CONVOLUTION ALGEBRAS AND ENVELOPING ALGEBRAS

BASED ON THE TALK BY JAN SCHRÖER

The talk is based on joint results with Christof Geiss and Bernard Leclerc. Throughout the talk K denotes a field.

1. Introduction

It is well-known that the finite dimensional simple complex Lie algebras correspond to the connected Cartan matrices. If C is a connected Cartan matrix, then we denote the corresponding Lie algebra by $\mathfrak{g}(C)$. Moreover, the connected Cartan matrices are in bijection with the Dynkin types. These are $\mathbb{A}_n$, $n \geq 1$, $\mathbb{D}_n$, $n \geq 4$, and $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, if C is symmetric, and $\mathbb{B}_n$, $n \geq 2$, $\mathbb{C}_n$, $n \geq 3$, $\mathbb{F}_4$, and $\mathbb{G}_2$, if C is not symmetric.

For a finite connected quiver Q without loops we define a symmetric generalized Cartan matrix $C_Q = (c_{ij})_{i,j \in Q_0}$ by the formulas: $c_{ii} := 2$, for $i \in Q_0$, and c_{ij} is the number of arrows between i and j (in both directions), for $i, j \in Q_0$ with $i \neq j$. Then we have the following.

Theorem 1 (Gabriel, 1970). *The path algebra KQ is representation finite if and only if C_Q is of (symmetric) Dynkin type. Moreover, if this is the case, then the dimension vectors of the indecomposable KQ-modules coincide with the positive roots of $\mathfrak{g}(C)$.*

Note that the finite dimensional path algebras of quivers form a subclass of the class of hereditary finite dimensional K-algebras. Using species and modulated graphs, Dlab and Ringel proved in 1971 an analogue of Theorem 1 covering all Dynkin types. However, there are restrictions of K in their theorem. For example, one cannot take as K the field of complex numbers, if C is non-symmetric.

If C a $n \times n$ connected Cartan matrix, then the algebra $\mathfrak{g}(C)$ decomposes $\mathfrak{g}(C) = \mathfrak{n}(C) \oplus \mathfrak{h}(C) \oplus \mathfrak{n}_-(C)$. If $\mathcal{U}(\mathfrak{n}(C))$ is the enveloping algebra, then

$$\mathcal{U}(\mathfrak{n}(C)) \simeq \mathbb{C}\langle e_1, \ldots, e_n \rangle / \langle (\operatorname{ad} e_i)^{1-c_{ij}}(e_j) : i \neq j \rangle,$$

where $(\operatorname{ad} x)(y) := [x, y] := xy - yx$. The above relations are called the Serre relations. We have the following.

Theorem 2 (Ringel, 1990). *If Q is a Dynkin quiver and $C := C_Q$, then the enveloping algebra $\mathcal{U}(\mathfrak{n}(C))$ is isomorphic to the convolution algebra $\mathcal{M}(\mathbb{C}Q, \{1_{S_1}, \ldots, 1_{S_n}\})$.*

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We have the following generalization.

Theorem 3 (Geiss/Leclerc/Schröer, 2015). *If C is a connected Cartan matrix, then the enveloping algebra $\mathcal{U}(\mathfrak{n}(C))$ is isomorphic to the convolution algebra $\mathcal{M}(\mathbb{C}Q/I, \{1_{E_1}, \dots, 1_{E_n}\})$ for some bound quiver (Q, I).*

We note that if C is non-symmetric, then Q has loops and $I \neq 0$.

2. Convolution algebras

Let Q be a finite connected quiver and I an admissible ideal in $\mathbb{C}Q$. For each vertex i of Q we denote by S_i the corresponding (1-dimensional) simple A-module, where $A := \mathbb{C}Q/I$. For a dimension vector $\mathbf{d}$ we denote by $\operatorname{rep}(A, \mathbf{d})$ the variety of A-modules with dimension vector $\mathbf{d}$. The group $G_{\mathbf{d}} := \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\mathbb{C})$ acts on $\operatorname{rep}(A, \mathbf{d})$ in such a way, that the orbits correspond to the isomorphism classes of A-modules with dimension vector $\mathbf{d}$.

A map $f\colon \operatorname{rep}(A, \mathbf{d}) \to \mathbb{C}$ is called constructible if $\operatorname{Im} f$ is finite, $f^{-1}(m)$ is constructible, for each $m \in \mathbb{C}$, and f is constant on $G_{\mathbf{d}}$-orbits. By $\mathcal{F}(A)_{\mathbf{d}}$ we denote the set of constructible maps $\operatorname{rep}(A, \mathbf{d}) \to \mathbb{C}$. This is a $\mathbb{C}$-vector space with natural addition. We put $\mathcal{F}(A) := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathcal{F}(A)_{\mathbf{d}}$. Examples of constructible functions are $1_{\mathbf{d}}$, for a dimension vector $\mathbf{d}$, and 1_M, for a representation M, where

$$1_{\mathbf{d}}(N) := \begin{cases} 1 & \text{if } \mathbf{dim}\, N = \mathbf{d}, \\ 0 & \text{else,} \end{cases} \qquad \text{and} \qquad 1_M(N) := \begin{cases} 1 & \text{if } N \simeq M, \\ 0 & \text{else.} \end{cases}$$

We define the convolution product in $\mathcal{F}(A)$ by the formula:

$$(f * g)(M) := \sum_{m \in \mathbb{C}} m \chi(\{U \subseteq M : f(U) g(M/U) = m\}).$$

The following proposition should be attributed to many authors, including Ringel/Schofield (1990) and Joyce (2007).

Proposition 4. *The convolution product $*$ gives $\mathcal{F}(A)$ a structure of $\mathbb{N}^{Q_0}$-graded $\mathbb{C}$-algebra, called the convolution algebras.*

If we fix a subset $\mathcal{S}_{\mathbf{d}} \subseteq \mathcal{F}(A)_{\mathbf{d}}$ for each dimension vector $\mathbf{d}$, then we denote by $\mathcal{M}(A, \mathcal{S})$ the subalgebra of $\mathcal{F}(A)$ generated by the elements which belong to $\mathcal{S} := \bigcup_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathcal{S}_{\mathbf{d}}$. We present some examples of this construction.

Theorem 5 (Schofield, 1990). *If Q is a quiver without oriented cycles, then*

$$\mathcal{U}(\mathfrak{n}(C_Q)) \simeq \mathcal{M}(A, \{1_{S_1}, \dots, 1_{S_n}\}),$$

where $\mathfrak{n}(C_Q)$ is the positive part of the Kac–Moody Lie algebra $\mathfrak{g}(C_Q)$ associated with C_Q and $A := \mathbb{C}Q$.

Similarly, we have the following.

Theorem 6 (Lusztig). *If Q is a quiver without oriented cycles, then*

$$\mathcal{U}(\mathfrak{n}(C_Q)) \simeq \mathcal{M}(A, \{1_{S_1}, \dots, 1_{S_n}\}),$$

where $A := \Pi(Q)$ is the preprojective algebra associated with Q.

Let $z\colon K_0(A) \to \mathbb{C}$ be a stability condition for an algebra A. For a dimension vector $\mathbf{d}$, we define $1_{z,\mathbf{d}} \in \mathcal{F}(A)$ by

$$1_{z,\mathbf{d}}(N) := \begin{cases} 1 & \text{if } N \text{ is semistable and } \mathbf{dim}\, M = \mathbf{d}, \\ 0 & \text{else.} \end{cases}$$

Theorem 7 (Reineke, 2003). *For each stability condition z we have*

$$\mathcal{M}(A, \{1_{\mathbf{d}} : \mathbf{d} \in \mathbb{N}^{Q_0}\}) = \mathcal{M}(A, \{1_{z,\mathbf{d}} : \mathbf{d} \in \mathbb{N}^{Q_0}\}).$$

We discuss now a question, where $\mathcal{M}(A, \mathcal{S})$ is isomorphic to the enveloping algebra $\mathcal{U}(L)$ for some Lie algebra L. We define a map

$$i\colon \mathcal{F}(A) \otimes_{\mathbb{C}} \mathcal{F}(A) \to \mathcal{F}(A \times A)$$

by $i(f \otimes g)(X, Y) := f(X)g(Y)$.

Lemma 1. *i is an injective algebra homomorphism.*

Next we define $c\colon \mathcal{F}(A) \to \mathcal{F}(A \times A)$ by $(c(f))(X, Y) := f(X \oplus Y)$.

Lemma 2. *c is an algebra homomorphism.*

Now let $\mathcal{M} := \mathcal{M}(A, \mathcal{S})$ for a set $\mathcal{S}$. We say that the Hopf condition is satisfied for $\mathcal{M}$ if $c(\mathcal{M}) \subseteq i(\mathcal{M} \otimes \mathcal{M})$. One verfies that the Hopf conditions is satisfied in the following cases:

(1) $A = \mathbb{C}Q$, for a quiver Q without oriented cycles, and $\mathcal{S} = \{1_{S_i} : i \in Q_0\}$;
(2) $A = \Pi(Q)$, for a quiver Q without oriented cycles, and $\mathcal{S} = \{1_{S_i} : i \in Q_0\}$;
(3) $\mathcal{S} = \{1_{\mathbf{d}} : \mathbf{d} \in \mathbb{N}^{Q_0}\}$;
(4) $\mathcal{S} = \{1_{\mathcal{P},\mathbf{d}} : \mathbf{d} \in \mathbb{N}^{Q_0}\}$, where

$$1_{\mathcal{P},\mathbf{d}}(N) := \begin{cases} 1 & \text{if } \operatorname{pdim} N < \infty \text{ and } \mathbf{dim}\, N = \mathbf{d}, \\ 0 & \text{else.} \end{cases}$$

Note that the first three cases correspond to Theorems 5–7.

The following theorem follows from a work of Ringel/Schofield (1990) and general Hopf theory. Recall that if $\mathcal{H}$ is a Hopf algebra with comultiplication c, then by $\mathcal{P}(\mathcal{H})$ we denote the set of primitive elements in $\mathcal{H}$, which consists of all $h \in \mathcal{H}$ such that $c(h) = h \otimes 1 + 1 \otimes h$. Then $\mathcal{P}(\mathcal{H})$ is a Lie algebra.

Theorem 8. *Assume that the Hopf condition is satisfied. Then*

(1) *$\mathcal{M}$ is a cocommutative Hopf algebra with comultiplication induced by c, and*
(2) *$\mathcal{M} \simeq \mathcal{U}(\mathcal{P}(\mathcal{M}))$.*

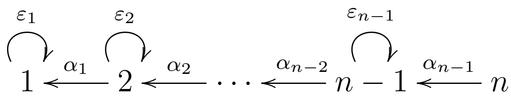
Lemma 3. *An element $f \in \mathcal{M}(A, \mathcal{S})$ is primitive if and only if the only A-modules M such that $f(M) \neq 0$ are indecomposable ones.*

An element g of a Hopf algebra $\mathcal{H}$ with comultiplication c is called grouplike if $c(g) = g \otimes g$.

Lemma 4. *The identify of $\mathcal{M}$ is the only grouplike element of $\mathcal{M}$.*

3. About Theorem 3

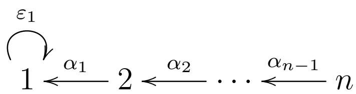
We describe now the objects appearing in Theorem 3. For each non-symmetric Dynkin type we define a bound quiver (Q, I). More precisely, for type $\mathbb{B}_n$, the quiver has the form



and the relations are

$$\varepsilon_i^2 = 0,\ i = 1, \ldots, n-1, \qquad \varepsilon_i \alpha_i = \alpha_i \varepsilon_{i+1},\ i = 1, \ldots, n-2.$$

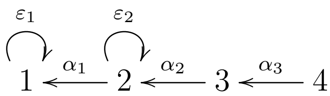
Next, for type $\mathbb{C}_n$, the quiver has the form



and the relation is

$$\varepsilon_1^2 = 0.$$

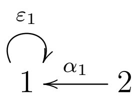
For type $\mathbb{F}_4$, the quiver has the form



and the relations are

$$\varepsilon_1^2 = 0,\ \varepsilon_2^2 = 0,\ \varepsilon_1 \alpha_1 = \alpha_1 \varepsilon_2.$$

Finally, for type $\mathbb{G}_2$, the quiver has the form



and the relation is

$$\varepsilon_1^3 = 0.$$

Finally, for a vertex i of Q, E_i denote the indecomposable A-module supported at i of maximal dimension, where $A := KQ/I$. In addition to Theorem 3, one also shows

$$\mathcal{M}(A, \{1_{E_i} : i \in Q_0\}) \simeq \mathcal{M}(A, \{1_{\mathcal{P},\mathbf{d}} : \mathbf{d} \in \mathbb{N}^{Q_0}\}).$$