When the sum equals the product

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The natural numbers $1, 2, 3$ have a special property: their sum is equal to their product.

$$1 + 2 + 3 = 1 \cdot 2 \cdot 3.$$

The numbers $1, 1, 2, 4$ possess the same property:

$$1 + 1 + 2 + 4 = 1 \cdot 1 \cdot 2 \cdot 4.$$

Look also at the following examples:

$$\begin{aligned} 1+1+1+2+5 &= 1\cdot 1\cdot 1\cdot 2\cdot 5 \\ 1+1+1+3+3 &= 1\cdot 1\cdot 1\cdot 3\cdot 3 \\ 1+1+2+2+2 &= 1\cdot 1\cdot 2\cdot 2\cdot 2. \end{aligned}$$

Let $n \geqslant 2$ be a natural number. We are interested in the sequences $(x_1, \dots, x_n)$ of natural numbers such that

$$x_1 + x_2 + \cdots + x_n = x_1 \cdot x_2 \cdot \cdots \cdot x_n \quad \text{and} \quad x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n.$$

We denote by $A(n)$ the set of all such sequences. Moreover, we denote by $a(n)$ the cardinality of the set $A(n)$, that is, $a(n)$ is the number of all elements of $A(n)$.

The sequence $(2, 2)$ is a unique element of $A(2)$. Thus $a(2) = 1$. The above examples deal with the cases when $n = 3, 4, 5$. Now we will prove that these are all the examples of such forms.

Theorem 1. *$a(3) = 1$, $a(4) = 1$, $a(5) = 3$.*

Proof. For $n = 3$ we have: $x_1x_2x_3 = x_1 + x_2 + x_3 \leqslant 3x_3$, so $x_1x_2 \leqslant 3$. Then (x_1, x_2) is one of the pairs $(1, 1)$, $(1, 2)$, $(1, 3)$. But only the case $(x_1, x_2) = (1, 2)$ is good and in this case $x_3 = 3$. Hence the set $A(3)$ has only one element $(1, 2, 3)$.

Let $n = 4$. Since $x_1x_2x_3x_4 = x_1 + x_2 + x_3 + x_4 < 4x_4$ (the case $x_1 = x_2 = x_3 = x_4$ is impossible), we have $x_1x_2x_3 \leqslant 3$. The triple (x_1, x_2, x_3) is then one of the triples $(1, 1, 1)$, $(1, 1, 2)$, $(1, 1, 3)$. But only the case $(x_1, x_2, x_3) = (1, 1, 2)$ is good. The set $A(4)$ has only one element $(1, 1, 2, 4)$.

For $n = 5$ we do the same. First we observe that $x_1x_2x_3x_4 \leqslant 4$. This implies that (x_1, x_2, x_3, x_4) is one of the sequences $(1, 1, 1, 1)$, $(1, 1, 1, 2)$, $(1, 1, 1, 3)$, $(1, 1, 1, 4)$, $(1, 1, 2, 2)$. The sequences $(1, 1, 1, 1)$ and $(1, 1, 1, 4)$ are not good. There is no number x_5 for such sequences. From the remaining sequences we obtain all the elements of $A(5)$: $(1, 1, 1, 2, 5)$, $(1, 1, 1, 3, 3)$ and $(1, 1, 2, 2, 2)$. □

One can find another proof of Theorem 1 in [1] (pp. 171 - 174).

Theorem 2. *For any $n \geqslant 2$ the set $A(n)$ is nonempty.*

Proof. $A(n)$ contains the sequence $(1, 1, \dots, 1, 2, n)$. □

Let us assume that the sequence $(x_1, \dots, x_n)$ belongs to $A(n)$. Then

$$x_i x_n \leqslant x_1 x_2 \dots x_n = x_1 + x_2 + \cdots + x_n \leqslant x_n + x_n + \cdots + x_n = n x_n$$

for all $i = 1, \dots, n-1$. Therefore, the numbers $x_1, \dots, x_{n-1}$ are smaller than $n+1$. But they determine the number x_n. Given $x_1, \dots, x_{n-1}$, we can find the number x_n from the equality $x_1 \cdots x_n = x_1 + \cdots + x_n$. So we get:

Theorem 3. *For any $n \geqslant 2$ the set $A(n)$ is finite.* □

Note several facts concerning $A(n)$ and $a(n)$.

Theorem 4. *If $(x_1, \dots, x_n) \in A(n)$ and $n \geqslant 3$, then $x_1 x_2 \cdots x_{n-1} \leqslant n - 1$.*

Proof. Observe that all the numbers $x_1, \dots, x_n$ are not equal. Suppose that $x_1 = \cdots = x_n = x$. Then $x^n = nx$ and so $x = \sqrt[n-1]{n}$. But $1 < \sqrt[n-1]{n} < 2$ for $n \geqslant 3$, so we have a contradiction. Therefore,

$$x_1 x_2 \cdots x_{n-1} x_n = x_1 + x_2 + \cdots + x_n < n x_n,$$

hence $x_1 x_2 \cdots x_{n-1} \leqslant n - 1$. □

Theorem 5 ([1] 175). *For any natural s there exists a natural n such that $a(n) > s$.*

Proof. Let $n = 2^{2s} + 1$ and let $x_1 = x_2 = \cdots = x_{n-2} = 1$. If $j \in \{0, 1, 2, \dots, s\}$, then we define:

$$x_{n-1} = 2^j + 1, \quad x_n = 2^{2s-j} + 1.$$

Every sequence $(x_1, \dots, x_n)$, of the above form, belongs to $A(n)$ and the number of such sequences equals $s + 1$. □

Theorem 6. *If $(x_1, \dots, x_n) \in A(n)$, $n \geqslant 2$, then $x_1 + \cdots + x_n \leqslant 2n$. The equality holds only in the case when $(x_1, \dots, x_n) = (1, 1, \dots, 1, 2, n)$.*

Proof. Let b_n denote the number of unit elements in $(x_1, \dots, x_n) \in A(n)$. Let k be the number of non-unit elements in $(x_1, \dots, x_n)$. We will denote the non-units elements by $y_1 + 1,\ y_2 + 1,\ \dots,\ y_k + 1$, respectively, where $1 \leqslant y_1 \leqslant y_2 \leqslant \cdots \leqslant y_k$. It is clear that $k \geqslant 2$, $b_n + k = n$ and

$$(y_1 + 1)(y_2 + 1) \dots (y_k + 1) = y_1 + y_2 + \cdots + y_k + k + b_n. \tag{1}$$

Let $k = 2$. Then $y_1 y_2 = n - 1$, hence $y_1 + y_2 \leqslant 1 + n - 1 = n$, yielding $x_1 + \cdots + x_n = n + y_1 + y_2 \leqslant 2n$. The equality is only in the case when $y_1 = 1$, $y_2 = n - 1$, that is, only when $(x_1, \dots, x_n) = (1, 1, \dots, 1, 2, n)$.

Let $k \geqslant 3$. Then the equality (1) implies:

$$\begin{aligned} y_1 + \cdots + y_k &\leqslant y_1y_2 + y_2y_3 + \cdots + y_ky_1 \\ &< (y_1+1)(y_2+1)\dots(y_k+1) - (y_1 + \cdots + y_k) \\ &= n. \end{aligned}$$

Therefore $x_1 + \cdots + x_n = y_1 + \cdots + y_k + n < 2n$. □

The above theorem was offered as a problem at the Polish Mathematical Olympiad in 1990.

Theorem 7. *Let $(x_1, \dots, x_n) \in A(n)$, $n \geqslant 2$. Denote by b_n the number of unit elements in $(x_1, \dots, x_n)$. Then*

$$b_n \geqslant n - 1 - [\log_2 n].$$

The equality holds, for example, in the case when n is of the form $2^s - s$ (where $s \geqslant 2$) and $(x_1, \dots, x_n) = (1, \dots, 1, \underbrace{2, 2, \dots, 2}_{s})$.

Proof. Theorem 6 implies that

$$2^{n-b_n} \leqslant x_1 \cdots x_n = x_1 + \cdots + x_n \leqslant 2n.$$

Hence $n - b_n \leqslant \log_2(2n) = 1 + \log_2 n$ and so $b_n \geqslant n - 1 - [\log_2 n]$. The remaining part of this theorem is obvious. □

Theorem 8. *If n is even and $(x_1, \dots, x_n) \in A(n)$ then the number $x_1 + \cdots + x_n$ is divisible by 4.*

Proof. Suppose that all the numbers $x_1, \dots, x_n$ are odd. Then we have an even number of odd numbers. The sum $x_1 + \cdots + x_n$ is then an even number, and the product $x_1 \cdots x_n$ is odd.

Therefore, at least one of the numbers $x_1, \dots, x_n$ is even. This means that the product is even and consequently the sum is also even. This implies that we have at least two even numbers. Thus the product, which is equal to the sum, is divisible by 4. □

n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$
1	1	11	3	21	4	31	4	41	7	51	4	61	9	71	6	81	7	91	6
2	1	12	2	22	2	32	3	42	2	52	3	62	3	72	3	82	4	92	3
3	1	13	4	23	4	33	5	43	5	53	7	63	4	73	9	83	5	93	6
4	1	14	2	24	1	34	2	44	2	54	2	64	4	74	4	84	2	94	3
5	3	15	2	25	5	35	3	45	4	55	5	65	7	75	3	85	10	95	6
6	1	16	2	26	4	36	2	46	4	56	4	66	2	76	3	86	5	96	5
7	2	17	4	27	3	37	6	47	5	57	5	67	5	77	6	87	4	97	6
8	2	18	2	28	3	38	3	48	2	58	4	68	5	78	3	88	5	98	5
9	2	19	4	29	5	39	3	49	5	59	4	69	4	79	5	89	8	99	4
10	2	20	2	30	2	40	4	50	4	60	2	70	3	80	2	90	2	100	5

The tables, obtained by a computer programme, present the numbers $a(n)$ for $1 \leqslant n \leqslant 100$. We see, for example, that $a(50) = 4$, $a(100) = 5$.

The set $A(50)$ has exactly 4 elements. We can prove that every sequence $(x_1, \dots, x_{50})$ belonging to $A(50)$ is such that $x_1 = x_2 = \cdots = x_{47} = 1$ and (x_{48}, x_{49}, x_{50}) is one of the triples:

$$(1, 2, 50), \quad (1, 8, 8), \quad (2, 2, 17), \quad (2, 5, 6).$$

The set $A(100)$ has exactly 5 elements. Every element is of the form $(x_1, \dots, x_{100})$, where $x_1 = x_2 = \cdots = x_{95} = 1$ and $(x_{96}, x_{97}, x_{98}, x_{99}, x_{100})$ is one of the sequences:

$$(1, 1, 1, 2, 100), \quad (1, 1, 1, 4, 34), \quad (1, 1, 1, 10, 12), \quad (1, 1, 4, 4, 7), \quad (2, 2, 3, 3, 3).$$

Using a computer we can prove that $a(1997) = 20$, $a(1998) = 8$, $a(1999) = 16$, $a(2000) = 10$.

We see, looking at the above tables, that 24 is the maximal two-digit number n such that $a(n) = 1$. There exist exactly 3 natural three-digit numbers n with the property $a(n) = 1$. They are: 114, 174 and 444. The authors do not know the answer to the following question:

Is there a natural n such that $a(n) = 1$ and $n > 444$?

Now we present some facts concerning the case $a(n) = 1$.

Theorem 9. *Let $n > 2$. If $a(n) = 1$ then $n - 1$ is prime.*

Proof. Suppose that $n - 1$ is not prime. Then $n = ab + 1$ for some natural a, b with $2 \leqslant a \leqslant b$. Then the two sequences $(1, 1, \dots, 1, 2, n)$ and $(1, 1, \dots, 1, a + 1, b + 1)$ are different and they belong to $A(n)$. □

As a consequence of the above theorem we get

Theorem 10. *If $n \geqslant 4$ and $a(n) = 1$, then $2 \mid n$.* □

Note also the following

Theorem 11. *If $n \geqslant 5$ and $a(n) = 1$, then $3 \mid n$.*

Proof. Theorem 9 implies that n is not of the form $3k + 1$. If $n = 3k + 2$ then the set $A(n)$ has two different sequences $(1, \dots, 1, 2, n)$ and $(1, 1, \dots, 1, 2, 2, k + 1)$. □

From the above facts we obtain

Theorem 12. *If $a(n) = 1$ and $n \geqslant 5$, then $6 \mid n$.* □

Note also the following

Theorem 13. *If $a(n) = 1$ and $n > 100$, then n is of the form either $7k$ or $7k + 2$ or $7k + 3$ or $7k + 6$ ($k \geqslant 14$).*

Proof. The set $A(n)$ contains the sequence $(1, \ldots, 1, 2, n)$. If $n = 7k+1$ or $7k+4$ or $7k+5$, then $A(n)$ contains also

$$(1, 1, \ldots, 1, 8, k+1), \quad (1, 1, \ldots, 1, 2, 4, k+1), \quad (1, 1, \ldots, 2, 2, 2, k+1),$$

respectively. □

Theorem 14. *If $a(n) = 1$ and $n > 100$, then n is of the form $30k$ or $30k+24$ $(k \geqslant 3)$.*

Proof. Since $6 \mid n$ (Theorem 12), the number n has one of the forms $30k$, $30k+6$, $30k+12$, $30k+18$ or $30k+24$.

If $n = 30k+6$, then $n-1$ is not prime; a contradiction with Theorem 9.

We know that the set $A(n)$ always contains the sequence $(1, \ldots, 1, 2, n)$. In the case when $n = 30k+12$ or $n = 30k+18$, the set $A(n)$ contains also

$$(1, 1, \ldots, 1, 2, 2, 2, 2, 2k+1), \quad (1, 1, \ldots, 1, 2, 3, 6k+4),$$

respectively. □

It follows from the above facts that if $n > 100$ and $a(n) = 1$ then the number n has one of the forms $210k$, $210k+24$, $210k+30$, $210k+84$, $210k+90$, $210k+114$, $210k+150$ or $210k+174$.

We proved (see Theorem 14) that if $a(n) = 1$ and $n \geqslant 5$ then n is of the form $30k$ or $30k+24$ $(k \geqslant 0)$. We think, however, that the case $n = 30k$ does not hold.

Conjecture 1. *If $n \geqslant 5$ and $a(n) = 1$, then n is of the form $30k+24$.*

Conjecture 2. *If $n > 100$ and $a(n) = 1$, then $n = 114$ or $n = 174$ or $n = 444$.*

References

[1] W. Sierpiński, *Number Theory*, Part II, (in Polish), PWN, Warszawa 1959.