Some new results on the analysis and simulation of bucket brigades (self-balancing production lines)

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The bucket brigades are self-balancing production lines where each human operator can move from one station to the next to continue working on a given part. When workers finish their job, they walk back to take over the ongoing work of their predecessor. For the bucket brigades, the line balancing emerges spontaneously. Several mathematical models have been proposed in literature for the analysis of bucket brigades, often where the basic assumptions seem to be too constraining. This paper analyses the behaviour of these lines when relaxing some of the usual restrictive assumptions. Two approaches are developed: analysis and simulation. An original analytical model is suggested and using this model, the sufficient condition of self-balancing for a finite backward velocity is studied. Then, a novel technique for the simulation of bucket brigades is proposed. This model uses the software Matlab/Simulink™. The results of simulation study are reported and analysed. The simulation model is flexible and easily extensible, in order to allow further embedding of complex dynamics which have not yet been considered.

Keywords: Production lines; Bucket brigades; Self-balancing; Dynamic behaviour; Simulation

1. Introduction

‘Bucket brigades’ are production lines which use fewer workers than work stations and where these workers are authorized to move between adjacent stations to continue their work. Each worker carries out his/her work on a item down the line until this item is acquired by a downstream (successor) colleague; then, he/she walks back to take over the item of its upstream (predecessor) colleague, and so on. The moments of hand-off are called ‘resets’. This principle is called the TSS protocol, from Toyota Sewn Products Management System, registered mark of Aisin Seiki Co. Ltd.: (www.aisin.com; Ohno 1988, Schroer et al. 1991, Bartholdi and Eisenstein 1996). The bucket brigades transform conventional lines by allowing work-sharing. Potential benefits of polyvalent workers with work-sharing in a more general context

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have recently been studied in Hopp et al. (2004), Anuar and Bukchin (2006) and Ahn and Righter (2006).

For traditional production lines, the main design problem is the assembly line balancing (ALB), which consists of assigning the tasks to employees minimizing the total worker idle time. ALB is a crucial problem at the line design or reconfiguration stage. A comprehensive survey on ALB procedures can be found, for example, in Baybars (1986), Erel and Sarin (1998), Rekiek et al. (2002) and Dolgui (2006). For the bucket brigades, thanks to the TSS protocol, the ALB is no longer needed to be solved. Since the workers are mobile, the tasks are ‘assigned’ and ‘reassigned’ to workers in real time.

Generally, a worker on a TSS line may be idled by his/her successor to the right, if this downstream worker is slower (this situation is termed ‘blocking’, i.e., the workers are impeded). It will be interesting to find out what initial condition would a bucket brigade need to fulfil in order to evolve to a steady-state optimal functioning, where there is no blocking. Under certain conditions, this profitable steady state is obtained. In this case, the cycle time for the given line is minimal (throughput is maximal) and all the workers are fully occupied (no blocking, no idle time).

For those unfamiliar with the topic of bucket brigades, a brief survey of the pertinent literature is to follow. Different analytical approaches of the bucket brigades have been investigated. The most significant results were obtained by Bartholdi III and Eisenstein (1996). The same authors have introduced the ‘normative model’, listed its assumptions (see section 2), and proved a sufficient condition for self-balancing using a Markov chain. In the normative model, the performance of a worker is measured by his/her work velocity which is assumed to be constant. The work on the next item restarts instantaneously and simultaneously for all the workers, i.e., the backward velocity is considered as infinite.

The behaviour of bucket brigades with two and three workers has been extensively studied in Bartholdi et al. (1999), whereas the influence of the stochastic operating times has been analysed in Bartholdi et al. (2001). Bucket brigades protocol of work sharing (TSS rules) has been revised in order to support the migration from craft manufacturing to assembly lines in Bartholdi and Eisenstein (2005), and adapted for a network of sub-assembly lines in Bartholdi et al. (2006).

Many other authors studied the bucket brigades. Armbruster and Gel (2006) have analysed the bucket brigade dynamics with two workers when in some sections of the line, the first worker is faster, but for rest of the line the second worker is faster. Their analysis relies upon the existence of fixed points and of limit cycles, attractors or repellers, in the state space. They have also considered cases with a modified TSS protocol, i.e., allowing passing (a worker is allowed to pass over a slower successor). They emphasized that, in such cases, a certain trade-off between throughput and self-organizing behaviour can be imposed as an initial requirement (design parameter). Armbruster et al. (2007) have studied the bucket brigades with worker learning, showing that both optimal workload assignment and optimal reset positions are reached, by means of an interchanging place policy during the transient phase of learning. Hirotani et al. (2006) have analysed the blocking conditions and the influence of initial operators’ positions on the efficiency of self-balancing lines under the normative model. They have also derived a procedure for the design of bucket brigades based upon this analysis.
Another interesting aspect of bucket brigades is the labour turnover effect (Villalobos et al. 1999, Quintana et al. 2000, Hutchinson et al. 2002, Muñoz and Villalobos 2002). The bucket brigade lines have proved more robust under high labour turnover when compared with traditional lines.

Buzacott (2002) has analysed how the worker differences influence production system output and has suggested that bucket brigades are, in many cases, the optimal policy to adopt. They constitute an environment suitable for exploiting the advantages of team working, i.e., the ability to change task allocation dynamically. In contrast, Schultz et al. (2003) have approached the negative side effects of worker flexibility in various production environments.

In Bratcu and Mînzu (1999, 2001) and Bratcu (2001), the authors used the normative model and showed that the bucket brigades may be regarded as non-linear dynamic systems. In this approach, the bucket brigade state trajectories are piecewise continuous in between the following discrete events: blocking (an operator catches up with a slower successor and is required to stop) and reset (each operator relinquishes his/her item to take the item of his/her predecessor). In terms of non-linear dynamic systems, these events can be modelled as autonomous switching and jumps, respectively. Therefore, hybrid dynamic systems with discontinuous motions can be used for the stability analysis. Simulations have been performed in the case of constant work velocities (both with and without blocking) that support this viewpoint.

For other publications on bucket brigades, please see our tentative state-of-the-art paper (Bratcu and Dolgui 2005) and the web site (www.bucketbrigades.com) of Bartholdi and Eisenstein.

This paper is based on our previous preliminary results in Bratcu and Dolgui (2003, 2004) which are more developed and better explained here. Two approaches are explored: analysis and simulation. An original analytical model is suggested. Based on this model, an analysis of the sufficient condition of self-balancing is given, where all workers move back with a common finite backward velocity. Then, a novel technique for the simulation of bucket brigades is proposed to study more general cases. The Matlab/Simulink™ software package is used. The proposed simulation model is flexible and easily extensible, in order to allow further embedding of complex dynamics which have not yet been considered.

The remaining text is organised as follows. In section 2, the normative model of bucket brigades is explained. Section 3 introduces our analytical model and analysis of the sufficient self-balancing condition and section 4 explains the simulation approach suggested. Simulation results are reported, which show the dynamic behaviour of bucket brigades under less restrictive assumptions for both deterministic and stochastic cases. Our concluding remarks are given in section 5.

2. The normative model

Bucket brigades are optimized instances of TSS lines. As aforementioned, the workers are not assigned to a fixed work station, they move from station to station to execute the tasks (operations) on an item. The order of the workers stays the same. The standard TSS protocol states that each time a worker finishes his/her tasks for an item, he/she walks back to take over the ongoing work of his/her predecessor (upstream). Who, in turn, moves backward to continue the work of his/her
predecessor, and so on, until the first worker relinquishes his/her part and picks up raw materials to start a new item. Workers are indexed by their order on the line (from left to right in figure 1). A worker with a superior index has priority, i.e., he/she interrupts the work of his/her predecessor. For each worker, the start of work on a new part is called a ‘reset’ and the time interval between two successive resets is called ‘iteration’.

Blockings appear on a TSS line when workers upstream ‘ram into’ their neighbours downstream. This can be due to the smaller velocity of the downstream worker or to non-uniformly distributed workloads on the line.

The normative model of bucket brigades as introduced in Bartholdi and Eisenstein (1996) is described here.

Let \( m \) be the number of work stations and \( n \) be the number of workers (\( n < m \)). The bucket brigades’ normative model assumptions are:

(a) Work required by a part is continuously and uniformly spread along the line (there are no work stations frontiers), and the line length is normalized to 1.
(b) Backward velocity is infinite: the working time is significantly larger than the hand-off time (the time between relinquishing a part and recuperating another), so the reset is supposed instantaneous and simultaneous for all the workers.
(c) The only characteristic of workers, that is considered, is their constant work velocity: each worker \( i \) is modelled by his/her instantaneous velocity \( v_i \); it is said that worker \( j \) is faster than worker \( i \) if \( v_j < v_i \).

Bartholdi and Eisenstein (1996) have proved a sufficient self-balancing condition for this model, which we will call the ‘total ordering of workers by velocity’ (TOWV): the placing of workers on the line from slowest (at the line’s start) to fastest (at the line’s end). The authors have shown that the normative model with the TOWV order of workers has neither blocking nor idle time. This steady-state optimal functioning is cost-effective because the cycle time for the given line is minimal (throughput is maximal) and all the workers are fully occupied.

This normative model and proofs of the sufficient condition are effective as demonstrated. This is an elegant abstraction of real world conditions. Several studies showed that this model is robust and can be applied for various situations with more general assumptions. Nevertheless, there remains a question: will the TOWV rule stay valid if we alter somewhat the basic assumptions (a)–(c)? In the next section, we will analytically examine this query for finite backward velocity. In addition, the stochastic forward and backward velocity cases will be analysed by simulation. For these endeavours, we have developed some analytical and simulation models of bucket brigades.

Figure 1. The standard workload split into stations \( p_i \) and the workers’ positions \( x_i \) (\( \sum_{j=1}^{m} p_j = 1 \)).
3. An analytical model of bucket brigade

In this section, we propose a new analytical model. Based on this model, we make an analysis of the bucket brigades with the relaxed second assumption of the normative model, i.e. considering a finite backward velocity. This analysis is an interesting issue for the exploration of the normative model. Here, the reset is neither instantaneous nor simultaneous, but will ‘propagate’ from the last to the first worker (Bratcu and Dolgui 2003, 2004).

An example which suggests that we cannot always use the infinite backward velocity is given in the paper by Bartholdi and Eisenstein (2005). The authors helped a company to transform a craft manufacturing system, where an entire tractor was fabricated by a sole worker, into an assembly line. For that, they proposed a new bucket brigade protocol specially adapted to this tractor manufacturer. In this protocol, some specific rules are used and not the TSS standard: a slower worker may pass a faster worker if the latter is delayed for any reason; hand-off time is not equal to zero; a worker can begin a walk-back only when there is no other walk-back or hand-off in progress, etc. However, what is truly interesting, is that the authors introduced in the suggested protocol fixed walking back times. Indeed, for this bucket brigade it was not possible to neglect the backward (walk-back) times. This may often be the case.

3.1 Modelling

In the following, we will study the problem of finite backward velocity for the standard rule of the normative model.

Let us introduce the following notation:

- $v_b$: Finite backward velocity, which is the same for all workers.
- $v_i$: Work velocity of worker $i$, $i \in \{1, 2, \ldots, n\}$.
- $x_i^k$: Reset position of worker $i$ at iteration $k$ (after $k - 1$ resets).
- $\lambda^{(k)}$: Vector of reset positions at iteration $k$.

It seems realistic enough to suppose that the backward velocity is larger than any of the work velocities. In addition, in most cases, to assume that walking speeds are the same throughout a workforce appears reasonable. Of course, the backward times can be different due to the differing distances spanned.

Figure 2 represents worker $i$ movements during an iteration according standard TSS protocol: (a) the forward movement after having met his/her predecessor $i - 1$, and (b) the backward walk after having met his/her successor $i + 1$.

![Figure 2](image-url)

Figure 2. Interaction of worker $i$ with $i - 1$ and $i + 1$: (a) forward and (b) backward movements.
The first contact of workers $i$ and $i-1$ happens at point $A$: worker $i$ takes the item of worker $i-1$ and starts his/her $k$th iteration, whereas $i-1$ starts the backward walk to meet his/her predecessor to the left. At the second point $C$, the worker $i$ finishes his/her $k$th iteration and takes the item of $i-1$, whereas $i-1$ ends the productive (forward) part of his $k$th iteration and begins his/her backward walk.

The duration, between two successive ‘rendezvous’ of workers $i$ and $i-1$, may be expressed thus:

- The worker $i$ walks from $A$ to $B$ with the work velocity $v_i$, then from $B$ to $C$ with the backward velocity $v_b$.
- Meanwhile, the worker $i-1$ moves from $A$ to $B'$ with the backward velocity and from $B'$ to $C$ with the work velocity $v_{i-1}$.

For an optimal solution where there is no blocking, the following equation for the durations is valid: $t_{AB}^i + t_{BC}^i = t_{AB}^{i-1} + t_{BC}^{i-1}$, so we deduce that (see figure 2):

$$
\frac{x_i^{(k+1)} - x_i^{(k)}}{v_i} + \frac{x_{i+1}^{(k+1)} - x_i^{(k+1)}}{v_b} = \frac{x_i^{(k)} - x_{i-1}^{(k)}}{v_b} + \frac{x_j^{(k+1)} - x_j^{(k)}}{v_{i-1}}.
$$

(1)

From the relation (1) one can deduce:

$$
\frac{x_i^{(k+1)} = x_i^{(k)} + \left(x_{i+1}^{(k+1)} - x_i^{(k+1)}\right) \cdot \frac{1/v_b + 1/v_i}{1/v_b + 1/v_{i-1}} + 1 
$$

for $2 \leq i < n$.

(2)

The same relation (1) gives for $i=n$, taking into account that $x_{n+1}^{(k+1)} = 1$ (remember that the line length is normalized to 1):

$$
\frac{x_n^{(k+1)} = x_n^{(k)} + \left(1 - x_n^{(k)}\right) \cdot \frac{1/v_b + 1/v_n}{1/v_b + 1/v_{n-1}}}{1/v_b + 1/v_{n-1}}.
$$

(3)

**Lemma:** For an optimal solution, the following expression is true:

$$
P(i) : x_i^{(k+1)} = x_i^{(k)} + \left(1 - x_n^{(k)}\right) \cdot \frac{1/v_b + 1/v_n}{1/v_b + 1/v_{i-1}}, \quad 2 \leq i \leq n.
$$

**Proof:** We show this claim by backward recurrence. $P(n)$ is true, see the relation (3). From the relations (2) and (3), we obtain:

$$
x_{n-1}^{(k+1)} = x_{n-1}^{(k)} + \left(1 - x_n^{(k)}\right) \cdot \frac{1/v_b + 1/v_n}{1/v_b + 1/v_{n-2}},
$$

therefore, $P(n-1)$ is also true. Let us suppose that $P(i+1)$ is true, $P(i+1)$:

$$
x_{i+1}^{(k+1)} = x_{i+1}^{(k)} + \left(1 - x_n^{(k)}\right) \cdot \frac{1/v_b + 1/v_n}{1/v_b + 1/v_{i+1}}.
$$

To prove this Lemma, we need to show that, in this case, $P(i)$ is also true. Using $P(i+1)$ to replace the quantity $x_{i+1}^{(k+1)} - x_i^{(k)}$ in the relation (2), one can successively write:

$$
x_i^{(k+1)} = x_i^{(k)} + \left(1 - x_n^{(k)}\right) \cdot \frac{1/v_b + 1/v_n}{1/v_b + 1/v_{i+1}} \cdot \frac{1/v_b + 1/v_i}{1/v_b + 1/v_{i-1}}.
$$

Therefore, $P(i)$ is true. □
The changes over time of the reset positions from iteration to the next may hence be described by the following recurrence:

\[
\begin{cases}
    x_1^{(k+1)} = 0 \\
    x_i^{(k+1)} = x_{i-1}^{(k)} + (1 - x_n^{(k)}) \cdot \frac{1/v_b + 1/v_n}{1/v_b + 1/v_{i-1}}, & 2 \leq i \leq n.
\end{cases}
\]  

(4)

3.2 The sufficient self-balancing condition

We shall use the following notation:

\[
r_i = \frac{1/v_b + 1/v_n}{1/v_b + 1/v_i}, \quad i = 1, 2, \ldots, n - 1
\]

(5)

As aforementioned, for an optimal solution there is no blocking, i.e., the workers are never impeded, thus we can formulate the following proposition.

**Proposition 1:** If the work velocities are constant, then, for the normative model with a finite backward velocity common for all workers, the sufficient condition of normative model (i.e., \(v_1 < v_2 < \cdots < v_n\)) for the convergence of the line’s behaviour to a unique fixed point is also true. For this case the fixed point is given by the following expression:

\[
x^{*}_b = \begin{bmatrix}
    0 & r_1 & \frac{1}{1 + \sum_{j=1}^{n-1} r_j} & \cdots & \frac{1}{1 + \sum_{j=1}^{n-1} r_j} & \cdots & \frac{1}{1 + \sum_{j=1}^{n-1} r_j} \\
    0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
    1 & 0 & \cdots & 0 & -r_1 & 0 & \cdots \\
    0 & 1 & \cdots & 0 & -r_2 & 0 & \cdots \\
    \vdots & & \ddots & & & \ddots & \ddots \\
    0 & \cdots & 1 & 0 & -r_{n-2} & 0 & \cdots \\
    0 & \cdots & 0 & 1 & -r_{n-1} & 0 & \cdots \\
\end{bmatrix}^T.
\]

(6)

Note: The fixed point with infinite backward velocity is denoted by \(x^*_b\).

**Proof:** We have to prove the convergence of the reset positions, that is, the convergence of the series

\[
\{x^{(k)}\}_{k=0,1,\ldots}
\]

For optimal solutions, we have already proved (see equation (4)) that the reset positions change according to the following recurrent equation:

\[
x^{(k+1)} = A \cdot x^{(k)} + r, \quad k = 0, 1, \ldots
\]

(7)

where:

\[
A = \begin{bmatrix}
    0 & 0 & \cdots & 0 & 0 \\
    1 & 0 & \cdots & 0 & -r_1 \\
    0 & 1 & \cdots & 0 & -r_2 \\
    \vdots & & \ddots & & \ddots \\
    0 & \cdots & 1 & 0 & -r_{n-2} \\
    0 & \cdots & 0 & 1 & -r_{n-1} \\
\end{bmatrix}, \quad r = \begin{bmatrix}
    0 & r_1 & r_2 & \cdots & r_{n-1}
\end{bmatrix}^T.
\]

The relation (7) describes a discrete-time linear time-invariant dynamic system having

\[
\{x^{(k)}\}_{k=0,1,\ldots}
\]
as state trajectory. Matrix $A$ is in the observer canonical form. Point $x_b^*$ verifies the equation (7). By introducing $y^{(k)} = x^{(k)} - x_b^*$, we obtain:

$$y^{(k+1)} = x^{(k+1)} - x_b^* = A \cdot (x^{(k)} - x_b^*) = A \cdot y^{(k)}, \quad k = 0, 1, \ldots$$

Now, for proving proposition 1, the convergence of the series

$$\left\{ \frac{y^{(k)}}{\Delta} \right\}_{k=0, 1, \ldots}$$

must be verified. This series represents the trajectory of a discrete-time linear time-invariant dynamic system:

$$y^{(k+1)} = A \cdot y^{(k)}, \quad k = 0, 1, \ldots$$

The convergence of the trajectory of this system is equivalent to its stability. A necessary and sufficient condition for the stability of a discrete dynamic system is that its matrix be Schur-stable (all its eigenvalues must be strictly placed inside the unit circle, see Levine (1996)). In our case, let us write this as $|\lambda_i(A)| < 1, i = 1, 2, \ldots n$. Therefore, the stability of matrix $A$ may be expressed by the convergence of its characteristic polynomial:

$$\Delta(\lambda) = \lambda^n + r_{n-1} \cdot \lambda^{n-1} + \cdots + r_1 \cdot \lambda.$$ 

According to Kakeya’s criterion, a sufficient condition for a polynomial to be convergent is as follows: $1 > r_{n-1} > \cdots > r_1 > 0$. So, it is sufficient to ensure that $v_n > v_{n-1} > \cdots > v_1 > 0$ for series

$$\left\{ \frac{y^{(k)}}{\Delta} \right\}_{k=0, 1, \ldots}$$

converge. Therefore:

$$\left\{ \frac{y^{(k)}}{\Delta} \right\}_{k=0, 1, \ldots} \rightarrow 0 \Leftrightarrow \left\{ \frac{y^{(k)}}{\Delta} \right\}_{k=0, 1, \ldots} \rightarrow x_b^*,$$

where $x_b^*$ is the fixed point, computed as

$$x_b^* = (I_n - A)^{-1} \cdot r$$

($I_n$ being the $n$ dimensional unity matrix), which is equivalent to the expression (6). $\square$

Thus, we have obtained the same sufficient condition of self-balancing, $v_n > v_{n-1} > \cdots > v_1$, as for the normative model, that is the slowest-to-fastest placing of workers along the line. A bucket brigade converges to a constant cycle time behaviour (steady-state, stabilized), even if the reset is no longer instantaneous, or simultaneous for all the workers. Each worker $i$ performs the same work at each iteration; that is:

$$\left[ \sum_{j=1}^{i-1} r_j \left( \frac{1}{1 + \sum_{j=1}^{n-1} r_j} \right) \cdot \sum_{j=1}^{i} r_j \left( \frac{1}{1 + \sum_{j=1}^{n-1} r_j} \right) \right].$$

The worker moves along this distance once with work velocity $v_i$ (remember, no blocking) and a second time with the backward velocity $v_b$. The duration of each
iteration is \( n \) times shorter than the cycle time of the line \( T_{cb} \). From the relations (5), we obtain:

\[
T_{cb} = \frac{n \cdot r_i}{1 + \sum_{j=1}^{n-1} r_j} \cdot \left( \frac{1}{v_i} + \frac{1}{v_b} \right) = \frac{n}{1 + \sum_{j=1}^{n-1} r_j} \cdot \left( \frac{1}{v_b} + \frac{1}{v_n} \right).
\]

The production rate (throughput) is obtained as:

\[
pr_b = \frac{v_b \cdot v_n}{v_b + v_n} \cdot \left( 1 + \sum_{j=1}^{n-1} r_j \right).
\] (8)

**Note:** If we replace the finite backward velocity \( v_b \) by an infinite one, we obtain the same expressions as in Bartholdi and Eisenstein (1996). Moreover, if we replace the velocity \( v_i \) by \( v'_i \) in their model, where \( 1/v'_i = 1/v_b + 1/v_i \), we obtain the same line performances as those given by equation (8).

### 3.3 Comparison with the infinite backward velocity case

In the normative model the backward velocity is infinite (instantaneous reset). For a slowest-to-fastest order of workers, the self-balancing is given by the following fixed point (Bartholdi and Eisenstein 1996):

\[
\mathbf{x}^* = \begin{bmatrix}
0 \\
\frac{n}{v_1} / \sum_{j=1}^{n} v_j \\
\cdots \\
\frac{n}{v_{n-1}} / \sum_{j=1}^{n} v_j \\
\frac{n}{v_n} / \sum_{j=1}^{n} v_j \\
\end{bmatrix}^T
\] (9)

with the production rate:

\[
pr = \sum_{j=1}^{n} v_j.
\] (10)

Note first that:

\[
\lim_{v_b \to \infty} r_i = \frac{v_i}{v_n}, \quad i = 1, 2, \ldots n - 1
\] (11)

next, by using the relations (6) and (9):

\[
\left\{ \begin{array}{l}
\lim_{v_b \to \infty} x_{b1}^* = x_1^* = 0 \\
\lim_{v_b \to \infty} x_{bi}^* = \sum_{j=1}^{i-1} v_j / \sum_{j=1}^{n} v_j = x_i^*, \quad i = 2, 3, \ldots n
\end{array} \right.
\]

that is:

\[
\lim_{v_b \to \infty} x_{bi}^* = x_i^*.
\] (12)

From the relations (8), (10) and (11), the relationship between the production rates in the two cases is:

\[
\lim_{v_b \to \infty} pr_b = \lim_{v_b \to \infty} \left( \frac{v_b \cdot v_n}{v_b + v_n} \right) \cdot \lim_{v_b \to \infty} \left( 1 + \sum_{j=1}^{n-1} r_j \right) = \sum_{j=1}^{n} v_j = pr.
\] (13)
The normative model with infinite velocity gives a superior limit for cases having a finite backward velocity, which we show in the next two propositions. For this, we use the following complementary notations:

\[ r_i = r_i(v_b), \quad i = 1, 2, \ldots n - 1, \quad x^*_i = x^*_i(v_b), \quad i = 2, 3, \ldots n, \quad pr_b = pr_b(v_b). \]

**Proposition 2:** If the workers are ranged from slowest to fastest (TOWV), then the production rate corresponding to the instantaneous reset (infinite backward velocity) is superior to the production rate for any other (common finite) backward velocity.

**Justification**: One can show that function \( pr_b(v_b) \) is strictly increasing in \( v_b \). That is, the production rate increases when the backward velocity \( v_b \) increases. Therefore, the ideal production rate is a superior limit for all the other cases, see equation (13).

The TOWV leads to functions \( r_i(v_b), i = 1, 2, \ldots n - 1, \) and \( x^*_i(v_b), i = 2, 3, \ldots n, \) strictly decreasing in \( v_b \). Then, considering the relations (11) and (12), it follows that the coordinates of the fixed point for infinite backward velocity are inferior limits to those of fixed points in the cases with a finite backward velocity:

\[
\begin{align*}
  r_i > & v_i/v_n, \quad i = 1, 2, \ldots n - 1 \\
  x^*_i(v_b) > & x^*_i, \quad i = 2, 3, \ldots n
\end{align*}
\]

Therefore, we can formulate the following proposition.

**Proposition 3:** In the case of constant work velocities, if the workers are in the optimal order (i.e., TOWV), then the relationships between the reset positions at self-balancing in the case of common finite backward velocity and those in the case of infinite backward velocity are the following:

\[ (1 + R(v_b)) \cdot x^*_i(v_b) > x^*_i(v_b), \quad i = 2, 3, \ldots n, \quad \text{where} \ R(v_b) = v_n/v_b. \]

To complete this section with the analytical approach, we present an example of the ideal fixed points and actual fixed points (reset positions) for a three-worker bucket brigades model backward velocity.

Figure 3 represents the position of the fixed point in the state-space for different backward velocities \( v_b \), to illustrate the behaviour of this model versus the case with infinite backward velocity (for the chosen configuration, the fixed point and production rate are \( x^* = [0 \ 0.1429 \ 0.4286]^T \) and \( pr = 7 \)).

This figure illustrates a nonlinear dependency of the fixed point coordinates vs. the backward velocity: the larger \( v_b \) is, the less these coordinates change.

**4. Simulation**

**4.1 Simulation model**

Backward finite velocity is not the only possible extension of the normative model, which could be interesting. Exploring the model under stochastic conditions is also a promising perspective. The analytical approaches are limited for the sake of simplicity. With simulation, this obstacle can be circumvented. Several papers have studied the development of simulation models for bucket brigades.
As early as 1991, Schroer et al. developed a C++ simulation model of a TSS line with identical workers. Bischak (1996) and Zavadlav et al. (1996) report simulation studies of performances of U-shaped lines staffed with moving workers. Villalobos et al. (1999), Muñoz and Villalobos (2002) have used the ProModel software for discrete-event simulation. Simulation based analysis was also conducted by Bartholdi et al. (1999) for two- and three-worker bucket brigades and in Bartholdi et al. (2001) for the analysis of stochastic work times. Wang et al. (2007) have recently used a manufacturing-focused simulation tool in order to determine the ratio between the number of workers and work stations that ensures the optimal performance of a linear bucket brigade.

While simulation results are reported in detail, literature is however poor in giving information on how the simulation models can actually be built. Moreover, mainly static performances (e.g., evolution of throughput or idle time versus number of workers or work stations) are discussed, whereas the dynamic behaviour of such lines is practically neglected.

In this section, we present a novel and original simulation approach, employing the Matlab/Simulink™ software tool. By the way, the model is quite versatile, and can be used for analysing more generalized cases of bucked brigades. Less restrictive assumptions may be applied (stochastic velocities depending on worker position, non-respect of continuously and uniformly spread of work along the line, etc.).

In our approach, simulations are carried out in continuous time, to illustrate the workers’ instantaneous change of position. Within this change over time, discrete reset moments appear in response to some discrete events. These events are the rendezvous of a worker with his/her two neighbours on the left or right. This continuous time simulation model illustrates the conditions in which the steady-state

![Figure 3. Excursion of the fixed point in the state space as the common walk-back velocity varies.](image-url)
periodical operation of a bucket brigade appears more clearly. This approach is less
time consuming in comparison with discrete event models.

Figure 4 illustrates an implementation of our model for three workers. In the
figure, there are three blocks, Worker 1, Worker 2, and Worker 3, which apply the
standard TSS protocol. In contrast with our analytical model, the simulation model
provides a distinct walk-back velocity for each worker $v_{bi}$. In addition, working and
backward velocities can be changed by means of the corresponding setting blocks.
Thus, various types of simulations can be obtained:

- By setting blocks $\text{stoch\_emb } v_i$ at 1, $i = 1, 2, \ldots, n$, this imposes that the
  working velocity of worker $i$ be the sum of a constant component (given by
  the value in the block ’mean $v_i$’) and a random component. The random
  component has the mean equal to zero and a variance that is adjustable
  (see block ‘Gaussian noise on work velocity $i$’). In this way, we obtain
  a normally distributed working velocity with the mean equal to $v_i$, $i = 1, 2, \ldots, n$. The case with constant working velocities is obtained when
  the blocks $\text{stoch\_emb } v_i$ are set to 0.
- The same for the blocks $\text{stoch\_emb } vb_i$, $i = 1, 2, \ldots, n$, which are used
  to provide either normally distributed or constant backward velocities.
- The case with infinite walk-back velocity is obtained by setting all parameters $v_{bi}$
to very large values (compared to work velocities).

Therefore, this simulation model is far more flexible and can be adapted to various
real life situations.

The implementation of the block ‘Worker’ uses the same basic principle as in the
simulation model from (Bratcu and Mînzu 1999, Bratcu 2001): a worker $i$ is modelled
as an integrator of his/her velocity. The commutations between the two velocities
(backward and forward) – ($v_i$ to $v_{bi}$) and ($v_{bi}$ to $v_i$) – are imposed by worker $i$ meeting
the worker with superior or inferior index, respectively. One can consider that each
integrator (worker) is driven by a reference signal $\text{ref}_i(t)$ which shuttles periodically
between the instantaneous position of the worker $i + 1$ (third input of the block) and
the position of the worker $i - 1$ (second input of the block):

$$
x_i(t) = \begin{cases} 
    v_i, & \text{if } \varepsilon_i(t) > 0 \\
    -v_{bi}, & \text{if } \varepsilon_i(t) < 0 
\end{cases}, \quad i = 1, 2, \ldots n
$$

$$
\varepsilon_i(t) = \text{ref}_i(t) - x_i(t)
$$

$$
\text{ref}_i(t) = \begin{cases} 
    x_{i+1}(t), & (x_{i+1}(t) \geq x_i(t)) \land (\dot{x}_i(t) > 0) \\
    x_{i-1}(t), & (x_{i-1}(t) \leq x_i(t)) \land (\dot{x}_i(t) < 0) 
\end{cases}, \quad i = 1, 2, \ldots n
$$

where $x_0(t) = 0$ and $x_{n+1}(t) = 1$. One can note that ‘dir’ (fourth input of the block)
is necessary in order to obtain a reference signal (directional information for
the successor). Reference signals are encapsulated into Worker blocks; however,
they are not presented in figure 4.

Note: Here and in the next tests, we consider only three workers, i.e., $n = 3$,
to simplify the diagrams; however, the simulation model obviously can be extended
to any number of workers.

In the previous section, we showed that the sufficient condition of the normative
model for the optimal steady-state performance of bucket brigades was also true for
Figure 4. Simulation model of a self-balancing line with three operators.

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the case of a finite backward velocity. This study will not be complete without examining the time to arrive at the steady-state regime. Now, with the simulation model, we can examine this dynamic behaviour. The simulation can deal with convergence or transient time, the influence of different parameters on this time, the relationship between the transient time for infinite and finite backward velocity, and finally, both the gap between dynamic performances as well as the sensitivity of the optimal orders for finite and infinite backward velocities.

4.2 Simulation results: the deterministic case

The simulation results for the ideal case (TOWV with infinite backward velocity: \(v_1 = 2, \ v_2 = 4, \ v_3 = 5\)) are given in figure 5(a). They correspond to the self-balancing condition. Work velocity is measured in ‘work unit per time unit’. These figures show the changes over time of the instantaneous positions of the workers. Here, the time variable denotes continuous time (measured in the same time units as the velocity). These positions are represented by a continuous line for the first worker, a dotted line for the second, and a dashed line for the third. We can see the convergence of the reset positions to the fixed point. This convergence depends on the backward velocity. After a short transition, a minimal cycle steady-state behaviour is reached,
which is periodic with the duration of an iteration, that is, equal to $T_{eb}/n$. Since the resets in the finite backward velocity case are not simultaneous but in series, the convergence time depends on the number of workers as well.

Figure 5 also shows the gap between dynamic performances of optimal order for finite and infinite backward velocities. Comparing figure 5(c) with 5(a) reveals how a TOWV bucket brigade with finite walk-back velocity behaves differently from its ideal instance. That is, the same TOWV bucket brigade with instantaneous reset. Figures from the second column illustrate the difference between infinite 5(b) and finite 5(d) backward velocity for the case where the order of workers is different than TOWV. One can remark on these two figures the presence of blockings affecting the bucket brigade cycle time.

These simulations illustrate that TOWV ensures a quick convergence to maximally productive steady-state behaviour, whereas an order other than TOWV may not even lead to a cyclic behaviour being identified.

4.3 Simulations results: the stochastic case

Now, we will present some simulations with the stochastic forward and backward velocities.

In the literature there are already publications where the work times were considered as stochastic. For example, in Bartholdi et al. (2001), the time required for worker $i$ to accomplish a task is exponentially distributed with the mean equal to $1/v_i$. The authors have proved by convergence analysis in topologic spaces and verified by simulation that the more the number of stations increases, the more the stochastic functioning is closer to the deterministic one, predicted by the normative model.

Now, to further study their conclusion, we will present the following simulations obtained with our model for the more complex case where both backward and forward velocities are stochastic. This also highlights the capacity of the proposed simulation technique to treat more sophisticated situations.

Figures 6 and 7 present simulation results for two stochastic cases where both working and backward velocities are normally distributed with 10% variance around their average values. In the two cases, the vector of workers’ initial positions is $x_0 = [0.2 \ 0.7 \ 0.9]$ and the duration to simulate has been chosen to be four time units. The same conventions as earlier are used to represent the instantaneous positions of workers across iterations: continuous line for the first, dotted line for the second and dashed line for the third.

In figure 6, the average values of working velocities respect the TOWV: $\bar{v}_1 = 2$, $\bar{v}_2 = 4$, $\bar{v}_3 = 6$ work units/time units. As seen in 6(b), the cycle time and reset positions vary stochastically. But, even if the backward velocities do not have the same mean ($\bar{v}_{b1} = 8$, $\bar{v}_{b2} = 7$, $\bar{v}_{b3} = 10$) and the working velocities happen to overlap occasionally, the line progresses without blockings.

Figure 7 illustrates the same bucket brigade as in figure 6, except that each worker now occupies his/her successor’s position and the last (the fastest) is now the first (i.e., $\bar{v}_1 = 6$, $\bar{v}_2 = 2$, $\bar{v}_3 = 4$). One can note the blocking situations, even when the backward velocities have all the same means:

$$(\bar{v}_{b1} = \bar{v}_{b2} = \bar{v}_{b3} = 8).$$
These results are not exhaustive; more simulations could have been executed to study the above properties. Finally, this model for bucket brigades can be easily generalized by integrating learning/forgetting processes and other complex models of human behaviour.

Figure 6. First case of stochastic bucket brigade: normally distributed working velocities with average values respecting TOWV ($v_1 = 2, v_2 = 4, v_3 = 6$) and normally distributed walkback velocities around the means $v_{b1} = 8, v_{b2} = 7$ and $v_{b3} = 10$.
In this paper, we proposed two novel approaches for bucket brigade modelling. The first is analytical and the second is based on continuous time simulation. Using our analytical model, we studied the normative model of bucket brigades without the restrictive assumption of infinite backward velocity of workers. For the case where a common finite backward velocity for all operators is assumed, we have shown that the sufficient condition to obtain a self-balancing behaviour is the same as for the normative model. We have also established relations between the performances of a bucket brigade under assumptions of finite and infinite backward velocities.

Then, we developed a novel technique for the design of simulation models to study more general cases of bucket brigades. The proposed approach employs the Matlab/Simulink™ software package. This model is not time consuming and can illustrate the sophisticated temporal behaviour of bucket brigades. Several simulations were accomplished. For the case of finite backward velocity, the simulation model showed the convergence time and the dynamic behaviour of the considered bucket brigades. It illustrated the gap between the behaviour of the normative model and those with non-zero backward times. Simulations have also been made with random working and/or walk-back velocities to illustrate the possibilities of the model. Furthermore, this model is designed to be adapted to study some more complex situations and to take into account different supplementary properties and parameters, as for example, velocities depending on worker position, blockings, etc. Therefore, other human interactions, not yet considered, can also be modelled with this tool.

Figure 7. Second case of stochastic bucket brigade: normally distributed working velocities with average values not respecting TOWV ($v_1 = 6$, $v_2 = 2$, $v_3 = 4$) and normally distributed walk-back velocities having the same mean ($v_{b1} = v_{b2} = v_{b3} = 8$).

5. Conclusion and future work

In this paper, we proposed two novel approaches for bucket brigade modelling. The first is analytical and the second is based on continuous time simulation. Using our analytical model, we studied the normative model of bucket brigades without the restrictive assumption of infinite backward velocity of workers. For the case where a common finite backward velocity for all operators is assumed, we have shown that the sufficient condition to obtain a self-balancing behaviour is the same as for the normative model. We have also established relations between the performances of a bucket brigade under assumptions of finite and infinite backward velocities.

Then, we developed a novel technique for the design of simulation models to study more general cases of bucket brigades. The proposed approach employs the Matlab/Simulink™ software package. This model is not time consuming and can illustrate the sophisticated temporal behaviour of bucket brigades. Several simulations were accomplished. For the case of finite backward velocity, the simulation model showed the convergence time and the dynamic behaviour of the considered bucket brigades. It illustrated the gap between the behaviour of the normative model and those with non-zero backward times. Simulations have also been made with random working and/or walk-back velocities to illustrate the possibilities of the model. Furthermore, this model is designed to be adapted to study some more complex situations and to take into account different supplementary properties and parameters, as for example, velocities depending on worker position, blockings, etc. Therefore, other human interactions, not yet considered, can also be modelled with this tool.
This careful study confirms the robustness of the normative model proposed by Bartholdi and Eisenstein and its sufficient condition of self-balancing under conditions of finite backward velocity and stochastic forward and backward times. Nevertheless, doubts about the normative model assumptions may appear, especially because the work velocities are considered as constant or with a constant average value. One can expect that psychological factors may eventually be significant enough to render impractical this assumption. Relatively few research papers address these issues. Our future work will concern an expansion of the analytical and simulation models to take into account these assumptions and industrial constraints which deal with learning/forgetting processes and other complex models of human behaviour.

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