A NOTE ON THE ISOMORPHISM OF CARTESIAN PRODUCTS OF ERGODIC FLOWS

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Abstract. We show an isomorphism stability property for Cartesian products of either flows with joining primeness property or flows which are $\alpha$-weakly mixing.

1. Introduction

Isomorphism problems in ergodic theory cover a wide spectrum of issues, e.g., the question when particular dynamical systems are metrically or spectrally isomorphic or the problem of a more general nature of finding invariants that distinguish dynamical systems. Our motivation in this note is a question posed by J.-P. Thouvenot, which lies slightly outside the scope of the above-mentioned problems:

(1) Whether an isomorphism of Cartesian squares of $T$ and $S$, respectively, implies an isomorphism of $T$ and $S$.

Thouvenot’s question still remains open in the class of all automorphisms. The first step toward a solution was made by Ryzhikov [14]. He proved that automorphisms $T$ and $S$ are isomorphic provided that their Cartesian powers $T^{\times d}$ and $S^{\times d}$ (for some $d \geq 1$) are isomorphic and $T$ is $\alpha$-weakly mixing. This result, taking into account that $\alpha$-weak mixing is generic [17], gives therefore the positive answer to (1) for a typical automorphism. In [16] Ryzhikov and Troitskaya strengthened the result from [14] by replacing $\alpha$-weak mixing with the existence of a polynomial in the weak closure of time automorphisms.

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Theorem 1 (see [16]). Let $T$ and $S$ be ergodic automorphisms of probability standard Borel spaces. Assume that for some $n_k \to \infty$
\[ T^{n_k} \to a \cdot \Pi + \sum_{i \in \mathbb{Z}} a_i T^i \] (2)
for $a_i \geq 0$, $i \in \mathbb{Z}$, and $a \geq 0$ such that
\[ a + \sum_{i \in \mathbb{Z}} a_i = 1 \]
with at least two summands positive.\(^1\) If, for some $d \geq 1$, $T^\times d$ is isomorphic to $S^\times d$ then $T$ is isomorphic to $S$.

Troitskaya [18] extended the above result to the case of $\mathbb{Z}^2$-actions.

It was known that the answer to question (1) is positive for simple systems as the structure of factors of their Cartesian products had been described thoroughly [6]. In general, however, the structure of factors of Cartesian products for a given system may be very complex which explains the difficulties in providing a full answer to Thouvenot’s question.

Theorem 1 gives a result for automorphisms. In a letter, Ryzhikov asked whether there is its natural counterpart for flows, i.e., we replace in (2) the sum
\[ \sum_{i \in \mathbb{Z}} a_i T^i \] with $(1 - a) \int T_t \, dP(t)$
for a Borel probability measure $P$ on $\mathbb{R}$. One of our aims is to give a partial positive answer to this question. In fact, we will deal with a more general problem (see (3) below).

It is not hard to see that the assumptions of Theorem 1 force $T$ to be weakly mixing, hence $S$ is also weakly mixing. Here, we will deal with weakly mixing flows. The problem we intend to consider is the following more general form of (1). Assume that $T_1, \ldots, T_d$ are weakly mixing flows.

Assume that $\mathcal{T}_1 \times \ldots \times \mathcal{T}_d$ is isomorphic to a product flow $\mathcal{S}_1 \times \ldots \times \mathcal{S}_d$. Is it true that there is a permutation $\pi$ of $\{1, \ldots, d\}$ such that $\mathcal{T}_i$ is isomorphic to $\mathcal{S}_{\pi(i)}$ for $i = 1, \ldots, d$? (3)

Remark 1. Note that while the answer to (1) remains unclear (and it is still plausible that it is positive), in general, the answer to (3) is negative. Indeed, assume that $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_1'$, and $\mathcal{T}_2'$ are Bernoulli flows such that
\[ h(\mathcal{T}_1) + h(\mathcal{T}_2) = h(\mathcal{T}_1') + h(\mathcal{T}_2') \]

\(^1\)The convergence in (2) is the weak convergence of Markov operators of the $L^2(X, \mathcal{B}, \mu)$-space on which the automorphism $T$ acts by $Tf = f \circ T$ and
\[ \Pi f = \Pi_{X, X} f = \Pi_X f := \int f \, d\mu. \]
but

\[ h(T_1) \neq h(T'_1) \neq h(T_2). \]

By Ornstein's theorem [12], \( T_1 \times T_2 \) is isomorphic to \( T'_1 \times T'_2 \), while \( T'_1 \) is neither isomorphic to \( T_1 \) nor to \( T_2 \). Another example can be found in the class of Gaussian systems with simple spectrum. Indeed, if \( \sigma, \sigma_1, \sigma_2, \sigma'_1, \) and \( \sigma'_2 \) are finite positive Borel measures on \( \mathbb{R} \) such that

\[
\sigma_1 \perp \sigma_2, \quad \sigma'_1 \perp \sigma'_2, \quad \sigma = \sigma_1 + \sigma_2 = \sigma'_1 + \sigma'_2, \quad \sigma'_1 \neq \sigma_1 \neq \sigma'_2
\]

and the Gaussian flow \( T_\sigma \) determined by \( \sigma \) has simple spectrum, then by the theory of Gaussian systems with ergodic self-joinings Gaussian [11] we have

\[
T_\sigma \simeq T_{\sigma_1} \times T_{\sigma_2} \simeq T_{\sigma'_1} \times T_{\sigma'_2}
\]

while \( T_{\sigma_1} \) is neither isomorphic to \( T_{\sigma'_1} \) nor to \( T_{\sigma'_2} \).

The main result of this note is the following theorem (see Sec. 2 for needed definitions).

**Theorem 2.** The answer for (3) is positive in the following cases:

(i) when \( T_1, \ldots, T_d \) are weakly mixing and satisfy the JP property;

(ii) when \( T_i \) are \( \alpha_i \)-weakly mixing for \( \alpha_i \in (0, 1) \) for \( 1 \leq i \leq d \).

Our main tool will be the theory of joinings. Apart from the JP property we will also study some joining properties of similar flavor for \( \alpha \)-weakly mixing flows. The generalization of part (i) of Theorem 2 to the actions of other abelian Polish groups is straightforward, as the notion of JP is independent of the acting group.

As Lemańczyk and Ryzhikov noted in a private communication, the methods used in this note are not sufficient to answer question (3) when we consider flows such that

\[
T^{t_n} \to a \cdot \Pi + (1 - a) \cdot \int T^t \, dP(t) \quad \text{for some } t_n \to \infty \text{ and } a \in [0, 1) \text{ without any further assumption on measure } P.
\]

Therefore, the question whether Theorem 1 has a full counterpart for flows remains open. It also seems to be an open problem whether (2) (or (4) in case of flows) implies the JP property or a weaker property in the spirit of Proposition 6. If \( a_i \)’s in (2) decrease to zero with exponential rate, then we obtain an analytic function in the weak closure of time automorphisms, which yields the CS property, hence the JP property (see [10]). The same mechanism works for flows and hence we have the following corollary of Theorem 2(i).
Corollary 1. The answer for (3) is positive whenever (4) holds and \( P \) is non-Dirac with analytic Fourier transform \( \hat{P} \). In particular, the answer for (3) is positive if \( P \) is continuous and has a bounded support.

This gives a positive partial answer to the original question by Ryzhikov.

In fact in the process of proving Theorem 2 we show more: the obtained isomorphisms between \( T_i \)'s and \( S_i \)'s are restrictions of the original isomorphism between the Cartesian products. In particular, this implies that the centralizer of the product \( T_1 \times \cdots \times T_d \) is the product of the centralizers of \( T_1, \ldots, T_d \) up to a permutation of the coordinates, see Corollary 2 and Corollary 4.

Similar problems to what we consider here were taken up in [5]. It was shown that for a typical automorphism and any \( k_1, \ldots, k_d, k'_1, \ldots, k'_d \in \mathbb{N} \) the convolutions \( \sigma_{T^{k_1}} \ast \cdots \ast \sigma_{T^{k_d}} \) and \( \sigma_{T^{k'_1}} \ast \cdots \ast \sigma_{T^{k'_d}} \) are mutually singular provided that \((k_1, \ldots, k_d)\) is not a rearrangement of \((k'_1, \ldots, k'_d)\). This property (which can be viewed as a variation of the CS property) has the following consequence: for a typical automorphism \( T \), the only way that \( T^l \) (for any \( l \in \mathbb{Z} \setminus \{0\} \)) can sit as a factor of \( T^{k_1} \times \cdots \times T^{k_n} \times \cdots \) is inside the \( i \)th coordinate \( \sigma \)-algebra for some \( i \) with \( k_i = l \). In particular (for a generic transformation),

\[
C(T^{k_1} \times \cdots \times T^{k_d}) = \bigcup_{\pi} C(T^{k_{\pi(1)}}) \times \cdots \times C(T^{k_{\pi(d)}}),
\]

where \( \pi \) runs over the set of permutations of \( \{1, \ldots, d\} \) such that \( \pi(i) = j \) implies \( k_i = k_j \).

2. Definitions and tools

2.1. Joinings. We recall the necessary information about joinings.\(^2\) Let \( \mathcal{T} = (T^t)_{t \in \mathbb{R}} \) and \( \mathcal{S} = (S^t)_{t \in \mathbb{R}} \) be measurable flows on \((X, \mathcal{B}, \mu)\) and \((Y, \mathcal{C}, \nu)\), respectively (by the measurability of the flow \((T^t)_{t \in \mathbb{R}}\) we mean that the map \( \mathbb{R} \ni t \mapsto \langle f \circ T^t, g \rangle \in \mathbb{C} \) is continuous for all \( f, g \in L^2(X, \mathcal{B}, \mu) \)). By \( J(\mathcal{T}, \mathcal{S}) \) we denote the set of all joinings between \( \mathcal{T} \) and \( \mathcal{S} \), i.e., the set of all \((T^t \times S^t)_{t \in \mathbb{R}}\)-invariant probability measures on \((X \times Y, \mathcal{B} \otimes \mathcal{C})\), whose projections on \( X \) and \( Y \) are equal to \( \mu \) and \( \nu \) respectively. For \( J(\mathcal{T}, \mathcal{T}) \) we write \( J(\mathcal{T}) \) and we denote the subspace of ergodic joinings by adding the superscript \( e \): \( J^e(\mathcal{T}, \mathcal{S}) \). Joinings are in one-to-one correspondence with Markov operators \( \Phi : L^2(X, \mathcal{B}, \mu) \to L^2(Y, \mathcal{C}, \nu) \) satisfying \( \Phi \circ T^t = S^t \circ \Phi \).

\(^2\)For more information on the theory of joinings, we refer the reader to [4,11,13].
for all \( t \in \mathbb{R} \):
\[
\Phi \mapsto \lambda_{\Phi} \in J(\mathcal{S}, \mathcal{T}), \quad \lambda_{\Phi}(A \times B) = \int_B \Phi(\mathbb{I}_A) \, d\nu,
\]
\[
\lambda \mapsto \Phi_{\lambda}, \quad \int \Phi_{\lambda} f(y) g(y) \, d\nu(y) = \int f(x) g(y) \, d\lambda(x, y).
\]

We denote by \( \Pi_{X,Y} \) the Markov operator corresponding to the product measure \( \mu \otimes \nu \). We denote the set of intertwining Markov operators by \( J(\mathcal{T}, \mathcal{S}) \). This identification allows us to view \( J(\mathcal{T}) \) endowed with the weak operator topology as a metrizable compact semi-topological semigroup.

It is said that \( \mathcal{T} \) and \( \mathcal{S} \) are disjoint (see [3]) if
\[
J(\mathcal{T}, \mathcal{S}) = \{ \mu \otimes \nu \};
\]
we write \( \mathcal{T} \perp \mathcal{S} \). Given a flow \( \mathcal{T} = (T^t)_{t \in \mathbb{R}} \) and a Borel probability measure \( P \) on \( \mathbb{R} \), we define the Markov operator
\[
\int_{\mathbb{R}} T^t \, dP(t)
\]
acting on \( L^2(X, \mathcal{B}, \mu) \) by
\[
\left\langle \left( \int_{\mathbb{R}} T^t \, dP(t) \right) f, g \right\rangle = \int_{\mathbb{R}} \left\langle T^t f, g \right\rangle \, dP(t)
\]
for all \( f, g \in L^2(X, \mathcal{B}, \mu) \).

2.2. JP property. We recall the notion of the joining primeness property (JP).

**Definition 1** (see [10]). An ergodic flow \( \mathcal{T} \) is said to have the joining primeness property\(^3\) (JP) if for any weakly mixing flows \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \), for every \( \lambda \in J_e(\mathcal{T}, \mathcal{S}_1 \times \mathcal{S}_2) \) we have
\[
\lambda = \lambda_{X,Y_1} \otimes \nu_2 \quad \text{or} \quad \lambda = \lambda_{X,Y_2} \otimes \nu_1,
\]
where \( \lambda_{X,Y_1} \) and \( \lambda_{X,Y_2} \) are the projections of \( \lambda \) onto the appropriate coordinates.

**Remark 2.** In terms of Markov operators, the JP property means that for every \( \Phi \in J_e(\mathcal{T}, \mathcal{S}_1 \times \mathcal{S}_2) \),
\[
\Phi = p_1 \circ \Phi \quad \text{or} \quad \Phi = p_2 \circ \Phi,
\]
where \( p_1 : L^2(Y_1 \times Y_2) \to L^2(Y_1) \otimes \mathbb{I} \) and \( p_2 : L^2(Y_1 \times Y_2) \to \mathbb{I} \otimes L^2(Y_2) \) are the orthogonal projections.

Recall some properties of the class of flows satisfying the JP property.

\(^3\)Property (5) defining the JP notion has been observed earlier for the class of quasi-simple systems in [15].
Proposition 1 (see [10]). The class of weakly mixing JP flows is closed under distal extensions which are weakly mixing.

The next result is a spectral criterion for the JP property.

Proposition 2 (see [10]). All weakly mixing flows whose maximal spectral type is singular with respect to the convolution of any two continuous measures (this is called in [10] the convolution simplicity property, i.e., the CS property) satisfy the JP property.

Recall that the CS property is generic in the class of flows on a fixed probability Borel space [10]. A stronger property than CS is the simple convolution property (SC) which, by definition, holds when the Gaussian action determined by the reduced maximal spectral type \( \sigma_T \) of the given system has simple spectrum. In a recent paper [9] it has been shown that there are natural classes of flows with the SC property: a typical flow on a fixed probability Borel space, special flows over a rotation by a “generic” \( \alpha \in [0,1) \) under a smooth roof function which is not a trigonometrical polynomial and special flows over rotations by \( \alpha \in [0,1) \) with unbounded partial quotients under some piecewise absolutely continuous roof function.

Note that by Propositions 1 and 2, whenever a weakly mixing flow satisfies the CS property, then its weakly mixing distal extension has the JP property.

A natural source of examples of flows with the CS property is given by the following result.

Proposition 3 (see [10]). Assume that for \( t_n \to \infty \),
\[
T^{t_n} \to \int T^t \, dP(t)
\]
for a probability Borel measure \( P \) on \( \mathbb{R} \) which is not a Dirac measure and such that \( \hat{P} \) is analytic.\(^4\) Then \( \sigma = \sigma_T \) is singular with respect to the convolution of any two continuous measures.

Such flows can be obtained in a natural way, namely as smooth flows on orientable surfaces. In [8], Kochergin proved that there is a natural class of flows on surfaces which are weakly mixing but not mixing. This result was later extended in [1] to a larger class of flows on surfaces. A property which (together with other properties of the flows under consideration) was used to prove the absence of mixing turned out to be one of the sufficient conditions to obtain in the weak closure of time automorphisms an operator of the form (6) satisfying the assumptions of Proposition 3 (see [2]).

Note also that by repeating word for word the proof of Proposition 3 (see [10]) we obtain the same result in the situation where (6) is replaced

\(^4\)This, for example, holds whenever \( P \) has bounded support.
with
\[ T^{t_n} \to a \cdot \Pi + (1 - a) \cdot \int T^t \, dP(t) \]
for some \( a \in (0, 1) \).

2.3. Partial mixing, partial rigidity and \( \alpha \)-weak mixing. We recall some basic definitions which will be used in the sequel.

Definition 2. A flow \( \mathcal{T} \) is called \( \alpha \)-partially mixing for \( \alpha \in (0, 1) \) along \( t_n \to \infty \) if
\[ T^{t_n} \to \alpha \cdot \Pi + (1 - \alpha) \cdot J \]
for some \( J \in \mathcal{J}(\mathcal{T}) \).

Definition 3. A flow \( \mathcal{T} \) is called \((1 - \alpha)\)-partially rigid for \( \alpha \in (0, 1) \) along \( t_n \to \infty \) if
\[ T^{t_n} \to \alpha \cdot J + (1 - \alpha) \cdot \text{Id} \]
for some \( J \in \mathcal{J}(\mathcal{T}) \).

Definition 4 (see [7,17]). A flow \( \mathcal{T} \) is called \( \alpha \)-weakly mixing for \( \alpha \in (0, 1) \) along \( t_n \to \infty \) if
\[ T^{t_n} \to \alpha \cdot \Pi + (1 - \alpha) \cdot \text{Id} \]
i.e., it is \( \alpha \)-partially mixing and \((1 - \alpha)\)-partially rigid along the same subsequence.

3. Results

3.1. Isomorphism problem and JP property. The following proposition is an immediate consequence of the definition of the JP property.

Proposition 4. Let a flow \( \mathcal{T} \) satisfy the JP property and let \( S_1 \) and \( S_2 \) be weakly mixing. Assume that \( \mathcal{T} \) is a factor of \( S_1 \times S_2 \). Then \( \mathcal{T} \) is a factor of \( S_1 \) or \( S_2 \).

Proof. Let
\[ \Phi : Y_1 \times Y_2 \to X \]
be such that
\[ \mathcal{T} \circ \Phi = \Phi \circ (S_1 \times S_2) \).
Then
\[ \Phi : L^2(X) \to L^2(Y_1 \times Y_2), \quad \Phi f = f \circ \Phi \]
determines an ergodic joining, whence by Remark 2, we have
\[ \Phi = p_1 \circ \Phi \quad \text{or} \quad \Phi = p_2 \circ \Phi, \]
which means that
\[ \Phi(L^2(X)) \subset L^2(Y_1) \otimes \mathbb{I} \text{ or } \Phi(L^2(X)) \subset \mathbb{I} \otimes L^2(Y_2). \]
This completes the proof. \( \square \)
Remark 3. In particular, the proof of Proposition 4 shows the following: if \( \mathcal{B} \subset \mathcal{C}_1 \otimes \mathcal{C}_2 \) is a factor-\( \sigma \)-algebra of \( \mathcal{S}_1 \times \mathcal{S}_2 \) representing the action \( \mathcal{T} \), then either \( \mathcal{B} \subset \mathcal{C}_1 \) or \( \mathcal{B} \subset \mathcal{C}_2 \). The extension of this fact to more than two flows \( \mathcal{S}_i \) is straightforward (proofs are similar). More precisely, for any \( d \geq 1 \) and \( \Phi_\lambda \in J^c(\mathcal{T}, \mathcal{S}_1 \times \cdots \times \mathcal{S}_d) \), the inclusion
\[
\Phi_\lambda(L^2(X)) \subset \sum_{i=1}^d L^2(Y_i)
\]
implies that
\[
\Phi_\lambda(L^2(X)) \subset L^2(Y_i) \quad \text{for some } 1 \leq i \leq d.
\]

Proof of Theorem 2(i). We will provide the proof for \( d = 2 \). The extension to the product of \( d \) JP flows is straightforward and (3) holds whenever \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) satisfy the JP property.

Let
\[
\Phi: Y_1 \times Y_2 \to X_1 \times X_2
\]
determine an isomorphism between \( \mathcal{S}_1 \times \mathcal{S}_2 \) and \( \mathcal{T}_1 \times \mathcal{T}_2 \). Using Proposition 4, we may assume without loss of generality that
\[
\Phi(L^2(X_1) \otimes \mathbb{I}) \subset L^2(Y_1) \otimes \mathbb{I}, \quad \Phi(\mathbb{I} \otimes L^2(X_2)) \subset \mathbb{I} \otimes L^2(Y_2).
\]
Since \( \Phi \) is an isomorphism and the image of \( \sigma \)-algebras \( \mathcal{B}_1 \otimes \{\emptyset, X_2\} \) and \( \{\emptyset, X_1\} \otimes \mathcal{B}_2 \) via \( \Phi^{-1} \) generates \( \mathcal{C}_1 \otimes \mathcal{C}_2 \), we have
\[
\Phi(L^2(X_1) \otimes \mathbb{I}) = L^2(Y_1) \otimes \mathbb{I}, \quad \Phi(\mathbb{I} \otimes L^2(X_2)) = \mathbb{I} \otimes L^2(Y_2),
\]
which completes the proof.

Moreover, the proof shows the following.

Corollary 2. Let \( \mathcal{T}_1, \ldots, \mathcal{T}_d \) be weakly mixing JP flows. Then the centralizer \( C(\mathcal{T}_1 \times \cdots \times \mathcal{T}_d) \) of \( \mathcal{T}_1 \times \cdots \times \mathcal{T}_d \) consists of transformations that have the form \( \pi \circ \tilde{S} \), where
\[
\tilde{S} \in C(\mathcal{T}_1) \times \cdots \times C(\mathcal{T}_d)
\]
and the permutation \( \pi \in S(d) \) acting on elements of \( X_1 \times \cdots \times X_d \) by
\[
\pi(x_1, \ldots, x_d) = (x_{\pi(1)}, \ldots, x_{\pi(d)})
\]
is such that \( \pi(i) = j \) implies \( \mathcal{T}_i \simeq \mathcal{T}_j \).

Thouvenot’s question has clearly a negative answer in the infinite case. Note that for any partition of \( \mathbb{N} \) into subsets \( \mathbb{N}_1, \mathbb{N}_2, \ldots \) we have
\[
\mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \simeq (\times_{i \in \mathbb{N}_1} \mathcal{T}_i) \times (\times_{i \in \mathbb{N}_2} \mathcal{T}_i) \times \cdots,
\]
so it suffices to consider, for example, \( \mathbb{N}_1 = \{1, 2\}, \mathbb{N}_i = \{i + 1\} \) for \( i \geq 2 \), a weakly mixing \( \mathcal{T} \) such that \( \mathcal{T} \neq \mathcal{T} \times \mathcal{T} \), and take \( \mathcal{T}_i = \mathcal{T} \) for all \( i \in \mathbb{N} \).

Therefore, the best infinite version of Theorem 2 we can hope for is that the isomorphism \( \Phi \) of \( \mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \) and \( \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \) implies that there exists
a partition of $\mathbb{N}$ into subsets $\mathbb{N}_1, \mathbb{N}_2, \ldots$ such that $\prod_{j \in \mathbb{N}_i} T_j$ is isomorphic to $S_i$ via $\Phi$. It turns out that this holds true if we assume that the flows $T_i$ satisfy the JP property.

**Proposition 5.** Let $T_i$ be weakly mixing flows satisfying the JP property for $i \geq 1$. For flows $S_i$, $i \geq 1$, such that $T_1 \times T_2 \times \ldots$ is isomorphic to $S_1 \times S_2 \times \ldots$ via $\Phi : Y_1 \times Y_2 \times \ldots \to X_1 \times X_2 \times \ldots$, there exists a partition of $\mathbb{N}$ into subsets $\mathbb{N}_1, \mathbb{N}_2, \ldots$ such that $\Phi$ determines an isomorphism between $\prod_{j \in \mathbb{N}_i} T_j$ and $S_i$ for $i \geq 1$. If we additionally assume that $S_i$ also satisfy the JP property for $i \geq 1$, then there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $\Phi$ determines an isomorphism between $T_i$ and $S_{\pi(i)}$.

**Proof.** Fix $i \in \mathbb{N}$. Let

$$\Phi^{-1}(B_i) \subset C_1 \otimes C_2 \otimes \ldots$$

be the factor-$\sigma$-algebra of $S_1 \times S_2 \times \ldots$ representing the action of $T_i$. We claim that there exists $k \geq 1$ such that

$$\Phi^{-1}(B_i) \subset C_1 \otimes \cdots \otimes C_k.$$

Indeed, note that for $k \geq 1$ we have

$$\Phi^{-1}(B_i) \subset (C_1 \otimes \cdots \otimes C_k) \otimes (C_{k+1} \otimes C_{k+2} \otimes \ldots).$$

By the JP property, either

$$\Phi^{-1}(B_i) \subset C_1 \otimes \cdots \otimes C_k$$

or

$$\Phi^{-1}(B_i) \subset C_{k+1} \otimes C_{k+2} \otimes \ldots.$$

If for all $k \geq 1$ we have

$$\Phi^{-1}(B_i) \subset C_{k+1} \otimes C_{k+2} \otimes \ldots,$$

then

$$\Phi^{-1}(B_i) \subset \bigcap_{k \geq 1} C_{k+1} \otimes C_{k+2} \otimes \ldots$$

and by the Kolmogorov zero-one law, $\Phi^{-1}(B_i)$ is trivial, which is impossible. Therefore, for some $k \geq 1$ we have

$$\Phi^{-1}(B_i) \subset C_1 \otimes \cdots \otimes C_k.$$

Using again the JP property, we obtain, as in the finite case, that for some $\pi(i) \geq 1$

$$\Phi^{-1}(B_i) \subset C_{\pi(i)}. \quad (7)$$

Setting for $i \geq 1$

$$\mathbb{N}_i := \{j \in \mathbb{N} : \Phi^{-1}(B_j) \subset C_i\},$$
we obtain a partition of \( N \); indeed, \( \Phi \) is an isomorphism, whence \( \bigcup_{i \in \mathbb{N}} \mathbb{N}_i = \mathbb{N} \) and the sets \( \mathbb{N}_i \) are disjoint by (7). Moreover, \( \sigma \)-algebras \( \Phi^{-1}(\mathcal{B}_j) \) for \( j \in \mathbb{N}_i \) are independent and generate the whole \( \sigma \)-algebra \( \mathcal{C}_i \), whence

\[
\mathcal{S}_i \cong \times_{j \in \mathbb{N}_i} \mathcal{T}_j,
\]

which completes the proof of the first assertion.

If we additionally assume that the flows \( \mathcal{S}_i \) satisfy the JP property for \( i \geq 1 \), then the sets \( \mathbb{N}_i \) for \( i \geq 1 \) are singletons and therefore determine a permutation \( \pi : \mathbb{N} \to \mathbb{N} \) such that \( \Phi \) is an isomorphism between \( \mathcal{T}_i \) and \( \mathcal{S}_{\pi(i)} \).

As a direct consequence of the above proposition we obtain the following result.

**Corollary 3.** Let \( \mathcal{T}_i, i \geq 1 \), be weakly mixing flows satisfying the JP property. In addition, assume that for \( i, j \in \mathbb{N} \) the isomorphism \( \mathcal{T}_i \cong \mathcal{T}_j \) implies that \( \mathcal{T}_i = \mathcal{T}_j \). Then the centralizer \( C(\mathcal{T}_1 \times \mathcal{T}_2 \times \ldots) \) of \( \mathcal{T}_1 \times \mathcal{T}_2 \times \ldots \) consists of transformations of the form \( \pi \circ \tilde{S} \), where

\[
\tilde{S} \in C(\mathcal{T}_1) \times C(\mathcal{T}_2) \times \ldots
\]

and the permutation \( \pi : \mathbb{N} \to \mathbb{N} \) acting on elements of \( X_1 \times X_2 \times \ldots \) by the rule

\[
\pi(x_1, x_2, \ldots) = (x_{\pi(1)}, x_{\pi(2)}, \ldots)
\]

is such that \( \pi(i) = j \) implies \( \mathcal{T}_i = \mathcal{T}_j \).

3.2. **JP property as a weaker version of disjointness.** Let \( \mathcal{T} \) and \( \mathcal{S} \) be ergodic flows on \((X, \mathcal{B}, \mu)\) and \((Y, \mathcal{C}, \nu)\), respectively. Now we compare the \( \alpha \)-partial rigidity and the \( \beta \)-partial mixing. First, we recall the following lemma.

**Lemma 1** (see [10]). Assume that \( \lambda \in J^e(\mathcal{T}, \mathcal{S}_1 \times \mathcal{S}_2) \) satisfies the condition

\[
\Phi_\lambda(L^2_0(X, \mathcal{B}, \mu)) \subset L^2_0(Y_1, \mathcal{C}_1, \nu_1) \oplus L^2_0(Y_2, \mathcal{C}_2, \nu_2).
\]

Then

\[
\lambda = \lambda_{X,Y_1} \otimes \nu_2 \quad \text{or} \quad \lambda = \lambda_{X,Y_2} \otimes \nu_1.
\]

**Proposition 6.** Let \( \alpha, \beta \in [0,1] \). Assume that \( \mathcal{T} \) is \( \alpha \)-partially rigid and \( \mathcal{S} \) is \( \beta \)-partially mixing along the same time-sequence, i.e., for some \( t_n \to \infty \) we have

\[
T^{t_n} \to \alpha \cdot \text{Id} + (1 - \alpha) \cdot J, \quad S^{t_n} \to \beta \cdot \Pi + (1 - \beta) \cdot K
\]

for some \( J \in \mathcal{F}(\mathcal{T}) \) and \( K \in \mathcal{F}(\mathcal{S}) \).

(i) If \( \alpha = \beta = 1 \), then \( \mathcal{T} \) and \( \mathcal{S} \) are spectrally disjoint.
(ii) If $\alpha + \beta > 1$, then $\mathcal{T} \perp \mathcal{S}$.\(^5\)

Let $\alpha \in (0, 1)$ and let $\mathcal{S}_1, \mathcal{S}_2$ be ergodic flows on $(Y_1, C_1, \nu_1)$ and $(Y_2, C_2, \nu_2)$, respectively.

(iii) If $\mathcal{T}$ is $(1 - \alpha)$-weakly mixing and $\mathcal{S}_1, \mathcal{S}_2$ are $(1 - \alpha)$-partially mixing along $t_n$, then for every $\lambda \in J^e(\mathcal{T}, \mathcal{S}_1 \times \mathcal{S}_2)$ we have

$$\lambda = \lambda_{X,Y_1} \otimes \nu_2 \quad \text{or} \quad \lambda = \lambda_{X,Y_2} \otimes \nu_1.$$

**Proof.** (i) is well-known.

We prove (ii). Suppose that there exists $\Phi \in J^e(\mathcal{T}, \mathcal{S})$ such that $\Pi \neq \Phi$. We have

$$S^{t_n} \Phi \to (\beta \cdot \Pi_{Y,Y} + (1 - \beta) \cdot K) \Phi = \beta \cdot \Pi_{X,Y} + (1 - \beta) \cdot K \Phi.$$ 

On the other hand,

$$S^{t_n} \Phi = \Phi T^{t_n} \to (\Phi \cdot \mathrm{Id} + (1 - \alpha) \cdot J) = \alpha \cdot \Phi + (1 - \alpha) \cdot \Phi J.$$ 

Therefore,

$$\beta \cdot \Pi + (1 - \beta) \cdot K \Phi = \alpha \cdot \Phi + (1 - \alpha) \cdot \Phi J. \quad (8)$$

We have that $\Pi, \Phi \in J^e(\mathcal{S}, \mathcal{T})$. Using (8), by the uniqueness of the ergodic decomposition we conclude that in the ergodic decomposition of $(1 - \alpha) \cdot \Phi J$ we will see $\beta \cdot \Pi$. Hence $1 - \alpha \geq \beta$, which yields a contradiction. Therefore,

$$J^e(\mathcal{S}, \mathcal{T}) = J(\mathcal{S}, \mathcal{T}) = \{\Pi\}.$$ 

Now we prove (iii). Take $\lambda \in J^e(\mathcal{T}, \mathcal{S}_1 \times \mathcal{S}_2)$. In view of Lemma 1, it suffices to show that for $\Phi = \Phi_\lambda$ we have

$$\Phi(L^2_0(X)) \subset (L^2_0(Y_1) \otimes \mathrm{I}) \oplus (\mathrm{I} \otimes L^2_0(Y_2)).$$

Take $f \in L^2_0(X)$. Since

$$L^2_0(Y_1 \times Y_1) = (L^2_0(Y_1) \otimes \mathrm{I}) \oplus (\mathrm{I} \otimes L^2_0(Y_2)) \oplus (L^2_0(Y_1) \otimes L^2_0(Y_2))$$

for some $f_1 \in L^2_0(Y_1)$, $f_2 \in L^2_0(Y_2)$, and $f_3 \in L^2_0(Y_1) \otimes L^2_0(Y_2)$, we have

$$\Phi f = f_1 \otimes \mathrm{I} + \mathrm{I} \otimes f_2 + f_3.$$ 

Therefore,

$$\langle (S_1 \times S_2)^{t_n} \Phi f, \Phi f \rangle = \langle S_1^{t_n} f_1, f_1 \rangle + \langle S_2^{t_n} f_2, f_2 \rangle + \langle (S_1 \times S_2)^{t_n} f_3, f_3 \rangle$$

$$\to \alpha \langle K_1 f_1, f_1 \rangle + \alpha \langle K_2 f_2, f_2 \rangle + \alpha^2 \langle (K_1 \otimes K_2) f_3, f_3 \rangle.$$ 

Hence

$$\left| \lim_{t_n \to \infty} \langle (S_1 \times S_2)^{t_n} \Phi f, \Phi f \rangle \right| \leq \alpha \|f_1\|^2 + \alpha \|f_2\|^2 + \alpha^2 \|f_3\|^2. \quad (9)$$

\(^5\)When $\alpha + \beta = 1$, it is not necessarily true that $\mathcal{T} \perp \mathcal{S}$. It suffices to consider the $\alpha$-weakly mixing $\mathcal{T} = \mathcal{S}$ and take their diagonal joining: $\Delta \in J(\mathcal{T}, \mathcal{S})$ given by $\Delta(A \times B) = \mu(A \cap B)$ for $A, B \in \mathcal{B}$. However, as item (iii) shows, in this case (provided that $\mathcal{T}$ is $(1 - \alpha)$-weakly mixing) we observe some kind of the JP property for $\mathcal{T}$. 

On the other hand,
\[
\langle (S_1 \times S_2)^{t_n} \Phi f, \Phi f \rangle = \langle \Phi T^{t_n} f, \Phi f \rangle = \langle T^{t_n} f, \Phi^* \Phi f \rangle
\]
\[
\rightarrow \alpha \langle f, \Phi^* \Phi f \rangle = \alpha \|\Phi f\|^2. \quad (10)
\]
Since
\[
\|\Phi f\|^2 = \|f_1\|^2 + \|f_2\|^2 + \|f_3\|^2,
\]
(9) and (10) may hold only when \( f_3 = 0 \), which completes the proof. \( \square \)

3.3. Isomorphism problem and \( \alpha \)-weak mixing. Proposition 7 below will complete the proof of Theorem 2.

**Proposition 7.** Let \( d \geq 1 \) and let \( T_i \) be \( \alpha_i \)-weakly mixing for some \( \alpha_i \in (0,1) \) for \( 1 \leq i \leq d \) along \( t_n \to \infty \). Let \( S_i \) be weakly mixing for \( 1 \leq i \leq d \).

If
\[
T_1 \times \cdots \times T_d \simeq S_1 \times \cdots \times S_d,
\]
then there exists a permutation \( \pi \in S(d) \) such that \( T_i \simeq S_{\pi(i)} \). More precisely, if
\[
\Phi : Y_1 \times \cdots \times Y_d \to X_1 \times \cdots \times X_d
\]
determines the isomorphism between \( S_1 \times \cdots \times S_d \) and \( T_1 \times \cdots \times T_d \), then \( \Phi|_{L^2(B_i)} \) yields an isomorphism between \( T_i \) and \( S_{\pi(i)} \).

Before we prove the result stated above, we need some auxiliary lemmas.

**Lemma 2.** Let \( R \in J(T) \), where \( T \simeq S \), i.e., \( S \circ Q = Q \circ T \) for some isomorphism \( Q : X \to Y \).

If
\[
R = \int V \, dP(V)
\]
corresponds to the ergodic decomposition of \( \lambda_R \), then
\[
Q^{-1} R Q = \int Q^{-1} V Q \, dP(V) = \int V \, d(\Phi^{-1} \circ \cdots \circ \Phi)_*(P)(V)
\]
corresponds to the ergodic decomposition of \( \lambda_{Q^{-1} R Q} \).

**Proof.** Note that
\[
\lambda_{Q^{-1} V Q} = \lambda_V \circ (Q \times Q)^{-1}.
\]
Indeed, for \( f, g \in L^2(Y) \) we have
\[
\int f \otimes g \, d\lambda_{Q^{-1} V Q} = \int Q^{-1} V Q f \cdot g \, d\nu = \int V Q f \cdot Q g \, d\mu
\]
\[
= \int Q f \otimes Q g \, d\lambda_V = \int (Q \otimes Q)(f \otimes g) \, d\lambda_V = \int f \otimes g \, d\lambda_V \circ (Q \otimes Q)^{-1}.
\]
It is clear that
\[
Q \times Q : X \times X \to Y \times Y
\]
yields an affine isomorphism of the simplices of joinings. \( \square \)
Lemma 3. When the assumptions of Proposition 7 hold, $S_i$ are $\beta_i$-weakly mixing for some $\beta_i \in (0, 1)$ and $1 \leq i \leq d$.

Proof. Let $\Phi : Y_1 \times \cdots \times Y_d \to X_1 \times \cdots \times X_d$ establish the isomorphism

$$(\mathcal{T}_1 \times \cdots \times \mathcal{T}_d) \circ \Phi = \Phi \circ (S_1 \times \cdots \times S_d).$$

Then

$$\Phi(S_1 \times \cdots \times S_d)^{tn} \Phi^{-1} = (\mathcal{T}_1 \times \cdots \times \mathcal{T}_d)^{tn}$$

$$\to \bigotimes_{i=1}^d ((1 - \alpha_i) \cdot \text{Id}_{X_i} + \alpha_i \cdot \Pi_{X_i}),$$

whence

$$(S_1 \times \cdots \times S_d)^{tn} \to \Phi^{-1} \circ \left( \bigotimes_{i=1}^d (1 - \alpha_i) \cdot \text{Id}_{X_i} + \alpha_i \cdot \Pi_{X_i} \right) \circ \Phi.$$

We may assume (passing to a subsequence if necessary) that $S_i^{tn} \to Q_i$ for some $Q_i \in J(S_i)$ for $1 \leq i \leq d$. Therefore

$$\bigotimes_{i=1}^d Q_i$$

$$= \Phi^{-1} \circ \left( \sum_{\epsilon_i \in \{0, 1\}, 1 \leq i \leq d} \bigotimes_{i=1}^d (1 - \alpha_i)^{1-\epsilon_i} \alpha_i^{\epsilon_i} \cdot \text{Id}^{1-\epsilon_i} \cdot \Pi^{\epsilon_i} \right) \circ \Phi$$

$$= \sum_{\epsilon_i \in \{0, 1\}, 1 \leq i \leq d} \Phi^{-1} \circ \left( \bigotimes_{i=1}^d (1 - \alpha_i)^{1-\epsilon_i} \alpha_i^{\epsilon_i} \cdot \text{Id}^{1-\epsilon_i} \cdot \Pi^{\epsilon_i} \right) \circ \Phi$$

$$= \sum_{\epsilon_i \in \{0, 1\}, 1 \leq i \leq d} \prod_{i=1}^d (1 - \alpha_i)^{1-\epsilon_i} \alpha_i^{\epsilon_i}$$

$$\cdot \left( \Phi^{-1} \circ \left( \bigotimes_{i=1}^d (\text{Id}^{1-\epsilon_i} \cdot \Pi^{\epsilon_i}) \right) \circ \Phi \right), \quad (11)$$

where for $1 \leq i \leq d$

$$\text{Id}^{1-\epsilon_i} \cdot \Pi^{\epsilon_i} = \begin{cases} \text{Id}_{X_i} & \text{if } \epsilon_i = 0, \\ \Pi_{X_i} & \text{if } \epsilon_i = 1. \end{cases}$$

\footnote{6We identify $\lambda_{Q_1 \otimes \cdots \otimes Q_d}$ with $\lambda_{Q_1 \otimes \cdots \otimes \lambda_{Q_d}}$.}
Since
\[ \bigotimes_{i=1}^{d} \text{Id}^{1-\varepsilon_i} \Pi_{\varepsilon_i}^{\varepsilon_i} \in \mathcal{J}(\mathcal{T}_1 \times \cdots \times \mathcal{T}_d), \]
by Lemma 2
\[ \Phi^{-1} \circ \left( \bigotimes_{i=1}^{d} \left( \text{Id}^{1-\varepsilon_i} \Pi_{\varepsilon_i}^{\varepsilon_i} \right) \right) \circ \Phi \in \mathcal{J}(\mathcal{S}_1 \times \cdots \times \mathcal{S}_d). \]
Note that
\[ \Phi^{-1} \circ (\text{Id}_{X_1} \otimes \cdots \otimes \text{Id}_{X_d}) \circ \Phi = \text{Id}_{Y_1} \otimes \cdots \otimes \text{Id}_{Y_d} \]
and
\[ \Phi^{-1} \circ (\Pi_{X_1} \otimes \cdots \otimes \Pi_{X_d}) \circ \Phi = \Pi_{Y_1} \otimes \cdots \otimes \Pi_{Y_d}. \]
Therefore, by taking the projection onto the first two coordinates in (11), we obtain
\[ Q_1 = \prod_{i=1}^{d} (1 - \alpha_i) \cdot \text{Id}_{Y_i} + \prod_{i=1}^{d} \alpha_i \cdot \Pi_{Y_i} + R_1, \]
where \( R_1 \) is the projection of the remaining Markov operator. Hence
\[ Q_1 = \beta_1 \cdot \text{Id}_{Y_1} + \gamma_1 \cdot \Pi_{Y_1} + \delta_1 \cdot \Phi_{\rho_i}, \]
where
\[ \beta_1 \geq (1 - \alpha_1) \ldots (1 - \alpha_d), \quad \gamma_1 \geq \alpha_1 \ldots \alpha_d, \quad \delta_i \geq 0, \quad \beta_1 + \gamma_1 + \delta_1 = 1, \]
and \( \Phi_{\rho_i} \) is such that the probability measure \( \rho_i \) is singular with respect to both \( \Delta_i \) and \( \nu_i \otimes \nu_i \).

In the same way
\[ Q_i = \beta_i \cdot \text{Id}_{Y_i} + \gamma_i \cdot \Pi_{Y_i} + \delta_i \cdot \Phi_{\rho_i}, \]
for some \( \beta_i, \gamma_i, \delta_i \geq 0 \) satisfying the conditions
\[ \beta_i + \gamma_i + \delta_i = 1, \quad \rho_i \perp \nu_i \otimes \nu_i, \quad \rho_i \perp \Delta_i. \]
Hence
\[ Q_1 \otimes \cdots \otimes Q_d = \sum_{\varepsilon_i^{\beta}, \varepsilon_i^{\gamma}, \varepsilon_i^{\delta} \in \{0,1\}, \varepsilon_i^{\beta} + \varepsilon_i^{\gamma} + \varepsilon_i^{\delta} = 1} \bigotimes_{i=1}^{d} \beta_i^{\varepsilon_i^{\beta}} \gamma_i^{\varepsilon_i^{\gamma}} \delta_i^{\varepsilon_i^{\delta}} \cdot \text{Id}^{\varepsilon_i^{\beta}} \Pi^{\varepsilon_i^{\gamma}} \Phi_{\rho_i}^{\varepsilon_i^{\delta}}, \tag{12} \]
where
\[ \text{Id}^{\varepsilon_i^{\beta}} \Pi^{\varepsilon_i^{\gamma}} \Phi_{\rho_i}^{\varepsilon_i^{\delta}} = \begin{cases} \text{Id}_{Y_i} & \text{if } (\varepsilon_i^{\beta}, \varepsilon_i^{\gamma}, \varepsilon_i^{\delta}) = (1,0,0), \\ \Pi_{Y_i} & \text{if } (\varepsilon_i^{\beta}, \varepsilon_i^{\gamma}, \varepsilon_i^{\delta}) = (0,1,0), \\ \Phi_{\rho_i} & \text{if } (\varepsilon_i^{\beta}, \varepsilon_i^{\gamma}, \varepsilon_i^{\delta}) = (0,0,1). \end{cases} \]

Both Eqs. (11) and (12) are some decompositions of the operator \( Q_1 \otimes \cdots \otimes Q_d \) and (11) is the ergodic decomposition. Decomposition (11) contains \( 2^d \) summands. Note that in decomposition (12), there are at least
2^d operators with nonzero coefficients corresponding to ergodic measures, namely,
\[
\bigotimes_{i=1}^{d} \beta_i^\varepsilon \cdot \gamma_i^\varepsilon \cdot \delta_i^\varepsilon \cdot \text{Id} \varepsilon^\rho \cdot \Phi^\varepsilon^\delta
\]
such that \( \varepsilon_i^\delta = 0 \). Each of the remaining operators is of the form
\[
\bigotimes_{i=1}^{d} \beta_i^\varepsilon \cdot \gamma_i^\varepsilon \cdot \delta_i^\varepsilon \cdot \text{Id} \varepsilon^\rho \cdot \Phi^\varepsilon^\delta,
\]
where for some \( 1 \leq i_0 \leq d \) we have \( \varepsilon_{i_0}^\delta = 1 \). Each of them is a convex combination of the 2^d operators corresponding to ergodic measures. However, this is impossible since all the measure in decomposition (12) are mutually singular, whence none of the operators is a convex combination of the remaining ones. Hence \( \delta_i = 0 \) for \( 1 \leq i \leq d \) and the proof is complete.

**Lemma 4.** Let \( a_1, \ldots, a_d, b_1, \ldots, b_d \in (0, 1) \). If the multisets (i.e., sets with elements of a multiplicity possibly greater than one)
\[
\{a_{i_1} \cdot \ldots \cdot a_{i_k} : i_1 < \ldots < i_k, \ 1 \leq k \leq d\}
\]
and
\[
\{b_{i_1} \cdot \ldots \cdot b_{i_k} : i_1 < \ldots < i_k, \ 1 \leq k \leq d\}
\]
are equal, then the multisets \( \{a_1, \ldots, a_d\} \) and \( \{b_1, \ldots, b_d\} \) are also equal.

**Proof.** We show how to determine \( a_1, \ldots, a_d \) knowing the multiset
\[
M = \{a_{i_1} \cdot \ldots \cdot a_{i_k} : i_1 < \ldots < i_k, \ 1 \leq k \leq d\}.
\]
Note that the largest number in \( M \) is equal to \( a_{j_1} \) for some \( 1 \leq j_1 \leq d \). Assume that we have found \( a_{j_1}, \ldots, a_{j_s} \) such that
\[
a_i \leq a_{j_s} \text{ for } i \notin \{j_1, \ldots, j_s\}.
\]
We show how to find \( a_{j_{s+1}} \) such that
\[
a_i \leq a_{j_{s+1}} \text{ for } i \notin \{j_1, \ldots, j_{s+1}\}.
\]
Consider the multiset
\[
M' = M \setminus \{a_{i_1} \cdot \ldots \cdot a_{i_k} : i_1 < \ldots < i_k, \ 1 \leq k \leq s, \ \{i_1, \ldots, i_k\} \subset \{j_1, \ldots, j_s\}\}.
\]
We claim that the largest number in \( M' \) is equal to \( a_{j_{s+1}} \) for some \( 1 \leq j_{s+1} \leq d \). Indeed, any other number in \( M' \) is a product of numbers between 0 and 1 with at least one factor being an element of the set \( \{a_i : i \notin \{j_1, \ldots, j_s\}\} \). By induction the proof is complete.

**Remark 4.** Note that if \( \Phi \) is a Markov operator on \( L^2(X, \mu) \) and \( t_n \to \infty \), then
\[
E_\Phi = \{ f \in L^2(X, \mu) : f \circ T^{t_n} \to \Phi f \ \text{weakly}\}.
\]
is a closed subspace. Indeed, if $f_k \to f$ in $L^2(X)$, $f_k \in E_\Phi$, then
\[
|\langle f \circ T^{t_n}, g \rangle - \langle \Phi f, g \rangle| \leq |\langle f \circ T^{t_n}, g \rangle - \langle f_k \circ T^{t_n}, g \rangle| + |\langle f \circ T^{t_n}, g \rangle - \langle f_k, g \rangle| + |\langle f_k, g \rangle - \langle \Phi f, g \rangle|.
\]
It follows that
\[
\{ \alpha \in (0, 1) : E_{(1-\alpha)1d+\alpha\Pi} \neq \{0\} \}
\]
is an isomorphism invariant.

**Lemma 5.** If When the assumptions of Proposition 7 hold, $S_i$ are $\alpha_{\pi(i)}$-weakly mixing for some permutation $\pi \in S(d)$.

**Proof.** We have
\[
L^2_0(X_1 \times \cdots \times X_d) = \bigoplus_{1 \leq k \leq d} \bigoplus_{1 \leq i_1 < \cdots < i_k \leq d} L^X_{i_1, \ldots, i_k}
\]
and
\[
L^2_0(Y_1 \times \cdots \times Y_d) = \bigoplus_{1 \leq k \leq d} \bigoplus_{1 \leq i_1 < \cdots < i_k \leq d} L^Y_{i_1, \ldots, i_k},
\]
where
\[
L^X_{i_1, \ldots, i_k} = \bigotimes_{j=1}^k L^2_0(X_{i_j}), \quad L^Y_{i_1, \ldots, i_k} = \bigotimes_{j=1}^k L^2_0(Y_{i_j}).
\]
Note that $(T_1 \times \cdots \times T_d)|_{L^X_{i_1, \ldots, i_k}}$ is $1 - (1 - \alpha_{i_1}) \cdots (1 - \alpha_{i_k})$-weakly mixing and $(S_1 \times \cdots \times S_d)|_{L^Y_{i_1, \ldots, i_k}}$ is $1 - (1 - \beta_{i_1}) \cdots (1 - \beta_{i_k})$-weakly mixing. Since $\Phi$ is an isomorphism, for every $f \in L^2_0(X_1 \times \cdots \times X_d)$ such that $(T_1^{t_n} \times \cdots \times T_d^{t_n})f \to \alpha \cdot f$ weakly, we have $(S_1^{t_n} \times \cdots \times S_d^{t_n})\Phi f \to \alpha \cdot \Phi f$ weakly. Hence the multisets
\[
\{(1 - \alpha_{i_1}) \cdots (1 - \alpha_{i_k}) : i_1 < \cdots < i_k, \ 1 \leq k \leq d\}
\]
and
\[
\{(1 - \beta_{i_1}) \cdots (1 - \beta_{i_k}) : i_1 < \cdots < i_k, \ 1 \leq k \leq d\}
\]
are equal and by Lemma 4, the proof is complete. \qed

**Proof of Proposition 7.** Without loss of generality, we may assume that $\alpha_1 \geq \cdots \geq \alpha_d$.

Let $i_1$ be the largest number among $1 \leq i \leq d$ such that $\alpha_1 = \cdots = \alpha_i$.

By the proof of Lemma 5, we have
\[
\Phi(L^2_0(X_i)) \subset L^2_0(Y_{\pi_1(i_1)}) \oplus \cdots \oplus L^2_0(Y_{\pi_1(i_1)})
\]
for $1 \leq i \leq i_1$, where $\pi_1 : \{1, \ldots, i_1\} \to \{1, \ldots, d\}$ is an injection such that $S_{\pi_1(i)}$ is $\alpha_1$-weakly mixing for $1 \leq i \leq i_1$. $L^2_0(Y_1 \times \cdots \times Y_d)$ is the largest subspace of $L^2_0(Y_1 \times \cdots \times Y_d)$ on which the flow $S_1 \times \cdots \times S_d$
is $\alpha_1$-weakly mixing). Note that (treating $L^2(Y_{\pi_1(1)} \times \cdots \times Y_{\pi_1(i_1)})$ as a subspace of $L^2(Y_1 \times \cdots \times Y_d)$)

$$\Phi \in J^e(\mathcal{T}_i, \mathcal{S}_{\pi_1(1)} \times \cdots \times \mathcal{S}_{\pi_1(i_1)}).$$

By Remark 3 we obtain that

$$\Phi(L_0^2(X_i)) \subset L_0^2(Y_{\pi(i)})$$

for a unique $\pi(i) \in \{\pi(1), \ldots, \pi(i_1)\}$. Since $\Phi$ is an isomorphism,

$$\Phi(L_0^2(X_i)) = L_0^2(Y_{\pi(i)})$$

for $1 \leq i \leq i_1$.

If $i_1 = d$, the proof is complete. If $i_1 < d$, let $i_2$ be the largest number among $i_1 < i \leq d$ such that $\alpha_{i_1+1} = \cdots = \alpha_i$. Arguing as in the proof of Lemma 5, we conclude that the only subspaces of

$$L_0^2(Y_1 \times \cdots \times Y_d) \ominus \Phi(L_0^2(X_1 \times \cdots \times X_{i_1}))),$$

which are of the form $L^Y_{i_1,...,i_k}$ for some $1 \leq i_1 < \cdots < i_k \leq d$ and on which the flow $\mathcal{S}_1 \times \cdots \times \mathcal{S}_d$ is $\alpha_{i_2}$-weakly mixing are spaces $L_0^2(Y_{\pi_2(i)})$ for $1 \leq \pi_2(i) \leq d$ such that the flow $\mathcal{S}_{\pi_2(i)}$ is $\alpha_{i_2}$-weakly mixing. As in the first part of the proof, we obtain

$$\Phi(L_0^2(X_i)) = L_0^2(Y_{\pi(i)})$$

for $i_1 + 1 \leq i \leq i_2$, where $\pi(i)$ is unique and such that $\mathcal{S}_{\pi(i)}$ is $\alpha_{i_2}$-weakly mixing.

In finitely many steps we obtain a permutation $\pi$ of $\{1, \ldots, d\}$ and we complete the proof. \hfill \Box

As a direct consequence of Proposition 7 we obtain the following corollary.

**Corollary 4.** Let $d \geq 1$ and let $\mathcal{T}_i$ be weakly mixing flows which are $\alpha_i$-weakly mixing for some $\alpha_i \in (0,1)$ for $1 \leq i \leq d$. In addition, assume that for $1 \leq i,j \leq d$ the isomorphism $\mathcal{T}_i \simeq \mathcal{T}_j$ implies that $\mathcal{T}_i = \mathcal{T}_j$. Then the centralizer $C(\mathcal{T}_1 \times \cdots \times \mathcal{T}_d)$ of $\mathcal{T}_1 \times \cdots \times \mathcal{T}_d$ consists of transformations of the form $\pi \circ \tilde{S}$, where

$$\tilde{S} \in C(\mathcal{T}_1) \times \cdots \times C(\mathcal{T}_d),$$

and the permutation $\pi \in S(d)$ acting on elements of $X_1 \times \cdots \times X_d$ by the rule

$$\pi(x_1, \ldots, x_d) = (x_{\pi(1)}, \ldots, x_{\pi(d)})$$

is such that $\pi(i) = j$ implies $\mathcal{T}_i = \mathcal{T}_j$.  

It is not clear whether it is possible to obtain a complete counterpart of Proposition 5 for $\alpha$-weakly mixing flows. We leave the following question open.

Does for $\alpha_i$-weakly mixing $T_i$ the isomorphism

$$T_1 \times T_2 \times \cdots \simeq S_1 \times S_2 \times \cdots$$

imply that each $S_i$ is a product of some $\alpha$-weakly mixing flows?

Using the methods which proved useful in the finite case, one can show, however, that some infinite version of Proposition 7 holds.

**Remark 5.** Note that $\{a_i \in (0,1) : i \in \mathbb{N}\}$ is well-ordered if and only if there are no infinite decreasing subsequences in $\{a_i : i \in \mathbb{N}\}$.

**Proposition 8.** Let $T_i$ be $\alpha_i$-weakly mixing and $S_i$ be $\beta_i$-weakly mixing along $t_n \to \infty$ for $i \geq 1$. Assume that $\{\alpha_i : i \in \mathbb{N}\}$ is well-ordered. If

$$\Phi : Y_1 \times Y_2 \times \cdots \to X_1 \times X_2 \times \cdots$$

determines an isomorphism of $S_1 \times S_2 \times \cdots$ and $T_1 \times T_2 \times \cdots$, then there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $\Phi$ determines an isomorphism of flows $S_i$ and $T_{\pi(i)}$ for all $i \geq 1$.

**Proof.** We will combine the arguments from the proofs of Lemmas 4 and 5 and Proposition 7. First, note that that as in (13) and (14) we have

$$L_0^2(X_1 \times X_2 \times \ldots) = \bigoplus_{k \geq 1} \bigoplus_{1 \leq i_1 < \ldots < i_k} L_{i_1,\ldots,i_k}^X$$

and

$$L_0^2(Y_1 \times Y_2 \times \ldots) = \bigoplus_{k \geq 1} \bigoplus_{1 \leq i_1 < \ldots < i_k} L_{i_1,\ldots,i_k}^Y,$$

where $(T_1 \times T_2 \times \ldots)|_{L_{i_1,\ldots,i_k}^X}$ is $1 - (1 - \alpha_{i_1}) \cdots (1 - \alpha_{i_k})$-weakly mixing and $(S_1 \times S_2 \times \ldots)|_{L_{i_1,\ldots,i_k}^Y}$ is $1 - (1 - \beta_{i_1}) \cdots (1 - \beta_{i_k})$-weakly mixing. Therefore, with each subspace $L_{i_1,\ldots,i_k}^X$ and $L_{i_1,\ldots,i_k}^Y$, we can associate numbers $(1 - \alpha_{i_1}) \cdots (1 - \alpha_{i_k})$ and $(1 - \beta_{i_1}) \cdots (1 - \beta_{i_k})$, respectively.

Let $i_1$ be such that $\alpha_{i_1} \leq \alpha_i$ for all $i \in \mathbb{N}$. Let

$$N_1 := \{i \in \mathbb{N} : \alpha_i = \alpha_{i_1}\}.$$

Note that for $a_i \in (0,1)$ for $1 \leq i \leq k$, we have

$$1 - (1 - a_1) \cdots (1 - a_k) > 1 - (1 - a_i) = a_i \quad \text{for} \quad 1 \leq i \leq k. \quad (17)$$

Therefore, the only $L_0^2$-subspaces of $L_0^2(Y_1 \times Y_2 \times \ldots)$ of the form $L_{i_1,\ldots,i_k}^Y$ on which we observe $\alpha_{i_1}$-weak mixing are $L_0^2(Y_j)$ such that $\beta_j = \alpha_{i_1}$ (on other spaces $L_{i_1,\ldots,i_k}^Y$ by (17) we observe $\alpha$-weak mixing for larger constants $\alpha$). By Remark 4 and Eqs. (15) and (16), the set

$$N'_1 := \{j \in \mathbb{N} : \beta_j = \alpha_{i_1}\}$$
is nonempty and for \( \beta_j \neq \alpha_{i_1} \) implies \( \beta_j > \alpha_{i_1} \) by the minimality of \( \alpha_{i_1} \). Hence

\[
\Phi(L_0^2(X_i)) \subset \bigoplus_{j \in \mathbb{N}_1'} L_0^2(Y_j) \quad \text{for } i \in \mathbb{N}_1.
\]

Using the arguments from the proof of [5, Theorem 2], we conclude that for all \( i \in \mathbb{N}_1 \), there exists \( \pi(i) \in \mathbb{N}_1' \) such that \( \Phi(L_0^2(X_i)) \subset L_0^2(Y_{\pi(i)}) \). Since \( \Phi \) is an isomorphism,

\[
\Phi(L_0^2(X_i)) = L_0^2(Y_{\pi(i)}) \quad \text{for } i \in \mathbb{N}_1
\]

and \( \pi : \mathbb{N}_1 \rightarrow \mathbb{N}_1' \) is bijective (we may reverse the roles of \( X_i \) and \( Y_i \) by considering \( \Phi^{-1} \) instead of \( \Phi \)).

If \( \mathbb{N} = \mathbb{N}_1 \), the proof is complete. Otherwise, let \( i_2 \in \mathbb{N} \) be such that \( \alpha_{i_2} \) is the smallest number in the set \( \{\alpha_i : i \notin \mathbb{N}_1\} \). Let

\[
\mathbb{N}_2 := \{i \in \mathbb{N} : \alpha_i = \alpha_{i_2}\}.
\]

Note that \( \Phi \) as an isomorphism that maps independent \( \sigma \)-algebras onto independent \( \sigma \)-algebras.

Moreover, the only subspaces of

\[
L_0^2(Y_1 \times Y_2 \times \ldots) \ominus \Phi(L_0^2(\times_{i \in \mathbb{N}_1} X_i))
\]

that are of the form \( L_0^2(Y_{i_1, \ldots, i_k}) \) on which \( S_1 \times S_2 \times \ldots \) is \( \alpha_{i_2} \)-weakly mixing are \( L_0^2(Y_j) \) for \( j \in \mathbb{N}_2' \), where

\[
\mathbb{N}_2' := \{j \in \mathbb{N} : \beta_j = \alpha_{i_2}\};
\]

it follows from (17) that on other subspaces

\[
L_0^2(Y_{i_1, \ldots, i_k}) \subset L_0^2(Y_1 \times Y_2 \times \ldots) \ominus \Phi(L_0^2(\times_{i \in \mathbb{N}_1} X_i))
\]

the flow \( S_1 \times S_2 \times \ldots \) is \( \alpha \)-weakly mixing for larger constants \( \alpha \). Therefore, in view of Remark 4, the set \( \mathbb{N}_2' \) is nonempty. Note that for \( j \notin \mathbb{N}_1' \cup \mathbb{N}_2' \), we have \( \beta_j > \alpha_{i_2} \) (otherwise, in view of Remark 4, \( \alpha_{i_2} \) would not be the smallest number in the set \( \{\alpha_i : i \notin \mathbb{N}_1\} \)). Hence

\[
\Phi(L_0^2(X_i)) \subset \bigoplus_{j \in \mathbb{N}_2'} L_0^2(Y_j) \quad \text{for } i \in \mathbb{N}_2.
\]

As in the first part of the proof, we obtain a bijection \( \pi : \mathbb{N}_2 \rightarrow \mathbb{N}_2' \) such that

\[
\Phi(L_0^2(X_i)) = L_0^2(Y_{\pi(i)}) \quad \text{for } i \in \mathbb{N}_2.
\]

We complete the proof using transfinite induction (recall that the set \( \{\alpha_i : i \in \mathbb{N}\} \) is well-ordered). \( \Box \)

The above proposition implies in particular (as in the finite case) that the centralizer of the infinite product \( \mathcal{T}_1 \times \mathcal{T}_2 \times \ldots \) is the product of the centralizers of \( \mathcal{T}_1, \mathcal{T}_2, \ldots \) up to a permutation of coordinates. More precisely, we have the following corollary.
Corollary 5. Let, for \( i \geq 1 \), \( T_i \) be \( \alpha_i \)-weakly mixing flows for some \( \alpha_i \in (0,1) \) such that \( \{ \alpha_i : i \in \mathbb{N} \} \) is well-ordered. In addition, assume that for \( i, j \in \mathbb{N} \), the isomorphism \( T_i \cong T_j \) implies \( T_i = T_j \). Then the centralizer \( C(T_1 \times T_2 \times \ldots) \) of the flow \( T_1 \times T_2 \times \ldots \) consists of transformations of the form \( \pi \circ \tilde{S} \), where

\[
\tilde{S} \in C(T_1) \times C(T_2) \times \ldots,
\]

and the permutation \( \pi : \mathbb{N} \to \mathbb{N} \) acting on elements of \( X_1 \times X_2 \times \ldots \) by the rule

\[
\pi(x_1, x_2, \ldots) = (x_{\pi(1)}, x_{\pi(2)}, \ldots)
\]

is such that \( \pi(i) = j \) implies \( T_i = T_j \).

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