Zadanie 1

Zadanie

Uzasadnić, że jeśli G jest grupą, to odwzorowanie

$$G \ni g \mapsto g^{-1} \in G,$$

jest homomorfizmem wtedy i tylko wtedy, gdy grupa G jest abelowa.

Rozważane odwzorowanie jest homomorfizmem wtedy i tylko wtedy, gdy

$$(ab)^{-1} = a^{-1}b^{-1}.$$

dla dowolnych $a, b \in G$.

W szczególności, gdy zachodzi powyższa równość, to stosując powyższą równość dla $a = g^{-1}$ i $b = h^{-1}$, mamy

$$gh = (g^{-1})^{-1}(h^{-1})^{-1} = (g^{-1}h^{-1})^{-1},$$

dla dowolnych $g, h \in G$.

Wiemy też, że

$$(ab)^{-1} = b^{-1}a^{-1}$$

dla dowolnych $a, b \in G$.

Zatem, kontynuując, otrzymujemy

$$gh = (g^{-1}h^{-1})^{-1} = (h^{-1})^{-1}(g^{-1})^{-1} = hg,$$

a więc grupa jest abelowa.

Z drugiej strony, gdy grupa jest abelowa, to $b^{-1}a^{-1} = a^{-1}b^{-1}$, więc

$$(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1},$$

a więc rozważane odwzorowanie jest homorfizmem na mocy uwagi z początku rozwiązania.

$\operatorname{Hom}(\mathbb{Z}_4, Q)$

$$Q = \{\mathrm{Id}, -\mathrm{Id}, I, -I, J, -J, K, -K\}.$$

$$I^2 = J^2 = K^2 = -\mathrm{Id}, \quad IJ = K, \quad JK = I, \quad KI = J, \quad JI = -K, \quad KJ = -I, \quad IK = -J.$$

Mamy

$$\operatorname{ord}(\mathrm{Id}) = 1,\ \operatorname{ord}(-\mathrm{Id}) = 2,\ \operatorname{ord}(I) = 4,\ \operatorname{ord}(-I) = 4,$$

$$\operatorname{ord}(J) = 4,\ \operatorname{ord}(-J) = 4,\ \operatorname{ord}(K) = 4,\ \operatorname{ord}(-K) = 4.$$

Zatem

$$|\operatorname{Hom}(\mathbb{Z}_4, Q)| = 8 \qquad \text{i} \qquad |\operatorname{Mono}(\mathbb{Z}_4, Q)| = 6.$$

		0	1	2	3
	φ_{Id}	Id	Id	Id	Id
	$\varphi_{-\mathrm{Id}}$	Id	$-$ Id	Id	$-$ Id
mono	φ_I	Id	I	$-$ Id	$-I$
mono	φ_{-I}	Id	$-I$	$-$ Id	I
mono	φ_J	Id	K	$-$ Id	$-J$
mono	φ_{-J}	Id	$-K$	$-$ Id	J
mono	φ_K	Id	J	$-$ Id	$-K$
mono	φ_{-K}	Id	$-J$	$-$ Id	K

$\mathrm{End}(\mathbb{Z}_4)$ i $\mathrm{Aut}(\mathbb{Z}_4)$

$$\mathbb{Z}_4 = \{0, 1, 2, 3\}.$$

Mamy

$$\mathrm{ord}(0) = 1, \ \mathrm{ord}(1) = 4, \ \mathrm{ord}(2) = 2, \ \mathrm{ord}(3) = 4.$$

Zatem

$$|\,\mathrm{End}(\mathbb{Z}_4)| = 4 \qquad \text{i} \qquad |\,\mathrm{Aut}(\mathbb{Z}_4)| = 2.$$

Mamy

		0	1	2	3
	φ_0	0	0	0	0
auto	φ_1	0	1	2	3
	φ_2	0	2	0	2
auto	φ_3	0	3	2	1

Uwaga

Gdy stosujemy notację addytywną, to $\varphi_h(k) = kh$.

$$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}.$$

Mamy

$$\mathrm{ord}(0) = 1,\ \mathrm{ord}(1) = 6,\ \mathrm{ord}(2) = 3,\ \mathrm{ord}(3) = 2,\ \mathrm{ord}(4) = 3,\ \mathrm{ord}(5) = 6.$$

Zatem

$$|\,\mathrm{End}(\mathbb{Z}_6)| = 6 \qquad \text{i} \qquad |\,\mathrm{Aut}(\mathbb{Z}_6)| = 2.$$

Mamy

		0	1	2	3	4	5
	φ_0	0	0	0	0	0	0
auto	φ_1	0	1	2	3	4	5
	φ_2	0	2	4	0	2	4
	φ_3	0	3	0	3	0	3
	φ_4	0	4	2	0	4	2
auto	φ_5	0	5	4	3	2	1

$$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}.$$

Mamy

$$\mathrm{ord}(0) = 1, \mathrm{ord}(1) = 6, \mathrm{ord}(2) = 3, \mathrm{ord}(3) = 2, \mathrm{ord}(4) = 3, \mathrm{ord}(5) = 6.$$

Zatem

$$|\,\mathrm{Hom}(\mathbb{Z}_4, \mathbb{Z}_6)| = 2 \qquad \text{i} \qquad |\,\mathrm{Mono}(\mathbb{Z}_4, \mathbb{Z}_6)| = 0.$$

	0	1	2	3
φ_0	0	0	0	0
φ_3	0	3	0	3

$\mathrm{Hom}(\mathbb{Z}_6, \mathbb{Z}_4)$

$$\mathbb{Z}_6 = \{0, 1, 2, 3\}.$$

Mamy

$$\mathrm{ord}(0) = 1, \mathrm{ord}(1) = 4, \mathrm{ord}(2) = 2, \mathrm{ord}(3) = 4.$$

Zatem

$$|\mathrm{Hom}(\mathbb{Z}_6, \mathbb{Z}_4)| = 2 \qquad \text{i} \qquad |\mathrm{Mono}(\mathbb{Z}_6, \mathbb{Z}_4)| = 0.$$

	0	1	2	3	4	5
φ_0	0	0	0	0	0	0
φ_2	0	2	0	2	0	2

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}.$$

$$\mathrm{ord}(0,0) = 1,\ \mathrm{ord}(0,1) = 2,\ \mathrm{ord}(1,0) = 2,\ \mathrm{ord}(1,1) = 2.$$

Zatem

$$|\,\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2)| = 4 \qquad \text{i} \qquad |\,\mathrm{Mono}(\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2)| = 3.$$

Mamy

		0	1
	$\varphi_{0,0}$	$(0,0)$	$(0,0)$
mono	$\varphi_{0,1}$	$(0,0)$	$(0,1)$
mono	$\varphi_{1,0}$	$(0,0)$	$(1,0)$
mono	$\varphi_{1,1}$	$(0,0)$	$(1,1)$

Zatem

$$|\,\mathrm{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)| = |\,\mathrm{Hom}(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2)| = 4 \cdot 4 = 16$$

i

$$|\,\mathrm{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)| = |\,\mathrm{Mono}(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2)| = 3 \cdot 3 - 3 = 6.$$

		0	1
	$\varphi_{0,0}$	$(0,0)$	$(0,0)$
mono	$\varphi_{0,1}$	$(0,0)$	$(0,1)$
mono	$\varphi_{1,0}$	$(0,0)$	$(1,0)$
mono	$\varphi_{1,1}$	$(0,0)$	$(1,1)$

		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
	$(\varphi_{0,0}, \varphi_{0,0})$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
	$(\varphi_{0,0}, \varphi_{0,1})$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,1)$
	$(\varphi_{0,0}, \varphi_{1,0})$	$(0,0)$	$(1,0)$	$(0,0)$	$(1,0)$
	$(\varphi_{0,0}, \varphi_{1,1})$	$(0,0)$	$(1,1)$	$(0,0)$	$(1,1)$
	$(\varphi_{0,1}, \varphi_{0,0})$	$(0,0)$	$(0,0)$	$(0,1)$	$(0,1)$
	$(\varphi_{0,1}, \varphi_{0,1})$	$(0,0)$	$(0,1)$	$(0,1)$	$(0,0)$
auto	$(\varphi_{0,1}, \varphi_{1,0})$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
auto	$(\varphi_{0,1}, \varphi_{1,1})$	$(0,0)$	$(1,1)$	$(0,1)$	$(1,0)$
	$(\varphi_{1,0}, \varphi_{0,0})$	$(0,0)$	$(0,0)$	$(1,0)$	$(1,0)$
auto	$(\varphi_{1,0}, \varphi_{0,1})$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
	$(\varphi_{1,0}, \varphi_{1,0})$	$(0,0)$	$(1,0)$	$(1,0)$	$(0,0)$
auto	$(\varphi_{1,0}, \varphi_{1,1})$	$(0,0)$	$(1,1)$	$(1,0)$	$(0,1)$
	$(\varphi_{1,1}, \varphi_{0,0})$	$(0,0)$	$(0,0)$	$(1,1)$	$(1,1)$
auto	$(\varphi_{1,1}, \varphi_{0,1})$	$(0,0)$	$(0,1)$	$(1,1)$	$(1,0)$
auto	$(\varphi_{1,1}, \varphi_{1,0})$	$(0,0)$	$(1,0)$	$(1,1)$	$(0,1)$
	$(\varphi_{1,1}, \varphi_{1,1})$	$(0,0)$	$(1,1)$	$(1,1)$	$(0,0)$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_4 = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3)\}.$$

$$\operatorname{ord}(0,0) = 1,\ \operatorname{ord}(0,1) = 4,\ \operatorname{ord}(0,2) = 2,\ \operatorname{ord}(0,3) = 4,$$

$$\operatorname{ord}(1,0) = 2,\ \operatorname{ord}(1,1) = 4, \operatorname{ord}(1,2) = 2,\ \operatorname{ord}(1,3) = 4.$$

Zatem

$$|\operatorname{Hom}(\mathbb{Z}_2, \mathbb{Z}_2 \oplus Z_4)| = 4 \qquad \text{i} \qquad |\operatorname{Mono}(\mathbb{Z}_2, \mathbb{Z}_2 \oplus Z_4)| = 3$$

oraz

$$|\operatorname{Hom}(\mathbb{Z}_4, \mathbb{Z}_2 \oplus Z_4)| = 8 \qquad \text{i} \qquad |\operatorname{Mono}(\mathbb{Z}_4, \mathbb{Z}_2 \oplus Z_4)| = 4,$$

więc

$$|\operatorname{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_4)| = |\operatorname{Hom}(\mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_4)| = 4 \cdot 8 = 32.$$

Mamy

	0	1
$\varphi_{0,2}$	$(0,0)$	$(0,2)$
$\varphi_{1,0}$	$(0,0)$	$(1,0)$
$\varphi_{1,2}$	$(0,0)$	$(1,2)$

	0	1	2	3
$\psi_{0,1}$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$
$\psi_{0,3}$	$(0,0)$	$(0,3)$	$(0,2)$	$(0,1)$
$\psi_{1,1}$	$(0,0)$	$(1,1)$	$(0,2)$	$(1,3)$
$\psi_{1,3}$	$(0,0)$	$(1,3)$	$(0,2)$	$(1,1)$

Zatem

$$|\operatorname{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_4)| = |\operatorname{Mono}(\mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_4)| = 3 \cdot 4 - 4 = 8.$$

	0	1
$\varphi_{0,2}$	(0, 0)	(0, 2)
$\varphi_{1,0}$	(0, 0)	(1, 0)
$\varphi_{1,2}$	(0, 0)	(1, 2)

	0	1	2	3
$\psi_{0,1}$	(0, 0)	(0, 1)	(0, 2)	(0, 3)
$\psi_{0,3}$	(0, 0)	(0, 3)	(0, 2)	(0, 1)
$\psi_{1,1}$	(0, 0)	(1, 1)	(0, 2)	(1, 3)
$\psi_{1,3}$	(0, 0)	(1, 3)	(0, 2)	(1, 1)

	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(1, 0)	(1, 1)	(1, 2)	(1, 3)
$(\varphi_{1,0}, \psi_{0,1})$	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(1, 0)	(1, 1)	(1, 2)	(1, 3)
$(\varphi_{1,0}, \psi_{0,3})$	(0, 0)	(0, 3)	(0, 2)	(0, 1)	(1, 0)	(1, 3)	(1, 2)	(1, 1)
$(\varphi_{1,0}, \psi_{1,1})$	(0, 0)	(1, 1)	(0, 2)	(1, 3)	(1, 0)	(0, 1)	(1, 2)	(0, 3)
$(\varphi_{1,0}, \psi_{1,3})$	(0, 0)	(1, 3)	(0, 2)	(1, 1)	(1, 0)	(0, 3)	(1, 2)	(0, 1)
$(\varphi_{1,2}, \psi_{0,1})$	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(1, 2)	(1, 3)	(1, 0)	(1, 1)
$(\varphi_{1,2}, \psi_{0,3})$	(0, 0)	(0, 3)	(0, 2)	(0, 1)	(1, 2)	(1, 1)	(1, 0)	(1, 3)
$(\varphi_{1,2}, \psi_{1,1})$	(0, 0)	(1, 1)	(0, 2)	(1, 3)	(1, 2)	(0, 3)	(1, 0)	(0, 1)
$(\varphi_{1,0}, \psi_{1,3})$	(0, 0)	(1, 3)	(0, 2)	(1, 1)	(1, 2)	(0, 1)	(1, 0)	(0, 3)

$$\mathbb{Z}_3 \oplus \mathbb{Z}_3 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}.$$

$$\operatorname{ord}(0,0) = 1, \qquad \operatorname{ord}(x,y) = 3, \quad (x,y) \neq (0,0).$$

Zatem

$$|\operatorname{Hom}(\mathbb{Z}_3, \mathbb{Z}_3 \oplus Z_3)| = 9 \qquad \text{i} \qquad |\operatorname{Mono}(\mathbb{Z}_3, \mathbb{Z}_3 \oplus Z_3)| = 8,$$

więc

$$|\operatorname{End}(\mathbb{Z}_3 \oplus \mathbb{Z}_3)| = |\operatorname{Hom}(\mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_3 \oplus \mathbb{Z}_3)| = 9 \cdot 9 = 81.$$

Mamy

	0	1	2
$\varphi_{0,1}$	$(0,0)$	$(0,1)$	$(0,2)$
$\varphi_{0,2}$	$(0,0)$	$(0,2)$	$(0,1)$
$\varphi_{1,0}$	$(0,0)$	$(1,0)$	$(2,0)$
$\varphi_{1,1}$	$(0,0)$	$(1,1)$	$(2,2)$
$\varphi_{1,2}$	$(0,0)$	$(1,2)$	$(2,1)$
$\varphi_{2,0}$	$(0,0)$	$(2,0)$	$(1,0)$
$\varphi_{2,1}$	$(0,0)$	$(2,1)$	$(1,2)$
$\varphi_{2,2}$	$(0,0)$	$(2,2)$	$(1,1)$

Zatem

$$|\operatorname{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_4)| = |\operatorname{Mono}(\mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_4)| = 8 \cdot 8 - 8 \cdot 2 = 48.$$

$$D_4 = \{\mathrm{Id}, O_{90^\circ}, O_{180^\circ}, O_{270^\circ}, S_k, S_l, S_m, S_n\}.$$

$\mathrm{ord}(\mathrm{Id}) = 1$, $\mathrm{ord}(O_{90^\circ}) = 4$, $\mathrm{ord}(O_{180^\circ}) = 2$, $\mathrm{ord}(O_{270^\circ}) = 4$, $\mathrm{ord}(S_k) = 2$, $\mathrm{ord}(S_l) = 2$, $\mathrm{ord}(S_m) = 2$, $\mathrm{ord}(S_n) = 2$.

Mamy $D_4 = N \rtimes K$, gdzie

$$N = \langle O_{90^\circ} \rangle = \{\mathrm{Id}, O_{90^\circ}, O_{180^\circ}, O_{270^\circ}\} \quad \text{i} \quad K = \langle S_k \rangle = \{\mathrm{Id}, S_k\}.$$

Wiadomo, że $N \simeq \mathbb{Z}_4$ i $K \simeq \mathbb{Z}_2$. Zatem

$$|\mathrm{Mono}(N, D_4)| = 2 \quad \text{i} \quad |\mathrm{Mono}(K, D_4)| = 5.$$

	$\mathrm{Id} = O_{90^\circ}^0$	$O_{90^\circ} = O_{90^\circ}^1$	$O_{180^\circ} = O_{90^\circ}^2$	$O_{270^\circ} = O_{90^\circ}^3$
$\varphi_{O_{90^\circ}}$	Id	O_{90°	O_{180°	O_{270°
$\varphi_{O_{270^\circ}}$	Id	O_{270°	O_{180°	O_{90°

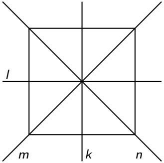
	$\mathrm{Id} = S_k^0$	$S_k = S_k^1$
$\psi_{O_{180^\circ}}$	Id	O_{180°
ψ_{S_k}	Id	S_k
ψ_{S_l}	Id	S_l
ψ_{S_m}	Id	S_m
ψ_{S_n}	Id	S_n

Musimy sprawdzić $2 \cdot 5 - 2 = 8$ par.

$\mathrm{Aut}(D_4)$ II

	Id	O_{90°	O_{180°	O_{270°
$\varphi_{O_{90^\circ}}$	Id	O_{90°	O_{180°	O_{270°
$\varphi_{O_{270^\circ}}$	Id	O_{270°	O_{180°	O_{90°

	Id	S_k
$\psi_{O_{180^\circ}}$	Id	O_{180°
ψ_{S_k}	Id	S_k
ψ_{S_l}	Id	S_l
ψ_{S_m}	Id	S_m
ψ_{S_n}	Id	S_n



$(\varphi_{O_{90^\circ}}, \psi_{S_k})$:

$$\varphi_{O_{90^\circ}}(S_k \circ O_{90^\circ} \circ S_k^{-1}) = \varphi_{O_{90^\circ}}(O_{270^\circ}) = O_{270^\circ}.$$
$$\psi_{S_k}(S_k) \circ \varphi_{O_{90^\circ}}(O_{90^\circ}) \circ \psi_{S_k}(S_k^{-1}) = S_k \circ O_{90^\circ} \circ S_k = O_{270^\circ}. \ \checkmark$$

$(\varphi_{O_{90^\circ}}, \psi_{S_l})$: $\checkmark$
$(\varphi_{O_{90^\circ}}, \psi_{S_m})$: $\checkmark$
$(\varphi_{O_{90^\circ}}, \psi_{S_n})$: $\checkmark$
$(\varphi_{O_{270^\circ}}, \psi_{S_k})$:

$$\varphi_{O_{270^\circ}}(S_k \circ O_{90^\circ} \circ S_k^{-1}) = \varphi_{O_{270^\circ}}(O_{270^\circ}) = O_{90^\circ}.$$
$$\psi_{S_k}(S_k) \circ \varphi_{O_{270^\circ}}(O_{90^\circ}) \circ \psi_{S_k}(S_k^{-1}) = S_k \circ O_{270^\circ} \circ S_k = O_{90^\circ}. \ \checkmark$$

$(\varphi_{O_{270^\circ}}, \psi_{S_l})$: $\checkmark$
$(\varphi_{O_{270^\circ}}, \psi_{S_m})$: $\checkmark$
$(\varphi_{O_{270^\circ}}, \psi_{S_n})$: $\checkmark$

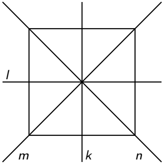
Zatem

$$|\,\mathrm{Aut}(D_4)| = 8.$$

$\mathrm{Aut}(D_4)$ III

	Id	O_{90°	O_{180°	O_{270°
$\varphi_{O_{90^\circ}}$	Id	O_{90°	O_{180°	O_{270°
$\varphi_{O_{270^\circ}}$	Id	O_{270°	O_{180°	O_{90°

	Id	S_k
$\psi_{O_{180^\circ}}$	Id	O_{180°
ψ_{S_k}	Id	S_k
ψ_{S_l}	Id	S_l
ψ_{S_m}	Id	S_m
ψ_{S_n}	Id	S_n



Mamy

$$\mathrm{Id} = \mathrm{Id} \circ \mathrm{Id},\ O_{90^\circ} = O_{90^\circ} \circ \mathrm{Id},\ O_{180^\circ} = O_{180^\circ} \circ \mathrm{Id},\ O_{270^\circ} = O_{270^\circ} \circ \mathrm{Id},$$
$$S_k = \mathrm{Id} \circ S_k,\ S_n = O_{90^\circ} \circ S_k,\ S_l = O_{180^\circ} \circ S_k,\ S_m = O_{270^\circ} \circ S_k.$$

	Id	O_{90°	O_{180°	O_{270°	S_k	S_l	S_m	S_n
$(\varphi_{O_{90^\circ}}, \psi_{S_k})$	Id	O_{90°	O_{180°	O_{270°	S_k	S_l	S_m	S_n
$(\varphi_{O_{90^\circ}}, \psi_{S_l})$	Id	O_{90°	O_{180°	O_{270°	S_l	S_k	S_n	S_m
$(\varphi_{O_{90^\circ}}, \psi_{S_m})$	Id	O_{90°	O_{180°	O_{270°	S_m	S_n	S_l	S_k
$(\varphi_{O_{90^\circ}}, \psi_{S_n})$	Id	O_{90°	O_{180°	O_{270°	S_n	S_m	S_k	S_l
$(\varphi_{O_{270^\circ}}, \psi_{S_k})$	Id	O_{270°	O_{180°	O_{90°	S_k	S_l	S_n	S_m
$(\varphi_{O_{270^\circ}}, \psi_{S_l})$	Id	O_{270°	O_{180°	O_{90°	S_l	S_k	S_m	S_n
$(\varphi_{O_{270^\circ}}, \psi_{S_m})$	Id	O_{270°	O_{180°	O_{90°	S_m	S_n	S_k	S_l
$(\varphi_{O_{270^\circ}}, \psi_{S_n})$	Id	O_{270°	O_{180°	O_{90°	S_n	S_m	S_l	S_k

$$D_4 = \{\mathrm{Id}, O_{90^\circ}, O_{180^\circ}, O_{270^\circ}, S_k, S_l, S_m, S_n\}.$$
$$\mathrm{ord}(\mathrm{Id}) = 1, \mathrm{ord}(O_{90^\circ}) = 4, \mathrm{ord}(O_{180^\circ}) = 2, \mathrm{ord}(O_{270^\circ}) = 4,$$
$$\mathrm{ord}(S_k) = 2, \mathrm{ord}(S_l) = 2, \mathrm{ord}(S_m) = 2, \mathrm{ord}(S_n) = 2.$$

Mamy $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = N \rtimes K$, gdzie

$$N = \{(0,0),(1,0)\} = \langle (1,0)\rangle \qquad \text{i} \qquad K = \{(0,0),(0,1)\} = \langle (0,1)\rangle.$$

Wiadomo, że $N \simeq \mathbb{Z}_2$ i $K \simeq \mathbb{Z}_2$. Zatem

$$|\mathrm{Mono}(N, D_4)| = 5 \qquad \text{i} \qquad |\mathrm{Mono}(K, D_4)| = 5.$$

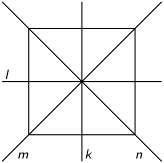
	$(0,0)$	$(1,0)$
$\varphi_{O_{180^\circ}}$	Id	O_{180°
φ_{S_k}	Id	S_k
φ_{S_l}	Id	S_l
φ_{S_m}	Id	S_m
φ_{S_n}	Id	S_n

	$(0,0)$	$(0,1)$
$\psi_{O_{180^\circ}}$	Id	O_{180°
ψ_{S_k}	Id	S_k
ψ_{S_l}	Id	S_l
ψ_{S_m}	Id	S_m
ψ_{S_n}	Id	S_n

Musimy sprawdzić $5 \cdot 5 - 5 = 20$ par.

	$(0,0)$	$(1,0)$
$\varphi_{O_{180^\circ}}$	Id	O_{180°
φ_{S_k}	Id	S_k
φ_{S_l}	Id	S_l
φ_{S_m}	Id	S_m
φ_{S_n}	Id	S_n

	$(0,0)$	$(0,1)$
$\psi_{O_{180^\circ}}$	Id	O_{180°
ψ_{S_k}	Id	S_k
ψ_{S_l}	Id	S_l
ψ_{S_m}	Id	S_m
ψ_{S_n}	Id	S_n



$(\varphi_{O_{180^\circ}}, \psi_{S_k})$: $\varphi_{O_{180^\circ}}((0,1)+(1,0)-(0,1)) = \varphi_{O_{180^\circ}}(1,0) = O_{180^\circ}$.
$\psi_{S_k}(0,1) \circ \varphi_{O_{180^\circ}}(1,0) \circ \psi_{S_k}(0,1)) = S_k \circ O_{180^\circ} \circ S_k = O_{180^\circ}$. ✓

$(\varphi_{O_{180^\circ}}, \psi_{S_l})$: ✓
$(\varphi_{O_{180^\circ}}, \psi_{S_m})$: ✓
$(\varphi_{O_{180^\circ}}, \psi_{S_n})$: ✓
$(\varphi_{S_k}, \psi_{O_{180^\circ}})$: $\varphi_{S_k}(1,0) = S_k$.
$\psi_{O_{180^\circ}}(0,1) \circ \varphi_{S_k}(1,0) \circ \psi_{O_{180^\circ}}(0,1) = O_{180^\circ} \circ S_k \circ O_{180^\circ} = S_k$. ✓

$(\varphi_{S_l}, \psi_{O_{180^\circ}})$: ✓
$(\varphi_{S_m}, \psi_{O_{180^\circ}})$: ✓
$(\varphi_{S_n}, \psi_{O_{180^\circ}})$: ✓
$(\varphi_{S_k}, \psi_{S_l})$: $\varphi_{S_k}(1,0) = S_k$.
$\psi_{S_l}(0,1) \circ \varphi_{S_k}(1,0) \circ \psi_{S_l}(0,1) = S_l \circ S_k \circ S_l = S_k$. ✓

$(\varphi_{S_k}, \psi_{S_m})$: $\varphi_{S_k}(1,0) = S_k$.
$\psi_{S_m}(0,1) \circ \varphi_{S_k}(1,0) \circ \psi_{S_m}(0,1) = S_m \circ S_k \circ S_m = S_l$. ✗

$(\varphi_{S_l}, \psi_{S_n})$: ✗
...

Zatem $|\mathrm{Mono}(\mathbb{Z}_2 \oplus \mathbb{Z}_2, D_4)| = 12$.