KOKSMA’S INEQUALITY AND GROUP EXTENSIONS OF KRONECKER TRANSFORMATIONS

Jon Aaronson¹, Mariusz Lemańczyk², Christian Mauduit³, and Hitoshi Nakada⁴

¹School of Mathematical Sciences, Tel Aviv University
69978 Tel Aviv, Israel
²Institute of Mathematics, Nicholas Copernicus University
ul. Chopina 12/18, 87-100 Toruń, Poland
³Laboratoire de Mathématiques Discrètes
Case 930, 163 avenue de Luminy
13288 Marseille Cedex 9, France
⁴Dept. Math., Keio University, Hiyoshi 3-14-1 Kohoku
Yokohama 223, Japan

ABSTRACT. We consider methods of establishing ergodicity of group extensions, proving that a class of cylinder flows are ergodic, coalescent and non-squashable. A new Koksmɑ-type inequality is also obtained.

Introduction

We study locally compact group extensions of Kronecker transformations.

Let $X$ be a compact monothetic group with Haar probability measure $m = m_X$, and $G$ a locally compact metric group with Haar measure $m_G$. Let $T$ be an ergodic translation on $X$, (called a Kronecker transformation) and set $\mu = m \times m_G$.

For $\varphi : X \to G$ measurable (called a cocycle), consider the skew product (or $G$-extension) which is the measure preserving transformation $T_\varphi : (X \times G, \mu) \to (X \times G, \mu)$ defined by

$$T_\varphi(x, g) = (Tx, \varphi(x)g).$$

Recall from [Aa1] that a measure preserving transformation $\tau : (Y, \nu) \to (Y, \nu)$ is called squashable if $\exists Q \ni Q\tau = \tau Q$ and $\nu Q^{-1} = c\nu$ for certain $c \neq 1$. It follows from [Aa2, Th3.4] that if the group $G$ is countable, and has no arbitrarily large finite normal subgroups (e.g. $G = \mathbb{Z}^k \times \mathbb{Q}'$) then no ergodic $G$-extension is squashable.

Aaronson and Lemańczyk would like to thank Keio University for hospitality provided while this research was done. Lemańczyk’s research was partly supported by KBN grant 512/2/91.

Algorithms, Fractals, and Dynamics
Most of the results in this paper are for the case $G = \mathbb{R}$. It is an open problem to decide if there is a conservative, ergodic, squashable $\mathbb{R}$-extension of a Kronecker transformation. Almost all of our results are in the other direction, showing that certain $\mathbb{R}$-extensions are nonsquashable.

We consider product-type cocycles for odometers in §1, obtaining conditions for ergodicity, nonsquashability, and coalescence (q.v.) Essentially the same ideas can be used in the context of [Kw-Le-Ru2] to obtain analytic cylinder flows (i.e. $\mathbb{R}$-extensions of rotations of the circle) which are ergodic, nonsquashable, and coalescent (see §4). We show in §5 that if $\varphi : T \to \mathbb{R}$ is $C^{1+\delta}$ then for a residual set of irrational rotations $T$, the cocycle is conservative and ergodic. We improve some recent results by D. Pask (in §6) [Pa1],[Pa2] on the ergodicity of cylinder flows also proving the non-squashability in this case.

One of our tools is a new Koksma-type inequality in $L^2(T)$ for functions whose Fourier coefficients are of order $O(1/n)$ (see §2) with possible speeds of convergence for smooth functions and irrational rotations admitting a speed of approximations by rationals (see §3).

The authors would like to thank E. Lesigne for a discussion on the proof of Theorem 5.1.

§1 Coalescence of group extensions, and ergodicity of product type cocycles

A non-singular transformation is called coalescent if all nonsingular commuting with it transformations are invertible. To begin this section, we study the form of non-singular transformations commuting with an ergodic, group extension of a Kronecker transformation.

Suppose that $T$ is an ergodic measure-preserving transformation of the probability space $(X, \mathcal{B}, m)$; let $(G, T)$ be an abelian, locally compact, second countable, topological group ($T = T(G)$ denotes the family of open sets in the topological space $G$), and let $\varphi : X \to G$ be a cocycle.

Let $T_\varphi : (X \times G, \mu) \to (X \times G, \mu)$,

$$T_\varphi(x, g) = (Tx, \varphi(x)g)$$

be ergodic (this implies that $G$ has to be amenable [Zim]), where $T$ is a Kronecker transformation on $X$, and $\varphi : X \to G$ is a cocycle.

Proposition 1.1. Suppose that $Q : X \times G \to X \times G$ is non-singular and $QT_\varphi = T_\varphi Q$. Then there exist a translation $S$ of $X$, and a continuous group homomorphism $w : G \to G$ which is non-singular in the sense that $m_G \circ w^{-1} \sim m_G$ and a measurable map $f : X \to G$ such that

$$Q(x, h) = (Sx, f(x)w(h)) \text{ for each } x \in X, h \in G.$$ 

Proof. Write $Q = (S, F)$, where $S : X \times G \to X$ and $F : X \times G \to G$. We have

$$S \circ T_\varphi = T \circ S \text{ and } F \circ T_\varphi = (\varphi \circ S) \cdot F.$$

28
Let \( U : X \times G \rightarrow X \) be defined by \( U(x, h) = x^{-1}S(x, h) \), then \( U \circ T_\varphi = U \), hence by ergodicity of \( T_\varphi \), \( U(x, h) = x_1 \), and \( S(x, g) = Sx = x x_1 = x_1 x \). Therefore

\[
FT_\varphi(x, h) = \varphi(Sx)F(x, h).
\]

Denote \( \sigma_g(x, h) = (x, hg) \) and note that for each \( g \in G \), \( \sigma_g T_\varphi = T_\varphi \sigma_g \). Hence

\[
\left( F^{-1} \circ (F \circ \sigma_g) \right) \circ T_\varphi(x, h) = F(T_\varphi(x, h))^{-1} F(T_\varphi(x, hg))
\]

\[
= \left( \varphi(Sx)F(x, h) \right)^{-1} \varphi(Sx)F(x, hg)
\]

\[
= \left( F^{-1} \circ \sigma_g \right)(x, h),
\]

whence there exists \( w : G \rightarrow G \) such that \( F^{-1}(F \circ \sigma_g) = w(g) \) for each \( g \in G \). It follows that \( w \) is a measurable homomorphism (and hence continuous).

Set \( \phi(x, h) = F(x, h)w(h)^{-1} \). By the above, \( \phi \circ \sigma_g = \phi \) for each \( g \in G \) whence there exists a measurable \( f : X \rightarrow G \) such that \( \phi(x, h) = f(x) \) a.e., and

\[
Q(x, g) = (Sx, f(x)w(g)).
\]

To see that \( w : G \rightarrow G \) is non-singular, note that \( \mu \circ S_f^{-1} = \mu \), and since \( QT_\varphi = T_\varphi Q \), \( \exists c > 0 \) such that \( \mu \circ Q^{-1} = c \mu \). Moreover

\[
\bar{w} := \text{Id} \times w = S_f^{-1} \circ Q
\]

whence \( \mu \circ \bar{w}^{-1} = c \mu \), and \( m \circ \bar{w}^{-1} = cm \).

\[\Box\]

Remarks

(1) If \( T \) is an invertible, ergodic probability preserving transformation and \( \varphi \) an ergodic cocycle, and \( Q(x, g) = (Sx, F(x, g)) \) is non-singular, and commutes with \( T_\varphi \), then \( Q \) has the above form.

(2) If \( w : G \rightarrow G \) is non-singular and measurable, then \( w \) is continuous, and onto. To see this, note that \( w(G) \) is a \( m_G \)-measurable subgroup of \( G \), whence

\[
\exists x \notin w(G) \quad \Rightarrow \quad x w(G) \subset G \setminus w(G)
\]

\[
\Rightarrow \quad m(w(G)) = m(x w(G)) \leq m(G \setminus w(G)) = 0.
\]

(3) If \( G \) is such that any continuous group non-singular homomorphism is 1-1 (e.g. \( G = \mathbb{Z}^k \times \mathbb{Q}^l \times \mathbb{R}^m \) then any ergodic \( G \)-extension of a Kronecker transformation is coalescent. For coalescence of other group extensions, see theorem 1.5 below.

(4) In case \( G = \mathbb{R} \) a skew product \( T_\varphi \) is squashable iff it commutes with a \( Q \) of form \( Q(x, t) = (Sx, ct + \psi(x)) \), where \( |c| \neq 1 \), or, in other words, \( c \varphi - \varphi \circ S \) is a coboundary for some \( |c| \neq 1 \) and \( S \) a translation of \( X \).
Next, we turn to methods of proving ergodicity of group extensions.

As in [Sch], the essential values of \( \varphi \) are defined as those group elements \( a \in G \) with the property that

\[
\forall A \in \mathcal{B}_+, \ U \in T(G) \text{ with } a \in U; \ \exists \ n \geq 1 \ \exists \ m(A \cap T^{-n}A \cap [\varphi^{(n)} \in U]) > 0
\]

where \( \varphi^{(n)}(x) = \varphi(T^{n-1}x) \cdots \varphi(x) \), \( n \geq 1 \).

The collection of essential values of \( \varphi \) is denoted by \( E(\varphi) \). It is shown in [Sch] that \( E(\varphi) \) is a closed subgroup of \( G \), and is the collection of periods for \( T_\varphi \)-invariant functions:

\[
E(\varphi) = \{ a \in G : f(x, y + a) = f(x, y) \ \text{a.e.} \ \forall f \circ T_\varphi = f \ \text{measurable} \}.
\]

In particular, \( T_\varphi \) is ergodic iff \( E(\varphi) = G \). Also,

**Lemma 1.2** [Sch]. For any compact set \( K \) which is disjoint from \( E(\varphi) \) there is a Borel set \( B, \mu(B) > 0 \), such that for each integer \( m > 0 \) we have

\[
\mu(B \cap T^{-m}B \cap [\varphi^{(m)} \in K]) = 0.
\]

**Definition.** A sequence \( q_n \in \mathbb{N} \ (n \geq 1) \), \( q_n \uparrow \infty \) is called a rigidity time for the probability preserving transformation \( T \) if \( T^{q_n \uparrow \infty} \xrightarrow{U(L^2(m))} \text{Id} \). Here \( U(L^2(m)) \) denotes the collection of unitary operators on \( L^2(m) \). Note that if \( T \) is a translation on the compact group \( X \) with Haar measure \( m \) then \( T^{q_n \uparrow \infty} \xrightarrow{U(L^2(m))} \text{Id} \) iff \( T^{q_n \uparrow \infty} X \rightarrow \text{Id} \).

**Lemma 1.3.** Suppose that \( K \subset \mathcal{M} \) is compact, and that \( \{ q_n \} \) is a rigidity time for \( T \) such that

\[
\forall A \in \mathcal{B}_+, \ \liminf_{n \to \infty} m(A \cap [\varphi^{(q_n)} \in K]) > 0,
\]

then

\[
K \cap E(\varphi) \neq \emptyset.
\]

**Proof.** Follows immediately from Lemma 1.2.

Let

\[
D(\varphi) = \{ a \in G : \exists q_n \to \infty, \ T^{q_n \uparrow \infty} \xrightarrow{U(L^2(m))} \text{Id} \ \exists \ \forall \ \ n_k \to \infty, \ a \in \{ \varphi^{(q_{n_k})} \}_{k \geq 1} \ \text{a.e.} \}.
\]

See also proofs of ergodicity in [Aa2, §4].

**Proposition 1.4.**

\[
D(\varphi) \subset E(\varphi).
\]

**Proof.** Suppose that \( y \in D \), and \( T^{q_n} \text{Id, } y \in \{ \varphi^{(q_{n_k})} : k \geq 1 \} \text{ a.e. } \forall n_k \to \infty \), then

\[
\forall A \in \mathcal{B}_+, \ y \in U \in T(G), \ \exists \ \delta > 0 \ \exists \ \liminf_{n \to \infty} m(A \cap [\varphi^{(m)} \in U]) \geq \delta,
\]
because if there were no such $\delta > 0$ we could choose $y \in U \in \mathcal{T}(G)$, and a subsequence $q_{n_k}$ $(k \geq 1)$ satisfying $m(A \cap [\varphi(q_{n_k}) \in U]) < 1/2^n$ and use the Borel-Cantelli lemma to get a contradiction to the definition of $y \in D(\varphi)$. Hence, since $T^{n_k} \to \text{Id}$, $\liminf_{n \to \infty} m(A \cap T^{-n} \cap [\varphi(q_{n_k}) \in U]) > \frac{\delta}{2} \forall n$ large, and therefore $y \in E(\varphi)$. \qed

Set

$$\overline{D}(\varphi) = \{a \in G : \exists q_n \ni T^{q_n} \cdot \frac{U(L^2(m))}{T^{L_2}} \text{Id, } \varphi(q_n) \to a \text{ a.e.}\}.$$ 

Clearly $\overline{D}(\varphi) \subset D(\varphi)$.

**Theorem 1.5.** Assume that $T$ is an ergodic translation. If $Gp(\overline{D}(\varphi))$ is dense in $G$, then $T_\varphi$ is ergodic, and

$$Q : X \times G \to X \times G \text{ nonsingular, } QT_\varphi = T_\varphi Q \Rightarrow Q(x, g) = (Sx, g + f(x))$$

where $ST = TS$ and $f : X \to G$ is measurable.

In particular, such a $T_\varphi$ is coalescent, and non-squashable.

**Proof.** By the previous proposition, $T_\varphi$ is ergodic. We know from proposition 1.1 that

$$Q : X \times G \to X \times G \text{ nonsingular, } QT_\varphi = T_\varphi Q \Rightarrow Q(x, g) = (Sx, w(g) + f(x))$$

where $ST = TS$, $f : X \to G$ is measurable, and $w : G \to G$ is a continuous nonsingular homomorphism. It follows that

$$w(\varphi) - \varphi \circ S = f - f \circ T,$$

whence

$$\overline{D}(w(\varphi) - \varphi \circ S) = \{0\}.$$ 

However, if $a \in \overline{D}(\varphi)$, and

$$q_n \to \infty, T^{q_n} \cdot \frac{U(L^2(m))}{T^{L_2}} \text{Id, } \varphi(q_n) \to a \text{ a.e.},$$

then

$$w(\varphi(q_n)) - \varphi(\varphi(q_n)) \circ S \to w(a) - a \text{ a.e.}$$

whence $w(a) - a \in \overline{D}(w(\varphi) - \varphi \circ S) = \{0\}$ and $w(a) = a \forall a \in \overline{D}(\varphi)$ and hence $\forall a \in G$. \qedsymbol

Set

$$C(\varphi) = \{a \in G : \liminf_{T^{q_n} \cdot \frac{U(L^2(m))}{T^{L_2}} \text{Id, } q_n \to 0} 1_{U(\varphi(q_n))} = 1 \text{ a.e.} \forall a \in \overline{U} \in \mathcal{T}(G)\}.$$ 

It is not hard to show that (for $T$ Kronecker)

$$E(\varphi) \subset C(\varphi) \subset \tilde{E}(\varphi)$$

where $\tilde{E}(\varphi) :=$

$$\{a \in G : \forall I \in \mathcal{T}(X), a \in U \in \mathcal{T}(G) \exists n \geq 1 \exists m(I \cap T^{-n}I \cap [\varphi(q_n) \in U]) > 0\}.$$
A popular misconception in the subject for the case \( G = \mathbb{R} \) ([Con, proposition 1] [He-La1, lemma 3]) seems to have been that \( C(\varphi) \subset E(\varphi) \).

This latter claim is wrong. A counterexample for a Kronecker transformation is given in example 1.7 (below). An analogous example for the case \( G = \mathbb{T} \) was given in [Furst]. See [Or, proposition 1] for a related method of proving ergodicity not based on the above.

The rest of this section is devoted to

Cocycles of product type for an odometer

For \( a_n \in \mathbb{N}, \ (n \in \mathbb{N}) \), set \( \Omega := \prod_{n=1}^{\infty} \{0, \ldots, a_n - 1\} \) equipped with the addition

\[
(\omega + \omega')_n = \omega_n + \omega'_n + \epsilon_n \mod a_n
\]

where \( \epsilon_1 = 0 \) and

\[
\epsilon_{n+1} = \begin{cases} 
0 & \omega_n + \omega'_n + \epsilon_n < a_n \\
1 & \omega_n + \omega'_n + \epsilon_n \geq a_n.
\end{cases}
\]

Clearly, \( \Omega \) equipped with the product discrete topology, is a compact Abelian topological group (called an odometer group), with Haar measure

\[
m = \prod_{n=1}^{\infty} \left( \frac{1}{a_n}, \ldots, \frac{1}{a_n} \right).
\]

Also if \( \tau = (1, 0, \ldots) \) then \( \Omega = \{n\tau\}_{n \in \mathbb{N}} \) whence \( x \mapsto Tx:= \tau + x \) (called an odometer transformation) is ergodic.

A cocycle of product type is a measurable function \( \varphi : \Omega \rightarrow G \) (where \( G \) is an Abelian topological group) of form

\[
\varphi(\omega) = \sum_{n=1}^{\infty} (b_n(T\omega) - b_n(\omega))
\]

where \( b_n(\omega) = \beta_n(\omega_n) \), where \( \beta_n : \{0, \ldots, q_n - 1\} \rightarrow G \) (notice that \( T\omega \) differs from \( \omega \) only in finitely many places whenever \( \omega \neq -\tau \), so \( \varphi \) is well-defined except for one point).

Set \( q_1 = 1, \ q_{n+1} = \prod_{k=1}^{n} a_k \), then

\[
(q_n\tau)_k = \begin{cases} 
1 & k = n \\
0 & k \neq n,
\end{cases}
\]

whence

\[
T^{q_n}\omega = (\omega_1, \ldots, \omega_{n-1}, \tilde{\omega}_n + (\omega_n, \ldots))
\]

where

\[
\tilde{\omega}_n = (1, 0, \ldots) \in \prod_{k=n}^{\infty} \{0, \ldots, a_k - 1\}.
\]

Note that

\[
\varphi^{(k)}(\omega) := \sum_{j=0}^{k-1} \varphi(T^j \omega) = \sum_{n=1}^{\infty} [b_n(T^n\omega) - b_n(\omega)],
\]

32
whence

\[ \varphi^{(n_k)}(\omega) = \sum_{n=1}^{\infty} \sum_{\ell_k(\omega) - 1}^{b_n(T^n\omega) - b_n(\omega)} \beta_k(0) - \beta_k(\alpha_k + 1) + \beta_k(\omega_k + \ell_k(\omega)) - \beta_k(\omega_k + \ell_k(\omega)), \]

where

\[ \ell_k(\omega) = \min\{n \geq 0 : \omega_k + 1 < \alpha_k + 1\}. \]

We begin by considering cocycles of form

\[ \beta_n(k) = k \lambda_n := \lambda_n + \cdots + \lambda_n \text{ for } 0 \leq k \leq a_n - 1, \text{ where } \lambda_n \in G. \]

Proposition 1.6. If \( r_n \in \mathbb{N} \) and \( \sum_{n=1}^{\infty} \frac{r_n}{a_n} < \infty \), then

\[ \{ k \lambda_n : n \geq 1, 1 \leq k \leq r_n \}' \subset \overline{D}(\varphi). \]

Proof. From the condition on \( \{ r_n \}_{n \in \mathbb{N}} \), for a.e. \( \omega \in \Omega \)

\[ \exists N_\omega \in \mathbb{N} : \omega_n < a_n - r_n - 1 \quad \forall n > N_\omega, \]

whence \( \forall n \geq N_\omega, 0 \leq k \leq r_n \),

\[ \varphi^{(kn_n)}(\omega) = \sum_{j=1}^{k} \varphi^{(\ell_n)}(T^{j-1}\omega), \]

and if \( k_n \lambda_n \to a \), then for a.e. \( \omega \in \Omega \),

\[ \varphi^{(kn_n)} \approx k_n \lambda_n \to a \text{ a.e.,} \]

and \( a \in \overline{D}(\varphi). \)

Theorem 1.5, and Proposition 1.6 facilitate easy constructions of conservative, ergodic, coalescent, non-squashable \( G \)-extensions of odometers.

Example 1.7. There is a continuous \( \mathbb{R} \)-valued cocycle of product type which is a coboundary, and satisfies

\[ \overline{\text{Gp}}(\mathcal{C}(\varphi)) = \mathbb{R}. \]

Proof. Assume that \( \sum_{n=1}^{\infty} \frac{1}{a_n} < +\infty \), \( a_n \geq 3 \). Let

\[ \varphi(\omega) = \sum_{n=1}^{\infty} (b_n(T^n\omega) - b_n(\omega)) \]
where, as before, \( b_n(\omega) = \beta_n(\omega_n) \). Set \( \beta_{2n+1} \equiv 0 \), and
\[
\beta_{2n}(k) = \begin{cases} \frac{1}{n} & k = 1, \\ 0 & \text{else.} \end{cases}
\]

By Borel-Cantelli lemma, since \( \mu \{ \omega : \omega_{2n} = 1 \} = \frac{1}{a_n} \), \( \varphi = \psi \circ T - \psi \) with
\[
\psi = \sum_{n=1}^{\infty} b_n.
\]

Note that \( \varphi(-\tau) = 0 \) (where \( -\tau = (a_1 - 1, a_2 - 1, \ldots) \)). For \( \omega \neq -\tau \), \( \ell(\omega) < \infty \)
\[
\varphi(\omega) = \sum_{n=0}^{a(\omega)-1} [\beta_n(0) - \beta_n(a_n - 1)] \\
+ \beta_{\ell(\omega)}(\omega_{\ell(\omega)} + 1) - \beta_{\ell(\omega)}(\omega_{\ell(\omega)}) \\
= \beta_{\ell(\omega)}(\omega_{\ell(\omega)} + 1) - \beta_{\ell(\omega)}(\omega_{\ell(\omega)}) ,
\]
since \( \beta_n(0) - \beta_n(a_n - 1) = 0 \), whence
\[
|\varphi(\omega)| \leq \frac{2}{\ell(\omega)}
\]
and the continuity of \( \varphi \) is ensured.

For a.e. \( \omega \in \Omega \), \( \exists n_\omega \) such that \( 2 < \omega_n < a_n - 2 \) \( \forall n > n_\omega \). Set 
\[
\kappa_n(\omega) = a_{2n} - \omega_{2n}
\]
for \( n > \frac{n_\omega}{2} \). Clearly \( \kappa_n(\omega)g_{2n} \tau \xrightarrow{\Omega} 0 \).

Moreover, for \( n > \frac{n_\omega}{2} \),
\[
(T^{(2n+1)\omega})_{2n} = \begin{cases} \omega_{2n} + j & 0 \leq j \leq \kappa_n(\omega) - 1, \\ 0 & j = \kappa_n(\omega), \end{cases}
\]
\[
(T^{(2n+1)\omega})_{2n+1} = \begin{cases} \omega_{2n+1} & 0 \leq j \leq \kappa_n(\omega) - 1, \\ \omega_{2n+1} + 1 & j = \kappa_n(\omega), \end{cases}
\]
and
\[
(T^{(2n+1)\omega})_k = \omega_k \forall 0 \leq j \leq \kappa_n(\omega), k \neq 2n, 2n + 1;
\]
whence
\[
\varphi((\kappa_n(\omega)+1)_{2n})_k(\omega) = \sum_{k=1}^{\infty} \left( b_k(T(\kappa_n(\omega)+1)_{2n}\omega) - b_k(\omega) \right) \\
+ \sum_{k=1}^{\infty} \left( \beta_k((T(\kappa_n(\omega)+1)_{2n}\omega)_k) - \beta_k(\omega_k) \right) \\
= \beta_{2n}(T((\kappa_n(\omega)+1)_{2n}\omega)_{2n}) - \beta_{2n}(\omega_{2n}) \\
= \beta_{2n}(1) = \frac{1}{n}.
\]
We use the fact that
\[ \forall y > 0, \ N \geq 1, \ \exists \ N < n_k(N) \uparrow \infty \ \exists \ \sum_{k=1}^{\infty} \frac{1}{n_k(N)} = y. \]
Now, for fixed \( \omega, y \), and \( N > \frac{n}{2} \) choose \( m_N \) such that
\[ |\sum_{k=1}^{m_N} \frac{1}{n_k(N)} - y| < \frac{1}{N} \]
and set
\[ Q_m^{(N)}(\omega) = \sum_{k=1}^{m} (\kappa_{n_k(N)} + 1)(\omega) q_{2n_k(N)}, \ \& \ Q_N = Q_N(\omega) := Q_m^{(N)}(\omega). \]
It follows that \( Q_N \xrightarrow{\Omega} 0 \) whence \( TQ_n \xrightarrow{\mathcal{U}(L^2(m))} \text{Id}. \) On the other hand,
\[ \varphi(Q_N)(\omega) = \sum_{k=1}^{m_N} \varphi((\kappa_{n_k+1})_{2n_k}) (TQ_{k-1}(N)\omega) = \sum_{k=1}^{m_N} \frac{1}{n_k(N)} \longrightarrow y. \]
Thus \( C(\varphi) \supset \mathbb{R}_+ \). With some minor adjustments, \( C(\varphi) = \mathbb{R} \) can be arranged. \( \square \)

§2 Homogeneous Banach spaces and Koksma inequalities

**Definition.** By a *pseudo-homogeneous* Banach space on \( T \) we mean a Banach space \( (B, \| \cdot \|_B) \) satisfying

1. \( B \subseteq L^1(T) \), and \( \| \cdot \|_B \geq \| \cdot \|_1 \),
2. if \( f \in B \) and \( t \in T \) then \( f_t \in B \), and \( \| f_t \|_B = \| f \|_B \), where \( f_t(x) = f(x-t), x \in T \).

A pseudo-homogeneous Banach space on \( T \) is called *homogeneous* if 
\( t \mapsto f_t \) is continuous 
\( T \longrightarrow B, \ \forall f \in B \).

The following properties of pseudo-homogeneous Banach spaces are either contained in, or can be easily deduced from [Katzn, chapter 1]:

1. there exists the largest homogeneous Banach subspace \( B_h \) contained in \( B \) defined by
   \[ B_h = \{ f \in B : t \mapsto f_t \text{ is continuous } T \rightarrow B \}; \]
2. the space \( B_h \) is the closure of trigonometric polynomials belonging to \( B \) (this is because \( B_h \) is homogeneous and hence if \( f \in B_h \) and \( g \in C(T) \) then the convolution of these two functions is an element of \( B_h \));
3. if \( f \in B \) then \( f \in B_h \) iff for each \( n \in \mathbb{Z} \) such that \( \hat{f}(n) \neq 0 \) there exists \( g \in B_h \) such that \( \hat{g}(n) \neq 0 \).
Suppose now that $B$ is a Banach space and $T$ is an isometry on it. Assume also that zero is the only fixed point of $T$. We say that for an $x \in B$ the ergodic theorem holds if

$$B - \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} T^j x = 0.$$  

The set of all elements of $B$ for which the ergodic theorem holds is denoted by $ET(B,T)$. An element $x \in B$ is said to be a $(B\text{-})$coboundary if $x = y - Ty$ for some $y \in B$ (called a transfer element). The following theorem is a version of the Mean Ergodic Theorem:

**Theorem 2.1** (von Neumann). An element $x \in ET(B,T)$ iff $x$ belongs to the closure of the subspace of $B$-coboundaries.

Suppose now that $B$ is a pseudo-homogeneous Banach space on $T$ (only functions with zero mean are considered). Let $T$ denote an irrational translation by $\alpha$, then $T$ acts as an isometry on $B$. Note that if $P$ is a trigonometric polynomial from $B$ then $P$ is a coboundary, in fact we have $P = Q - Q \circ T$, where $Q$ is another trigonometric polynomial, hence $P, Q \in B_h$. This proves

**Corollary 2.2**

$$B_h \subset ET(B,T).$$

Let $\alpha = [0; a_1, a_2, \ldots]$ be the continued fraction expansion of $\alpha$. The positive integers $a_n$ are called the partial quotients of $\alpha$. Put

$$q_0 = 1, \quad q_1 = a_1, \quad q_{n+1} = a_{n+1} q_n + q_{n-1}, \quad p_0 = 0, \quad p_1 = 1, \quad p_{n+1} = a_{n+1} p_n + p_{n-1}.$$  

The rationals $p_n/q_n$ are called the convergents of $\alpha$ and the inequality

$$|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}$$

holds. A denominator $q_n$ is said to be a Legendre denominator if $|\alpha - \frac{p_n}{q_n}| < \frac{1}{2q_n^2}$. We’ll sometimes denote the set of Legendre denominators of $\alpha$ by $L(\alpha)$.

Note that if $q \in L(\alpha)$ is a Legendre denominator then

$$\|j\alpha - j'\alpha\| > \frac{1}{2q} \quad \text{whenever} \quad 0 \leq j \neq j' \leq q - 1. \quad (2.1)$$

Here, for $t \in \mathbb{R}$,

$$\|t\| = d(t, \mathbb{Z}) = \min_{n \in \mathbb{Z}} |n - t|.$$  

We recall that one of any two consecutive denominators of an irrational $\alpha$ must be a Legendre denominator i.e. $(\forall \alpha \notin \mathbb{Q}, n \geq 1), \{q_n, q_{n+1}\} \cap L(\alpha) \neq \emptyset$.

Let $B$ be a pseudo-homogeneous Banach space on $T$. We say that *Koksma’s inequality* holds for the pair $(B,T)$ provided that there exists a positive sequence
\( \tilde{D}_N = \tilde{D}_N(\alpha), \ N \geq 1, \) satisfying \( \tilde{D}_{q_n} = O(1/q_n) \) where \( \{q_n\} \) is the sequence of denominators of \( \alpha \) and

\[
\| \frac{1}{N} f^{(N)}(\cdot) - \int_0^1 f(t)\,dt \|_{L^1} \leq \| f \|_B \tilde{D}_N(\alpha) \quad \forall \ f \in B,
\]

where \( f^{(N)}(x) = \sum_{j=0}^{N-1} f(T^j x), \ x \in T. \) For the classical cases where Koksma inequality is satisfied for functions with bounded variation or Lipschitz continuous functions we refer to [Ku-Ni], chapter 2.

The proposition below (essentially due to M. Herman, [He], p.189) will play a role in the proofs of ergodicity of certain cylinder flows.

**Proposition 2.3.** If Koksma's inequality is satisfied for the pair \( (B, T) \) then for each \( f \in B_h \) with \( \int_0^1 f(t)\,dt = 0 \) we have

\[
\lim_{n \to \infty} f(q_n) = 0 \quad \text{in} \quad L^1(T).
\]

**Proof.** Denote by \( B_0 \) the subspace of \( B \) consisting of functions with zero mean. Then define a map \( S : B_0 \to l^\infty \) by

\[
Sg = (\|g(q_n)\|_{L^1})_{n \geq 1}.
\]

Note that by the Koksma inequality, \( S \) is well-defined and continuous. Hence, the set \( S^{-1}(c_0) \) is closed as \( c_0 \) is a closed subspace of \( l^\infty \). Each coboundary \( f = h - hT, \ h \in B \) is in \( S^{-1}(c_0) \) since for each function \( u \in L^1(T) \) we have

\[
uT^u \to u \quad \text{in} \quad L^1(T).
\]

It follows from this, theorem 2.1 and corollary 2.2, that

\[
B_h \subset ET(B, T) = \{h - h \circ T : h \in B \} \subset S^{-1}(c_0).
\]

\( \square \)

We will now pass to a proof of Koksma's inequality in the space \( B = O(1/n) \) (of functions whose Fourier coefficients are of order \( O(1/n) \)), where the norm is defined as \( \|f\|_B = \|f\|_{L^1} + \sup_{n \neq 0} |n \hat{f}(n)|. \) If \( \{x_1, \ldots, x_N\} \) is a finite set of points from \( [0,1) \) then by discrepancy \( D_N = D_N(x_1, \ldots, x_N) \) we mean

\[
D_N = \sup_{x, y} \left\{ \frac{\# \{1 \leq j \leq N \ x_j \in [x, y)\} }{N} - (y - x) \right\}.
\]

**Lemma 2.4.**

\[
\sup_x \# \{1 \leq j \leq N \ x_j \in [x, x + \frac{1}{N})\} \leq ND_N + 1.
\]

**Proof.** For an arbitrary \( x \in [0,1), \)

\[
\left| \frac{\# \{1 \leq j \leq N \ x_j \in [x, x + \frac{1}{N})\} }{N} - (x + \frac{1}{N} - x) \right| \leq D_N,
\]
whence the assertions follows immediately. 

Lemma 2.5. There exists $C > 0$ such that

$(\forall m \geq 1)(\forall a \geq 1)(\forall x_{1}, \ldots, x_{m-1} \in [0, 1))$ if in each interval of length $\frac{1}{m}$: there are at most $a$ points of the form $x_{i}$ then $\sum_{x_{i} \in (\frac{1}{2m}, 1 - \frac{1}{2m})} \frac{1}{\|x_{i}\|^2} \leq Cam^{2}$. 

Proof. Denote by $I$ the set of those $1 \leq i \leq m - 1$ so that $x_{i} \in (\frac{1}{2m}, 1 - \frac{1}{2m})$. Then define a map $i \mapsto j(i)$, $i \in I, 1 \leq j(i) \leq m - 1$, by

\[ |x_{i} - \frac{j(i)}{m}| \leq \frac{1}{2m}. \]

Since $\|x_{i}\| > \frac{1}{2m},$

\[ \frac{1}{2} \leq \frac{\|x_{i}\|}{\|j(i)/m\|} \leq 2. \]

Note that if $k$ is in the image of the function $j$ then

$\# j^{-1}(k) \leq a$

by our assumption and (2.4). Hence by (2.5)

\[ \sum_{i \in I} \frac{1}{\|x_{i}\|^2} \leq 2a \sum_{k \in \mathbb{N}, j} \frac{1}{\|k/m\|^2} \leq 4a \sum_{i=1}^{m-1} \frac{1}{(i/m)^{2}} = Cam^{2}. \]

Combining this with Lemma 2.4, we obtain

Corollary 2.6. Under the conditions of lemma 2.5,

\[ \sum_{i \in I} \frac{1}{\|x_{i}\|^2} \leq C(mD_{m} + 1)m^{2}, \]

where $I$ is the same as in the proof of Lemma 2.5.

Now, suppose that $f \in O(\frac{1}{n})$,

\[ f(x) = \sum_{k=-\infty}^{\infty} f_{k} e^{2\pi i k x}. \]

We have

\[ f^{(m)}(x) = \sum_{i=0}^{m-1} f(x + i\alpha) = f^{(m)}(x) = \sum_{k=-\infty}^{\infty} f_{k} e^{2\pi i k \alpha} - \frac{1}{e^{2\pi i k \alpha}}. \]

Theorem 2.7 (Koksma's Inequality in $O(\frac{1}{n})$). There is a constant $K > 0$ such that if we denote

\[ \hat{B}_{m} = \sqrt{K \left( \sum_{k \in \mathcal{A}_{m}} \frac{1}{k^{2}} + (mD_{m} + 1)(\|m\alpha\|^{2} + \frac{1}{m^{2}}) \right)} \]

\[ 38 \]
then $\forall f \in O(\frac{1}{n})$,

$$
\left\| \frac{1}{m} \sum_{i=0}^{m-1} f(\cdot + i\alpha) - \int_0^1 f(t) dt \right\|_{L^2}^2 \leq \|f\|^2 O(\frac{1}{\sqrt{m}}) D_m,
$$

where $D_m = D_m(0, \alpha, 2\alpha, \ldots, (m-1)\alpha)$, and $A_m = \{0 \leq j \leq m - 1 : 0 < \|j\alpha\| < \frac{1}{2m}\}$. Moreover,

$$
D_{qn} = O(1/q_n).
$$

Proof. Without loss of generality we will assume that $\int_0^1 f(t) dt = 0$ and it is enough to prove that

$$
(2.6) \quad \|f^{(m)}\|_{L^2}^2 \leq C_2\|f\|^2 O(\frac{1}{\sqrt{m}}) (m^2 \sum_{k \in A_m} \frac{1}{k^2} + C(mD_m + 1)m^2\|m\alpha\|^2 + C_3(mD_m + 1)),
$$

where $C_2, C, C_3$ are some absolute constants. Since $f$ is real,

$$
\|f^{(m)}\|_{L^2}^2 \leq 2C_1 \sum_{k=1}^{\infty} |f_k|^2 \frac{\|km\alpha\|^2}{\|k\alpha\|^2} = C_2(S_1 + S_2),
$$

where

$$
S_1 = \sum_{k=1}^{m-1} \frac{|f_k|^2 \|km\alpha\|^2}{\|k\alpha\|^2}, \quad S_2 = \sum_{k=m}^{\infty} \frac{|f_k|^2 \|km\alpha\|^2}{\|k\alpha\|^2}.
$$

Now,

$$
S_1 = \sum_{k=1}^{m-1} \frac{|f_k|^2 \|km\alpha\|^2}{k^2\|k\alpha\|^2} \leq \|f\|^2 O(\frac{1}{\sqrt{m}}) \sum_{k=1}^{m-1} \frac{\|km\alpha\|^2}{k^2\|k\alpha\|^2} = \|f\|^2 O(\frac{1}{\sqrt{m}}) (S_{11} + S_{12}),
$$

where

$$
S_{11} = \sum_{k \in A_m} \frac{\|km\alpha\|^2}{k^2\|k\alpha\|^2}, \quad S_{12} = \sum_{k \in A_m} \frac{\|km\alpha\|^2}{k^2\|k\alpha\|^2}.
$$

We have, $S_{11} \leq m^2 \sum_{k \in A_m} \frac{1}{k^2}$, and $S_{12} \leq \|m\alpha\|^2 \sum_{k \in A_m} \frac{1}{k\alpha\|^2}$.

By Corollary 2.6,

$$
S_{12} \leq \|m\alpha\|^2 C(mD_m + 1)m^2.
$$

We pass now to estimate $S_2$. We have

$$
S_2 = \sum_{k=m}^{\infty} \frac{|f_k|^2 \|km\alpha\|^2}{\|k\alpha\|^2} = \sum_{p=1}^{\infty} \sum_{r=0}^{m-1} |f_{pm+r}|^2 \|(pm + r)m\alpha\|^2 \leq \|f\|^2 O(\frac{1}{\sqrt{m}}) \sum_{p=1}^{\infty} \frac{1}{p^2} \sum_{r=0}^{m-1} \frac{\|(pm + r)m\alpha\|^2}{m^2\|(pm + r)\alpha\|^2} \leq \frac{1}{m^2\|f\|^2 O(\frac{1}{\sqrt{m}})} \sum_{p=1}^{\infty} \frac{1}{p^2} \sum_{r=0}^{m-1} \min(m^2, \frac{1}{\|pm\alpha + r\alpha\|^2}).
$$
Denote \( x = pm\alpha \). In the interval \((-\frac{1}{2m}, \frac{1}{2m}) = [0, \frac{1}{2m}) \cup [1 - \frac{1}{2m}, 1) \mod 1\) we have at most \( mD_m + 1 \) points of the form \( x + r\alpha \) because \( D_m = D_m(x, x + \alpha, \ldots, x + (m-1)\alpha) \). By Corollary 2.6 we thus have

\[
S_2 \leq \frac{1}{m^2} \|f\|_2^2 O(\frac{1}{\alpha}) \sum_{p=1}^{\infty} \frac{1}{p^2} ((mD_m + 1)m^2 + C(mD_m + 1)m^2) \leq C_3 \|f\|_2^2 O(\frac{1}{\alpha}) (mD_m + 1).
\]

To complete the proof we have to show that the sequence \( \{q_n D_{q_n}\} \) is bounded. But classically, \( D_{q_n} = O(1/q_n) \) and also \( q_n \|q_n\alpha\| \) is bounded. Now, note that in the interval \( M_n = [0, \frac{1}{2q_n}) \cup [1 - \frac{1}{2q_n}, 1) \) we can have at most one point of the form \( j\alpha \), where \( j = 1, \ldots, q_n - 1 \). Moreover, \( |j\alpha - j\alpha_{q_n}| < \frac{1}{q_n} \), so if \( j\alpha \in M_n \), then we must have \( j^2 \frac{1}{q_n} > \frac{1}{2q_n} \). In particular, \( j > q_n/2 \), so \( \sum_{k \in A_{q_n}} \frac{1}{k^2} = O(1/q_n^2) \). □

Now, proceeding as in the proof of Proposition 2.3, we obtain the following extension of the main result from [Le-Ma]

**Corollary 2.8.** If \( f \in O(\frac{1}{n}) \), \( \int_0^1 f(t) \, dt = 0 \) and \( \{q_n\} \) is the sequence of all denominators of \( \alpha \) then

\[
\|f(q_n)\|_{L^2} \rightarrow 0.
\]

§3 Speed of approximation in Koksma’s Inequality for spaces \( O(1/a(n)) \)

Assume that \( a : \mathbb{N} \rightarrow \mathbb{R}^+ \) satisfies

\[
(3.1) \quad a(k) \geq k,
\]

\[
(3.2) \quad a(pm + r) \geq a(p)a(m), \quad \text{for arbitrary } p, m \geq 1, r = 0, \ldots, m - 1.
\]

We will now concentrate on a pseudo-homogeneous Banach space \( B = O(1/a(n)) \) of functions

\[
f(x) = \sum_{k=-\infty}^{\infty} f_k e^{2\pi ikx},
\]

with \( f_k = O(1/a(k)) \). The norm is defined as

\[
\|f\|_{O(1/a(n))} = \|f\|_{L^2} + \sup_{n \neq 0} |a(n)f_n|.
\]

Notice that in this case \( B_k = o(1/a(n)) \) the subspace of functions whose Fourier coefficients are of order \( o(1/a(n)) \). Keeping the notation from the proof of Theorem 2.7 and proceeding as before we obtain that

\[
S_1 \leq \|f\|^2_{O(1/a(n))}(S_{11} + S_{12}),
\]

where

\[
S_{11} = m^2 \sum_{k \in A_m} \frac{1}{a(k)^2},
\]

40
and by (3.1)
\[ S_{12} \leq \|m\alpha\|^2 \sum_{k \in A_m} \frac{k^2}{a(k)^2} \frac{1}{\|k\alpha\|^2} \leq \|m\alpha\|^2 m^2(D_m m + 1) \cdot C. \]

In view of (3.2),
\[ S_2 \leq \|f\|^2 O(1/a(\alpha)) \sum_{p=1}^{\infty} \frac{1}{a(p)^2} \sum_{r=0}^{m-1} \frac{\|(pm + r)m\alpha\|^2}{a(m)^2 (pm + r) \alpha\|^2} \leq \]
\[ \frac{1}{a(m)^2} \|f\|^2 O(1/a(\alpha)) m^2 C_4(mD_m + 1) \sum_{p=1}^{\infty} \frac{1}{a(p)^2} \leq \left( \frac{m}{a(m)} \right)^2 \|f\|^2 O(1/a(\alpha)) (mD_m + 1) C_5. \]

For a function $a(\cdot)$ satisfying (3.1) and (3.2) denote
\[ I(a) = \{ \alpha \in [0,1) \setminus Q : \lim_{q \to \infty, q \in \alpha(\alpha)} a(q)\|q\alpha\| < \infty \}. \]

Lemma 3.1. If $f = gT - g$, $g \in O(1/a(\alpha))$, $\alpha \in I(a)$ and $q_{n_k} \in \mathcal{L}(\alpha)$ with $a(q_{n_k})\|q_{n_k}\alpha\| = O(1)$, then
\[ \|f(q_{n_k})\|_{L^2} = o\left( \frac{q_{n_k}}{a(q_{n_k})} \right). \]

Proof. All we need to show is that $\sum_{s=1}^{\infty} \|g\|^2 O(1/a(\alpha)) (\sum_{s=1}^{q_{n_k}-1} \frac{\|q_{n_k}s\alpha\|^2}{a(s)^2} + \sum_{s=q_{n_k}}^{\infty} \frac{\|q_{n_k}s\alpha\|^2}{a(s)^2}) \leq \]
\[ \|g\|^2 O(1/a(\alpha)) (q_{n_k}\|q_{n_k}\alpha\|^2 + q_{n_k} \sum_{p=1}^{\infty} \frac{1}{a(p)a(q_{n_k})^2}) = \]
\[ \|g\|^2 O(1/a(\alpha)) \left( \frac{q_{n_k}}{a(q_{n_k})} \right)^2 a(q_{n_k})\|q_{n_k}\alpha\|^2 + q_{n_k} \sum_{p=1}^{\infty} \frac{1}{a(p)^2} \right) = o\left( \frac{q_{n_k}}{a(q_{n_k})} \right)^2. \]

Corollary 3.2. If $f \in O(1/a(\alpha))$, $\int_1^t f(t) dt = 0$ and $\alpha \in I(a)$ and $q_{n_k} \in \mathcal{L}(\alpha)$ with $a(q_{n_k})\|q_{n_k}\alpha\| = O(1)$, then
\[ \|f(q_{n_k})\|_{L^2} \leq \text{const.} \|f\| O(1/a(\alpha)) \frac{q_{n_k}}{a(q_{n_k})}. \]

Moreover, if in addition $f \in o\left( \frac{1}{a(\alpha)} \right)$ then
\[ \|f(q_{n_k})\|_{L^2} = o\left( \frac{q_{n_k}}{a(q_{n_k})} \right). \]

Proof. Since (3.3) is satisfied for all coboundaries by Lemma 3.1, the mechanism described in the proof of Proposition 2.3 works well. The map $S$ is defined as $Sf = \left( \frac{a(q_{n_k})}{q_{n_k}} \|f(q_{n_k})\|_{L^2} \right)_{k \geq 1}$. \qed
Suppose now that $a(n) = \frac{1}{n}$ for certain natural number $t \geq 1$. Hence $I(a) = I(t)$ is the set of those irrationals $\alpha$ for which $(q_n, \|q_n, \alpha\|)$ is bounded for certain subsequence of Legendre denominators of $\alpha$.

**Corollary 3.3.** If $f \in o\left(\frac{1}{n^2}\right)$, $\int_0^1 f d\lambda = 0$ then for an arbitrary $\alpha \in I(t)$ and $q_{n_k} \in \mathcal{L}(\alpha)$ with $q_{n_k} \|q_{n_k}, \alpha\| = O(1)$, we have

(i) $\|f(q_{n_k})\|_{L^2} = o\left(\frac{1}{q_{n_k}^2}\right)$,

(ii) the sequence $(q_{n_k})$ is a rigidity time for $\alpha$ and

\[
\lim_{k \to \infty} f(q_{n_k}) = 0 \text{ in } L^2(T).
\]

**Proof.** It is enough to notice that $f^{q_{n_k}^t} = f^{q_{n_k}^t q_{n_k}^{-1}}$ and that $\|f^{q_{n_k}^t q_{n_k}^{-1}}\|_{L^2} \leq q_{n_k}^{-1} \|f^{q_{n_k}}\|_{L^2}$. $\square$

§4 Constructions of ergodic analytic cylinder flows

Constructions which are known of ergodic cylinder flows are rather based on some irregularities in the smoothness of the cocycle (e.g. [He-La1], [He-La2], [Pa1], [Pa2], [Ba-Me1], [Ba-Me2]). Below, we will show a new method coming from [Kw-Le-Ru2] for constructing analytic cylinder flows which are ergodic.

Assume that $T x = x + \alpha$, where $\alpha = [0; a_1, a_2, \ldots]$. From the continued fraction expansion of $\alpha$ we obtain, for each $n$, two Rokhlin towers $\xi_n, \bar{\xi}_n$ whose union coincides with the whole circle. For $n$ even

\[
\xi_n = \{[0, \{q_n \alpha\}], T[0, \{q_n \alpha\}], \ldots, T^{(q_{n+2n+1} q_{n+1})-1}[0, \{q_n \alpha\}]\},
\]

\[
\bar{\xi}_n = \{\{q_{n+1} \alpha\}, 1\}, \ldots, T^{q_{n}}[\{q_{n+1} \alpha\}, 1]\}.
\]

Given a subsequence \{n_k\} of natural numbers we will denote

\[I_k = [0, \{a_{2n_k+1} q_{2n_k} \alpha\}], \quad J_k^t = T^{(t-1) q_{2n_k}}(0, \{q_{2n_k} \alpha\}),\]

\[t = 1, \ldots, a_{2n_k+1}.\] Notice that

\[I_k = \bigcup_{i=1}^{a_{2n_k+1}} J_i^t,\]

and

\[|J_i^t| < \frac{1}{a_{2n_k+1} q_{2n_k}}.\]

We will recall here a notion of an a.a.c.c.p. (almost analytic cocycle construction procedure) from [Kw-Le-Ru2] which is to construct a real 1-periodic cocycle $\bar{\varphi} : \mathcal{R} \to \mathcal{R}$ such that in its $\mathcal{R}$-cohomology class (for certain $\alpha$) there is an analytic cocycle.

An a.a.c.c.p. is given by a collection of parameters as follows. We are given a sequence \{M_k\} of natural numbers and an array \{(d_{k,1}, \ldots, d_{k,M_k})\}, $d_{k,i} \in \mathcal{R}$ satisfying for each $k$

\[\sum_{i=1}^{M_k} d_{k,i} = 0.\]
Denote $D_k = \max_{1 \leq i \leq M_k} |d_{k,i}|$. Choose a sequence $\{\varepsilon_k\}$ of positive real numbers satisfying
\[
\sum_{k=1}^{\infty} \sqrt{\varepsilon_k} M_k < +\infty,
\]
\[
\sum_{k=1}^{\infty} \varepsilon_k < 1,
\]
\[
\varepsilon_k < \frac{1}{D_k^2}, \quad k = 1, 2, \ldots.
\]
Finally, we are given $A > 1$ completing the parameters of the a.a.c.c.p.

We say that this a.a.c.c.p. is realized over an irrational number $\alpha$ with continued fraction expansion $[0; a_1, a_2, \ldots]$ and convergents $p_n/q_n, n \geq 1$ if there exists a strictly increasing sequence $\{n_k\}$ of natural numbers such that
\[
A^{n_k} D_{k} M_{k} \| P_{k} \|_X < \frac{1}{2^k}
\]
and
\[
\frac{D_k \| P_k' \|_\infty}{a_{2n_k+1} q_{2n_k}} < \sqrt{\varepsilon_k},
\]
where $\{P_k\}$ is a sequence of "bump" real trigonometric polynomials, i.e.

(i) $\int_0^1 P_k(t) dt = 1$,

(ii) $P_k \geq 0$,

(iii) $P_k(t) < \varepsilon_k$ for each $t \in (\eta_k/2, 1)$,

where the $\eta_k$'s are chosen in such a way that

\[
4 M_k \eta_k < \frac{\varepsilon_k}{q_{2n_k}}
\]
and $N_k$ is the degree of $P_k$. Finally, $a_{2n_k+1} > 1$ and

\[
\frac{1}{a_{2n_k+1} q_{2n_k}} < \frac{1}{2^k} \eta_k
\]

Using the above parameters define a cocycle
\[
\varphi = \sum_{k=1}^{\infty} \varphi(k)
\]
as follows. In view of (4.2), (4.3) and (4.1), in the interval $I_k = [0, \{a_{2n_k+1} q_{2n_k} \alpha\})$ we can choose $w_{k,1}, \ldots, w_{k,M_k}$ to be consecutive pairwise disjoint intervals of the same length contained between $\eta_k$ and $2\eta_k$ such that each $w_{k,i}$ consists of say $e_k$ consecutive subintervals $J_{k,i}^k$, where $e_k$ is an odd number. Let $J_{k,i}^k$ be the central subinterval in $w_{k,i}$ and now define
\[
\varphi(k)(x) = \begin{cases} d_{k,i} & \text{if } x \in J_{k,i}^k \\ 0 & \text{otherwise.} \end{cases}
\]
Note that the $\varphi(k)$'s have disjoint supports so $\varphi$ is well defined.
As proved in [Kw-Le-Ru2]

(A) The set of \( \alpha \)'s over which an a.a.c.c.p. is realized is a \( G_\delta \) and dense subset of the circle.

(B) If an a.a.c.c.p. is realized over \( \alpha \) then there exists an analytic cocycle \( f : \mathcal{T} \rightarrow \mathbb{R} \) which is \( \alpha \)-cohomologous to \( \varphi \).

We will need an additional property of an a.a.c.c.p. which is not explicitly formulated in [Kw-Le-Ru2]. Namely,

\begin{equation}
\varphi|_{T^s I_k} \text{ is constant for } s = 1, \ldots, q_{2n_k} - 1, \quad \sum_{s=1}^{q_{2n_k}} \varphi|_{T^s I_k} = 0
\end{equation}

which is Lemma 3 from [Kw-Le-Ru1].

**Example 4.1.** There is an a.a.c.c.p. with \( \operatorname{Gp}(\overline{D}(\varphi)) = E(\varphi) = \mathbb{Z} \lambda \).

**Proof.** Assume that \( \lambda \in \mathbb{R} \) is given. We will assume that an a.a.c.c.p. satisfies the following additional requirements:

\[ a_{2n_k+1} = M_k r_k + N_k, \]

with \( 0 \leq N_k < r_k \) and both \( M_k, r_k \) tending to infinity. We put \( d_{k,1} = 0, d_{k,i} = \lambda \) for \( i = 2, \ldots, M_k - 1 \) and \( d_{k,M_k} = -(M_k - 1) \lambda \). In the definition of \( \varphi_k \) we require that \( \varphi_k |_{J^k_{r_k+1}} = d_{k,i} \lambda \) for \( i = 0, \ldots, M_k - 1 \) and zero for all others subintervals \( J^k_i \), \( k \geq 1 \).

Notice that \( E(\varphi) \subset \mathbb{Z} \lambda \), because the values of \( \varphi \) are from the group \( \mathbb{Z} \lambda \). It is then enough to show that \( \lambda \in \overline{D}(\varphi) \). Define

\[ X_k = \bigcup_{s=0}^{q_{2n_k} - 1} \bigcup_{i=r_k+1}^{(M_k-1) r_k} T^s J^k_i. \]

By our definition of \( \varphi \) and a basic property of an a.a.c.c.p. (see (4.4)) we have \( \varphi^{(M_k r_k)}(x) = \lambda \) for all \( x \in X_k \). It is clear also that \( M_k r_k \) is a rigidity time for \( T \). Therefore \( \lambda \in \overline{D}(\varphi) \).

**Example 4.2.** An a.a.c.c.p. with \( \overline{\operatorname{Gp}(D(\varphi))} = \mathbb{R} \).

This is an obvious modification of the previous construction. We divide the sequence \( \{n_k\} \) into two disjoint subsequences say \( \{n^1_k\} \) (\( i = 1, 2 \)) and repeat the previous construction for rationally independent \( \lambda_1, \lambda_2 \in \mathbb{R} \), with the sequences \( \{n^i_k\}, i = 1, 2 \). From the previous arguments we find \( \lambda_1, \lambda_2 \in \overline{D}(\varphi) \). The group generated by \( \lambda_1, \lambda_2 \) is dense in \( \mathbb{R} \) and the advertised condition is attained.

**Remark** It follows from proposition 1.5 that the cocycles of example 4.2 are ergodic, coalescent, and nonsquashable.

---

§5 Ergodicity of smooth cylinder flows. Generic point of view

Suppose that \( f : T \rightarrow \mathbb{R} \) is smooth. We shall prove that under certain assumptions, the set of those irrational translations for which the corresponding cylinder flow is ergodic is residual. For similar results see [Kr], [Ka].
Assume that $f(x) = \sum_{n=1}^{\infty} b_n e^{2\pi i n x}$ with zero mean is in $A(T')$, that is its Fourier transform is absolutely summable. Put $f_m(x) = f(x) + f(x + \frac{1}{m}) + \ldots + f(x + \frac{m-1}{m}) = m \sum_{n=1}^{\infty} b_n e^{2\pi i n x}$, $m = 1, \ldots$.

Theorem 5.1. Suppose that there exist an infinite subsequence $\{q_n\}$ and a constant $C > 0$ such that

1. $q_n \sum_{n=-\infty}^{\infty} |b_{q_n}| \leq C \|f_{q_n}\|_{L^2}$, $n = 1, 2, \ldots$,
2. $0 < \|f_{q_n}\|_{L^2} \to 0$,

then there exists a dense $G_\delta$ set of irrational numbers $\alpha$ such that the corresponding cylinder flow $T_f$, $T_f x = x + \alpha$ is ergodic.

Proof. We will need the following

Lemma 5.2. Given $C > 0$ there exist positive numbers $K, L, M$ such that $0 < K < 1 < L$, $0 < M < 1$ and for each $h \in L^4(T')$ if $\|h\|_4 \leq C \|h\|_2$, then

$$\mu \{x \in T' : K \|h\|_2 \leq |h(x)| \leq L \|h\|_2 \} > M$$

We will prove the lemma later. Denote

$$g_n(x) = q_n \sum_{n=-\infty}^{\infty} b_{q_n} e^{2\pi i n x}.$$

In view of (1) we have that

(5.1) $$g_n(b_n x) = f_{q_n}(x), x \in T'$$

and

$$\|g_n\|_{L^\infty} \leq q_n \sum_{n=1}^{\infty} |b_{q_n}| \leq C \|g_n\|_{L^2},$$

in particular, $\|g_n\|_4 \leq C \|g_n\|_2$. Hence by Lemma 5.2

$$\mu \{x \in T' : K \|g_n\|_2 \leq |g_n(x)| \leq L \|g_n\|_2 \} > M.$$ By (2) we have $\|g_n\|_2 = \|f_{q_n}\|_2 \to 0$.

Let $\{D_n\}$ be a family of pairwise disjoint closed intervals, $D_n = [c_n, d_n]$, with

$$d_n/c_n = 100 \frac{L}{K} \quad \text{and} \quad d_n \to 0.$$

Assume that $\{D'_n\}$ is a sequence of the above intervals with the property that each $D_n$ repeats infinitely many times in $\{D'_n\}$.

Now, fix $n$, that is we have the interval $D'_n$. Choose a natural number $k_n$ so that for some natural $s_n$

$$[s_n K \|g_{q_n}\|_{L^2}, s_n L \|g_{q_n}\|_{L^2}] \subset \tilde{D}'_n,$$

where $\tilde{D}'_n$ is a strict subinterval of $D'_n$. This gives us a subsequence $\{k_n\}$. For it we have that

$$\mu \{x \in T' : |s_n g_{k_n}(x)| \in \tilde{D}'_n \} \geq M.$$
From this and (5.1) we obtain that for each interval \( I \) of length being a multiple of \( \frac{1}{g_n} \)

\[
(5.2) \quad \mu \{ x \in I : |s_n f_{q_n}(x)| \in \mathcal{D}'_n \} \geq M |I|.
\]

We will also use the following lemma whose proof is contained in [Kw-Le-Ru2].

**Lemma 5.3.** Given an infinite set \( \{Q_n\} \) of natural numbers and a positive real valued function \( \delta = \delta(Q_n) \) the set

\[
A = \{ \alpha \in [0,1) : \# \{ n : P_n \equiv \frac{P_n}{Q_n} \text{ a convergent of } \alpha, \quad |\alpha - \frac{P_n}{Q_n}| < \delta(Q_n) \} = \infty \}
\]

is a dense \( G_\delta \).

Let us fix \( r \). So we have infinitely many \( n = n(r) \) with \( D'_n = D_r \). Consider now those \( \alpha \) which are approximated by \( \frac{p_{n(t)}}{q_{n(t)}} \) so well to have

\[
|s_{n(t)} q_{n(t)} \alpha| \rightarrow 0
\]

and

\[
(5.3) \quad \mu \{ x \in I : |f^{(s_n q_n)}(x)| \in D'_n \} \geq \frac{M}{2} |I|
\]

for each interval \( I \) with \( |I| = \frac{1}{q_{n(t)}} \), \( t = 1, \ldots, q_{kn(r)} \) (remember that we know the modulus of continuity of \( f \) and that

\[
\sum_{i=0}^{s-1} \sum_{j=0}^{q-1} f(x + \frac{j}{q}) - \sum_{k=0}^{q-1} f(x + i\alpha + k\alpha)) = \]

\[
\sum_{i=0}^{s-1} \sum_{k=0}^{q-1} f(x + \frac{k\alpha}{q}) - f(x + i\alpha + k\alpha)) \leq \sum_{i=0}^{s-1} \sum_{k=0}^{q-1} \omega(f, i\alpha + k\alpha - \frac{p}{q}),
\]

where \( \gcd(p, q) = 1, p = p_{n(t)}, q = q_{n(t)} \) and \( \omega(f, h) \) stands for the modulus of the continuity of \( f \); now given \( s, q \) the size of the above quantity depends on the distance between \( \alpha \) and \( \frac{p}{q} \).

In view of Lemma 5.3 we have a \( G_\delta \) and dense subset of \( \alpha \), say \( Y_r \), for which (5.3) holds true for an infinite subsequence of \( \{q_{kn(r)}\} \). Finally take

\[
Y = \bigcap_{r=1}^{\infty} Y_r
\]

which is \( G_\delta \) and dense. If we take \( \alpha \in Y \) then for each \( r \) we have an infinite subsequence \( n(\alpha) \) such that

\[
\mu \{ x \in I : |f^{(s_{n(\alpha)} q_{n(\alpha)})}(x)| \in D'_{n(\alpha)} \} \geq \frac{M}{2} |I|
\]

for each interval \( I \) with \( |I| = \frac{1}{q_{n(\alpha)}} \) and \( D'_{n(\alpha)} = D_r \).
It remains to prove that if $Tz = z + \alpha$, where $\alpha \in Y$ then the cylinder flow $T_f$ is ergodic. Suppose that $E(f) = \lambda \mathbb{Z}$. Choose $r$ so big to have that the compact set $K_r := D_r \cup (-D_r)$ is disjoint with $\lambda \mathbb{Z}$. By Lemma 1.2 there exists a Borel set $B$, with $\mu(B) > 0$ such that for all $m \geq 1$

\[(5.4) \quad \mu(B \cap T^{-m}B \cap \{x \in \mathcal{T} : f^{(m)}(x) \in K_r\}) = 0.\]

If $m = s_m q_{m_n}$, $n = n(\alpha)$, then $\mu(B \Delta T^{-s_m q_{m_n}}B) \to 0$ since $s_m q_{m_n}$ is a rigidity time for $T$. If $y$ is a density point of $B$ then for an interval $I$ of length $t/q_{m_n}$ containing $y$ we will have $\mu(B \cap I) > (1 - t/q_{m_n})|I|$. Hence a subset $A_n$ of $B$ of measure at least $\frac{\mu(B)}{t/q_{m_n}}$ has the property that $f^{(s_m q_{m_n})}(x) \in K_r$ whenever $x \in A_n$. This contradicts (5.4).

**Proof of Lemma 5.2.** It is enough to consider the case $\|h\|_2 = 1$. Take two real numbers $K, L$ satisfying $0 < K < 1 < L$. From Tchebycheff inequality we have

$$
\mu\{|h| \leq L\} \geq \mu\{|h|^2 - 1 \leq L^2 - 1\} \geq 1 - \text{Var}(\|h\|^2)/(L^2 - 1)^2 \geq 1 - (C^4 - 1)(L^2 - 1)^{-2}.
$$

On the other hand, from Cauchy-Schwartz inequality

$$
1 = \int_{|h| > K} h^2 + \int_{|h| \leq K} h^2 \leq (\int h^4)^{1/2}(\mu\{|h| > K\})^{1/2} + K^2;
$$

whence $\mu\{|h| > K\} \geq (1 - K^2)^2/C^4$. Now to have the conclusion of the lemma it is enough to choose $\epsilon > 0$, put $M = 1/C^4 - 2\epsilon$, then find $K$ small enough to have $(1 - K^2)^2/C^4 > M + \epsilon$ and finally select $L$ sufficiently big to have $(C^4 - 1)(L^2 - 1)^{-2} < \epsilon$. \hspace{1cm} \Box

**Remarks**

As shown in [Kw-Le-Ru2], the assumptions of Theorem 5.1 are satisfied for each zero mean function $f \in C^{\alpha+\delta}(\mathcal{T})$, $\delta > 0$ which is not a trigonometric polynomial. Recall that a subset $E \subseteq \mathbb{Z}$ is called of type $\Lambda(2)$ if for every $q \geq 2$ there exists a constant $C = C(q, E)$ such that for every function $h \in L^q(\mathcal{T})$ we have $\|h\|_q \leq C\|h\|_2$ whenever $\text{supp}(h) \subseteq E$. For instance, each lacunary subset is of that type ([Katzn], Chapter 5.). Now, if $f \in L^2(\mathcal{T})$ with the absolutely summable Fourier transform has the property that the support of its Fourier transform is an infinite $\Lambda(2)$ type set and moreover that $\hat{f}(n) = o(1/n)$ then the assumptions of Theorem 5.1 are also satisfied.

§6 Ergodicity of a class of cylinder flows

This section will be devoted to a generalization of a result of Pask [Pa1].

A function $f : \mathcal{T} \rightarrow \mathbb{R}$ is called **piecewise linear** (piecewise absolutely continuous) if there are points $x_1 < x_2 < \ldots < x_K$ such that $f$ restricted to $[x_j, x_{j+1})$ is linear (absolutely continuous), $j = 1, 2, \ldots$ (mod $K$). Denote by $d_j$ the jump of the values of $f$ at $x_j$. It is clear that if $f$ is piecewise absolutely continuous then

$$
\int_0^1 f'(t) dt = \sum_{j=1}^K d_j.
$$
Lemma 6.1. Suppose that $f : \mathbb{T} \rightarrow \mathbb{R}$, $\int_0^1 f(t) \, dt = 0$ is piecewise linear, and $\sum_{j=1}^K d_j \neq 0$, then for each irrational number $\alpha$ the corresponding cylinder flow $T_f$ is ergodic.

Proof. There is no loss of generality in assuming that $\sum_{j=1}^K d_j > 0$. Since $f'$ is Riemann integrable, the ergodic theorem holds uniformly, so

$$\frac{1}{q} \sum_{j=0}^{q-1} f'(x + j\alpha) \rightarrow \int_0^1 f'(t) \, dt > 0$$

uniformly in $x$. Hence, we can find two constants $0 < C_1 < C_2$ such that for all $q$ sufficiently large,

$$(6.1) \quad C_1 q \leq f^{(q)}(x) \leq C_2 q \quad \forall x \in \mathbb{T}.$$ 

On the other hand, $f^{(q)}$ is still piecewise linear with the discontinuity points of the form $x_i + j\alpha$, with the jump at it equal to $d_i$, where $i = 1, \ldots, K$, $j = 0, \ldots, q - 1$. Substitute from now on $q = q_n$ a Legendre denominator of $\alpha$. Take the division of the circle given by the points of the form $x_i + j\alpha$. It may happen that for $i \neq i'$ we will have for some $j \neq j'$ that $x_i + j\alpha = x_{i'} + j'\alpha$. This gives rise to a partition, say $\xi_n$, of the circle into closed-open subintervals. Consequently the number of atoms in $\xi_n$ is not bigger than $Kq_n$. Note that no subinterval in $\xi_n$ can be longer than $1/q_n$, so $\xi_n$ is tending to the point partition. Let us call a subinterval in $\xi_n$ long if its length is at least $\frac{1}{100q_n}$. Hence there must exist a constant $D = D(K) > 0$ such that for all $n \geq 1$ the number of long subintervals is at least $Dq_n$. Finally, by the classical Koksma inequality, we have

$$|f^{(n)}(x) - f^{(n)}(y)| \leq \text{Var}(f) \quad \text{for all} \quad x, y \in \mathbb{T}.$$ 

Suppose now that $E(f) = ZZ\lambda$. Choose a very small $\varepsilon = \varepsilon(\lambda, \text{Var}(f), C_1, C_2, D) > 0$ and let

$$K = \{ r \in [-2 \text{Var}(f), 2 \text{Var}(f)] : \text{dist}(r, ZZ\lambda) \geq \varepsilon \}.$$ 

It is clear that $K$ is compact. If $\varepsilon$ is small enough, in view of (6.1) and (6.2), there exists a constant $F > 0$ such that for each long subinterval of $\xi_n$ there exists a subset with measure at least $F \frac{1}{q_n}$ such that for each $x$ from this subset we have $f^{(n)}(x) \in K$. It is now sufficient to apply Lemma 1.3 to obtain an obvious contradiction to $K \cap E(f) = \emptyset$.

It is clear that the arguments from the above proof persist if instead of a piecewise continuous function we consider a function $g = f + h$, where $f$ is piecewise linear with $\int_0^1 f'(t) \, dt \neq 0$, $h$ is integrable, $\int_0^1 f \, dt = \int_0^1 h \, dt = 0$ and $h^{(n)}$ is tending to zero in measure along the sequence of Legendre denominators of $\alpha$. In particular, because of Proposition 2.3, we have proved the following

Theorem 6.2. Let $B$ be a homogeneous Banach space on $\mathbb{T}$ and $T$ an irrational translation. If for the pair $(B, T)$ the Koksma inequality holds true then for each cocycle $g = f + h$, where $f$ is piecewise linear with $\int_0^1 f'(t) \, dt \neq 0$, $h \in B_h$, $\int_0^1 f \, dt = \int_0^1 h \, dt = 0$ the corresponding cylinder flow $T_f$ is ergodic.

In particular (see Corollary 2.8)
Corollary 6.3. Suppose that \( g = f + h \) where \( f \) is piecewise linear with \( \int_0^1 f'(t) \, dt \neq 0 \), and \( h(n) = o(1/n) \). If \( \int_0^1 f \, dt = \int_0^1 h \, dt = 0 \) then for each irrational translation \( T \) the corresponding cylinder flow \( T_f \) is ergodic.

Remarks 1. Assume as in [Pa1] that \( g : T \to R \) is piecewise absolutely continuous, with \( \int_0^1 g'(t) \, dt \neq 0 \) and \( \int_0^1 g(t) \, dt = 0 \). Denote by \( x_1, \ldots, x_K \) the discontinuity points and let \( a_j \) be the jump at \( x_j \). Take any piecewise linear function \( f \) with the same discontinuity points and the same jumps as \( g \); in particular \( \int_0^1 f'(t) \, dt \neq 0 \). By adding a constant if necessary we can assume that \( \int_0^1 f(t) \, dt = 0 \). Define \( h = g - f \). We have that \( h \) has zero mean and is absolutely continuous. Now, the result from [Pa1] directly follows from Corollary 6.3.

2. Notice that if \( g \) is of the form as in Corollary 6.3 then for each \( \beta \in T, c \neq 1 \) the cocycle \( g(\cdot + \beta) - cg(\cdot) \) is still of the same form, hence ergodic. We have proved that all ergodic cocycles from Corollary 6.3 are not squashable. In particular, piecewise absolutely continuous cocycles with a nonzero sum of the jumps are not squashable.

3. Using our result on the speed in Koksma's inequality (see Corollary 3.3) and the technique from [Pa2], we can slightly improve the main result of that paper by requiring that the functions from this paper can be modified by those whose Fourier coefficients are of order \( O(1/n) \) with an additionally remark that all those cocycles are not squashable.

References


