Prediction of kth records

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Abstract
We consider a sequence of independent identically distributed random
variables with a continuous distribution function and evaluate the ex-
pectations of future $k$th records values based on the sequence under the
condition that some previous ones are known. The bound is expressed
in terms of quantiles and absolute moments centered about the quan-
tiles of the parent distribution.

Keywords: $k$th record, quantile, conditional expectation, Hölder in-
equality, bound.

1 Introduction.
Consider a sequence $X_1, X_2, \ldots$ of i.i.d. random variables with common con-
tinuous distribution function $F$ and quantile function

$$F^{-1}(x) = \sup \{y : F(y) \leq x\}, \quad 0 \leq x < 1.$$ (1)

Following Dziubdziela and Kopociński [4], for $k \geq 1$ we define $k$th record
statistics $R_n^{(k)}$ as:

$$R_n^{(k)} = X_{L_n^{(k)}:L_n^{(k)}+k-1}, \quad n \geq 0,$$ (2)

where $L_n^{(k)}, \ n \geq 0,$ are the $n$th occurrence times of $k$th records defined as:

$$L_0^{(k)} = 1,$$

$$L_{n+1}^{(k)} = \min \{j > L_n^{(k)} : X_{L_n^{(k)}:L_n^{(k)}+k-1} < X_{j:j+k-1}\},$$

and $X_{j:n}, 1 \leq j \leq n < \infty,$ denote the $j$th order statistic of sample $X_1, \ldots, X_n.$

Accordingly, we assume that the time starts when the first $k$ observations oc-
cur. By definition which is the $k$th greatest order statistic from $X_1, X_2, \ldots X_k.$

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The 0th value of \( k \)th record is \( X_{1,k} \). The consecutive \( k \)th records occur, when new observations exceed the \( k \)th greatest order statistics from the preceding samples, and their values amount to new \( k \)th greatest order statistics, which are strictly greater than previous ones.

In this paper we evaluate the expectations of future \( k \)th record values \( R^{(k)}_n \) under the condition that some previous ones \( R^{(k)}_{m_1}, R^{(k)}_{m_2}, \ldots, R^{(k)}_{m_l}, R^{(k)}_m \), \( 1 \leq m_1 < \ldots < m_l < m \) are known. By the Markov property of record values in continuous models, we have

\[
E_F(R^{(k)}_n | R^{(k)}_{m_1} = y_1, \ldots, R^{(k)}_{m_l} = y_l, R^{(k)}_m = y) = E_F(R^{(k)}_n | R^{(k)}_m = y).
\]

Precisely we present sharp bounds of the conditional expectations

\[
E_F(R^{(k)}_n - R^{(k)}_m | R^{(k)}_m = y = F^{-1}(\xi)), \quad m < n,
\]

for arbitrarily given \( 0 < \xi < 1 \), expressed in scale units \( \sigma_p(\xi) \), \( 1 \leq p < \infty \), defined as follows

\[
\sigma_p(\xi) = (E_F|X - F^{-1}(\xi)|^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty. \tag{3}
\]

Here and later on, considering \( \sigma_p(\xi) \) we tacitly assume that

\[
E_F|X|^p = \int_0^1 |F^{-1}(x)|^p dx
\]

is finite. We also consider bounds in the absolute quantile deviations units

\[
\sigma_\infty(\xi) = \text{ess sup} |X - F^{-1}(\xi)| = \max\{F^{-1}(\xi) - F^{-1}(0), F^{-1}(1) - F^{-1}(\xi)\} \tag{4}
\]

for the random variables which are bounded almost surely. The bounds and distributions attaining them are presented in Section 2. We followed a common and convenient convention of writing that the bounds are attained by some discontinuous distributions. In fact, the bounds are attained by the sequences of continuous distributions that tend weakly to the unique discontinuous distributions in limit. Section 3 contains the proofs.

Earlier results for the increments of records or \( k \)th records can be found in literature. Rychlik (1997) and Danielak (2004) presented distribution-free bounds on the standard record increments. Results for increments of \( k \)th records from general and restricted populations were established by Danielak and Raqab (2004 a,b), respectively. Klimczak and Rychlik (2005) provided optimal upper bounds for the increments of order and record statistics under the condition that the values of future order statistics and records are known. Rychlik (2002) presented upper bounds on expectations of order and record statistics increments under the condition that previous values are known. The bounds were expressed in terms of units more complicated than (3) and (4).
2 Auxiliary results

We assume that all the integrals presented below are finite. First we mention the Moriguti inequality (see Moriguti (1953)).

If \( h(x) \) is an integrable function on an interval \([a, b]\) and \( \bar{h}(x) \) stands for the right derivative of the greatest convex function \( \bar{H}(x) \), that is not greater than the indefinite integral \( H(x) = \int_{a}^{x} h(t)dt \) of \( h(x) \), then we have

\[
\int_{a}^{b} g(x)h(x)dx \leq \int_{a}^{b} g(x)\bar{h}(x)dx
\]

for all nondecreasing functions \( g \). The equality holds in (5) iff \( g(x) \) is constant on every interval contained in the open set \( \{ x : H(x) > \bar{H}(x) \} \).

We also recall the well known Hölder inequality

\[
\int_{a}^{b} f(x)g(x)dx \leq \left( \int_{a}^{b} |f(x)|^{p}dx \right)^{\frac{1}{p}} \left( \int_{a}^{b} |g(x)|^{q}dx \right)^{\frac{1}{q}} = ||f||_{p} ||g||_{q}
\]

for arbitrary \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). The equality holds iff either \( g = 0 \) or

\[
f(x) = ||f||_{p} \frac{|f(x)|^{\frac{1}{p-1}} \text{sgn}(x)}{||g||^{\frac{1}{q-1}}}
\]

almost everywhere.

It is well known (see, e.g., Rychlik 2001, p. 134) that the cumulative distribution function of \( R_{n}^{(k)} \) is \( F_{n}^{(k)}(F(x)) \), where

\[
F_{n}^{(k)}(x) = 1 - [1 - x]^{k} \sum_{j=0}^{n} \frac{k^{j}}{j!} [-\ln(1-x)]^{j}
\]

denotes the cdf of \( n \)th value of \( k \)th record of the standard uniform sequences and

\[
f_{n}^{(k)}(x) = [F_{n}^{(k)}(x)]' = \frac{k^{n+1}}{n!} [-\ln(1-x)]^{n}(1-x)^{k-1}
\]

is the respective density function.

Since for continuous distribution, (1) is strictly increasing and so preserves record values of sequences we obtain

\[
\mathbb{E}_{F}R_{n}^{(k)} = \int_{0}^{1} F^{-1}(x) f_{n}^{(k)}(x) dx.
\]
We also use the following properties of record statistics. The first one can be found for instance, in Nevzorov (2001, p.92).

If we consider two sequences of i.i.d. random variables: \( X_1, X_2, \ldots \) with continuous distribution function \( F \) and

\[
Y_1 = \min\{X_1, \ldots, X_n\}, \quad Y_2 = \min\{X_{k+1}, \ldots, X_{2k}\}, \ldots
\]

with distribution function \( G(x) = 1 - (1 - F(x))^k \) and \( R_n^{(k)} = 1 \), \( n = 1, 2, \ldots \) denotes the \( k \)th record values related to the sequence \( X_1, X_2, \ldots \) and \( R_n \), \( n = 1, 2, \ldots \) denotes the ordinary record values in the sequence \( Y_1, Y_2, \ldots \), then for any \( k = 1, 2, \ldots \) sequences \( \{R_n^{(k)}\}_{n=1}^{\infty} \) and \( \{R_n^{(1)}\}_{n=1}^{\infty} \) are identically distributed.

Standard record values have the following Markov property (cf. Nevzorov and Balakrishnan (1998, p. 527)) Namely, if \( R_n^{(1)} \), \( n \geq 0 \), is a sequence of standard record values based on an i.i.d. sequence with a continuous distribution, then the distribution of \( R_n^{(1)} \) under condition that \( R_1^{(1)} = y_1, R_2^{(1)} = y_2, \ldots, R_{m-1}^{(1)} = y_{m-1}, R_m^{(1)} = y \), \( 0 \leq m_1 < m_2 < \ldots < m_l < m < n \), depends only on the last one and coincides with the distribution of \( R_n^{(1)} \) from population with parent distribution function

\[
F_{\mid y}(x) = \frac{F(x) - F(y)}{1 - F(y)} 1_{[y, \infty)}(x),
\]

where \( 1_A(x) \) denotes the indicator function of set \( A \). This is the distribution function of original \( X_1 \) under condition that this exceeds level \( y \).

Since

\[
[1 - (1 - F)^k](y) = 1 - \left[ \frac{1 - F(x)}{1 - F(y)} \right]^k = 1 - [1 - F_{\mid y}(x)]^k,
\]

we have

\[
\mathbb{E} F_{\mid y}(P_n^{(k)}|P_{m_1}^{(1)} = y_1, \ldots, P_m^{(1)} = y) = \mathbb{E}_{1 - (1 - F)^k}(P_n^{(1)}|P_{m_1}^{(1)} = y_1, \ldots, P_m^{(1)} = y) = \mathbb{E}_{1 - (1 - F)^k}(P_n^{(1)}|P_m^{(1)} = y) = \mathbb{E}_{1 - (1 - F)^k}_{\mid y}(R_n^{(1)} - R_{n-m-1}^{(1)}) = \int_0^1 F_{\mid y}^{-1}(x)f_{n-m-1}^{(k)}(x)dx,
\]

where \( f_{n-m-1}^{(k)}(x) \) is given in (7). By (8) and (10), it follows that

\[
\mathbb{E} F_{\mid y}(P_n^{(k)} - P_m^{(k)}|P_m^{(k)} = F^{-1}(\xi)) = \mathbb{E}_{\mid y} f_{n-m-1}^{(k)}(x)dx.
\]

This is the distribution function of original \( X_1 \) under condition that this exceeds level \( y \).
Plugging \( y = F^{-1}(\xi) \) into (9), we have

\[
F_y(x) = \frac{F(x) - \xi}{1 - \xi} 1_{[F^{-1}(\xi), \infty)}(x). \tag{12}
\]

Combining (11) and (12) and changing the variables we obtain

\[
\mathbb{E}(R^{(k)}_n - R^{(k)}_m | R^{(k)}_m = F^{-1}(\xi)) = \\
= \int_0^1 [F^{-1}(x) - F^{-1}(\xi)] f_{n-m-1}^{(k)} \left( \frac{x - \xi}{1 - \xi} \right) \frac{1}{1 - \xi} 1_{[\xi, 1]}(x) dx. \tag{13}
\]

Let us denote

\[
g^{(k)}_{n-m-1}(x) = f^{(k)}_{n-m-1} \left( \frac{x - \xi}{1 - \xi} \right) \frac{1}{1 - \xi} 1_{[\xi, 1]}(x). \tag{14}
\]

Consider the mappings \( x \mapsto F^{-1}(x) - F^{-1}(\xi), \ 0 < x < 1 \), for the quantile functions \( F^{-1} \). These functions form the convex cone of nondecreasing right continuous functions on \([0, 1]\) that change the sign at \( \xi \).

We denote the antiderivative of (14) by

\[
G^{(k)}_{n-m-1}(x) = \int_0^x g^{(k)}_{n-m-1}(t) dt. \tag{15}
\]

Using the method described by Moriguti (1953) we construct the greatest convex minorant \( \bar{G}^{(k)}_{n-m-1}(x) \) of the function \( G^{(k)}_{n-m-1} \) and its derivative \( \bar{g}^{(k)}_{n-m-1} \), which is the projection of \( g^{(k)}_{n-m-1} \) on the family nondecreasing functions in \( L^2([0, 1], dx) \).

To this end, we first analyze the behavior of functions \( g^{(k)}_{n-m-1}, k \geq 1, m < n \). In case \( k = 1 \) and \( m = n - 1 \) we see that

\[
g^{(1)}_{n-m-1}(x) = \frac{1}{(n - m - 1)!} \left[ -\ln \frac{1 - x}{1 - \xi} \right]^{n-m-1} 1_{[\xi, 1]}(x). \tag{16}
\]

is equal to 0 on \([0, \xi]\) and positive increasing on \([\xi, 1]\). Because \( G^{(1)}_{n-m-1} \) is equal to 0 on \([0, \xi]\) and convex increasing on \([\xi, 1]\), then the greatest convex minorant satisfies \( \bar{G}^{(1)}_{n-m-1} = G^{(1)}_{n-m-1} \). Therefore the projection has the form

\[
\bar{g}^{(1)}_{n-m-1}(x) = f^{(1)}_{n-m-1} \left( \frac{x - \xi}{1 - \xi} \right) \frac{1}{1 - \xi} 1_{[\xi, 1]}(x). \]

Now consider the case \( k \geq 2 \) and \( m = n - 1 \). Function \( g^{(k)}_0 \) is equal to 0 on \([0, \xi]\) and positive decreasing on \([\xi, 1]\). Therefore \( G^{(k)}_0 \) is equal to 0 on \([0, \xi]\).
and concave increasing on \([\xi, 1]\). The greatest convex minorant \(\tilde{G}_0^{(k)}\) of the function \(G_0^{(k)}\) is piecewise linear

\[
\tilde{G}_0^{(k)}(x) = \begin{cases} 
0, & 0 \leq x \leq \xi, \\
\frac{x-\xi}{1-\xi}, & \xi \leq x \leq 1.
\end{cases}
\]

We conclude that the projection of \(g_0^{(k)}\) is of the form

\[
g_0^{(k)}(x) = \frac{1}{1-\xi} \mathbf{1}_{[\xi, 1]}(x). \tag{17}
\]

Note that for \(k = 1\), (16) and (17) are identical.

In the third case, \(m < n - 1\) and \(k \geq 2\), function \(g_{n-m-1}^{(k)}\) is vanishing on \([0, \xi]\) and increasing in \((\xi, t)\) and decreasing in \((t, 1)\), where

\[
t = 1 - (1 - \xi) e^{-\frac{n-m-1}{x-\xi}} \in (\xi, 1). \tag{18}
\]

Moreover \(g_{n-m-1}^{(k)}(\xi) = 0\) and \(g_{n-m-1}^{(k)}(1) = 0\). The antiderivative \(G_{n-m-1}^{(k)}\) is vanishing on \((0, \xi]\), convex increasing on \([\xi, t]\) and concave increasing on \([t, 1]\) with \(t\) defined in (18). Moreover, \(G_{n-m-1}^{(k)}(0) = 0\) and \(G_{n-m-1}^{(k)}(1) = 1\).

Now we determine the greatest convex minorant \(\tilde{G}_{n-m-1}^{(k)}(x)\) of (15) in this case. It is obvious that \(G_{n-m-1}^{(k)}(x)\) is equal zero on \([0, \xi]\), then coincides with \(G_{n-m-1}^{(k)}\) and finally is linear increasing. Consider the lines tangent to \(G_{n-m-1}^{(k)}\) at points \(\alpha \in [\xi, t]\). The slopes of \(l_\alpha(x)\) continuously increase for \(\alpha \in [\xi, t]\). Moreover \(l_\alpha(1)\) amounts to 0 for \(\alpha = \xi\) and \(l_\alpha(1) > 1 = G_{n-m-1}^{(k)}(1)\). Also, for a unique \(\alpha^* \in (\xi, t)\) we get \(l_{\alpha^*}(1) = 1\). This defines the greatest convex minorant that has the formula

\[
\tilde{G}_{n-m-1}^{(k)}(x) = \begin{cases} 
0, & 0 \leq x \leq \xi, \\
G_{n-m-1}^{(k)}(x), & \xi \leq x \leq \alpha^*, \\
g_{n-m-1}^{(k)}(\alpha^*)(x-1) + 1, & \alpha^* \leq x \leq 1,
\end{cases}
\]

where \(\alpha^* \in (\xi, t)\) is such that

\[
g_{n-m-1}^{(k)}(\alpha^*) = \frac{1 - G_{n-m-1}^{(k)}(\alpha^*)}{1 - \alpha^*}. \tag{19}
\]

Therefore the projection of \(g_{n-m-1}^{(k)}\) is

\[
g_{n-m-1}^{(k)}(x) = \begin{cases} 
0, & 0 \leq x \leq \xi, \\
g_{n-m-1}^{(k)}(x), & \xi \leq x \leq \alpha^*, \\
g_{n-m-1}^{(k)}(\alpha^*), & \alpha^* \leq x \leq 1.
\end{cases}
\]

\[6\]
3 Main results

We now present bounds for increments of \(k\)th records values if the previous one is known in units defined in (3) and (4).

**Theorem 1** Let \(1 \leq m < n\), \(k \geq 1\), \(0 < \xi < 1\), \(1 < p, q < \infty\) and \(q = p/(q-1)\).

If \(m = n - 1\) and \(k \geq 1\) then

\[
\mathbb{E}_F \left( \frac{R_n^{(k)} - R_{n-1}^{(k)} | R_{n-1}^{(k)} = F^{-1}(\xi)}{\sigma_p(\xi)} \right) \leq (1 - \xi)^{-\frac{1}{p}} \tag{20}
\]

and the equality in (20) holds for the two-point distribution

\[
\mathbb{P}(X = F^{-1}(\xi) + \sigma_p(\xi)(1 - \xi)^{-\frac{1}{p}}) = 1 - \xi,
\]

\[
\mathbb{P}(X = F^{-1}(\xi)) = \xi. \tag{21}
\]

If \(1 \leq m \leq n - 2\) and \(k \geq 2\), the inequality

\[
\mathbb{E}_F \left( \frac{R_n^{(k)} - R_{n-1}^{(k)} | R_{n-1}^{(k)} = F^{-1}(\xi)}{\sigma_p(\xi)} \right) \leq B_q = B(m, n, \xi, q) \tag{22}
\]

is sharp for

\[
B_q = \frac{1}{(1 - \xi)^{q-1}} \int_0^{\frac{x - F^{-1}(\xi)}{\sigma_p(\xi)}} \left( \frac{x - F^{-1}(\xi)}{\sigma_p(\xi)} \right)^q dx + \frac{1 - \alpha^*}{(1 - \xi)^q} \left( f_{n-m-1} \left( \frac{\alpha^* - \xi}{1 - \xi} \right) \right)^q, \tag{23}
\]

with \(\alpha^* \in (\xi, 1)\) satisfying equation (19).

Moreover, the bound is achieved for

\[
F(x) = \begin{cases} 
0, \\
(1 - \xi) f_{n-m-1}^{-1} \left( (1 - \xi) B_q \left( \frac{x - F^{-1}(\xi)}{\sigma_p(\xi)} \right)^{p-1} \right) + \xi, & 0 \leq \frac{x - F^{-1}(\xi)}{\sigma_p(\xi)} < c(\alpha^*), \\
1, & c(\alpha^*) \leq \frac{x - F^{-1}(\xi)}{\sigma_p(\xi)} \tag{24}
\end{cases}
\]

for

\[
c(\alpha^*) = \left( f_{n-m-1} \left( \frac{\alpha^* - \xi}{1 - \xi} \right) \right)^{\frac{1}{p-1}},
\]

where \(f_{n-m-1}^{(k)}\) is defined in (7).

If \(k = 1\) and \(m < n - 1\) then

\[
\mathbb{E}_F \left( \frac{R_m^{(1)} - R_{m-1}^{(1)} | R_{m-1}^{(1)} = F^{-1}(\xi)}{\sigma_p(\xi)} \right) \leq (1 - \xi)^{-\frac{1}{p}} \left( \frac{\Gamma((n - m - 1) q + 1)}{(n - m - 1)!} \right)^{\frac{1}{1/p}} \tag{25}
\]

and bound (25) is achieved by (24) with \(\alpha^* = \xi\).
Now, we consider case if $p = 1$.

**Theorem 2** Let $1 \leq m < n$, $k \geq 1$, $0 < \xi < 1$ and $p = 1$.
If $k \geq 1$, $m = n - 1$, we have the bound
\[
\mathbb{E}_F \left( \frac{R_n^{(k)} - R_{n-1}^{(k)} | R_{n-1}^{(k)} = F^{-1}(\xi)}{\sigma_1(\xi)} \right) \leq (1 - \xi)^{-1}.
\] (26)

The equality holds for the two point distribution
\[
\mathbb{P}(X = F^{-1}(\xi)) = \xi, \\
\mathbb{P}(X = F^{-1}(\xi) + \sigma_1(\xi)(1 - \xi)^{-1}) = 1 - \xi.
\] (27)

If $k \geq 2$, $1 \leq m \leq n - 2$, then
\[
\mathbb{E}_F \left( \frac{R_n^{(k)} - R_m^{(k)} | R_m^{(k)} = F^{-1}(\xi)}{\sigma_1(\xi)} \right) \leq \frac{f^{(k)}_{n-m-1} \left( \frac{\alpha^* - \xi}{1 - \xi} \right)}{1 - \xi}
\] (28)

and the equality holds for two point distribution
\[
\mathbb{P}(X = F^{-1}(\xi)) = \alpha^*, \\
\mathbb{P}(X = F^{-1}(\xi) + \sigma_1(\xi)f^{(k)}_{n-m-1} \left( \frac{\alpha^* - \xi}{1 - \xi} \right)) = 1 - \alpha^*.
\] (29)

If $k = 1$ and $m < n - 1$, we have not a finite bound. It follows from the fact proven in Nagaraja (1978) that there exist distributions $F$ with finite expectations such that the expectations of standard records based on i.i.d. sequences with distribution $F$ are infinite.

Finally we present the bound for conditional expectation of $k$th record values in case $p = \infty$.

**Theorem 3** For arbitrary $1 \leq m < n$, $k \geq 1$, $0 < \xi < 1$ and $p = \infty$, we have
\[
\mathbb{E}_F \left( \frac{R_n^{(k)} - R_{n-1}^{(k)} | R_{n-1}^{(k)} = F^{-1}(\xi)}{\sigma_{\infty}(\xi)} \right) \leq 1,
\] (30)

and the equality in (30) is attained iff
\[
\mathbb{P}(X = F^{-1}(\xi) + \sigma_{\infty}(\xi)) = 1 - \xi, \\
\mathbb{P}(F^{-1}(\xi) - \sigma_{\infty}(\xi) \leq X \leq F^{-1}(\xi)) = \xi.
\] (31)
4 Proofs

Proof of Theorem 1.
Applying (13) and Moriguti’s inequality we obtain

\[ \mathbb{E}_F(R_n^{(k)} - R_m^{(k)}) \leq \int_0^1 [F^{-1}(x) - F^{-1}(\xi)] \bar{g}_{n-m-1}^{(k)}(x) \, dx, \quad (32) \]

where

\[ \bar{g}_{n-m-1}^{(k)}(x) = \begin{cases} \frac{1}{1-x} 1_{[\xi,1)}(x), & k \geq 1, \ m = n - 1; \\ \frac{1}{1-x} f_{n-m-1}^{(k)}(\min\{x,\alpha^*-\xi\}) 1_{[\xi,1)}(x), & k \geq 2, \ 1 \leq m \leq n - 2, \\ f_{n-m-1}^{(1)}(\frac{x-\xi}{1-\xi}) \frac{1}{1-x} 1_{[\xi,1)}(x), & k = 1, \ 1 \leq m < n - 1. \end{cases} \]

Applying Hölder inequality to (32) and noting that function \( \bar{g}_{n-m-1}^{(k)}(x) \) is nonnegative on \([0,1]\), we obtain

\[ \int_0^1 [F^{-1}(x) - F^{-1}(\xi)] \bar{g}_{n-m-1}^{(k)}(x) \, dx \leq \left( \int_0^1 [F^{-1}(x) - F^{-1}(\xi)]^p \, dx \right)^{\frac{1}{p}} \left( \int_0^1 (\bar{g}_{n-m-1}^{(k)}(x))^q \, dx \right)^{\frac{1}{q}} = \sigma_p(\xi) ||\bar{g}_{n-m-1}^{(k)}||_q. \quad (33) \]

The equality in (33) holds iff

\[ \frac{F^{-1}(x) - F^{-1}(\xi)}{\sigma_p(\xi)} = (\bar{g}_{n-m-1}^{(k)}(x))_{p^{-1}} \frac{1}{||\bar{g}_{n-m-1}^{(k)}||_q^{\frac{1}{q^{-1}}}}. \quad (34) \]

Observe that in all the cases (34) defines \( F^{-1}(x) \) which is nondecreasing right continuous, vanishes at \( \xi \) and its \( q \)th norm is equal to \( \sigma_p(\xi) \). In particular, if \( m = n - 1 \) and \( k \geq 1 \), then

\[ ||\bar{g}_{0}^{(k)}||_q = (1 - \xi)^{-\frac{1}{p}}. \]

Hence the equality in (33) is attained if

\[ F^{-1}(x) - F^{-1}(\xi) = \sigma_p(\xi) (1 - \xi)^{-\frac{1}{p}} 1_{[\xi,1)}(x) \]

which implies the two point distribution given in (21).

For \( k \geq 2 \) and \( 1 \leq m \leq n - 2 \), we have

\[ ||\bar{g}_{n-m-1}^{(k)}||_q = B_q, \]
where

\[ B_q = \left\{ \begin{array}{ll}
\frac{1}{(1-\xi)^{q-1}} \int_0^{(\alpha^*-\xi)/(1-\xi)} |f^{(k)}(n-m-1)(x)|^q dx + \frac{1-\alpha^*}{(1-\xi)^q} \left( f^{(k)}(n-m-1) \left( \frac{\alpha^*-\xi}{1-\xi} \right) \right)^q \end{array} \right\}^{\frac{1}{q}}.
\]

The equality is achieved iff

\[
\frac{F^{-1}(x) - F^{-1}(\xi)}{\sigma_p(\xi)} = \begin{cases} 
0, & 0 \leq x < \xi; \\
\left\lceil \frac{f^{(k)}(n-m-1)(\xi-\xi)}{(1-\xi)B_q} \right\rceil^\frac{1}{p}, & \xi \leq x < \alpha^*, \\
\left\lceil \frac{f^{(k)}(n-m-1)(\alpha^*-\xi)}{(1-\xi)B_q} \right\rceil^\frac{1}{p}, & \alpha^* \leq x < 1,
\end{cases}
\]

which leads us to (24).

In case \( k = 1 \) and \( 1 \leq m < n - 1 \), we obtain

\[
||\tilde{g}^{(1)}_{n-m-1}||_q = (1-\xi)^{-\frac{1}{p}} \frac{1}{(n-m-1)!} (\Gamma((n-m-1)q + 1))^\frac{1}{q}.
\]

The equality is attained by (35) with \( \alpha^* = \xi \).

\[ \square \]

Proof of Theorem 2.
Using representation (32) and the Hölder inequality in case \( p = 1 \), we have

\[
\mathbb{E}_F(R^{(k)}_n - R^{(k)}_m | R^{(k)}_m = F^{-1}(\xi)) \leq \int_0^1 [F^{-1}(x) - F^{-1}(\xi)] \tilde{g}^{(k)}_{n-m-1}(x) dx
\]

\[
\leq \sup_{0 \leq x \leq 1} \tilde{g}^{(k)}_{n-m-1}(x) \int_0^1 |F^{-1}(x) - F^{-1}(\xi)| dx.
\]

If \( k \geq 1 \) and \( m = n - 1 \), we have

\[
\sup_{0 \leq x \leq 1} \tilde{g}^{(k)}_0(x) = \frac{1}{1-\xi},
\]

and so we obtain the bound given in (26).

Case \( m < n - 1 \) with \( k \geq 2 \), yields

\[
\sup_{0 \leq x \leq 1} \tilde{g}^{(k)}_0(x) = g^{(k)}_{n-m-1}(\alpha^*),
\]

and we have the bound of (28). In these two cases, the first inequality becomes in (36) equality if \( F^{-1}(x) \) is constant on \((\xi, 1)\) and \((\alpha^*, 1)\), respectively. For \( m = n - 1 \) this provides the equality in the second inequality.
as ell. If $m < n - 1$, then $F^{-1}(x) = F^{-1}(\xi)$ on $(\xi, 1)$ is needed as well. In particular (36) becomes equality for two point distribution defined in (27) and (29), which satisfy quantile and moment conditions as well. If $k = 1$ and $1 \leq m < n - 1$, then the right side of (36) is infinite.

Proof of Theorem 3.
Using (32) and Hölder’s inequality in case $p = \infty$, we obtain

$$E_F(R^{(k)}_n - R^{(k)}_m | R^{(k)}_m = F^{-1}(\xi))$$

$$\leq \sup_{0 \leq x \leq 1} |F^{-1}(x) - F^{-1}(\xi)| \int_0^{\bar{g}_{n-m-1}(x)} dx$$

$$= \bar{G}_{m-n-1}(1) \sigma_{\infty}(\xi) = \sigma_{\infty}(\xi).$$

The equality is attained if

$$F^{-1}(x) - F^{-1}(\xi) = \sigma_{\infty}(\xi) \quad \text{for} \quad x \in \{x : \bar{g}_{n-m-1}(x) \neq 0\}$$

and

$$|F^{-1}(x) - F^{-1}(\xi)| < \sigma_{\infty}(\xi) \quad \text{for} \quad x \in \{x : \bar{g}_{n-m-1}(x) = 0\}$$

except for a set of Lebesgue measure 0.

Here

$$\{x : \bar{g}_{n-m-1}(x) \neq 0\} = (\xi, 1],$$

$$\{x : \bar{g}_{n-m-1}(x) = 0\} = [0, \xi],$$

which implies that

$$F^{-1}(x) = F^{-1}(\xi) + \sigma_{\infty}(\xi) \quad \text{for} \quad x \in (\xi, 1]$$

and

$$-\sigma_{\infty}(\xi) + F^{-1}(\xi) \leq F^{-1}(x) \leq F^{-1}(\xi) \quad \text{for} \quad x \in [0, \xi].$$

This implies (31) and completes the proof.

References


