Homomorfizmy z grup cyklicznych

Oznaczenia

Jeśli G i H są grupami, to

$$\mathrm{Hom}(G,H) := \{\varphi\colon G \to H : \varphi \text{ jest homomorfizmem grup}\}$$

i

$$\mathrm{Mono}(G,H) := \{\varphi\colon G \to H : \varphi \text{ jest monomorfizmem grup}\}.$$

Opis $\mathrm{Hom}(\mathbb{Z}_n, H)$ i $\mathrm{Mono}(\mathbb{Z}_n, H)$

Mamy

$$\mathrm{Hom}(\mathbb{Z}_n, H) \leftrightarrow \{h \in H : \mathrm{ord}(h) \mid n\}$$

oraz

$$\mathrm{Mono}(\mathbb{Z}_n, H) \leftrightarrow \{h \in H : \mathrm{ord}(h) = n\}.$$

Jeśli $h \in H$ i $\mathrm{ord}(h) \mid n$, to odpowiedni homomorfizm $\varphi_h\colon \mathbb{Z}_n \to H$ jest dany wzorem

$$\varphi_h(k) := h^k \qquad (k \in \mathbb{Z}_n).$$

Oznaczenia

Jeśli G jest grupą, to $\mathrm{End}(G) := \mathrm{Hom}(G,G)$ i

$$\mathrm{Aut}(G) := \{\varphi\colon G \to G : \varphi \text{ jest izomorfizmem grup}\}.$$

Uwaga

Jeśli $|G| < \infty$, to $\mathrm{Aut}(G) = \mathrm{Mono}(G,G)$.

Przykład: $\mathrm{Hom}(\mathbb{Z}_4, D_4)$

Przypomnijmy, że
$$D_4 = \{\mathrm{Id}, O_{90^\circ}, O_{180^\circ}, O_{270^\circ}, S_k, S_l, S_m, S_n\}.$$
Mamy
$$\mathrm{ord}(\mathrm{Id}) = 1,\ \mathrm{ord}(O_{90^\circ}) = 4,\ \mathrm{ord}(O_{180^\circ}) = 2,\ \mathrm{ord}(O_{270^\circ}) = 4,$$
$$\mathrm{ord}(S_k) = 2,\ \mathrm{ord}(S_l) = 2,\ \mathrm{ord}(S_m) = 2,\ \mathrm{ord}(S_n) = 2.$$
Ponieważ $1, 2, 4 \mid 4$, wiec $|\mathrm{Hom}(\mathbb{Z}_4, D_4)| = 8$.

Ponadto $|\mathrm{Mono}(\mathbb{Z}_4, D_4)| = 2$.

Dokładniej mamy

		0	1	2	3
	φ_{Id}	Id	Id	Id	Id
mono	$\varphi_{O_{90^\circ}}$	Id	O_{90°	O_{180°	O_{270°
	$\varphi_{O_{180^\circ}}$	Id	O_{180°	Id	O_{180°
mono	$\varphi_{O_{270^\circ}}$	Id	O_{270°	O_{180°	O_{90°
	φ_{S_k}	Id	S_k	Id	S_k
	φ_{S_l}	Id	S_l	Id	S_l
	φ_{S_m}	Id	S_m	Id	S_m
	φ_{S_n}	Id	S_n	Id	S_n

Definicja

Jeśli G i H są grupami (abelowymi), to definiujemy grupę $G \oplus H$ wzorem:

$$G \oplus H := (G \times H, +),$$

gdzie

- $G \times H$ jest iloczynem kartezjańskim zbiorów G i H, tj.
$$G \times H := \{(g, h) : g \in G \text{ i } h \in H\};$$
- $(g_1, h_1) + (g_2, h_2) := (g_1 +_G g_2, h_1 +_H h_2)$, gdzie $+_G$ i $+_H$ są działaniami w grupach G i H odpowiednio.

Grupę $G \oplus H$ nazywamy sumą prostą grup G i H.

Mamy
$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}.$$

$+$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,0)$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,1)$	$(0,1)$	$(0,2)$	$(0,0)$	$(1,1)$	$(1,2)$	$(1,0)$
$(0,2)$	$(0,2)$	$(0,0)$	$(0,1)$	$(1,2)$	$(1,0)$	$(1,1)$
$(1,0)$	$(1,0)$	$(1,1)$	$(1,2)$	$(0,0)$	$(0,1)$	$(0,2)$
$(1,1)$	$(1,1)$	$(1,2)$	$(1,0)$	$(0,1)$	$(0,2)$	$(0,0)$
$(1,2)$	$(1,2)$	$(1,0)$	$(1,1)$	$(0,2)$	$(0,0)$	$(0,1)$

Opis $\mathrm{Hom}(G_1 \oplus G_2, H)$ i $\mathrm{Mono}(G_1 \oplus G_2, H)$

Jeśli G_1, G_2 i H są grupami abelowymi, to

$$\mathrm{Hom}(G_1 \oplus G_2, H) \leftrightarrow \mathrm{Hom}(G_1, H) \times \mathrm{Hom}(G_2, H).$$

Jeśli $\varphi_1 \in \mathrm{Hom}(G_1, H)$ i $\varphi_2 \in \mathrm{Hom}(G_2, H)$, to odpowiedni homomorfizm $G_1 \oplus G_2 \to H$ ma postać

$$(g_1, g_2) \mapsto \varphi_1(g_1) + \varphi_2(g_2).$$

Ponadto, jeśli $\varphi \leftrightarrow (\varphi_1, \varphi_2)$, to $\varphi \in \mathrm{Mono}(G_1 \oplus G_2, H)$ wtedy i tylko wtedy, gdy

- $\varphi_1 \in \mathrm{Mono}(G_1, H)$, $\varphi_2 \in \mathrm{Mono}(G_2, H)$, oraz
- $|\operatorname{Im} \varphi_1 \cap \operatorname{Im} \varphi_2| = 1$.

$$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}.$$
$$\mathrm{ord}(0) = 1, \mathrm{ord}(1) = 6, \mathrm{ord}(2) = 3, \mathrm{ord}(3) = 2, \mathrm{ord}(4) = 3, \mathrm{ord}(5) = 6.$$

Stąd

$$|\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_6)| = 2,$$
$$|\mathrm{Mono}(\mathbb{Z}_2, \mathbb{Z}_6)| = 1,$$

	0	1
φ_0	0	0
φ_3	0	3

$$|\mathrm{Hom}(\mathbb{Z}_3, \mathbb{Z}_6)| = 3,$$
$$|\mathrm{Mono}(\mathbb{Z}_3, \mathbb{Z}_6)| = 2,$$

	0	1	2
ψ_0	0	0	0
ψ_2	0	2	4
ψ_4	0	4	2

Zatem

$$|\mathrm{Hom}(\mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z}_6)| = 2 \cdot 3 = 6 \qquad \text{i} \qquad |\mathrm{Mono}(\mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z}_6)| = 1 \cdot 2 - 0 = 2.$$

Uwaga

Ponieważ $|\mathbb{Z}_2 \oplus \mathbb{Z}_3| = |\mathbb{Z}_6| < \infty$, więc

$$\mathrm{Mono}(\mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z}_6) = \mathrm{Iso}(\mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z}_6),$$

gdzie

$$\mathrm{Iso}(G, H) := \{\varphi \colon G \to H : \varphi \text{ jest izomorfizmem}\}.$$

W szczególności, $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \simeq \mathbb{Z}_6$, co wynika też z Chińskiego Twierdzenia o Resztach.

	0	1
φ_0	0	0
φ_3	0	3

	0	1	2
ψ_0	0	0	0
ψ_2	0	2	4
ψ_4	0	4	2

		(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)
	(φ_0, ψ_0)	$0+0=0$	$0+0=0$	$0+0=0$	$0+0=0$	$0+0=0$	$0+0=0$
	(φ_0, ψ_2)	$0+0=0$	$0+2=2$	$0+4=4$	$0+0=0$	$0+2=2$	$0+4=4$
	(φ_0, ψ_4)	$0+0=0$	$0+4=4$	$0+2=2$	$0+0=0$	$0+4=4$	$0+2=2$
	(φ_3, ψ_0)	$0+0=0$	$0+0=3$	$0+0=0$	$3+0=3$	$3+0=0$	$3+0=3$
maizo	(φ_3, ψ_2)	$0+0=0$	$0+2=5$	$0+4=4$	$3+0=3$	$3+2=2$	$3+4=1$
maizo	(φ_3, ψ_4)	$0+0=0$	$0+4=1$	$0+2=2$	$3+0=3$	$3+4=4$	$3+2=5$

Homomorfizmy z produktów półprostych

Definicja

Niech $K \leq G$ i $N \trianglelefteq G$.

Grupę G nazywamy produktem półprostym dzielnika N i podgrupy K, jeśli:

$$NK = G \qquad \text{i} \qquad N \cap K = \{1\}.$$

W powyższej sytuacji piszemy $G = N \rtimes K$.

Opis $\mathrm{Hom}(N \rtimes K, H)$

Jeśli $G = N \rtimes K$, to homomorfizmy $G \to H$ są dane przez pary (φ, ψ) takie, że:

- $\varphi \in \mathrm{Hom}(N, H)$ i $\psi \in \mathrm{Hom}(K, H)$;
- $\varphi(knk^{-1}) = \psi(k)\varphi(n)\psi(k^{-1})$, dla wszystkich $n \in N$ i $k \in K$.

Jeśli $\phi \leftrightarrow (\varphi, \psi)$, to

$$\phi(nk) := \varphi(n)\psi(k).$$

Ponadto, w powyższej sytuacji $\phi \in \mathrm{Mono}(G, H)$ wtedy i tylko wtedy, gdy

- $\varphi \in \mathrm{Mono}(N, H)$, $\psi \in \mathrm{Mono}(K, H)$, oraz
- $|\operatorname{Im}\varphi \cap \operatorname{Im}\psi| = 1$.

Uwaga

- Jeśli $|G| < \infty$ i $N \cap K = \{1\}$, to $NK = G$ wtedy i tylko wtedy $|N||K| = |G|$.
- Warunek
$$\varphi(knk^{-1}) = \psi(k)\varphi(n)(\psi(k^{-1})$$
wystarczy sprawdzać dla generatorów.

$$D_3 = \{\mathrm{Id}, O_{120^\circ}, O_{240^\circ}, S_k, S_l, S_m\}$$

$$\mathrm{ord}(\mathrm{Id}) = 1,\ \mathrm{ord}(O_{120^\circ}) = 3,\ \mathrm{ord}(O_{240^\circ}) = 3,\ \mathrm{ord}(S_k) = 2,\ \mathrm{ord}(S_l) = 2,\ \mathrm{ord}(S_m) = 2.$$

Mamy $D_3 = N \rtimes K$, gdzie

$$N = \langle O_{120^\circ} \rangle = \{\mathrm{Id}, O_{120^\circ}, O_{240^\circ}\} \qquad \text{i} \qquad K = \langle S_k \rangle = \{\mathrm{Id}, S_k\}.$$

Wiadomo, że $N \simeq \mathbb{Z}_3$ i $K \simeq \mathbb{Z}_2$. Zatem

$$|\mathrm{Hom}(N, D_3)| = 3, |\mathrm{Mono}(N, D_3)| = 2, |\mathrm{Hom}(K, D_3)| = 4, |\mathrm{Mono}(K, D_3)| = 3.$$

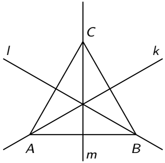
	$\mathrm{Id} = O_{120^\circ}^0$	$O_{120^\circ} = O_{120^\circ}^1$	$O_{240^\circ} = O_{120^\circ}^2$
φ_{Id}	Id	Id	Id
$\varphi_{O_{120^\circ}}$	Id	O_{120°	O_{240°
$\varphi_{O_{240^\circ}}$	Id	O_{240°	O_{120°

	$\mathrm{Id} = S_k^0$	$S_k = S_k^1$
ψ_{Id}	Id	Id
ψ_{S_k}	Id	S_k
ψ_{S_l}	Id	S_l
ψ_{S_m}	Id	S_m

Zatem musimy sprawdzić $3 \cdot 4 = 12$ par.

	Id	O_{120°	O_{240°
φ_{Id}	Id	Id	Id
$\varphi_{O_{120^\circ}}$	Id	O_{120°	O_{240°
$\varphi_{O_{240^\circ}}$	Id	O_{240°	O_{120°

	Id	S_k
ψ_{Id}	Id	Id
ψ_{S_k}	Id	S_k
ψ_{S_l}	Id	S_l
ψ_{S_m}	Id	S_m



$(\varphi_{\mathrm{Id}}, \psi_{\mathrm{Id}})$: $\varphi_{\mathrm{Id}}(S_k \circ O_{120^\circ} \circ S_k^{-1}) = \varphi_{\mathrm{Id}}(O_{240^\circ}) = \mathrm{Id}.$
$\psi_{\mathrm{Id}}(S_k) \circ \varphi_{\mathrm{Id}}(O_{120^\circ}) \circ \psi_{\mathrm{Id}}(S_k^{-1}) = \mathrm{Id} \circ \mathrm{Id} \circ \mathrm{Id} = \mathrm{Id}.$ ✓

$(\varphi_{\mathrm{Id}}, \psi_{S_k})$: $\varphi_{\mathrm{Id}}(S_k \circ O_{120^\circ} \circ S_k^{-1}) = \mathrm{Id}.$
$\psi_{S_k}(S_k) \circ \varphi_{\mathrm{Id}}(O_{120^\circ}) \circ \psi_{S_k}(S_k^{-1}) = S_k \circ \mathrm{Id} \circ S_k = \mathrm{Id}.$ ✓

$(\varphi_{\mathrm{Id}}, \psi_{S_l})$: ✓

$(\varphi_{\mathrm{Id}}, \psi_{S_m})$: ✓

$(\varphi_{O_{120^\circ}}, \psi_{\mathrm{Id}})$: $\varphi_{O_{120^\circ}}(S_k \circ O_{120^\circ} \circ S_k^{-1}) = \varphi_{O_{120^\circ}}(O_{240^\circ}) = O_{240^\circ}.$
$\psi_{\mathrm{Id}}(S_k) \circ \varphi_{O_{120^\circ}}(O_{120^\circ}) \circ \psi_{\mathrm{Id}}(S_k^{-1}) = \mathrm{Id} \circ O_{120^\circ} \circ \mathrm{Id} = O_{120^\circ}.$ ✗

$(\varphi_{O_{120^\circ}}, \psi_{S_k})$: $\varphi_{O_{120^\circ}}(S_k \circ O_{120^\circ} \circ S_k^{-1}) = O_{240^\circ}.$
$\psi_{S_k}(S_k) \circ \varphi_{O_{120^\circ}}(O_{120^\circ}) \circ \psi_{S_k}(S_k^{-1}) = S_k \circ O_{120^\circ} \circ S_k = O_{240^\circ}.$ ✓

$(\varphi_{O_{120^\circ}}, \psi_{S_l})$: ✓

$(\varphi_{O_{120^\circ}}, \psi_{S_m})$: ✓

$(\varphi_{O_{240^\circ}}, \psi_{\mathrm{Id}})$: ✗

$(\varphi_{O_{240^\circ}}, \psi_{S_k})$: ✓

$(\varphi_{O_{240^\circ}}, \psi_{S_l})$: ✓

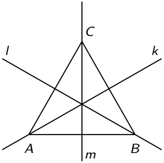
$(\varphi_{O_{240^\circ}}, \psi_{S_m})$: ✓

Zatem

$$|\mathrm{End}(D_3)| = 10 \qquad \text{i} \qquad |\mathrm{Aut}(D_3)| = 6.$$

	Id	O_{120°	O_{240°
φ_{Id}	Id	Id	Id
$\varphi_{O_{120^\circ}}$	Id	O_{120°	O_{240°
$\varphi_{O_{240^\circ}}$	Id	O_{240°	O_{120°

	Id	S_k
ψ_{Id}	Id	Id
ψ_{S_k}	Id	S_k
ψ_{S_l}	Id	S_l
ψ_{S_m}	Id	S_m



Mamy

$$\mathrm{Id} = \mathrm{Id} \circ \mathrm{Id},\ O_{120^\circ} = O_{120^\circ} \circ \mathrm{Id},\ O_{240^\circ} = O_{240^\circ} \circ \mathrm{Id},\ S_k = \mathrm{Id} \circ S_k,\ S_m = O_{120^\circ} \circ S_k,\ S_l = O_{240^\circ} \circ S_k.$$

	Id	O_{120°	O_{240°	S_k	S_l	S_m
$(\varphi_{O_{120^\circ}}, \psi_{S_k})$	$\varphi(\mathrm{Id}) \circ \psi(\mathrm{Id}) = \mathrm{Id}$	$\varphi(O_{120^\circ}) \circ \psi(\mathrm{Id}) = O_{120^\circ}$	$\varphi(O_{240^\circ}) \circ \psi(\mathrm{Id}) = O_{240^\circ}$	$\varphi(\mathrm{Id}) \circ \psi(S_k) = S_k$	$\varphi(O_{240^\circ}) \circ \psi(S_k) = S_l$	$\varphi(O_{120^\circ}) \circ \psi(S_k) = S_m$
$(\varphi_{O_{120^\circ}}, \psi_{S_l})$	Id	O_{120°	O_{240°	S_l	S_m	S_k
$(\varphi_{O_{120^\circ}}, \psi_{S_m})$	Id	O_{120°	O_{240°	S_m	S_k	S_l
$(\varphi_{O_{240^\circ}}, \psi_{S_k})$	Id	O_{240°	O_{120°	S_k	S_m	S_l
$(\varphi_{O_{240^\circ}}, \psi_{S_l})$	Id	O_{240°	O_{120°	S_l	S_k	S_m
$(\varphi_{O_{240^\circ}}, \psi_{S_m})$	Id	O_{240°	O_{120°	S_m	S_l	S_k

Automorfizmy wewnętrzne

Definicja

Automorfizmami wewnętrznymi grupy G nazywamy funkcje postaci $\sigma_g \colon G \to G$, gdzie $g \in G$ oraz

$$\sigma_g(h) := ghg^{-1} \qquad (h \in G).$$

Notacja

Grupę automorfizmów wewnętrznych grupy G oznaczamy $\mathrm{Inn}(G)$.

Zadanie

Ile jest automorfizmów grupy G?

Obserwacja

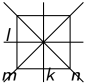
Jeśli $G = \langle x_1, \ldots, x_n \rangle$, to $\sigma_g = \sigma_h$ wtedy i tylko wtedy, gdy

$$\sigma_g(x_i) = \sigma_h(x_i) \text{ dla wszystkich } i.$$

Uwaga

Jeśli grupa G jest abelowa, to $|\mathrm{Inn}(G)| = 1$.

Przykład: $\mathrm{Inn}(D_4)$



Wiemy, że

$$D_4 = \langle S_k, S_m \rangle = \{\mathrm{Id}, O_{90^\circ}, O_{180^\circ}, O_{270^\circ}, S_k, S_l, S_m, S_n\}.$$

- $\sigma_{\mathrm{Id}}(S_k) = \mathrm{Id} \circ S_k \circ \mathrm{Id}^{-1} = S_k$
  $\sigma_{\mathrm{Id}}(S_m) = \mathrm{Id} \circ S_m \circ \mathrm{Id}^{-1} = S_m$
- $\sigma_{O_{90^\circ}}(S_k) = O_{90^\circ} \circ S_k \circ O_{90^\circ}^{-1} = S_l$
  $\sigma_{O_{90^\circ}}(S_m) = O_{90^\circ} \circ S_m \circ O_{90^\circ}^{-1} = S_n$
- $\sigma_{O_{180^\circ}}(S_k) = O_{180^\circ} \circ S_k \circ O_{180^\circ}^{-1} = S_k$
  $\sigma_{O_{180^\circ}}(S_m) = O_{180^\circ} \circ S_m \circ O_{180^\circ}^{-1} = S_m$
- $\sigma_{O_{270^\circ}}(S_k) = O_{270^\circ} \circ S_k \circ O_{270^\circ}^{-1} = S_k$
  $\sigma_{O_{270^\circ}}(S_m) = O_{270^\circ} \circ S_m \circ O_{270^\circ}^{-1} = S_m$
- $\sigma_{S_k}(S_k) = S_k \circ S_k \circ S_k^{-1} = S_k$
  $\sigma_{S_k}(S_m) = S_k \circ S_m \circ S_k^{-1} = S_n$
- $\sigma_{S_l}(S_k) = S_l \circ S_k \circ S_l^{-1} = S_k$
  $\sigma_{S_l}(S_m) = S_l \circ S_m \circ S_l^{-1} = S_n$
- $\sigma_{S_m}(S_k) = S_m \circ S_k \circ S_m^{-1} = S_l$
  $\sigma_{S_m}(S_m) = S_m \circ S_m \circ S_m^{-1} = S_m$
- $\sigma_{S_n}(S_k) = S_n \circ S_k \circ S_n^{-1} = S_l$
  $\sigma_{S_n}(S_m) = S_n \circ S_m \circ S_n^{-1} = S_m$

Zatem $|\,\mathrm{Inn}(D_4)| = 4$.

Przypomnijmy

	Id	O_{90°	O_{180°	O_{270°	S_k	S_l	S_m	S_n
$(\varphi_{O_{90^\circ}}, \psi_{S_k})$	Id	O_{90°	O_{180°	O_{270°	S_k	S_l	S_m	S_n
$(\varphi_{O_{90^\circ}}, \psi_{S_l})$	Id	O_{90°	O_{180°	O_{270°	S_l	S_k	S_n	S_m
$(\varphi_{O_{90^\circ}}, \psi_{S_m})$	Id	O_{90°	O_{180°	O_{270°	S_m	S_n	S_l	S_k
$(\varphi_{O_{90^\circ}}, \psi_{S_n})$	Id	O_{90°	O_{180°	O_{270°	S_n	S_m	S_k	S_l
$(\varphi_{O_{270^\circ}}, \psi_{S_k})$	Id	O_{270°	O_{180°	O_{90°	S_k	S_l	S_n	S_m
$(\varphi_{O_{270^\circ}}, \psi_{S_l})$	Id	O_{270°	O_{180°	O_{90°	S_l	S_k	S_m	S_n
$(\varphi_{O_{270^\circ}}, \psi_{S_m})$	Id	O_{270°	O_{180°	O_{90°	S_m	S_n	S_k	S_l
$(\varphi_{O_{270^\circ}}, \psi_{S_n})$	Id	O_{270°	O_{180°	O_{90°	S_n	S_m	S_l	S_k

- $\sigma_{\mathrm{Id}} = \sigma_{O_{180^\circ}} : S_k \mapsto S_k,\ S_m \mapsto S_m \implies \sigma_{\mathrm{Id}} = (\varphi_{90^\circ}, \psi_{S_k})$.
- $\sigma_{O_{90^\circ}} = \sigma_{O_{270^\circ}} : S_k \mapsto S_l,\ S_m \mapsto S_n \implies \sigma_{O_{90^\circ}} = (\varphi_{90^\circ}, \psi_{S_l})$.
- $\sigma_{S_k} = \sigma_{S_l} : S_k \mapsto S_k,\ S_m \mapsto S_n \implies \sigma_{S_k} = (\varphi_{270^\circ}, \psi_{S_k})$.
- $\sigma_{S_m} = \sigma_{S_n} : S_k \mapsto S_l,\ S_m \mapsto S_m \implies \sigma_{S_m} = (\varphi_{270^\circ}, \psi_{S_l})$.