **43** (1978) 144-148.

[Mei56] Meizler, D.G., *On limit distributions for the maximum term of a variational series*, N. Zap. Lwow. Polit. Inst. (ser.math.-phys.), **38** (1956) 90–109 (in Russian).

[Mor76] Mori, T.,*Limit laws for maxima and second maxima from strong-mixing processes*, Ann. Probab., **4** (1976) 122–126.

[New64] Newell, G. F., *Asymptotic extremes for m-dependent random variables*, Ann. Math. Statist., **35** (1964) 1322-1325.

[New80] Newman, C.M., *Normal fluctuations and the FKG inequalities*, Comm. Math. Phys., **74** (1980) 119-128.

[New84] Newman, C.M., *Asymptotic independence and limit theorems for positively and negatively dependent random variables*, *in:* **Inequalities in Statistics and Probability**, pp. 127–140, *IMS Lecture Notes, vol. 5*, (1984).

[OBr74a] O'Brien, G.L., *Limit theorems for the maximum term of a stationary processes*, Ann. Probab.,**2** (1974) 540-545.

[OBr74b] O'Brien, G.L., *The maximum term of uniformly mixing stationary processes*, Z. Wahr. verw. Gebiete, **30** (1974) 57-63.

[OBr86] O'Brien, G.L., *Extreme values for stationary sequences*, *in:* Eberlein, E. & Taqqu, M.S., Eds., **Dependence in Probability and Statistics** pp. 429–437, *Progress in Probab. and Statist.*, Vol. 11., Birkhäuser, Boston 1986.

[OBr87] O'Brien, G.L., *Extreme values for stationary and Markov sequences*, Ann. Probab., **15** (1987) 281-291.

[Pel82] Peligrad, M., *Invariance principles for mixing sequences of random variables*, Ann. Probab., **10** (1982) 968–981.

[Pel83] Peligrad, M., *A note on two measures of dependence and mixing sequences*, Adv. Appl. Prob., **15** (1983) 461–464.

[Pel85] Peligrad, M., *An invariance principle for ϕ-mixing sequences*, Ann. Probab., **13** (1985) 1304–1313.

[Pel86] Peligrad, M., *Recent advances in the Central Limit Theorem and its weak Invariance Principle for mixing sequences of random variables (A survey)*, *in:* Eberlein, E. & Taqqu, M.S., Eds., **Dependence in Probability and Statistics** pp. 193–223, *Progress in Probab. and Statist.*, Vol. 11., Birkhäuser, Boston 1986.

[Pel90] Peligrad, M., *On Ibragimov-Iosifescu conjecture for ϕ-mixing sequences*, Stoch. Proc. Appl., **35** (1990) 293–308.

[Phi86] Philipp, W., *Invariance principles for independent and weakly dependent random variables*, *in:* Eberlein, E. & Taqqu, M.S., Eds., **Dependence in Probability and Statistics** pp. 225–268, *Progress in Probab. and Statist.*, Vol. 11., Birkhäuser, Boston 1986.

[Rog76] Rogozin, B.A., *Relatively stable walks*, Theory Proba. Appl, **21** (1976) 375-379.

[Roo88] Rootzén, H., *Maxima and exceedances of stationary Markov chains*, Adv. Appl. Probab., **20** (1988) 371-390.

[Ros56] Rosenblatt, M., *A central limit theorem and a strong mixing condition*, Proc. Natl. Acad. Sci. USA, **42** (1956) 43-47.

[Sam84] Samur, J. D., *Convergence of sums of mixing triangular arrays of random vectors with stationary rows*, Ann. Prob., **12** (1984) 390–426.

[Sam85] Samur, J. D., *A note on the convergence to Gaussian laws of sums of stationary ϕ-mixing arrays*, *in:* **Probability on Banach spaces V**, Lecture Notes in Math. **1153**, pp. 387–399, Springer, Berlin 1985.

[Sam87] Samur, J. D., *On the Invariance Principle for stationary ϕ-mixing triangular arrays with infinitely divisible limits*, Prob. Th. Rel. Fields, **75** (1987) 245–259.

[Sen76] Seneta, E., **Regularly Varying Functions**, *Lecture Notes in Mathematics*, Vol. **506**, Springer, Berlin 1976.

[Sha86] Shao, Q., *An invariance principle for stationary ρ-mixing sequences with infinite variance*, *preprint* (1986+).

[Sze88] Szewczak, Z. S., *On a Central Limit Theorem for m-dependent Sequences*, Bull. of Pol. Ac. of Sc. Math., **36**, No 5–6, (1988), 327-331.

[Sze89] Szewczak, Z. S., *An Invariance Principle for strongly mixing stationary sequences when* $EX_1^2 = +\infty$, *to appear in* Probability and Mathematical Statistics, (1989+).

[Wat54] Watson, G.S., *Extreme values in samples from* m*-dependent stationary stochastic processes*, Ann. Math. Statist., **25** (1954) 798–800.

[Wel72] Welsch, R.E., *Limit laws for extreme order statistics from strong-mixing processes*, Ann. Math. Statist., **43** (1972) 439-446.

[Zol83] Zolotarev, W.M., **One-dimensional stable distributions**, Nauka, Moscow 1983, (in Russian).