THE INTEGRAL CLUSTER CATEGORY

BASED ON THE TALK BY SARAH SCHEROTZKE

For an autoequivalence F of an additive category $\mathcal{A}$ we define the orbit category $\mathcal{A}/F$ in the following way. The objects of $\mathcal{A}/F$ coincide with the objects of $\mathcal{A}$, however if X and Y are objects of $\mathcal{A}$, then

$$\mathrm{Hom}_{\mathcal{A}/F}(X,Y) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X, F^n Y).$$

The canonical functor $\pi : \mathcal{A} \to \mathcal{A}/F$ is universal among the functors $G : \mathcal{A} \to \mathcal{B}$ such that $G \circ F \simeq G$.

The following example show that $\mathcal{A}/F$ does not have to be a triangulated category even if $\mathcal{A}$ is a triangulated category and F is an exact functor. Namely, let $\mathcal{A}$ be the bounded derived category of the modules over the algebra $k[x]/x^2$ and $F := [2]$. If μ is the following map

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A & \xrightarrow{\cdot x} & A & \xrightarrow{\cdot x} & A & \xrightarrow{\cdot x} & A & \longrightarrow & 0 & \longrightarrow & \cdots, \\
 & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A & \xrightarrow{\cdot x} & A & \xrightarrow{\cdot x} & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

then $\mathrm{End}_{\mathcal{A}/F}(k) = k[\mu]$. On the other hand, μ is an monomorphism, i.e. if $\mu \circ f = 0$, then $f = 0$. However, in triangulated categories monomorphisms split, which leads to a contradiction.

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