MATRIX IDENTITIES WITH FORMS

BASED ON THE TALK BY ARTEM LOPATIN

Throughout the talk we fix an infinite field $\mathbb{F}$ and positive integers d and n. Next, we denote by V the product of d copies of the space of the $n \times n$ matrices. The group $G := \mathrm{GL}(n)$ acts on V by conjugation and this action induces the action of G on the ring R of the polynomial functions on V. We are interested in the ring R^G of the G-invariants.

For each $k \in \{1, 2, \ldots, d\}$ we denote by X_k the k-th generic matrix, i.e., $X_k \in \mathbb{M}_n(R)$ and

$$X_k(M_1, M_2, \ldots, M_d) = M_k$$

for any $(M_1, M_2, \ldots, M_d) \in V$. Let X be the free monoid generated by the letters $x_1, x_2, \ldots, x_d$ and $\mathbb{F}X$ the vector space with basis X. Let $\psi : \mathbb{F}X \to \mathbb{M}_n(R)$ be the k-linear extension of the homomorphism $X \to \mathbb{M}_n(R)$ defined by the condition $\psi(x_k) = X_k$ for each $k \in \{1, 2, \ldots, d\}$. If we define the functions $\sigma_1, \ldots, \sigma_n : \mathbb{M}_n(R) \to R$ by the condition

$$T^n - \sigma_1(A) \cdot T^{n-1} + \ldots + (-1)^n \cdot \sigma_n(A) = \det(T \cdot \mathrm{Id}_n - A)$$

for each $A \in \mathbb{M}_n(R)$, then $R_G(T)$ is generated by the elements $\sigma_t(\psi(a))$, where $t \in \{1, 2, \ldots, d\}$ and $a \in \mathbb{F}X$. In other words, if we denote by $\sigma(X)$ the polynomial algebra generated by the elements $\sigma_{t,a}$, $t \in \mathbb{N}_+$ and $a \in \mathbb{F}X$, then we have the epimorphism $\Psi : \sigma(X) \to R^G$ given by the condition

$$\Psi(\sigma_{t,a}) = \sigma_t(\psi(a))$$

for any $t \in \mathbb{N}_+$ and $a \in \mathbb{F}X$, where we put $\sigma_t := 0$ for $t > n$. Our next aim will be to understand the kernel K of this map.

The first class of relations comes from a formula due to Amitsur, which we describe now. Let Λ be the free monoid generated by the letters x and y. By a cycle which mean every element c of Λ of positive length such that there is no element c_0 of Λ and a positive integer m such that $c = c_0^m$. We call two cycles equivalent if one is obtained form the other by a rotation. We denote by $\mathcal{C}$ a set of representatives of the equivalence classes of the cycles. If $A, B \in \mathbb{M}_n(R)$, then by $\phi_{A,B}$ we denote the homomorphism $\Lambda \to \mathbb{M}_n(R)$ induced by the conditions

$$\psi_{A,B}(x) := A \qquad \text{and} \qquad \psi_{A,B}(y) := B.$$

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Then

$$\sigma_t(A+B) = \sum_{s\geq 1} \sum_{\substack{c_1,c_2,\ldots,c_s\in\mathcal{C}\\ \text{pairwise different}}} \sum_{\substack{k_1,k_2,\ldots,k_s\in\{1,2,\ldots,n\}\\ k_1\cdot\ell(c_1)+k_2\cdot\ell(c_2)+\ldots+k_s\cdot\ell(c_s)=t}}$$

$$(-1)^{t-(k_1+k_2+\ldots+k_s)}\cdot\sigma_{k_1}(\psi_{A,B}(c_1))\cdot\sigma_{k_2}(\psi_{A,B}(c_2))\cdot\ldots\cdot\sigma_{k_s}(\psi_{A,B}(c_s))$$

for any $t\in\mathbb{N}_+$ and $A,B\in\mathbb{M}_n(R)$. For example,

$$\sigma_1(A+B) = \sigma_1(A)+\sigma_1(B),$$

$$\sigma_2(A+B) = \sigma_2(A)+\sigma_2(B)-\sigma_1(A\cdot B)+\sigma_1(A)\cdot\sigma_1(B)$$

and

$$\sigma_3(A+B) = \sigma_3(A)+\sigma_3(B)+\sigma_1(A^2\cdot B)+\sigma_1(A\cdot B^2)+\sigma_2(A)\cdot\sigma_1(B) + \sigma_1(A)\cdot\sigma_2(B)-\sigma_1(A)\cdot\sigma_1(A\cdot B)-\sigma_1(A\cdot B)\sigma_1(B).$$

Inspired by the above formula, we define the element $F_t(a,b)$ of $\sigma(X)$ for $t\in\mathbb{N}_+$ and $a,b\in\mathbb{F}X$ by the formula

$$F_{t,a,b} := \sum_{s\geq 1} \sum_{\substack{c_1,c_2,\ldots,c_s\in\mathcal{C}\\ \text{pairwise different}}} \sum_{\substack{k_1,k_2,\ldots,k_s\in\{1,2,\ldots,n\}\\ k_1\cdot\ell(c_1)+k_2\cdot\ell(c_2)+\ldots+k_s\cdot\ell(c_s)=t}}$$

$$(-1)^{t-(k_1+k_2+\ldots+k_s)}\cdot\sigma_{k_1,\varphi_{a,b}(c_1)}\cdot\sigma_{k_2,\varphi_{a,b}(c_2)}\cdot\ldots\cdot\sigma_{k_s,\varphi_{a,b}(c_s)},$$

where $\varphi_{a,b}:\Lambda\to\mathbb{F}X$ is the homomorphism induced by the conditions

$$\varphi_{a,b}(x) := a \qquad \text{and} \qquad \varphi_{a,b}(y) := b.$$

Next, observe that for each $t\in\mathbb{N}_+$ and $l>0$, there exists a polynomial $P_{t,l}\in\mathbb{F}[x_1,\ldots,x_n]$ such that

$$\sigma_t(A^l) = P_{t,l}(\sigma_1(A),\sigma_2(A),\ldots,\sigma_n(A)).$$

Finally, let $\sigma'(X)$ be the quotient of $\sigma(X)$ by the ideal generated by the elements:

(1) $\sigma_{t,a\cdot b}-\sigma_{t,b\cdot a}$ for $a,b\in X$ and $t\in\mathbb{N}_+$.
(2) $\sigma_{t,\alpha\cdot a} = \alpha^t\cdot\sigma_{t,a}$ for $\alpha\in\mathbb{F}$, $a\in X$ and $t\in\mathbb{N}_+$.
(3) $\sigma_{t,a+b} = F_{t,a,b}$ for $a,b\in\mathbb{F}X$ and $t\in\mathbb{N}_+$.
(4) $\sigma_t(a^l) = P_{t,l}(\sigma_{1,a},\sigma_{2,a},\ldots,\sigma_{t,a})$.

Let $\pi:\sigma(X)\to\sigma'(X)$ be the canonical injection. If $t\in\mathbb{N}_+$ and $\mathbf{a} = (a_1,a_2,\ldots,a_r)\in X^r$, then there exist $\sigma'_{\mathbf{t},\mathbf{a}}\in\sigma'(X)$, $\mathbf{t}=(t_1,t_2,\ldots,t_r)\in\mathbb{N}^r$, $|\mathbf{t}| := t_1+t_2+\ldots+t_r = t$, such that

$$\pi(\sigma_{t,\lambda_1\cdot a_1+\lambda_2\cdot a_2+\ldots+\lambda_r\cdot a_r}) = \sum_{\mathbf{t}\in\mathbb{N}:|\mathbf{t}|=t}\lambda_1^{t_1}\cdot\lambda_2^{t_2}\cdot\ldots\cdot\lambda_r^{t_r}\cdot\sigma'_{\mathbf{t},\mathbf{a}}$$

for all $\lambda_1,\lambda_2,\ldots,\lambda_r\in\mathbb{F}$. We denote by $\sigma_{\mathbf{t},\mathbf{a}}$ a fixed inverse image of $\sigma'_{\mathbf{t},\mathbf{a}}$.

The main theorem says that K is generated by the following elements:

(0) $\sigma_{t,a}$ for $t>n$ and $a\in\mathbb{F}X$.

(1) $\sigma_{t,a\cdot b} - \sigma_{t,b\cdot a}$ for $a, b \in X$ and $t \in \{1, 2, \ldots, n\}$.
(2) $\sigma_{t,\alpha\cdot a} = \alpha^t \cdot \sigma_{t,a}$ for $\alpha \in \mathbb{F}$, $a \in X$ and $t \in \{1, 2, \ldots, n\}$.
(3) $\sigma_{t,a+b} = F_{t,a,b}$ for $a, b \in \mathbb{F}X$ and $t \in \{1, 2, \ldots, n\}$.
(4) $\sigma_t(a^l) = P_{t,l}(\sigma_{1,a}, \sigma_{2,a}, \ldots, \sigma_{t,a})$.
(5) $\sigma_{\mathbf{t},\mathbf{a}}$ for $|\mathbf{t}| \in \mathbb{N}^r$ and $\mathbf{a} \in X^r$, $r \in \mathbb{N}_+$, such that $|\mathbf{t}| \in \{n+1, n+2, \ldots, 2 \cdot n\}$.